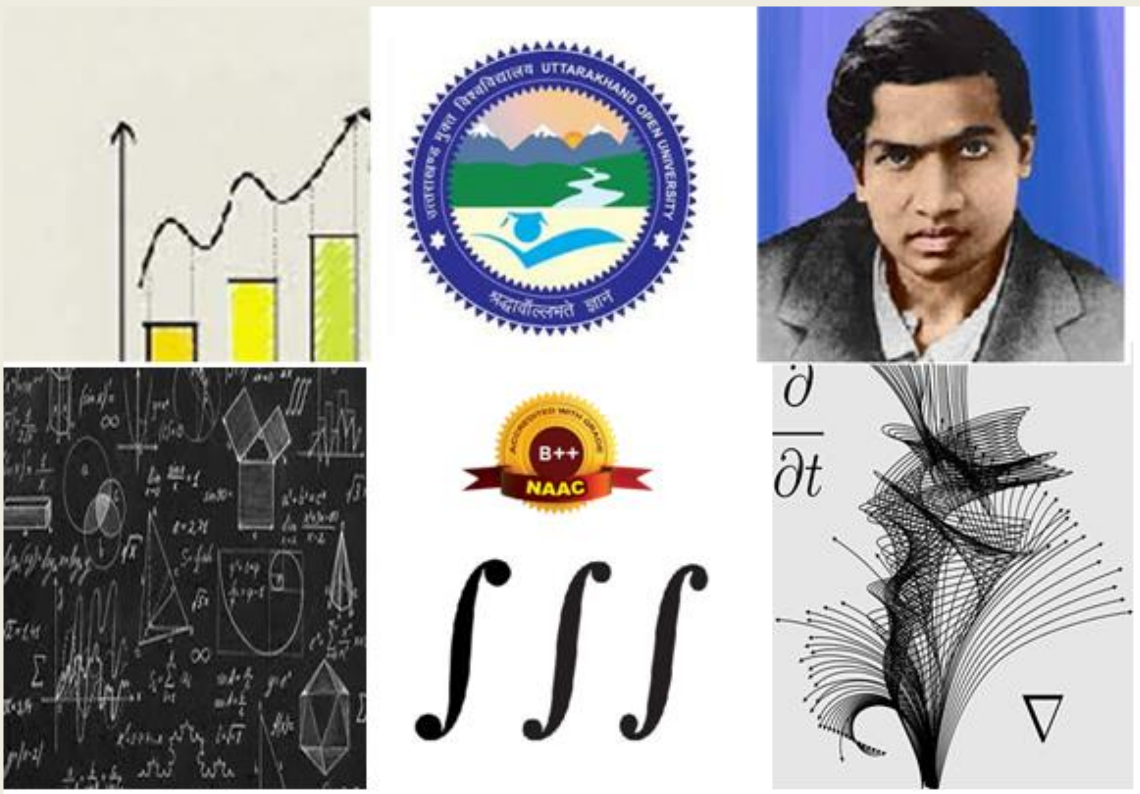
# Master of Science

## Mathematics

## Fourth Semester

# MAT 610
# MATHEMATICAL MODELLING




**DEPARTMENT OF MATHEMATICS**
**SCHOOL OF SCIENCES**
**UTTARAKHAND OPEN UNIVERSITY**
**HALDWANI, UTTARAKHAND**
**263139**

# COURSE NAME: MATHEMATICAL MODELLING

# COURSE CODE: MAT 610

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# MAT - 610

# COURSE INFORMATION

The present self-learning material **"MATHEMATICAL MODELLING"** has been designed for **M.Sc. (Fourth Semester)** learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a Mixture of Four Block.

First block is **Mathematical Modelling - I,** in this block Introduction of Mathematical Modelling, Linear – Non-Linear Growth –Decay Model, Mathematical Modelling in Population Dynamics & Mathematical Modelling of Epidemics defined Clearly.

Second block is **Mathematical Modelling - II,** in this block Mathematical Modelling Through Difference Equations, Linear Models, Non-Linear Models & Continuous Models Using Ordinary Differential Equations defined clearly.

Third block is **Mathematical Modelling - III,** in this block Spatial Models Using Partial Differential Equations, Modeling with Delay Differential Equations, Modeling with Stochastic Differential Equations & Mathematical Modelling through Graph are defined.

Fourth block is **Mathematical Modelling - IV,** in this block Mathematical Modelling through Calculus of Variation & Mathematical Modelling through Dynamic Programming are defined.

Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

# COURSE NAME: MATHEMATICAL MODELLING

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## BLOCK-I

## MATHEMATICAL MODELLING - I

# UNIT 1: INTRODUCTION OF MATHEMATICAL MODELLING

## CONTENTS:



## *1.1 INTRODUCTION*

A mathematical model is an abstract description of a particular system using mathematical and linguistic concepts. The process of designing a mathematical model is called, Mathematical modelling. In this unit our main focus on explain the need, techniques, classifications and simple illustrations of Mathematical modelling. Simply in other words in mathematical modelling, we select a real-world problem and formulate a equivalent mathematical problem. We then solve the mathematical problem, and explain its solution in terms of the real-world problem. After this we feel to what extent the solution is valid in the sense of the real-world problem. So, the *stages* involved in mathematical modelling are formulation, solution, interpretation and validation So if we will just go for this importance of mathematical modeling, why we do need mathematics to represent our system is it going to help us out? Well, the

| | |
|---|---|
| **Alexander Wilhelm von Brill**<br>**Born:**<br>20,September 1842, Darmstadt, Germany<br>**Died:**<br>8 June 1935<br>Tübingen, Germany<br>**Summary:**<br>**Alexander von Brill** was a German mathematician who worked in algebraic geometry. He also made mathematical models and was interested in the history of mathematics. | <br>***Fig 1.1***<br>Ref:<br>https://mathshistory.st-andrews.ac.uk/Biographies/Brill/ |

answer is yes, since the mathematical model plays a vital role in mathematics and other subjects the system can be divided into two types, one you can just say it is abstract model, another one it is like a real experimental model, so in the abstract model we can just simulate or we can just try to find this solution in two forms, either we can have a like physical model, another one we can just say it is a mathematical model, and if we will have a mathematical model then we can just find the solution of the system in two forms, so that is either analytical solution or a simulated result. If we will have a simulated result then we can just compare this simulated result with the actual system whether the system is preserving the actual behavior or nature of the system or not. So especially if we will just do the simulation on models rather experiments on actual system. So if the models can be constructed so then we can just try to understand the actual or the physical behavior of the system in a concrete sense, so sometimes it is necessary to find the economic way for the measurements or control on parameters or allow us to predict that future nature which have not seen so far.

Suppose if we will see a population level, so the total population size it depends on like the birth rate and death rate at that time level, so if learner just construct like the mathematical model based on this population birth rate or death rate which is proportional to the total population size, so we have to consider the past data to verify the model, and based on that past data we can just to say that how the future model will be, so this can be predictor or it can be made through mathematical modeling only, so this is our last point that allow us to predict future nature which have not been seen so far.

## *1.2 OBJECTIVES*

After studying this unit, learner will be able to

**i.** Described the concept of simple situations requiring mathematical modelling.
**ii.** Explained the technique of mathematical modelling
**iii.** Define the classification of mathematical models.
**iv.** Analyze the mathematical modelling through geometry, algebra, trigonometry and calculus.
**v.** Justify the limitations of mathematical modelling.

## *1.3 WHY MATHEMATICAL MODELLING IS NEEDED*

**Model :** The dictionary meaning of model is to represent something. Thus modelling is a process or way of representation. Thus Mathematical modelling is a process of representation of system with the help of mathematical formulas and model is the structure obtained.

A model is an object or concept that is used to represent something else. It is reality scaled down and converted to a form we can comprehend. For example, a model airplane, made of wood, plastic, and glue, is a model of a real airplane. Another example is the idea that, in politics, public opinion is like a pendulum because it changes periodically from left- to right-wing ideas then back again in a way which reminds us of a pendulum swinging back and forth. In our terminology we would say that a pendulum is a model for public opinion. A model airplane and pendulum are physical objects; so they are not mathematical models. A mathematical model is a model whose parts are mathematical concepts, such as constants, variables, functions, equations, inequalities, etc.

## *1.3.1 SIMPLE SITUATIONS REQUIRING MATHEMATICAL MODELLING*

Suppose we have a problem that without climbing find the height of a tower. For solving it we try to express the height of the river in terms of some distances and angles which can be measured on the ground.

For solve a given physical, biological or social problem, we first develop a mathematical model for it, then solve the model and finally interpret the solution in terms of the original problem. Whenever we want to find the value of an entity which cannot be measured directly, we introduce symbols $x, y, z, \dots\dots..$ to represent the entity and some others which vary with it, then we appeal to laws of physics, chemistry, biology or economics and use whatever information is available to us to get relations between these variables, some of which can be measured or are known and others which cannot be directly measured and have to be found out.

We use the mathematical relations developed to solve for the substance which cannot be measured directly in terms of those substance whose values can be measured or are known.

The mathematical relations we get may be in terms of algebraic, transcendental, differential, difference, integral, integro-differential, differential-difference equations or even in terms of in equalities.

**Simple** Ram travelled 432 kilometers on 48 litres of petrol in his car. Ram have to go by his car to a place which is 180 km away. How much petrol do Ram need?

**Step 1 :**

Petrol needed for travelling 432 km = 48 litres

Petrol needed for travelling 180 km = ?

Mathematical Description : Let

x = distance Ram travel

y = petrol Ram need

y varies directly with x.

So, y = kx, where k is a constant.

Ram can travel 432 kilometers with 48 litres of petrol.

So, y = 48, x = 432.

Therefore $k = \frac{y}{x} = \frac{48}{432} = \frac{1}{9}$

Since y = kx, therefore, $y = \frac{1}{9}x$ ………………**(1)**

Equation or Formula (1) describes the relationship between the petrol needed and distance travelled.

**Step 2: Solution :** Ram want to find the petrol Ram need to travel 180 kilometers; so, Ram have to find the value of y when x = 180. Putting x = 180 in (1), Ram have

$$y = \frac{180}{9} = 20$$

**Step 3: Interpretation :** Since y = 20, Ram need 20 litres of petrol to travel 180 kilometers. Did it occur to Ram that Ram may not be able to use the formula (1) in all situations? For example, suppose the 432 kilometers route is through mountains and the 180 kilometers route is through flat plains. The car will use up petrol at a faster rate in the first route, so we cannot use the same rate for the 180 kilometers route, where the petrol will be used up at a slower rate. So the formula works if all such conditions that affect the rate at which petrol is used are the same in both the trips. Or, if there is a difference in conditions, the effect of the difference on the amount of petrol needed for the car should be very small. The petrol used will vary directly with the distance travelled only in such a situation.

## *1.3.2 THE TECHNIQUE OF MATHEMATICAL MODELLING*

The technique in mathematical modelling is first we select a real-world problem and analyze about the problem and formulate a equivalent mathematical problem. We then solve the mathematical problem, and do interpretation of mathematical problem in simple words. After this we check the validation of interpretation, if it is correct and we feel that the mathematical problem is possible. Then our mathematical model exists. If validation of interpretation, is not correct and we feel that the mathematical problem is not possible then again create mathematical formulation (Remove the fault) and apply the same procedure.

In other words, we can divide the modelling process into three main steps: formulation, finding solution and interpretation and evaluation.

1. **Formulation:** Formulation can, in turn, be divided into three steps

**i. Stating the question:**

Understanding natural phenomena requires explaining them. A clear explanation can answer questions like:

How long will it take? How fast? How loud? And so on. But the questions we start with shouldn't be vague or too difficult. F or real world problems, this should be done by explaining the problem definition and then putting the problem in context.

## LAY OUT OF MATHEMATICAL MODELLING

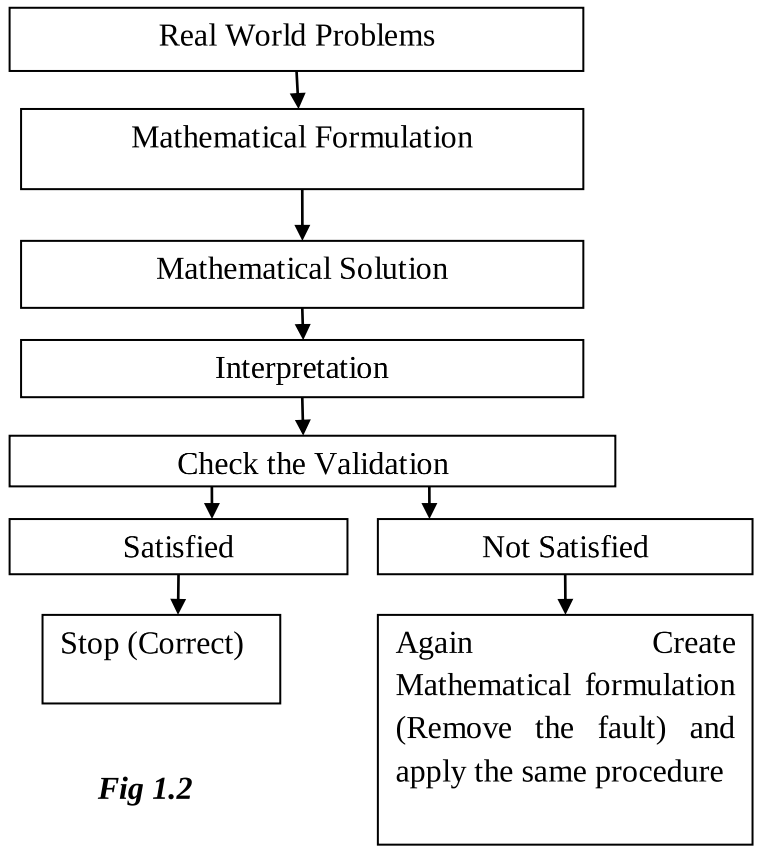


***Fig 1.2***

ii. **Identifying relevant factors:** Decide which quantities and relationships are important for your question and which are unimportant for your question and can be neglected. The unimportant quantities are those that have very little or no effect on the process. For example, in studying the motion of a falling body, its colour is usually of little interest.

iii. **Mathematical description:** Each important quantity should be represented by a suitable mathematical entity e.g. a variable, a function, a geometric figure etc. Each relationship should be represented by an equation, inequality, or other suitable mathematical assumption.

**iv.** **Finding the Solution:** The mathematical formulation rarely gives us answers directly. We usually have .to do some operations. This may involve a calculation, solving an equation, proving a theorem etc.

**v.** **Evaluation:** Since a model is a simplified representation of a real problem, by its very nature, has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is a good one or not i.e., when the obtained results are interpreted physically whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new Modelling evaluation.

The following twelve – point procedure for solving problems through mathematical modelling:

**i.** Be clear about the real world situation to be investigated.

**ii.** Think about all the physical, chemical, biological, social, economic laws that may be relevant to the situation.

**iii.** Formulate the problem in program language (PL).

**iv.** Think about all the variables $x_1, x_2 \dots \dots \dots x_n$ and $a_1, a_2 \dots \dots \dots a_n$ involved. Classify these into known and unknown ones.

**v.** Think of the most appropriate mathematical model and translate the problem suitably into mathematical language (ML) in the form

$$f_i\left(x_i, a_h, \frac{\partial}{\partial x_i}, \int \dots \dots dx_i, d\right) \leq 0 \quad \text{………….(2)}$$

$i.e.$, in terms of algebraic, transcendental, differential, difference, integral, integro-differential, differential-difference equations or inequations.

**vi.** Think of all possible ways of solving the equations of the model.

**vii.** If a reasonable change in the assumptions makes analytical solution possible, investigate the possibility.

**viii.** Make an error analysis of the method used. If the error is not within acceptable limits, change the method of solution.

**ix.** Translate the final solution into P.L.

**x.** If the agreement is not good, examine the assumptions and approximations and change them in the light of the discrepancies observed and proceed as before.

**xi.** Continue the process till a satisfactory model is obtained which explains all earlier data and observations.

**xii.** Deduce conclusions from your model and test these conclusions against earlier data and additional data that may be collected and see if the agreement still continues to be good.

We can use geometry, algebra, trigonometry and calculus for mathematical modelling.

## *1.3.3 CLSSIFICATION OF MATHEMATICAL MODELS*

**a.** May be classified according to the subject matter of the models. Thus we have mathematical method in Physics, mathematical method in Chemistry, mathematical method in Biology, mathematical method in Medicine, mathematical method in Economics, mathematical method in Sociology, mathematical method in Engineering and so on. For mathematical modelling the two aspects are essential first theoretical, mathematical, statistical and computer – based and other of which is empirical, experimental and observational.

**b.** May also classify Mathematical models according to the mathematical techniques used in solving them. Thus we have mathematical modelling through classical algebra, mathematical modelling linear algebra and matrices, mathematical model through ordinary and partial difference equation, mathematical model through functional equations, mathematical model through graphs, Mathematical models through mathematical programming, Mathematical models through maximum principle and so on.

**c.** Mathematical Models may also be classified according to the purpose, we have for the model. Thus we have Mathematical Models for description, Mathematical Models for prediction, Mathematical Models for optimization, Mathematical Model for Control and Mathematical Model for action.

**d.** Mathematical Models may also be classified according to their Nature:

**i.** **Linear or Non-Linear Models:** A function or an operator is called linear if it follows the principle of superposition i.e.

$$T(ax + by) = aT(x) + bT(y)$$

If all the functions and operators involved in the model are linear, then it's called a linear model otherwise a non - linear models. Linear models are relatively simple to analyze as compare to the non-linear. In order to analyze the non-linear models, some times linearization techniques are used to convert it into equivalent linear model.

**Example:** $x_{n+1} = kx_n$ is a linear equation and $x_{n+1} = kx_n(1 - x_n)$ is a non linear equation where $k$ is any constant.

a. **Static or Dynamic** Static systems (models) accounts only for steady state i.e. system in an equilibrium state and hence it is the time in-variant. Dynamic models deal with time – dependent changes in the state of system. They are typically represented by difference or differential equations.

**Example :** A person sitting beside you in static with respect to you while dynamic with respect to earth. $2y = x$ is a static system while $y = x(t)$ is dynamic.

b. **Discrete or Continuous** Discrete time model treats object at countable time steps example $x_{n+1} = kx_n$. Continuous time model deals for continuous time example $dy/dt = y$.

c. **Deterministic or Stochastic** If every variable state involved in system can be uniquely determined by parameters in the model, it's termed as deterministic.

**Example.** The length of hypotenuse can be determined from length of base and perpendicular in an right angled triangle. If any one of the variable state shows random nature than unique, then this model is called stochastic.

**Example.** Prediction for raining in next month.

d. **Autonomous and Non – Autonomous Models:**

Any autonomous model is one in which system of ODE's does not explicitly depend on independent variable. When variable is time, the model is also referred as time-invariant model.

**Example.** $\frac{dy}{dt} = y$.

Any autonomous system can be transformed into dynamical system and vice – versa (with a very weak assumption). A system which is not autonomous is called non-autonomous.

**Example.** $\frac{dy}{dt} = sin(yt)$.

e. **Analytical model**

An analytical model is a mathematical model that uses a closed-form solution to represent a system's changes.

**Example.** The computation of the mass of the system from the mass of it's parts, or the computation of the static geometric properties of a system, such as its length or volume. The mass or geometric relationships may vary with time, but the computation is given for a single point in time.

**Simulation.** Simulation is used to evaluate a new design, diagnose problems with an existing design, and test a system under conditions that are hard to reproduce in an actual system.

**Example.** Car manufacturing: Simulations can help car manufacturers virtually crash test new vehicles to see how they might perform in different accidents. This can help determine if the car is safe to drive without having to physically crash dozens of cars.

## *1.4 CHALLENGES AND LIMITATIONS OF MATHEMATICAL MODELLING*

While mathematical modeling is a powerful tool, it comes with challenges and limitations:

i. **Data Availability:** Models heavily rely on data, and their accuracy is limited by the quality and quantity of available data. In some cases, data may be scarce or unreliable.
ii. **Assumption Sensitivity:** Models are built on assumptions, and their results can be highly sensitive to these assumptions. Small changes in assumptions can lead to significantly different outcomes.
iii. **Complexity:** Real-world systems are often highly complex, and simplifications are necessary for modelling. However, overly simplistic models may not capture important nuances.
iv. **Uncertainty:** Models cannot eliminate uncertainty entirely, especially in stochastic systems. They can only provide estimates of probabilities and outcomes.
v. **Computational Resources:** Some models, particularly those involving large-scale simulations, require significant computational resources, which can be expensive and time-consuming.

## *1.5 SOLVED EXAMPLES*

**Example 1.** How would you rnodel speed and velocity?

**Solution.** We can say the definition of $speed/velocity$ is equal to the rate of change of distance travelled. Since speed is a scalar, we model it as $L/T$ where $L$ is the distance travelled and $T$ is the time required to travel. While modelling velocity, the direction too should be specified and hence, the model for velocity is $v = L/T$ where the velocity is the vector quantity. Using Calculus, the model can be further improved by writing the elementary distance as $ds = (dx, dy, dz)$, so that $v = \frac{ds}{dt}$.

**Example 2.** Finding the width of a river or a canal (without crossing it).

**Solution.**

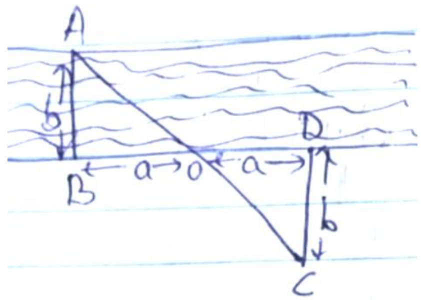


***Fig 1.1***

For this we try to express the width of river or canal in term of some distance and angle which can be measured from one side of the river. From the Figure 1.1 consider $B$ and $D$are two points on the bank of the river. $A$ is any point on the other side of the river. $AB$ is perpendicular at $BD$ line. Again $C$ is any fix point and we draw perpendicular from $C$ to $BD$ line and measure an angle between $CD$ and $AC$. $AC$ line crosses the lines $AB$ and $CD$ passes from point $O$. Hence $\Delta COD$ and $\Delta AOB$ are congruent therefore $CD$ will be the breadth of the river $AB$.

**Example 3.** Estimate the population of a fish in a pond.

**Solution.** Let $x(t)$ be the population of fish in a pond at time $t$ and $x(t+\Delta t)$ be the population after time $(t+\Delta t)$. Again "$b$" be the birth per individual per unit time and "$d$" be the death per individual per unit time. Then population after time $t+\Delta t$ is,

$$x(t+\Delta t) = x(t) + b.x.\Delta t - d.x.\Delta t + 0(\Delta t)$$
$$x(t+\Delta t) - x(t) = (b-d)x(t).\Delta t + 0(\Delta t)$$
$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = (b-d)x(t) + 0(\Delta t)$$

as $\Delta t \to 0$ and let $b - d = a$

$$x'(t) = ax(t)$$
$$\frac{x'(t)}{x(t)} = a$$

On integration, $\log x(t) = at + logc$,$x(t) = ce^{at}$

Let $t = 0$, $x(0) = ce^0$. It implies that ,$x(0) = c$, $x(t) = x(0)\, e^{at}$,where $a = b - d$.

## 1.6 SUMMARY

In this unit we have explain the following main points.

**a)** Mathematical models transform real life problems into mathematical explanations. Trying to understand and solve real world problems can be risky and expensive sometimes testing is not possible. In this case, mathematical modelling is the only solution and can be very cheap if we can represent and solve real problems with appropriate equations.

**b)** The mathematical modeling process has three main steps: desig n, problem solving, interpretation, and evaluation.

**c)** Mathematical models can be divided into linear/nonlinear, stati c/dynamic, discrete/continuous, and deterministic/stochastic.

**d)** Many types of modeling require one or more of these during design or problem solving.

**e)** One thing to keep in mind in mathematics is that when interpre ting real world problems in mathematical terms, many simplifications are needed. Everyone who is learner should know this.

## 1.7 GLOSSARY

i. **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

ii. **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

iii. **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

iv. **Objective function:** In mathematical modelling, an objective function is defined as a linear equation that characterizes and addresses optimization problems. It is a function dependent on decision variables, which can be selected to either maximize or minimize the objective. Typically, the objective function is expressed in the form $Z = ax + by$, where $(a)$ and $(b)$ are constants, while $(x)$ and $(y)$ are the variables that need to be optimized. Additional constraints, such as $(x > 0)$ or $(y > 0)$, may also impose limits on the objective.

**CHECK YOUR PROGRESS**

**1.** Which system/model applies deductive reasoning of mathematical theory to solve a model:-
- **a.** Dynamic Model
- **b.** Static Model
- **c.** Analytical Model
- **d.** Numerical Model

**2.** Which model follows the changes over time that results from the system activities:-
- **(a)** Dynamic Model
- **(b)** Static Model
- **(c)** Analytical Model
- **(d)** Numerical Model

**3.** **…………………………….** is considered to be a numerical computation technique used in conjunction with dynamic mathematical models.
- **(a)** Analysis
- **(b)** System simulation
- **(c)** Dynamic computation
- **(d)** None of the above**.**

**4.** Which system/model applies computational procedures to solve equations:-
- **(a)** Dynamic Model
- **(b)** Static Model
- **(c)** Analytical Model
- **(d)** Numerical Model

**5.** Which model can any show the values that system attributes takes when the system is in balance:-
- **(a)** Dynamic Model
- **(b)** Static Model
- **(c)** Analytical Model
- **(d)** Numerical Model

## *1.8REFERENCE*

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## *1.9 SUGGESTED READINGS*

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## *1.10 TERMINAL QUESTIONS*

**TQ 1:** Formulate the model for estimating the population of lions in a forest.

**TQ2:** How would you model acceleration of a particle?

## *1.11 ANSWERS*

### CHECK YOUR PROGRESS

1. c.
2. a.
3. b.
4. d.
5. b.

# UNIT 2 LINEAR - NON LINEAR GROETH –DECAY MODEL

## CONTENTS:



## *2.1 INTRODUCTION*

In previous unit we have discussed why mathematical modelling needed? Simple situations requiring mathematical modelling, The technique of mathematical modelling, Classification of mathematical models and challenges and limitations of mathematical modelling. In present unit we are discussing about mathematical modelling in terms of differential equations arises when the situation modeled involves some continuous variable (s) and we have some reasonable hypothesis about the rate of change of dependent variable (s) with respect to independent variables (s). When we have one dependent variable x (say population size) depending on one independent variable (say time t), we get a mathematical model in terms of an ordinary differential equation of the first order, if the hypothesis is about the rate of change $dx/dt$. If there are a number of dependent continuous variables and only one independent variable, the hypothesis may give a mathematical model in terms of a system of first or higher order ordinary differential equations. If there is one dependent continuous variable (say velocity of fluid $u$) and a number of independent continuous variables (say space coordinates $x, y, z$ and time $t$ ), we get a mathematical model in terms of a partial differential equation. If there are a number of dependent continuous variables and a number of independent continuous variables, we can get a mathematical model in terms of systems of partial differential equations.

## 2.2 OBJECTIVES

After studying this unit, learner will be able to

i. Described the concept of Linear growth and decay models
ii. Explained the Population growth models
iii. Define the Non-linear growth and decay models
iv. Analyze the Logistic law of population growth
v. Justify the limitations of Population growth models and Logistic law of population growth

## 2.3 LINEAR GROWTH AND DECAY MODELS

### 2.3.1 POPULATIONAL GROWTH MODELS

Population growth models are mathematical representations of how a population's size changes over time.

Let $x(t)$ be the population size at time $t$ and let $b$ and $d$ be the birth and death rates, i.e., the number of individuals born or dying per individual per unit time then in time interval $(t, t+\Delta t)$, the numbers of births and deaths would be $bx\Delta t + 0(\Delta t)$ and $dx\Delta t + 0(\Delta t)$, where $0(\Delta t)$is an infinitesimal which approaches zero, so that

$$x(t+\Delta t) - x(t) = \big(bx(t) - dx(t)\big)\Delta t + 0\Delta t \ldots\ldots\ldots\ldots (1)$$

so that dividing by $\Delta t$ and proceeding to the limit as $\Delta t \to 0$, we get

$$\frac{dx}{dt} = (b-d)x = ax \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating equation (2), we get

$$x(t) = x(0)\exp(at) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The population grows exponentially if $a > 0$, decays exponentially if $a < 0$ and remains constant if $a = 0$.

**i.** If $a > 0$, the population will become double its present size at time $T$, where

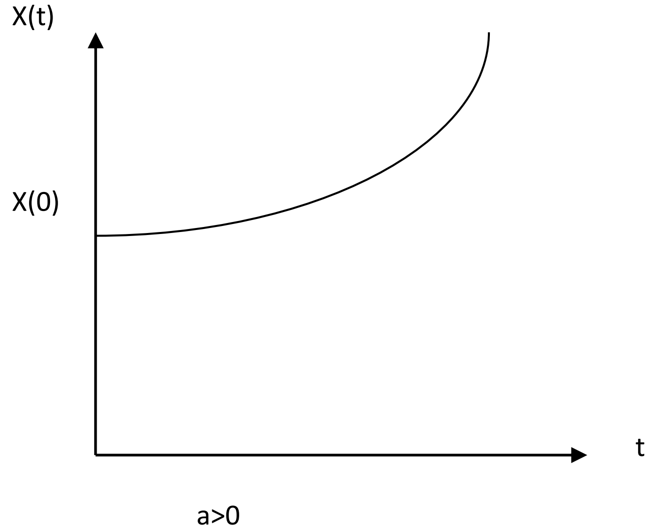



***Fig 2.3.1***

$2x(0) = x(0)\exp(aT)$ or $\exp(aT) = 2$

Or

$T = \frac{1}{a}\ln 2 = (0.69314118)a^{-1}$….(4)

$T$ is called the doubling period of the population and it may be noted that this doubling period is independent of $x(0)$. It depends only on $a$ (i.e., greater the difference between birth and death rates), the smaller is the doubling period.

**ii.** If $a < 0$, the population will become half its present size in time $T'$, when

$\frac{1}{2}x(0) = x(0)\exp(aT')$ or $\exp(aT') = \frac{1}{2}$

or

$T' = \frac{1}{a}\ln\frac{1}{2} = -(0.69314118)a^{-1}$….(5)

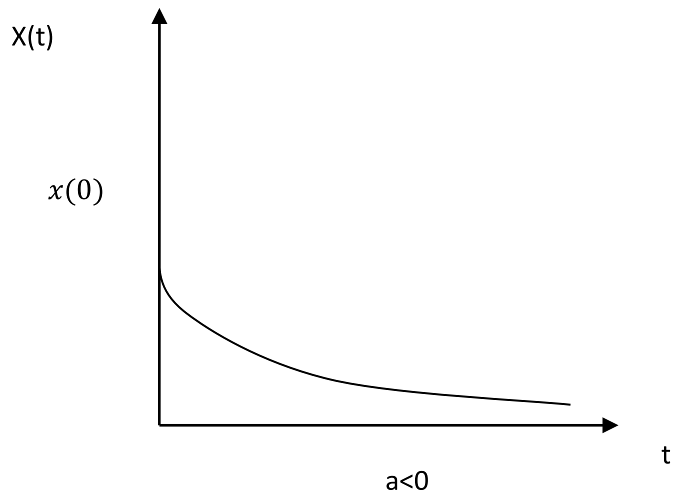



***Fig 2.3.2***

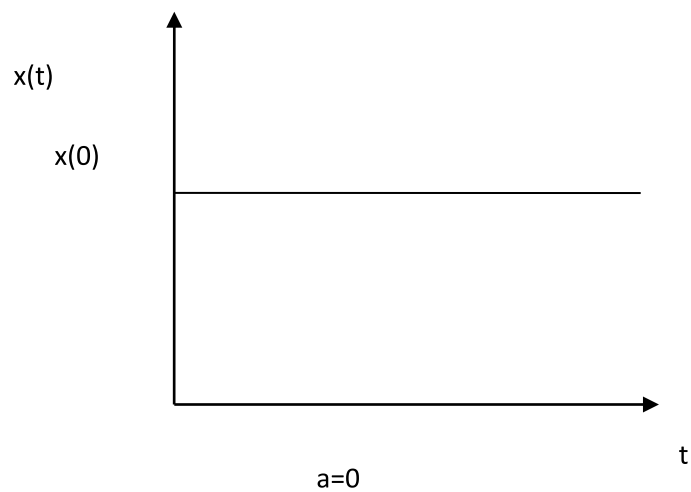



***Fig 2.3.3***

It may be noted that $T'$ is also independent of $x(0)$ and since $a < 0, T' > 0$. $T'$ may be called the half – life (period) of the population and it decreases as the excess of death rate over birth rate increases.

## 2.4 NON LINEAR GROWTH AND DECAY MODELS

### 2.4.1 LOGISTIC LAW OF POPULATION GROWTH MODELS

Logistic growth describes a pattern of data whose growth rate gets smaller and smaller as the population approaches a certain maximum - often referred to as the carrying capacity. The graph of logistic growth is a sigmoid curve. As population increases, due to overcrowding and imitations of resources the birth rate $b$ decreases and death rate $d$ increases with the population size $x$.

We are taking an assumption:

$$b = b_1 - b_2 x, d = d_1 + d_2 x, \quad b_1, b_2, d_1, d_2 > 0 \dots\dots\dots (6)$$

equation (2) becomes

$$\frac{dx}{dt} = \big((b_1 - d_1) - (b_2 + d_2)x\big) = x(a - bx), a > 0, b > 0 \dots\dots.. (7)$$

Integrating equation (7), we get

$$\frac{x(t)}{a - bx(t)} = \frac{x(0)}{a - bx(0)} e^{at} \dots\dots\dots\dots.. (8)$$

Equations (7) and (8) show that

**i.** $x(0) < a/b \Rightarrow dx/dt > 0 \Longrightarrow x(t)$ is a monotonic increasing function of $t$ which approaches $a/b$ as $t \to \infty$

**ii.** $x(0) > a/b \Rightarrow x(t) > a/b \Rightarrow dx/dt < 0 \Longrightarrow x(t)$ is a monotonic decreasing function of $t$ which approaches $a/b$ as $t \to \infty$.

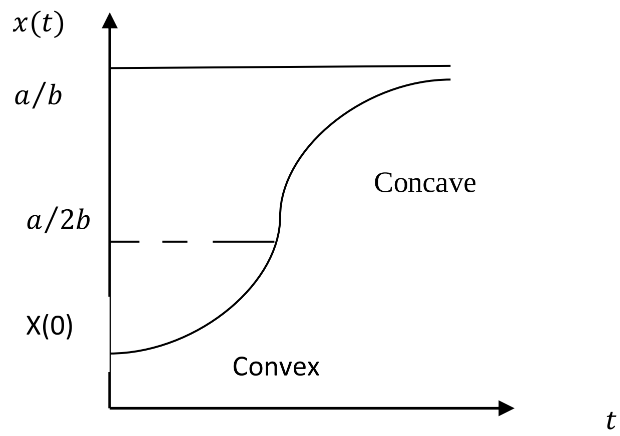
Now from (7)

$$\frac{d^2x}{dt^2} = a - 2bx \dots\dots\dots\dots (9)$$

So that $\frac{d^2x}{dt^2} \lesseqgtr 0$ according as $\lesseqgtr a/2b$.

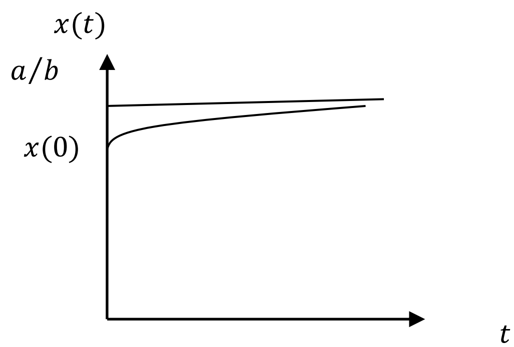
Thus in case (i) the growth curve is convex if $x < a/2b$ and is concave if $x > a/2b$ and it has a point of inflexion at $x = a/2b$. Thus the graph of $x(t)$ against $t$ is as given in below $x(0) < a/b \Rightarrow dx/dt > 0 \Longrightarrow x(t)$ is a monotonic increasing function of $t$ which approaches $a/b$ as $t \to \infty$ $x(0) > a/b \Rightarrow x(t) > a/b \Rightarrow dx/dt < 0 \Longrightarrow x(t)$ is a monotonic decreasing function of $t$ which approaches $a/b$ as $t \to \infty$. If $x(0) > a/b, x(t)$ is always equal to $a/b$. If $x(0) > a/b, x(t)$

decreases at decreasing absolute rate and approaches $a/b$, as $t \to \infty$.



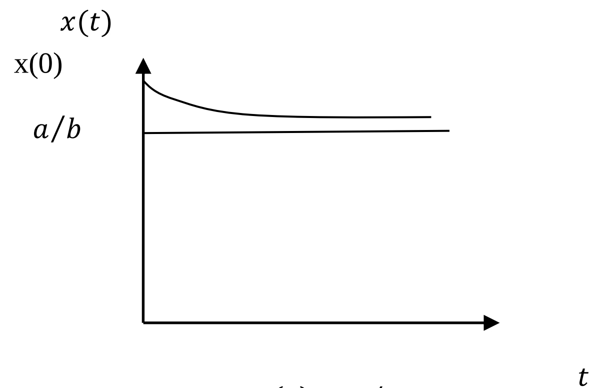

$x(0) < a/2b$

***Fig.2.4.1***




$$\frac{a}{2b} < x(0) < \frac{a}{b}$$

***Fig 2.4.2***




$x(0) > a/b$

**Fig 2.4.3**

**MOTIVATING QUESTION**

- How can we use differential equations to realistically model the growth of a population?
- How can we assess the accuracy of our models?

## *2.5 PROBLEM*

**Application: Radioactive Decay:**

Application areas include Physics, Biology and Nuclear Engineering.

**Question 1:** If the half – life of a particular radioactive substance is known to be 10 days and there are 25 miligrams initially, how much is present after 8 days?

To answer this question we use our five step procedure.

**Step 1: Understand the Concepts in the Application Area Where the Questions are Asked:**

We first describe the phenomenon to be modeled, including the laws it must follow (e.g., that are imposed by nature, by an entrepreneurial environment or by the modeler). To understand radioactive decay, we consider the following empirical physical law.

**Physical Law:** From physical experiments, it is found that radioactive substances decay at a rate that is proportional to the amount present. It is useful to draw a sketch to help visualize the process being modeled. Try to visualize the radioactive substance on a table radiating out into the room. That is, the room is a sink. The amount of substance on the table is constantly decreasing. (Obviously, in a physics lab, safety precautions must be taken to protect against personal injury and pollution.) Now let us consider the sentence "Radio active substances decay at a rate which is proportional to the amount present." Rate means time rate of change which implies derivative with respect to time. Thus our model will include a first order ODE that is a rate equation. (This is a special one-dimensional or scalar version of our quintessential model.)Always make a list of the variables and parameters you use. In an engineering research paper, this is called the nomenclature section. Begin with those stated in the problem. If we need a variable not given, choose one that is appropriate and helps us to remember what it stands for.

We begin our list:

**Nomenclature:**

$Q =$ quantity of the radioactive substance (state variable).

$t =$ time (independent variable)

To understand the concept of half life, we must first develop and solve the model.

**Step 2: Understand the Needed Concepts in Mathematics:**

**iii.** High School Algebra.

**iv.** Calculus.

**v.** Solution techniques in this part of the notes.

**Step 3: Develop the Mathematical Model:**

If the problem is not complicated, a general model may be developed. By this we mean that arbitrary constants (parameters) are used instead of specific data. This general model may then be used for any specific problem where the modeling assumptions used to obtain the general model are satisfied. If the assumptions are changed, a new model must be formulated. If a general model can be developed and solved, the results can be recorded and used for any specific data. However, you may wish to redevelop the same model for different specific data in order to develop your modeling skills. Let us more carefully analyze the sentence "Radio active substances decay at a rate which is proportional to the amount present." Rate means time rate of change which implies derivative with respect to time. Decay implies that the derivative is negative. Proportional means multiply the quantity by a proportionality constant, say k. Hence this sentence means the appropriate rate equation (first order ODE) to model radioactive decay is

$$\frac{dQ}{dt} = -kQ, \quad k > 0 \text{...........(1)}$$

For this model we have followed the standard convention of putting in the minus sign explicitly since we know that the substance is always decaying (i.e., its time derivative is negative). This is not necessary, but forces the physical constant k to be positive. Physical constants are normally listed in reference books as positive quantities. We can and should check that the value you obtain for k in a specific problem is positive. If not, we can check our computations to find our mistake. Also, k > 0 makes the model more intuitive. We emphasize that the equation is a rate equation with units of mass per unit time (M/T e.g. grams per second, gm/sec). Thus it can be viewed as a conservation law. We only have a sink so that the rate of change is equal to the rate out. To determine the amount present at all times, we must also know the amount present initially (or at some time). Since no initial condition is given, we assume an arbitrary value, say $Q_0$as a parameter. We add k and $Q_0$ to our nomenclature list.

**NOMENCALTURE:**

$Q =$ quantity of the radioactive substance (state variable).
$t =$ time (independent variable)
$k =$ positive constant of proportionality (parameter).
$Q_0 =$ initial amount of the radioactive substance (parameter).

The IVP that models radioactive decay is:

**MATHEMATICAL MODEL:**

Radio active decay
Ordinary differential equation

$$\frac{dQ}{dt} = -kQ \; \ldots \ldots (2)$$

Initial condition

$$Q(0) = Q_0 \; \ldots \ldots \ldots \ldots (3)$$

Note that the model is "general" in that we have not explicitly given the proportionality constant $k$ or the initial amount $Q_0$ of the substance. The parameter can be given or found using specific (e.g., experimental data). However, we do not need to know the values of $k$ and $Q_0$ to solve the model.

**Step 4: Solve the Mathematical Model:**

Once the model is developed, it is not necessary that the solver of the model understand any of the application concepts in order to solve the model. What is required now is not an understanding of the physics, but an understanding of the mathematics. To solve the ODE in this model, we note that it is both linear and separable. We choose to solve it as a separable problem, but recall that since it is linear, we can (and must) solve for Q explicitly. Separating variables we obtain the sequence of equivalent equations

$$\frac{dQ}{Q} = -k\,dt,$$
$$\int \frac{dQ}{Q} = -k \int dt,$$
$$ln|Q| = -Kt + c,$$
$$|Q| = e^{-Kt+c} = e^{c}e^{-kt}.$$

Letting $A = \pm e^{c}(+e^{c}\ if\ Q > 0, -e^{c} if\ Q < 0)$ we obtain $Q = Ae^{kt}$. Although the physics implies $Q \geq 0$, the mathematics does not require this in order for a unique solution to the IVP to exist. Applying the initial condition $Q(0) = Q_0$, we obtain $Q_0 = A$. Hence the unique solution to the IVP is

$$Q = Q_0 e^{kt} \ldots\ldots\ldots\ldots\ldots (4)$$

It is the solution to the general model for radioactive decay for $Q_0 \geq 0$. Radioactive substances are said to experience exponential decay.

The formula (4) is found in physics and biology texts. There are two constants (parameters) to be determined and we need further data to evaluate them. Known values of the constant k ( with units 1/T e.g. 1/days) or its (multiplicative) inverse 1/k (which is referred to as a time constant since it has units of time) for specific substances could be given in reference books. (Usually half lives are given instead as explained below. The existence and uniqueness theory says that exactly one solution exists for the IVP given by (2) and (3) and that the interval of validity is **R**. If we have any doubts that we have found it, we can check that it satisfies both the IC and the ODE for all $x \in \mathbb{R}$.

**Step 5: Interpret the Results**. Although interpretation of results can involve different things, in the context of this course it means "After you have solved the model (IVP) in whatever generality is appropriate, apply the specific data given to answer the questions that motivated our study". This may require additional solution of algebraic equations (e.g. the formula that you have derived for the general solution of the model). The term general solution is used since arbitrary values of k and $Q_0$ are used. (Recall that the term general solution is also used to indicate the family of functions which are solutions to an ODE before an initial condition is imposed. We could argue that since the initial condition is arbitrary, we really have not imposed an initial condition, but again, general here means not only an arbitrary initial condition, but also an arbitrary value of k.) This brings us to the concept of half life. For an arbitrary value of $Q_0$, let $t_{hl}$ be the time when only half of $Q_0$ is left. From (4) we obtain the sequence of equivalent scalar equations:

$$(1/2)Q_0 = Q_0 e^{-kt_h}, (1/2) = e^{-kt_n}, ln(1/2) = -kt_n, t_h = -\frac{ln(1/2)}{k}$$

First note that the half life depends only on the value of $k$ and not on $Q_0$. In fact there is a **one to one correspondence** between values for k and values for $t_{hl}$. Thus we also have,

$$k = -\frac{ln(1/2)}{t_h}.$$

Note that although it may appear that k is negative, in fact $ln(1/2)$is negative and

$$k = -\frac{ln(2)}{t_h} \ldots\ldots\ldots\ldots (6)$$

Reference books generally give half lives. The value of k can then be computed using (6).

**APPLICATION TO SPECIFIC DATA**

Once a general model has been formulated and solved, it can be applied to specific data. Alternately, the model can be written in terms of the specific data and then solved (again). If a general solution of the model has been obtained, this is redundant. However, resolving the model provides practice in the process of formulating and solving models and hence is useful in preparing for exams. Solutions of general models are not normally given on exams and are usually not memorized. Also specific data may simplify the process and the formulas obtained.

Suppose that the following specific information is given:

**SPECIFIC DATA**.

If the half-life (the time required for a given amount to decrease to half that amount) of a particular radioactive substance is known to be 10 days and there are 25 milligrams initially, then find the amount present after 8 days. We develop a data chart so that the specific data and the questions to be answered are at our finger tips.

**Data Chart:**

| $t$ | $t_0 = 0$ | $t_1 = 0$ | $t_h = 10$ |
|---|---|---|---|
| $Q$ | $Q_0 = 25$ | $Q_1 = ?$ | $Q_h = 1/2\, Q_0$ |

All of the information in the sentence is now contained in the data chart for easy access. Recall that the "general" solution of the model (IVP) is given by $Q = Q_0 e^{kt}$. We need to apply the information in the data chart to obtain specific values for the constants (parameters) $Q_0$ and $k$, thus completing the model for this specific data. It is certainly acceptable (and indeed desirable since it gives practice in formulating and solving models) to formulate and solve the model using this specific data.. The advantage of formulating and solving a model in a general context is that the solutions can be recorded in textbooks in physics, biology, etc. (and programmed on personal computers) for those not interested in learning to solve differential equations. However, if the model assumptions change, a new model must be formulated and solved. Practice in formulating and solving specific models will help you to know when a different model is needed and in what generality a model can reasonably be developed. General models are useful when their results can be easily recorded (or can be programmed). On the other hand, trying to use the results of a complicated model can unduly complicate a simple problem. Applying the data in the data chart we obtain:

At $t = 0, Q = 25 \Rightarrow Q_0 = 25$.
At $t = 10, Q = \frac{1}{2} Q_0 = \frac{1}{2}(25) = Q_0 e^{-k(10)} = 25e^{-k(10)}$.
Hence $\ln\left(\frac{1}{2}\right) = -k(10)$.
Note that this result is independent of the value $Q_0$. So that,
$k = \frac{\ln(2)}{10}$. Hence, $Q = 25e^{-\frac{\ln(2)}{10}t} = 25\exp\left(-\frac{ln2}{10}\,t\right)$.
Thus after 8 days

$$Q = 25e^{-\frac{\ln(2)}{10}8} = 25\exp\left(-\frac{ln2}{10}\,8\right)$$

$$= 25\exp\left(2^{-(4/5)}\right) = 25/\left(2^{4/5}\right) = \frac{25}{\sqrt[5]{16}}.$$

**Application: Continuous Compounding**

Application Areas include Business and Economics.

**Question2.** If $1000 is invested at 6% annually compounded continuously how much will the investment be worth in 6 years. How long before the investment doubles?

We again apply our four step procedure to solve this applied math or application problem:

**Step 1:** Understand the concepts in the application area where the questions are asked. This we describe the phenomenon to be modeled, including the laws it must follow. We consider the following "economic definition" for continuous compounding and develop a general model (IVP) that governs this process with no deposits or withdrawals after the initial investment.

**Economic Definition** Continuous compounding means that the time rate of change of the total investment (principle plus interest) is increasing in proportion to the amount present. (Money grows like rabbits. The more there is, the faster it grows.)Unfortunately (in terms of understanding continuous compounding, but not in terms of understanding bank accounts) most people already have some understanding of discrete compounding. Boyce and Diprima develops discrete as well as continuous compounding and compares them. As a first effort at trying to understand continuous compounding, it is probably better not to worry about discrete compounding (what banks do) and just focus on the model of continuous compounding as described in the above "economic definition". We note however that continuous compounding is in fact the limit of discrete compounding as the interval of compounding (i.e., $\Delta t$) is allowed to go to zero.

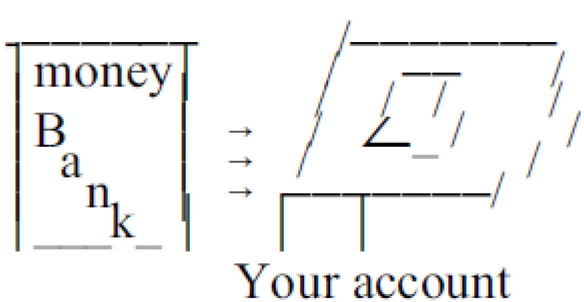



*Fig 2.5.1*

**Step 2:** Understand the Needed Concepts in Mathematics. 1. High School Algebra, 2. Calculus, 3. Solution Techniques covered in this Part of the Notes.

**Step3:** Develop the Mathematical Model. If the problem is not complicated, a **"general" model** may be developed and solved first. This "general" model may then be used for any specific problem where the modeling assumptions used to obtain the "general " model are satisfied. If they are not, a new model must be formulated and solved. To develop a model, let us analyze the sentence. "Continuous compounding means the time rate of change of the total investment (principle plus interest) is increasing in proportion to the amount present." This is just like radioactive decay, except that the rate is increasing, instead of decreasing. Hence we omit the minus sign and obtain

$$\frac{dS}{dt} = kS, k > 0.$$

Where $S =$ Total investment (Principle plus interest).
$t =$ time
$k =$ positive constant of proportionality (text uses $r$).

Always make a list of the variables and parameters you use. Begin with those stated in the problem. If you need a variable not given, choose one that is appropriate and helps you to remember what it stands for. Note that this rate equation has units of money per unit time (e.g. dollars per year, $\$per\ year$). To determine the amount present at all times, we must also know the amount present initially (or at some time). Since no initial condition is given, we assume an arbitrary value, say $S_0$.

Hence the IVP that models this phenomenon is given by:

**MATHEMATICAL MODEL**: Continuous Compounding.

ODE $\frac{dS}{dt} = kS,$
IVP IC $S(0) = S_0$

Note that the model is "general" in that we have not explicitly given the proportionality constant k or the initial investment $S_0$. These will have to be given or found using data. More needs to be said about the proportionality constant k. Unlike radioactive substances whose decay rates are set by nature, growth rates for money are set by bankers or the government). By looking at the definition of interest rates for discrete compounding and taking the limit as the time interval for compounding goes to zero (or simply by assuming this as a definition of continuous compounding) we agree that the constant k expressed as a fraction (e.g. 6% = 0.06) is the **rate of interest**. It is the (multiplicative) inverse of a time constant and has units of fractional portion (from the percentage rate) per unit time (1/T, e.g. one over years, 1/yrs).Since it is a "rate of interest", we now replace k by r.

**Step 4: Solve the Mathematical Model.**

To solve the ODE, we note that it is essentially the same equation as for radioactive decay with )-k replaced by r. Hence the solution is given by

$$S = S_0\, e^{rt}.$$

There are two constants (parameters) to be determined and we need data to evaluate them. If the assumptions of the model are violated (e.g. if we add the additional assumption that we are adding to the original investment or withdrawing money on a continuous basis) the model must be reformulated and resolved.

**Step 5: Interpret the Results**. Although interpretation of results can involve a number of things, in the context of this course it usually means "After you have formulated and solved the "general" model (IVP) for the conditions presented, use your results and the specific data given to answer the specific questions asked". This may require additional solution of algebraic equations obtained in solving the model (IVP), for example, the equation obtained as the "general" solution of the model. The term "general" solution is used here since arbitrary values of $r$ and $S_0$ are used. (Recall that the term general solution is also used to indicate the family of functions which are solutions to an ODE before an initial condition is imposed. We could argue that since the initial condition is arbitrary, we really have not imposed an initial condition, but again, "general" here means not only an arbitrary initial condition, but also an arbitrary value of r.)

**APPLICATION OF SPECIFIC DATA**

Once a general model has been formulated and solved, it can be applied using specific data. Alternately, the model can be written in terms of the specific data and resolved. Although redundant, this resolving of the model provides much needed practice in the process of formulating and solving models. This is useful in preparation for exams since solutions of general models are not normally given on exams and are usually not memorized. Also specific data may simplify the process and the formulas obtained. Suppose that the following specific information is given:

**SPECIFIC DATA**. If $1000 is invested at 6% annually compounded continuously how much will the investment be worth in 6 years. How long before the investment doubles?

We develop a **data chart** for: r = 6% = 0.06

| $t$ | $t_0 = 0$ | $t_1 = 6$ | $t_d = ?$ |
|---|---|---|---|
| $S$ | $S_0 = 1000$ | $S_1 = ?$ | $S_d = 2S_0$ |

All of the information in the sentence except r = 6% = 0.06 is now contained in the data chart for easy access. Recall that the "general" solution of the model (IVP) is given by $S = S_0 e^{-rt}$ .We need to apply the information in the data chart to obtain values for the ?'s in the chart. It is certainly acceptable to include computation of $S_0$ (i.e. writing the formula with the value given) as part of the solution process with Step 2, but normally Step 2 involves the solution of the model (IVP) in the most general form that is reasonable.) Letting r = 6% = 0.06 and applying the data in the data chart we obtain:

At t = 0, S = 1000 which implies $S_0 = 1000$. Hence S = 1000 $e^{0.06t}$. Hence at t = 6 years, S = 1000$e^{0.06(6)}$ = ($ 1433.33 using a calculator). At t = $t_d$ , S = 2 $S_0$ = 2000 =$S_0$ $e^{0.06t}$ = $1000\, e^{0.06t}$ so that 2 2 $= e^{0.06 t_d}$and hence 0.06 $t_d$ = Rn(2). Thus$t_d$ = (100/6)Rn(2) ( ≈11.55 years using a calculator). Similar to half life, the **doubling time** for the initial investment (or the doubling time for rabbits) is not dependent on value of the initial investment (or the initial number of rabbits). It is important to emphasize that if the modeling assumptions are changed, the result (i.e. formula for the solution) derived for the above model in Steps 1 and 2 and applied in Step 3, is not valid. The model must be reformulated and re-solved.

## 2.6 *SUMMARY*

In this unit we have explained linear growth and decay models, Population growth models, Non-linear growth and decay models and Logistic law of population growth models. Demographers developed an array of models to measure population growth; of these models are usually utilized. A population growing arithmetically would increase by a constant number of people in each period.

## 2.7 *GLOSSARY*

i. **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

ii. **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

iii. **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

iv. **Objective function:** In mathematical modelling, an objective function is defined as a linear equation that characterizes and addresses optimization problems. It is a function dependent on decision variables, which can be selected to either maximize or minimize the objective. Typically, the objective function is expressed in the form $Z = ax + by$, where $(a)$ and $(b)$ are constants, while $(x)$ and $(y)$ are the variables that need to be optimized. Additional constraints, such as $(x > 0)$ or $(y > 0)$, may also impose limits on the objective.

**CHECK YOUR PROGRESS**

**True/False Questions**

i. Logistic growth describes a pattern of data whose growth rate gets smaller and smaller as the population approaches a certain maximum - often referred to as the carrying capacity. The graph of logistic growth is a sigmoid curve. **True/False**.

ii. Population models can be used to determine whether alterations in reproductive rates, mortality rates, or length of time needed to mature resulting from exposure to environmental pollutants will significantly alter the ability of a population to sustain itself over time. **True/False**.

iii. A population model is not a type of mathematical model that is applied to the study of population dynamics. **True/False**.

iv. Ecological population modelling is not concerned with the changes in parameters such as population size and age distribution within a population. **True/False**.

v. Logistic growth model Used in agriculture to model crop response to changes in growth factors. The model can be used to describe positive or negative growth curves. **True/False**.

## *2.8 REFERENCE*


1. J.N. Kapoor: (2009). Mathematical Modelling, New Age International (P) Limited Publishers.
2. J. N. Kapoor (1985). Mathematical Models in Biology and Medicine, Affiliated East West Press.
3. J.D. Murrey (2014). Mathematical Biology (3rd Edition), Springer.
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## 2.9 *SUGGESTED READINGS*


1. E.A. Bender (1978) An Introduction to Mathematical Modeling, New York, John Wiley and Sons.
2. W.E. Boyce (1981) Case Studies in Mathematical Modeling, Boston, Pitman.
3. A. Friedman and W. Littman, (1994) Industrial Mathematics: A Course in Solving Real World Problems, Philadelphia, SIAM.
4. F.R. Giordano and M.D. Weir (1985). A First Course in Mathematical Modeling, Monterey, Brooks/Cole.


## *2.10 TERMINAL QUESTIONS*

**TQ1.** Suppose a tank initially has 10 pounds of salt dissolved in 100 gallons of water. If brine at a concentration of 1/4 pound of salt per gallon is entering the tank at the rate of 3 gallons per minute and the well stirred mixture leaves the tank at the same rate, how much salt is left in the tank after 30 minutes? What is the maximum amount of salt which accumulates in the tank?

**TQ2.** A body with mass 5 grams falls from rest in a medium offering resistance proportional to the square of the velocity. If the limiting velocity is 2 centimeters per second, find the velocity v as a function of time t.

**TQ3.** The rate of change of a culture of bacteria is proportional to the population itself when $t = 0$, there are 100 bacteria. Two minutes later, at $t = 2$, there are 300 bacteria. How many bacteria are there at 4 minutes?

**TQ4.** A population of rabbits has a rate of change of

$$\frac{dN}{dt} = 0.05N\left(1 - \frac{N}{500}\right)$$

where $t$ is a measure in years.

**i.** What is the size of the population of rabbits at four years?
**ii.** How many rabbits will there be at 10 years?
**iii.** When will the rabbit population reach 400?

## *2.11 ANSWERS*

**TQ1.** As $t \to \infty$ the amount of salt in the tank approaches 25 lbs. This is called the steady-state.

**TQ2:** $k \approx 34.1$ gr/cm.

**TQ3:** Therefore, at 4 minutes, the bacteria population is 900.

**TQ4: i.** After four years, the rabbit population will be about 117.
**ii.** After 10 years, the rabbit population will be about 146.
**iii.** the rabbit population about 55.5 years to reach a population of 400.

## CHECK YOUR PROGRESS

**CYP1.** True.
**CYP2.** True.
**CYP3.** False.
**CYP4.** False.
**CYP5.** True.

# UNIT 3: MATHEMATICAL MODELLING IN POPULATION DYNAMICS

## CONTENTS:



## *3.1 INTRODUCTION*

In previous unit we have discussed Linear growth and decay models: Population growth models and Non-linear growth and decay models: Logistic law of population growth. In present unit we explained mathematical modelling in population dynamics. Mathematical modeling is a useful tool for studying population dynamics, which is the study of how populations change over time. The study of the dynamics of populations is a tool of fundamental importance in this area, notably in Genetics, Ecology, and Epidemiology, to name just a few. In general, deterministic models in this field concern global or averaged features of the population, typically the size of certain sub-populations, or the proportion of individuals sharing certain characteristics. That is, the features of the population are averaged and the model aims at depicting the evolution of those averaged quantities as time passes. They are based on the implicit assumption that, roughly speaking, all individuals in a given sub-population behave essentially the same. Dynamics are usually modeled in discrete times through some difference equations, and through differential equations in continuous times.

## 3.2 OBJECTIVES

After studying this unit, learner will be able to

**i.** Described the concept of Prey-Predator Models.
**ii.** Explained the technique of Competition Models.
**iii.** Identify different types of interactions between the populations of two species.

## 3.3 PREY-PREDATOR MODELS

Alfred Lotka, an American biophysicist who studied the predator-prey model in 1920 and published his findings in his 1925 book Elements of Physical Biology and Vito Volterra, an Italian mathematician who developed the predator-prey model in 1926. The model is based on several assumptions, including:

- Predators can eat without limit.
- The food supply for prey depends on the prey population size.
- The rate of population change is directly related to the population size.
- The environment is constant.
- There are no genetic adaptations for either species.
- The prey has an unlimited food supply.

Let $x(t), y(t)$ be the populations of the prey and predator species at time $t$. We assume that

**i.** If there are no predators, the prey species will grow at a rate proportional to the population of the prey species.
**ii.** If there are no prey, the predator species will decline at a rate proportional to the population of the predator species.
**iii.** The presence of both predators and preys is beneficial to growth of predator species and is harmful to growth of prey species. More specifically the predator species increases and the prey species decreases at rates proportional to the product of the two populations.

These assumptions give the systems of non-linear first order ordinary differential equations:

$$\frac{dx}{dt} = ax - bxy = x(a - by), a, b > 0 \ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

$$\frac{dy}{dt} = -py + qxy = -y(p - qx), \qquad p, q > 0 \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Now, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ both vanish if,

$$x = x_e = \frac{p}{q}, y = y_e = \frac{a}{b} \dots\dots\dots\dots\dots\dots (3)$$

If the initial populations of prey and predator species are $\frac{p}{q}$ and $\frac{a}{b}$ respectively, the populations will not change with time. These are the equilibrium sizes of the populations of the two species. Of course $x = 0, y = 0$ also gives another equilibrium position.

From equation (1) and (2), $\frac{dy}{dx} = -\frac{y(p-qx)}{x(a-by)} \dots\dots\dots. (4)$

Or

$$\frac{a-by}{y}dy = -\frac{p-qx}{x}dx; x_0 = x(0), y_0 = y(0), \dots\dots\dots\dots\dots (5)$$

Integrating,

$$aln\frac{y}{y_0} + pln\frac{x}{x_0} = b(y-y_0) + q(x-x_0) \dots\dots\dots\dots\dots\dots. (6)$$

$$ln\frac{y}{y_0}^a + ln\frac{x}{x_0}^p = b(y-y_0) + q(x-x_0) \dots\dots\dots. (6-a)$$

$$\frac{y}{y_0}^a \frac{x}{x_0}^p = e^{b(y-y_0)+q(x-x_0)}$$

$$\frac{x^p y^a}{x_0 y_0} = e^{b(y-y_0)+q(x-x_0)}$$

$$\frac{x^p y^a}{x_0 y_0} = e^{by}e^{qx}e^{-(x_0+y_0)}$$

$$\frac{y^a x^p}{e^{by}e^{qx}} = x_0 y_0 e^{-(x_0+y_0)}$$

$$\frac{y^a}{e^{by}}\frac{x^p}{e^{qx}} = k_1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6\text{-b})$$

A **critical point** of the system of equations $\frac{dx}{dt} = F(x,y), \frac{dy}{dt} = G(x,y)$ is a point $(x^*, y^*)$ such that $F(x^*, y^*) = G(x^*, y^*) = 0$. Also then the constant valued functions $x(t) = x^*,\ y(t) = y^*$ satisfying the system is called an equilibrium solution.

Thus Eqn.(6-b) which represents a family of closed curves gives the solution of system of Eqn (1).Thus through every point of the first quadrant of the $x-y$ plane, there is a unique trajectory. No two trajectories can intersect, since intersection will imply two different slopes at the same point. If we start with $(0,0)$ or $\left(\frac{p}{q}, \frac{a}{b}\right)$, we get point trajectories.

If we start with $x = x_0, y = 0$, from equations (1) and (2), we find that $x$ increases while $y$ remains zero. Similarly if we start with $x = 0, y = y_0$, we find that $x$ remains zero while $y$ decreases.Thus positive axes of $x$ and $y$ give two trajectories.

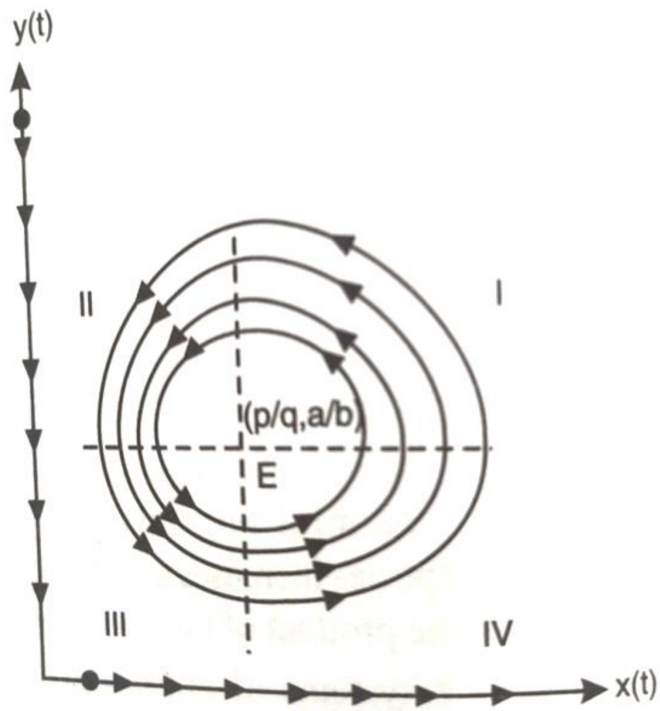


**Fig 3.3.1**

Since no two trajectories intersect, no trajectory starting from a point situated within the first quadrant will intersect the $x-$ axis and $y-$axis trajectories. Thus all trajectories corresponding to positive initial populations will lie strictly within the first quadrant. Thus if the initial populations are positive, the population will be always positive. If the population of one (or both) species is initially zero, it will always remain zero.The lines through $\left(\frac{p}{q}, \frac{a}{b}\right)$ parallel to the axes of coordinates divide the first quadrant into four parts I,II,III and IV. Using (1), (2), we find that

| | | | |
|---|---|---|---|
| in I | dx/dt<0 | dy/dt>0 | dy/dx<0 |
| in II | dx/dt>0 | dy/dt<0 | dy/dx>0 |
| in III | dx/dt>0 | dy/dt<0 | dy/dx<0 |
| in IV | dx/dt>0 | dy/dt>0 | dy/dx>0 |

| Region | $\frac{dx}{dt}$ | $\frac{dy}{dt}$ |
|---|---|---|
| I | < 0 | > 0 |
| II | < 0 | < 0 |
| III | > 0 | < 0 |
| IV | > 0 | > 0 |

This gives the direction field at all points as shown in figure 3.3.1. Each trajectory is a closed convex curve. These trajectories appear relatively cramped near the axes. In I and II, prey species decreases and in III and IV, it increases. Similarly in IV and I, predator species increases and in II and III, it decreases. After a certain period, both species return to their original sizes and thus both species sizes vary periodically with time.

## Stability

To check the stability of the critical point $P\left(\frac{p}{q}, \frac{a}{b}\right)$ and get an idea of the' pattern of the orbits near the critical point, i.e. whether the orbits are moving towards the critical point or moving away from it or exhibiting some other type of behaviour, we use the perturbation technique. The basic idea of this technique is to perturb or disturb the equilibrium slightly and then to see whether the system remains in the neighbourhood of the equilibrium or deviates far away from it. Mathematically, we change the equilibrium values of x and y slightly by adding to them very small quantities,

Let $x = \frac{p}{q}(1+u), y = \frac{a}{b}(1+v)$

…………………………………………………(a)

where u, v are very small quantities. This transformation indicates small departure from the equilibrium point $\left(\frac{p}{q}, \frac{a}{b}\right)$.

From equation (1) and (a),

$$\frac{du}{dt} = au - auv$$
$$\frac{dv}{dt} = pu + puv$$

……………………………………………………………(b)

Clearly the system of Equations. (b) is almost linear system and has (0, 0) as the critical point corresponding to the critical point $\left(\frac{p}{q}, \frac{a}{b}\right)$of the system of eqns. (1). In order to check the nature, and stability of the critical point of system (a) we consider the related linear system,

$$\frac{du}{dt} = -av$$
$$\frac{dv}{dt} = pu$$

$|A - \lambda I| = 0$, Where $A = \begin{pmatrix} 0 & -a \\ p & 0 \end{pmatrix}$.

$$\begin{vmatrix} -\lambda & -a \\ p & -\lambda \end{vmatrix} = 0$$

This implies that $\lambda^2 + ap = 0$

$$\lambda = \pm i\sqrt{ap}$$
$$\frac{d^2u}{dt^2} = -a\frac{dv}{dt} = -apu$$
$$\frac{d^2v}{dt^2} = p\frac{du}{dt} = -apv$$
$$T = \frac{2\pi}{(ap)^{1/2}}$$
$$pudu + avdv = 0$$

On integration, $pu^2 + av^2 = \lambda$

$$\frac{u^2}{\lambda/p} + \frac{v^2}{\lambda/a} = 1$$

where $\lambda$ is an arbitrary non negative constant of integration. Thus the trajectories of the system'(b) are ellipses around the critical point (0,0). Some of these ellipses are shown in Fig. **3.3.1.**We have shown that the critical point (0, 0) is a stable center of the linear system (b). We now need to assess its character for the almost linear system ('7). Here as we know, our theory for almost linear systems. The effect of the nonlinear terms may be to change the center into a stable spiral point, or into an unstable spiral point, or it may remain as a stable center. Fortunately, in this case we have actually solved the nonlinear Eqns.(l) and seen, what happens. We have shown in Fig. **3.3.1.** that the graph of this equation for a fixed value of C in Eqn. (5) is a closed curve enclosing the critical point $\left(\frac{p}{q}, \frac{a}{b}\right)$. Thus the predator and prey have a cyclic variation about the critical point $\left(\frac{p}{q}, \frac{a}{b}\right)$ and the critical point $(0,0)$ is also the center of the system (1).

## *3.4 COMPETITION MODELS*

Competition models in mathematical modeling are used to study the interactions between competing species or entities:

- **Species competition:** When two species compete, the growth of one species reduces the resources available to the other. A mathematical model for species competition can include parameters to account for the unknown impact of each species on the other.

- **Competitive environments :** Mathematical models can be used to analyze competitive environments, such as tournaments. For example, a model can include success gates that promote the winner to the next level, and failure gates that either deny the winner or promote the loser.
- **Bio-mathematical models:** A bio-mathematical model for competition between two species can include factors such as consumption, population density, and birth and death rates. Mathematical models are used to generate answers to questions about complex systems and processes that can't be answered through observation. The answers can then be used to help understand, manage, and predict future behavior.

Let $x(t), y(t)$ be the populations of two species competing for the same resources, then each species grows in the absence of the other species, and the rate of growth of each species decreases due to the presence of the other species.

This gives the system of differential equations

$$\frac{dx}{dt} = ax - bxy = bx\left(\frac{a}{b} - y\right); a > 0, b > 0 \dots\dots(7)$$

$$\frac{dy}{dt} = px - qxy = y(p - qx) = qy\left(\frac{p}{q} - x\right); p > 0, q > 0 \dots\dots(8)$$

There are two equilibrium positions viz. $(0,0)$ and $\left(\frac{p}{q}, \frac{a}{b}\right)$.

There are two point trajectories viz. $(0,0)$ and $\left(\frac{p}{q}, \frac{a}{b}\right)$ and there are two line trajectories viz. $x = 0$ and $y = 0$.

| | | | | |
|---|---|---|---|---|
| in I | dx/dt<0 | dy/dt<0 | dy/dx>0 | (9) |
| in II | dx/dt<0 | dy/dt<0 | dy/dx<0 | |
| in III | dx/dt>0 | dy/dt>0 | dy/dx>0 | |
| in IV | dx/dt>0 | dy/dt<0 | dy/dx<0 | (10) |

This gives the direction field as shown in *Fig.3.3.2.* From equations (7) and (8),

$\frac{dy}{dx} = \frac{y(p-qx)}{x(a-by)}$ $or$ $\frac{a-by}{y} dy = \frac{p-qx}{x} dx$……………………..(11)

Integrating

$$a ln \frac{y}{y_0} - b(y - y_0) = p ln \frac{x}{x_0} - q(x - x_0) \dots\dots\dots\dots(12)$$

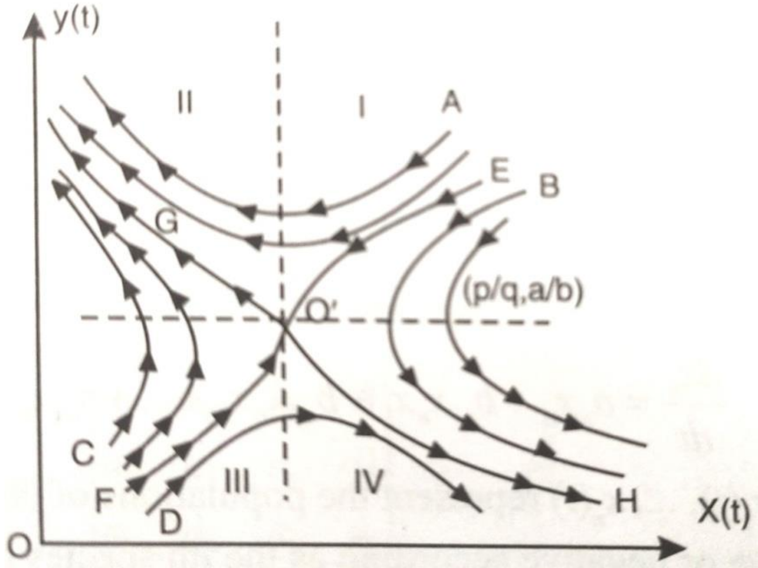


***Fig.3.3.2***

If the initial populations correspond to the point $A$, ultimately the first species dies out and the second species increases in size to infinity.

If the initial populations correspond to the point $B$, then ultimately the second species dies out and the first species tends to infinity.

Similar if the initial populations correspond to point $C$, the first species dies out and the second species goes to infinity and if the initial populations correspond to point $D$, the second species dies out and the first species goes to infinity.

If the initial populations correspond to the point $E$ or $F$, the species populations converge to equilibrium populations $p/q, a/b$ and if the initial populations correspond to point $G$, $H$, the first and second species die out respectively.

Thus except when the initial populations correspond to points on curve $O'E$ and $O'F$, only one species will survive in the competition process and the species can coexist only when the initial population sizes correspond to point on the curve $E\ F$.

It is also interesting to note that while the initial populations corresponding to $A, E, B$ are quite closer to one another, the ultimate behavior of these populations are drastically different. For populations starting at $A$, the second species alone survives, for populations starting at $B$, the first species alone survives, while for population starting at $E$, both species can coexist.

Thus a slight change in the initial population sizes can have a catastrophic effect on the ultimate behaviour.

It may also be noted that for both prey-predator and competition models, we have obtained a great deal of insight into the models without using the solution of these equations (1),(2),(7) and (8).

By using numerical methods of integration with the help of computers, we can draw some typical trajectories in both cases an can get additional insight into the behaviour of these models.

## Stability:

We now examine the stability of the steady state $(x^*, y^*)$by using the perturbation technique
Let $x = x^*(1 + u), y = y^*(1 + v)$
……………………………………………….(13)
where u and v are very said and indicate small deviations from the equilibrium.

From equation (7), (8) and (13),

$$\frac{du}{dt} = -av - auv$$
$$\frac{dv}{dt} = -pu - puv$$

……………………………………………………….(14)
System of Equations. (14) is almost linear system and has (0, 0) as the critical point corresponding to the critical point $\left(\frac{p}{q}, \frac{a}{b}\right)$of the system of eqns. (7) and (8).
To examine the stability of the critical point $(0,0)$ of the system (14) we consider the related linear system,

$$\frac{du}{dt} = -av$$
$$\frac{dv}{dt} = -pu$$

……………………………………………………….(15)
$|A - \lambda I| = 0$, Where $A = \begin{pmatrix} 0 & -a \\ -p & 0 \end{pmatrix}$.

$$\begin{vmatrix} -\lambda & -a \\ -p & -\lambda \end{vmatrix} = 0$$

This implies that $\lambda^2 - ap = 0$

$$\lambda = \pm\sqrt{ap}$$
$$\frac{d^2u}{dt^2} = -a\frac{dv}{dt} = apu$$
$$\frac{d^2v}{dt^2} = -p\frac{du}{dt} = apv$$

…………………………………………………….(16)
The general solution of equations (15) and (16),

$$u = c_1 e^{\sqrt{(ap)t}} + c_2 e^{\sqrt{(ap)t}}$$

where $c_1$ and $c_2$ are arbitrary constants.

We thus and that $u \to \infty$ $as$ $t \to \infty$.

Similarly, on solving Eqn.(16), we find that $v \to \infty \, as \, t \to \infty$. Thus critical point (0, 0) is unstable saddle point of system (16) and hence of the system (15).

It is, therefore, clear that the steady state $(x^*, y^*)$ of the system (14) is unstable. The point $(x^*, y^*)$ moves on to either x-axis or y-axis in the $(x, y)$ -plane, depending on the initial conditions. We may be wondering in this case why we did not solve the system of Eqns.(16) analytically like we did for the prey-predator model. Yes! we can solve the system (14) and find its analytical solution in this case also.

# *3.5* SOLVED EXAMPLES

**Example 1:** For the system of equations:

$$\frac{dx}{dt} = x - y + xy$$

$$\frac{dy}{dt} = 3x - 2y - xy$$

……………………………………(13)

verify that $(0,0)$ is a critical point. Show that the system is almost linear and discuss the type and stability of the critical point $(0,0)$.

**Solution:** Clearly $(0,0)$ is a critical point of the system (13) can be written in the form,

$$\frac{dx}{dt} = x - y + f(x, y)$$

$$\frac{dy}{dt} = 3x - 2y + g(x, y)$$

where $f(x, y) = xy$ and $g(x, y) = -xy$

For checking the condition for almost linear system it is convenient to use polar coordinates by letting $x = rcos\theta, y = rsin\theta$.

Now $\frac{f(x,y)}{r} = \frac{r^2 cos\theta sin\theta}{r} = rcos\theta sin\theta \to 0$ as $r \to 0$.

Also $\frac{g(x,y)}{r} = -\frac{r^2 cos\theta sin\theta}{r} = -rsin\theta cos\theta \to 0$ as $r \to 0$.

Thus system (13) is almost linear. The related linear system in the neighbourhood of $(0,0)$ is:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

…………………………………………………..(14)

Eigenvalues of (15) are the roots of the equation

$$\begin{vmatrix} 1-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

This implies that $\lambda^2 + \lambda + 1 = 0$ …………………….(15)

$or\ \lambda = \frac{-1 \pm i\sqrt{3}}{2}$.

Therefore,

$$\lambda_1 = \frac{-1 + i\sqrt{3}}{2}, \lambda_2 = \frac{-1 - i\sqrt{3}}{2}.$$

Since the eigenvalues are conjugate complex of the form $\lambda \pm i\mu, \lambda, \mu$ real. Critical point $(0,0)$ of the system (14) is a spiral.

Also since $\lambda < 0$, it is asymptotically stable point. Since the system (14) is almost linear, critical point (0,0) of the system is also asymptotically stable spiral point.

**Example 2:** For the system of equations:

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -x + 2y$$

……………………………………(16)

Find the critical point of the system. Discuss the type and stability of the critical point. Write down the general solution of the system (16) and sketch the graph of its trajectories.

**Solution:** Clearly $(0,0)$ is a critical point of the system (16).

Eigenvalues of (16) are the roots of the equation

$$\begin{vmatrix} 1-\lambda & -0 \\ -1 & -2-\lambda \end{vmatrix} = 0$$

This implies that $\lambda^2 - 3\lambda + 2 = 0$

Therefore,

$$\lambda_1 = 1, \lambda_2 = 2.$$

Eigenvalues are real, distinct and of the same sign so the critical point is a mode. Also since $\lambda_1 > 0, \lambda_2 > 0$ it is unstable.

To find the general solution of the system (16), we find the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

Eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ is the solution of the equation

$$\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We see that $\binom{1}{1}$ is one possible eigenvector. Similarly $\binom{0}{1}$ is one possible eigenvector corresponding to the eigenvalue $\lambda_2 = 2$.

Then the general solution of the system (16) can be written as,

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

It implies that $x = c_1 e^t$

$x = c_1 e^t + c_2 e^{2t}$……………………….(17)

where $c_1, c_2$ are arbitrary constants.
For $c_1 = 0, x = 0$ and $x = c_2 e^{2t}$. In this case the trajectory is positive $y$ axis when $c_2 > 0$ and it is negative $y$ axis when $c_2 < 0$ and also since $y \to \infty\ as\ t \to \infty$, each path approaches $\infty$ as $t \to \infty$.
For $c_2 = 0, x = c_1 e^t; y = c_1 e^t$. This trajectory is half line $y = x, x > 0$ when $c_1 > 0$ and the half line $y = x, x < 0$, when $c_1 < 0$ and again both paths$\to \infty$ as $t \to \infty$.
When both $c_1$ and $c_2$ are $\neq 0$, the trajectories are parabolas

$$y = x + (c_2/c_1^2)x^2$$

which passes through the origin with slope 1. Each of these trajectories also approach $\infty$ as $t \to \infty$.
The sketch of the trajectories is shown,

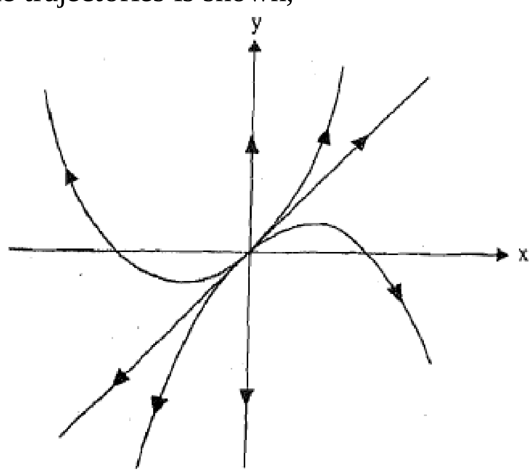


***Fig 3.5.1***

**Example 3:** Determine the type and stability of the critical point $(0,0)$ of the almost linear system

$$\frac{dx}{dt} = 4x + 2y + 2x^2 - 3y^2$$
$$\frac{dy}{dt} = 4x - 3y + 7xy$$

……………………………………(18)

Find the general solution of the corresponding linear system and sketch it's trajectories.

**Solution :** The auxiliary equation of the associated linear system

$$\frac{dx}{dt} = 4x + 2y$$
$$\frac{dy}{dt} = 4x - 3y$$

…………………………………………………………..(19)

is $(4 - \lambda)(-3 - \lambda) - 8 = (\lambda - 5)(\lambda + 4) = 0$.

The roots $\lambda_1 = -4$ and $\lambda_2 = 5$ are real unequal and have opposite sign. So the critical point (0,0) is an unstable saddle point of the system (19) and hence of the system (18).

Eigenvector corresponding to the eigenvalue $\lambda_1 = -4$ is the solution of the equation $\begin{pmatrix}1\\-4\end{pmatrix}$ and the corresponding to the eigenvalue $\lambda_2 = 5$ is $\begin{pmatrix}2\\1\end{pmatrix}$.

So the general solution of (19) can be written as,

$x = c_1 e^{-4t} + 2\, c_2 e^{5t}$

$x = -4c_1 e^{-4t} + c_2 e^{5t}$……………………….(20)

where $c_1, c_2$ are arbitrary constants.

For $c_1 = 0, x = 2c_2 e^{5t}, y = c_2 e^{5t}$ .

This trajectory is the half line $y = \frac{x}{2}, x > 0$ when $c_2 > 0$ and half line $y = \frac{x}{2}, x > 0$ when $c_2 < 0$.

Also x→ ∞ , $y \to \infty$, as $t \to \infty$.

For $c_2 = 0, x = c_1 e^{t}; y = -4c_1 e^{-4t}$.

This, trajectory is half line $y = -4x, x > 0$ when $c_1 > 0$ and the half line $y = -4x, x < 0$, when $c_1 < 0$.

Both the trajectories approach and enter the origin as $t \to \infty$. $c_1$ and $c_2$ are $\neq 0$, the trajectories are parabolas

$$y = x + (c_2/c_1^2)x^2$$

which passes through the origin with slope 1.

Each of these trajectories also approach $\infty$ as $t \to \infty$. If $c_1 \neq 0$, $c_2 \neq 0$, solution (20) represents curved trajectories none of which approaches $(0,0)$ as $t \to \infty$.

The sketch of the trajectories is shown,

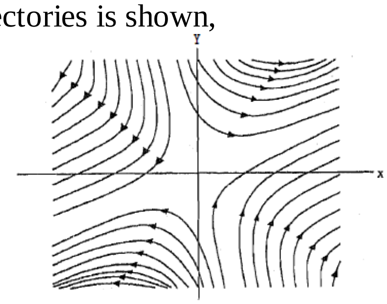

***Fig 3.5.2***

# ***3.6 LIMITATIONS***

## **Prey-Predator Models:**

From the above discussion it is clear that there is no strategy for the Volter ra system to maintain its trivial state.As can be seen from Figure 3.3.1, for a small change in the control phase the prey predator system changes from one orbit to another**.** In mathematical terms, we explain the behavior of these systems by saying that Volterra orbits have no "roughness". We also observed that

i. in the absence of predators, the prey population grow, unbounded exponentially and
ii. in the absence of prey, the predator population goes' to extinction due to lack of food.

These phenomena arc not found to occur in reality.
In the absence of predators, the prey population is expected to increase rapidly to start with; after considerable increase in its size, its growth must be retarded due to crowding effects and ultimately, it cannot increase beyond a limiting level.

On the other hand, when prey (food) is not available, the predator population is expected to decrease rapidly in the beginning; after some time, the predators are likely to adjust themselves with the situation by finding alternative sources of food. So far we discussed mathematical model for two 'species in which one species preyed upon the other. In contrast to this, we shall now consider two species. Which compete with each other for the food available in their common environment.

## Competition models

The major limitation of this model lies in the extreme outcome that one species may be such a strong conipetitor that it, may force the other species to go extinct. In the natural environment, however populations are distributed over space, and space is strongly inhomogeneous. A species that is completely out-competed by another species, may find various refuges where it can continue to survive, at least in small numbers.

It is also found in natural environment that two species competing for a common resource for their survival coexist. This model fails to exhibit such coexistence of two competing species.

Another limitation of the model lies in the observation that each species grows unbounded in the absence of the other. This can never happen in reality -.there must be carrying capacity for the growing species.

# *3.7 SUMMARY*

In the present unit we explain the prey – predator model In briefly the predator-prey equations is an ecological system. Two linked equations model the two species which depend on each other: One is the prey, which provides food for the other, the predator. Both prey and predator populations grow if conditions are right.

We also explain competition models relates the population density and carrying capacity of two species to each other and includes their overall effect on each other.

## 3.8 GLOSSARY

i. **Population dynamics:** The population dynamics is a description (and prediction) of the size and age composition of a group of individuals of one particular species, and how the number and age composition of individuals in a population change over time.

ii. **Predators in mathematics:** A boundary – value problem for a system of two non-linear differential equations in partial derivatives. A stationary state stability is studied. A variational method is used to build a numerical solution.

iii. **Trajectory:** A trajectory is a path taken up by a moving object that is following through space as a function of time. Mathematically, a trajectory is described as a position of an object over a particular time.

iv. **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

v. **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

vi. **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

**CHECK YOUR PROGRESS**

1. **Fill in the blanks:**
   **i.** Assumptions in the predator-prey model, including:
   **a.** Predators can eat …….
   **b.** The environment is…………..
   **ii.** Competition models in mathematical modelling are used to study the interactions between …………………..
2. **True/False**
   **i.** In predator-prey model we assume that there are no predators, the prey species will grow at a rate proportional to the population of the prey species. **True\False**.

**ii.** In competition model if there is populations of two species competing for the same resources, then each species grows in the absence of the other species, and the rate of growth of each species increases due to the presence of the other species. **True\False**.

**3.** This Question consist of two statements – Assertion (A) and Reason (R). Answer these questions selecting the appropriate option given below:

**Assertion (A):** In the absence of a predator, the prey population growth will always be exponential.

**Reason (R):** Exponential growth is when the resources and the environment allow an organism to realise fully its innate potential to grow in numbers.

**i.** Both A and R are true and R is the correct explanation of A.
**ii.** Both A and R are true and R is not the correct explanation of A.
**iii.** A is true but R is false.
**iv.** A is False but R is true.

## *3.9 REFERENCE*

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## *3.10 SUGGESTED READINGS*

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3. A. Friedman and W. Littman, (1994) Industrial Mathematics: A Course in Solving Real World Problems, Philadelphia, SIAM.

**4.** F.R. Giordano and M.D. Weir (1985). A First Course in Mathematical Modeling, Monterey, Brooks/Cole.

## *3.11 TERMINAL QUESTIONS*

**TQ1:** If each of the following problems, verify that (0, 0) is a critical point, show that the system is almost linear, arid discuss the type and stability of the critical point (0, 0).

a) $$\frac{dx}{dt} = y + x(1 - x^2 - y^2)$$
$$\frac{dt}{dx} = -x + y(1 - x^2 - y^2)$$

b) $$\frac{dx}{dt} = 2x + y + xy^3$$
$$\frac{dy}{dt} = x - 2y - xy$$

**TQ2:** Find the critical point of the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x$$

and discuss its nature and stability. Find the general solution of the system and sketch its trajectories.

## *3.12 ANSWERS*

**TQ1** a) Spiral point, unstable b) Saddle point, unstable

**TQ2** Critical point (0, 0) is a center. General solut.ion of the system is

$$x = -c_1 \sin t + c_2 \cos t$$
$$y = c_1 \cos t + c_2 \sin t$$

So x(t) and y(t) are periodic and, each trajectory is a closed curve surrounding the origin. Also we have from given system $\frac{dy}{dx} = -\frac{x}{y}$ whose general solution $x^2 + y^2 = c^2$ this yields all the curve which are circles. Also from the differential equations, from the region x>0, y>0 we see that $\frac{dx}{dt} < 0$, means $x$ decreases with $t$, $\frac{dy}{dt} > 0$ means $y$ increases with $t$. This the trajectories are anticlockwise the circle.

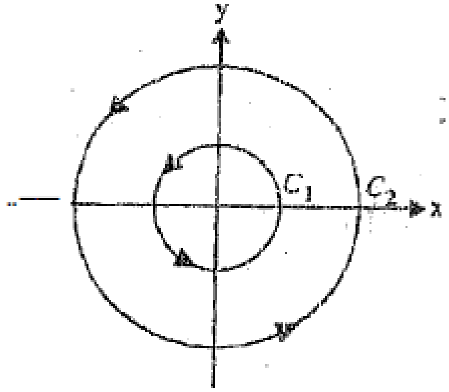



*Fig 3.12.1*

## CHECK YOUR PROGRESS

**CHQ1. ia)** without limit. **ib)** constant. **ii** competing species.

**CHQ2**. **i)** True. **ii** False.

**CHQ3. i.** Both A and R are true and R is the correct explanation of A.

# UNIT 4: MATHEMATICAL MODELLING OF EPIDEMICS

## CONTENTS:



## *4.1 INTRODUCTION*

In previous units we have discussed about why mathematical modelling needed and what is a role of linear - Non-linear growth and decay models. We have discussed in previous unit mathematical modelling in population dynamics.

This unit is a presentation of epidemic model. Epidemiology is a discipline, which deals with the study of infectious diseases in a population. It is concerned with all aspects of epidemic, e.g. spread, control, vaccination strategy etc.

What is epidemic principle?

Epidemic refers to an increase, often sudden, in the number of cases of a disease above what is normally expected in that population in that area. Outbreak carries the same definition of epidemic, but is often used for a more limited geographic area.

The aim of epidemic modeling is to understand and if possible control the spread of the disease. In this context following questions may arise:

- How fast the disease spreads?
- How much of the total population is infected or will be infected?
- Control measures!
- Effects of Migration/ Environment/ Ecology, etc. Persistence of the disease.

Infectious diseases are basically of two types:

**Acute (Fast Infectious):** Stay for a short period (days/weeks) e.g. Inuenza, Chickenpox etc.

**Chronic Infectious Disease:** Stay for larger period (month/year) e.g. hepatitis.

In general the spread of an infectious disease depends upon:

- Susceptible population,
- Infective population,
- The immune class, and the mode of transmission.

**Assumptions:**

We shall make some general assumptions, which are common to all the models and then look at some simple models before taking specific problems.

- The disease is transmitted by contact (direct or indirect) between an infected individual and a susceptible individual.
- There is no latent period for the disease, i.e., the disease is transmitted instantaneously when the contact takes place.
- All susceptible individuals are equally susceptible and all infected ones are equally infectious.
- The population size is large enough to take care of the fluctuations in the spread of the disease, so a deterministic model is considered.
- The population, under consideration is closed, i.e. has a fixed size.

## 4.2 OBJECTIVES

After studying this unit, learner will be able to

**i.** Explain the Susceptible-Infected (SI) Epidemic Model.
**ii.** Describe the Susceptible-Infected-Susceptible (SIS) Model.
**iii.** Define the SIS Model with constant number of carriers.
**iv.** Analyze the Simple epidence model with carriers and other models.

## 4.3 SUSCEPTIBLE – INFECTIVE (SI) EPIDEMIC MODEL

Let $S(t)$ be the number of susceptible (i.e., those who can get a disease) and $I(t)$ infected persons (i.e., those who have already got the disease) and recovered class, denoted by $R(t)$(i.e., persons who have recovered from the disease).

The steps we are following here:

- Construct ODE (Ordinary Differential Equation) models
- Relationship between the diagram and the equations
- Alter models to include other factors.

ODEs deal with populations, not Individuals. We assume the population is well-mixed. We keep track of the inflow and the outflow.

Initially let there be $n$ susceptible and one infected person in the system, so that,

$$S(t) + I(t) = n + 1, S(0) = n,\ \ I(0) = 1 \dots\dots\dots\dots (1)$$

A susceptible person gets infected when he comes in contact with an infected one. Mathematically, we can say that the rate of increase of the infected class is proportional to the product of the susceptible and infected persons. Hence, the susceptible class also decreases at the same rate.

So that we get the system of differential equations:

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI \dots\dots\dots (2)$$

Where $\beta > 0$.

So that,

$$\frac{dS}{dt} + \frac{dI}{dt} = 0,$$

$$S(t) + I(t) = Constant = S(0) + I(0) = n + 1$$

$$\Rightarrow S(t) + I(t) = n + 1 \ \ldots\ldots\ldots\ldots (3)$$

And

$$\frac{dS}{dt} = -\beta SI(n + 1 - S),$$
$$\frac{dI}{dt} = \beta I(n + 1 - I) \ldots . (4)$$

Integrating, the differential equation we obtain

$$\int \frac{dS}{S(n + 1 - S)} = -\beta dt \Rightarrow \frac{1}{n + 1}\int \left[\frac{1}{n + 1 - S} + \frac{1}{S}\right] ds$$
$$= -\int \beta dt,$$

This implies

$-\ln(n + 1 - S) + \ln(S) = -(n + 1)\beta t + \quad A$(constant).

At $t = 0, S(0) = n$.

This implies $A = \ln(n)$.

$$\Rightarrow ln\left[\frac{S}{n(n+1-S)}\right] = -(n + 1)\beta t \Rightarrow \frac{S}{n(n+1-S)} = e^{-(n+1)\beta t},$$

$$S(t) = \frac{n(n + 1)}{n + e^{(n+1)\beta t}},$$
$$I(t) = (n + 1) - S(t) = n + 1 - \frac{n(n + 1)}{n + e^{(n+1)\beta t}}$$
$$= \frac{n + 1}{1 + ne^{-(n+1)\beta t}}$$
$$I(t) = \frac{n + 1}{1 + ne^{-(n+1)\beta t}}$$

….…………………………(5)

so that

$$\lim_{n\to\infty} S(t) = 0, \ \lim_{n\to\infty} I(t) = n + 1 \ \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Therefore, we conclude that as time increases, all the susceptible persons will become infected.

**Example.** If the contact rate $(\beta)$ be $0.001$ and the number of susceptiles $(n)$ be $2000$ initially, determine:

**a.** The number of susceptible left after $3$ weeks;
**b.** The density of susceptible when the rate of appearance of new cases is maximum;
**c.** The time (in weeks) at which the rate of appearance of new cases is a maximum.
**d.** The maximum rate of appearance of new cases and
**e.** The epidemic curve.

**Solution:**

a) The number of susceptible left after 3 weeks is:

$$x = \frac{n+1}{1+ne^{-(n+1)\beta t}}$$

$$x = \frac{2000+1}{1+2000e^{-(2000+1)0.001\times 3}} = \frac{2001}{1+2000e^{-2001\times 0.001\times 3}}$$

$$= \frac{2001}{1+2000e^{-6.003}} = \frac{2000\times 2001}{2000+e^{-6.003}} \approx 1664.$$

b) When the rate of appearance of new cases is maximum, the density of susceptibles is

$$x = \frac{n+1}{2} = \frac{2001}{2} = 1000.5 \approx 1001$$

c) The time at which the rate of appearance of new cases is a maximum is

$$t = \frac{\ln n}{\beta(n+1)} = \frac{\ln 2000}{(.001)(2001)} = \frac{ln 2000}{2.001} = 1.6 \text{ weeks}$$

d) The maximum rate of appearance of new cases is

$$-\frac{dx}{dt} = \beta\left(\frac{n+1}{2}\right)^2 = (.001)(1000.5)^2 \approx 1001$$

The epidemic curve is obtained by plotting $-\frac{dx}{dt}$ against $t$ in the relation

$$x = \frac{n+1}{1+ne^{-(n+1)\beta t}}$$

$$-\frac{dx}{dt} = \frac{\beta(n+1)^2 ne^{(n+1)\beta t}}{\{1+ne^{-(n+1)\beta t}\}^2}$$

We plot $-\frac{dx}{dt}$ against $t$ using the values $\beta = 0.001, n = 200$ and $t = 1,2,3,4,5,6,7,8,9,10.$

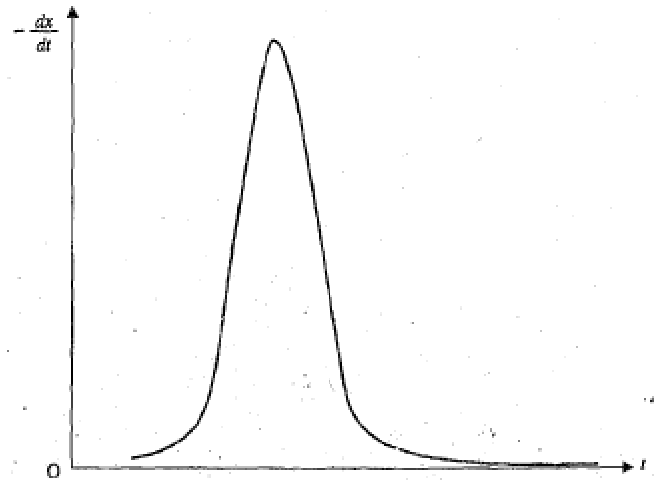



***Fig 4.5.1***

## 4.4. SUSCEPTIBLE - INFECTED SUSCEPTIBLE (SIS) MODEL

We model such a system by dividing a population into three distinct groups: susceptible ( ), -infected ( ) and -infected ( ), based on the Susceptible-Infectious-Susceptible (SIS) model. Once the individuals in the -infected group recover from the disease, they gain no permanent immunity.

The SIS (Susceptible – Infectious - Susceptible) Model was introduced by Weiss and Dishon to study infections in a closed population of n individuals, infections that do not confer any long lasting immunity (gonorrhea, or the common cold, for example).

Here, a susceptible person can become infected at a rate proportional to $SI$ and an infected person can recover and become susceptible again at a rate $\gamma I$.

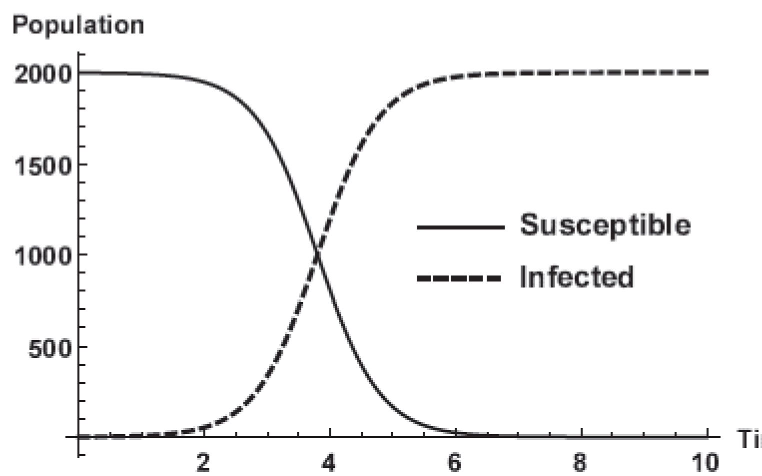
So that,

$$\frac{dS}{dt} = -\beta SI + \gamma I, \frac{dI}{dt} = \beta SI - \gamma I, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

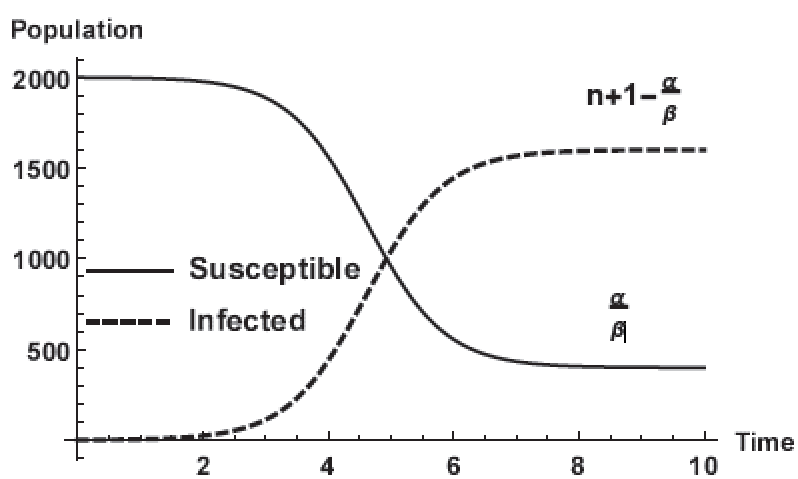
which gives

$$\frac{dI}{dt} = (\beta(n+1) - \gamma)I - \beta I^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Recovery rate $\alpha = \frac{1}{\text{Infectious period}}$ is taken to be constant though infection period (the time spent in the infectious class) is distributed about a mean value & can be estimated from the clinical data. In the above model, it is assumed that the whole population is divided into two classes susceptible and infective; and that if one is infected then it remains in that class. However, this is not the case, as an infected person may recover from the disease.




(a) SI model.




(b) SIS model.

***Fig.4.3.1***

The above figures show the dynamics of epidemic models.

**a)** SI model with $\beta = 0.001, I(0) = 1$,

**b)** SIS model with $\beta = 0.001, \alpha = 0.4$,

$$S(0) = 2000, I(0) = 1.$$

$$\frac{dS}{dt} + \frac{dI}{dt} = 0,$$

$S(t) + I(t) = K$ (constant).

This implies that

$K = S(0) + I(0) = n + 1 \Rightarrow S(t) + I(t) = n + 1.$

$$\frac{dS}{dt} = -[(n+1)\beta + \alpha]S + \beta S^2 + (n+1)\alpha,$$

$\frac{dI}{dt} = [(n+1)\beta - \alpha]I - \beta I^2 = cI - \beta I^2,$

where $c = (n+1)\beta - \alpha$.

$$\Rightarrow \frac{dI}{I\left(1 - \frac{\beta}{c}I\right)} = cdt \Rightarrow \frac{\frac{c}{\beta}dI}{I\left(\frac{c}{\beta} - I\right)} = cdt$$

$$\Rightarrow \left[\frac{1}{I} + \frac{1}{\frac{c}{\beta} - I}\right] = cdt.$$

Integrating we obtain,

$$\ln(I) - ln\left(\frac{c}{\beta} - I\right) = ct + B(constant).$$

Now, $I(0) = 1 \Rightarrow B$

$= ln(I) - \ln\left(\frac{c}{\beta} - I\right) + ln\left(\frac{c}{\beta} - I\right) = ct \Longrightarrow \frac{I\left(\frac{c}{\beta} - I\right)}{\left(\frac{c}{\beta} - I\right)} = e^{ct},$

This implies that

$$I(t) = \frac{\frac{c}{\beta}}{1 + \left(\frac{c}{\beta} - 1\right)e^{-ct}}$$

$$= \frac{(n+1) - \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{c}{\beta} - 1\right)e^{-[(n+1)\beta - \alpha]t}}.$$

$S(t) = n + 1 - I(t)$

$$= n + 1 - \frac{(n+1) - \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{c}{\beta} - 1\right)e^{-[(n+1)\beta - \alpha]t}},$$

$$\Rightarrow S(t) = \frac{(n+1)\left(n+1-\frac{\alpha}{\beta}-1\right)e^{-[(n+1)\beta-\alpha]t}+\frac{\alpha}{\beta}}{1+\left(n+1-\frac{\alpha}{\beta}-1\right)e^{-[(n+1)\beta-\alpha]t}}$$

As $t \to \infty, S \to \frac{\alpha}{\beta}$ and $I \to n+1-\frac{\alpha}{\beta}$;
provided $(n+1)\alpha - \beta > 0$.

Hence in this case, a fraction of susceptible persons will be there, which have not been infected or a fraction of infected persons have recovered and becomes susceptible again.

## 4.5. SUSCEPTIBLE - INFECTED – RECOVERED (SIR) MODEL

This model was developed by Kermack and McKendrick and is given by the set of differential equations as follows:

$$\frac{dS}{dt} = -\beta SI,$$

$$\frac{dI}{dt} = \beta SI - \alpha I,$$

$$\frac{dI}{dt} = \alpha I, (\alpha, \beta > 0).$$

It Is assumed that the susceptible become infected when they come in contact with one another $(\beta SI)$ and a fraction of the infected class $(\alpha I)$ recovers from the disease and moves to the recovered class. Now,

$$\frac{dS}{dR} = \frac{dS}{dt}.\frac{dt}{dR} = \frac{-\beta SI}{\alpha I} = \frac{-\beta}{\alpha}S,$$

$$\Rightarrow \frac{dS}{S} = \frac{-\beta}{\alpha}dR \Rightarrow ln(S) = \frac{-\beta}{\alpha}R + constant = \ln(n).$$

Initially,

$$\frac{dI}{dS} = \frac{dI}{dt}.\frac{dt}{dS} = \frac{\beta SI - \alpha I}{-\beta SI} = -1 + \frac{\alpha}{\beta}ln(S) + constant.$$

Initially at $t = 0$,

$$S(0) = n, I(0) = 1 \Rightarrow 1 + n - \frac{\alpha}{\beta}\ln(n).$$

$$\Rightarrow I(t) = -S + \frac{\alpha}{\beta} ln(S) + n + 1 - \frac{\alpha}{\beta} ln(n)$$
$$= n + 1 - S + \frac{\alpha}{\beta} ln\left(\frac{S}{n}\right).$$

Since $S = ne^{\frac{-\beta}{\alpha}\boldsymbol{R}}$, we have,

This implies that $\frac{dR}{dt} = \ \alpha\left[n + 1 - n\left(e^{-\frac{\beta}{\alpha}R}\right) - R\right]$,

$$\Rightarrow \frac{dR}{dt} = \alpha\left[n + 1 - n\left(1 - \frac{\beta}{\alpha}R + \frac{\beta^2}{\mathbf{2}\alpha^2}\right) - R\right],$$

(Assuming $\frac{R}{\frac{\alpha}{\beta}}$ is small),

$$\Rightarrow \frac{dR}{dt} = \alpha\left[1 - n\frac{\beta^2}{2\alpha^2}\left\{R^2 - \frac{2\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)\right\}R\right],$$
$$= \alpha\left[1 - n\frac{\beta^2}{2\alpha^2}\left\{R - \frac{\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)\right\}^2\right] + \frac{\alpha^2}{2n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)^2,$$
$$= \frac{n\beta^2}{2\alpha^2}\left[\frac{2\alpha^2}{n\beta^2} + \frac{\alpha^4}{n^2\beta^4}\left(\frac{n\beta}{\alpha} - 1\right)^2 - \left\{R - \frac{\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)\right\}^2,\right]$$

$= \frac{n\beta^2}{2\alpha^2}[B^2 - (R - A)^2]$ where $A = \frac{\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)$, and

$B^2 = \frac{2\alpha^2}{n\beta^2} + \frac{\alpha^4}{n^2\beta^4}\left(\frac{n\beta}{\alpha} - 1\right)^2$. Integrating we get,

$$\int \frac{dR}{B^2 - (R - A)^2} = \int n\frac{\beta^2}{2\alpha}dt \Rightarrow \frac{1}{B}tanh^{-1}\left(\frac{R - A}{B}\right)$$
$$= n\frac{\beta^2}{2\alpha}t + constant.$$

Initially at $t = 0, R(0) = 0$.

This implies constant $= \frac{1}{B}tanh^{-1}\left(\frac{-A}{B}\right) = -\frac{1}{B}tanh^{-1}\left(\frac{A}{B}\right)$.

$$\Rightarrow \frac{1}{B}tanh^{-1}\left(\frac{R - A}{B}\right) = \frac{n\beta^2}{2\alpha}t - \frac{1}{B}tanh^{-1}\left(\frac{A}{B}\right).$$
$$\frac{R - A}{B} = \ \tanh\left[B\frac{n\beta^2}{2\alpha}t - \frac{1}{B}tanh^{-1}\left(\frac{A}{B}\right)\right],$$

This implies that

$$R(t) = A + B\tanh\left[B\frac{n\beta^2}{2\alpha}t - \frac{1}{B}tanh^{-1}\left(\frac{A}{B}\right)\right].$$

Therefore,

$$S(t) = ne^{-\frac{\beta}{\alpha}\left[A + B\tanh\left[B\frac{n\beta^2}{2\alpha}t - \frac{1}{B}tanh^{-1}\left(\frac{A}{B}\right)\right],\right]}$$

and

$$I(t) = n + 1 - S(t) - R(t).$$

The numerical solution of the model shows that both susceptible and infected goes to zero and there is a full recovery.

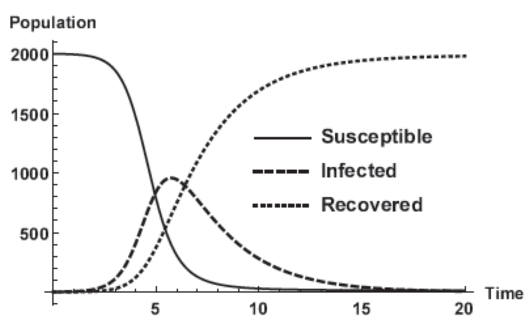
## 4.6 SUSCEPTIBLE - INFECTED – RECOVERED - SUSCEPTIBLE (SIRS) MODEL

A refinement of the SIR model can be made by assuming that the recovered person becomes susceptible again due to loss of immunity at a rate proportional to the population in recovery class $R$, with proportionality constant $\gamma$. The following differential equations describe the model:

$$\frac{dS}{dt} = -\beta SI + \gamma R,$$

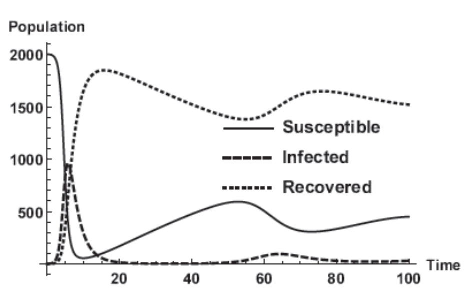
$$\frac{dI}{dt} = \beta SI - \alpha I,$$

$$\frac{dI}{dt} = \alpha I - \gamma R, (\alpha, \beta, \gamma > 0).$$




(a) SIR model.

***Fig 4.6.1***




(b) SIRS model.

***Fig 4.6.2***

Figure 4.6.2 shows the dynamics of SIRS model for $\beta = 0.001$, $\alpha = 0.4$ and $\gamma = 0.01$.

The figures show the dynamics of epidemic models.

(a) SIR model with $\beta = 0.001$, $\alpha = 0.4, S(0) = 2000, I(0) = 1, R(0) = 0$.

(b) SIRS model with $\beta = 0.001$, $\alpha = 0.4, S(0) = 2000, I(0) = 1, R(0) = 0$.

We can use the SIRS model to capture the dynamics of COVID-19. The Susceptible population becomes infected by COVID-19 at a rate $\beta$ (per-capita Effective contact rate), which is the number of effective contacts made by a given individual per unit time. We are trying to minimize the value of $\beta$ by practicing social distancing. Once infected, the susceptible population ($\beta$ moves to the infected class. The infected class recovers from the virus by hard immunity of individual (since no vaccine is available) but have the chance to reinfection.

## *4.4 LIMITATIONS*

The limitations of above methods are as following:

- **i.** The accuracy of a model is dependent on the assumptions it's based on.
- **ii.** Some models are too simple to explain all types of epidemics.
- **iii.** Some models work well for certain diseases, but not for others.
- **iv.** Different models are needed to account for different ways diseases spread, such as through water or by a vector like a mosquito

## 4.5 SUMMARY

In the present unit we explain the following models

i. **Susceptible – Infective (SI) Model:** The SI model is a simple mathematical model that describes the spread of an infectious disease through a population that is divided into two groups: susceptible and infected.

ii. **Susceptible – Infected - Susceptible (SIS) Model:** The SIS (Susceptible-Infectious-Susceptible) model is a mathematical model that describes how infectious diseases spread through a population. It's a fundamental tool in epidemiology that helps predict the spread of disease and assess public health interventions.

iii. **Susceptible - Infected-Recovered (SIR) Model:** In a standard SIR model, the host population is divided into susceptible, infected and recovered individuals, denoted by S(t), I(t) and R(t), respectively.

iv. **Susceptible – Infected – Removed - Susceptible (SIRS) Model:** The improve form of the SIR model can be made by assuming that the recovered person becomes susceptible again due to loss of immunity at a rate proportional to the population in recovery class , with proportionality constant.

## 4.6 GLOSSARY

i. **Population dynamics:** The population dynamics is a description (and prediction) of the size and age composition of a group of individuals of one particular species, and how the number and age composition of individuals in a population change over time.

ii. **Predators in mathematics:** A boundary – value problem for a system of two non-linear differential equations in partial derivatives. A stationary state stability is studied. A variational method is used to build a numerical solution.

iii. **Trajectory:** A trajectory is a path taken up by a moving object that is following through space as a function of time. Mathematically, a trajectory is described as a position of an object over a particular time.

iv. **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

v. **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

vi. **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

## CHECK YOUR PROGRESS

1. In susceptible – infective epidemic model (SI) the rate of increase of the infected class is ………….. to the product of the susceptible and infected persons. Hence, the susceptible class also ………… at the same rate.
2. The SIS (Susceptible – Infectious - Susceptible) Model is to study infections in a …………… of n individuals, infections that do not confer any long lasting immunity.
3. The numerical solution of the Susceptible - Infected–Recovered (SIR) Model shows that both susceptible and infected goes to ……. and there is a full recovery.
4. In Susceptible - Infected–Recovered- Susceptible (SIRS) model the recovered person becomes ……… again due to loss of immunity at a rate ……….. to the population in recovery class $R$, with proportionality constant $\gamma$.

## 4.7 REFERENCE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## 4.9 TERMINAL QUESTIONS

1. What is an epidemic model?
   ......................................................
2. What are the different types of epidemic models?
   .......................................................
3. What is epidemic principle?
   .........................................................
4. Explain SI and SIS epidemic model?
   ............................................
5. Explain the SIR and SIRS models?

## *4.10 ANSWERS*

**CHECK YOUR PROGRESS**

**CYP1**. Proportional, decreases
**CYP2.** Closed Population.
**CYP3.** Zero.
**CYP4.** Susceptible, Proportional.

# COURSE NAME: MATHEMATICAL MODELLING

# COURSE CODE: MAT 610

## BLOCK-II

## MATHEMATICAL MODELLING - II

# UNIT 5: MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS

## CONTENTS:



## *5.1 INTRODUCTION*

In the previous unit we explain the Susceptible – Infective Model, Susceptible – Infected - Susceptible (SIS) Model, Susceptible – Infected - Recovered (SIR) Model and Susceptible-Infected - Removed-Susceptible (SIRS**)** Model. Present unit is the explanation of mathematical modelling through difference equations. In using of modelling with difference equations we may avoid modelling through differential equations. The biological and social scientist who do not know calculus and transcendental number like $e$ can still work with difference equation models and some important consequences of these models can be deduced with the help of calculators by even high school students.

What is the meaning of difference equation?

A difference equation is an equation involving differences. One can define a difference equation as a sequence of numbers that are generated recursively using a rule to the previous numbers in the sequence the difference equation $T_{n+1} = T_n + (n+1)$, with $T_0 = 0$ is a sequence of triangular numbers $0, 1, 3, 6, 10, 15, 21, \ldots$. where $n = 0,1,2,3,4,5.6$ ....One can also define difference equation as an iterated map $x_{n+1} = f x_n$.

## 5.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Explain the mathematical modelling through difference equation.
- **ii.** Describe the basic theory of linear difference equations with constant coefficients.
- **iii.** Evaluate the Solution of Homogeneous Equations
- **iv.** Analyze Difference Equations: Equilibria and Stability
- **v.** Discuss Linear Difference Equations
- **vi.** Identify System of Linear Difference

## 5.3 LINEAR DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

Consider the linear difference equation of the form:

$$c_0 u_n + c_1 u_{n-1} + c_2 u_{n-2} = f(n) \ldots\ldots\ldots..(1)$$

The difference equation is homogeneous if $f(n) = 0$, otherwise it is non-homogeneous. The order of the difference equation is the difference between the largest $(n)$ and smallest $(n-2)$ arguments appearing in the difference equation with unit interval. Thus, the order of the equation (1) is $2$. Equation (1) is a linear difference equation with constant coefficients as the coefficients of the successive differences are constants and the differences of successive orders are of the first degree.

## 5.4 SOLUTION OF HOMOGENEOUS EQUATIONS

**a.** Consider the first – order homogeneous linear difference equation

$$u_n - k(u_{n-1}) = 0 \ldots\ldots\ldots.(2)$$

Putting $n = 1,2,3$ ....

$$u_1 = ku_0,$$
$$u_2 = ku_1 = k(ku_0) = k^2u_0,$$
$$u_3 = ku_{n-1} = k^nu_0,$$

……………………………………………………………..

Therefore,

$u_n = c\,k^n$, ($c$ is an arbitrary constant) is a general solution of (2).

**b.** Consider the first – order homogeneous linear difference equation

$$u_n = au_{n-1} + b = 0\ (n = 1,2 \ldots\ldots),$$

where $a$ and $b$ are constants. Now,

$$u_n = a(u_{n-1} + b) + b = a^2(u_{n-1}) + b(a+1)$$
$$= a^2(u_{n-3} + b) + b(a+1) = a^3u_{n-3} + b(a^2 + a + 1)$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= a^nu_0 + b(a^{n-1} + a^{n-2} + \cdots a^2 + a + 1)$$
$$= a^nu_0 + b\left(\frac{1-a^n}{1-a}\right)(\text{if } a < 1),$$
$$= a^nu_0 + nb(\text{if } a = 1),$$

which is the required solution of the first-order linear difference equation, $u_n = au_{n-1} + b$.

**c.** Consider the second – order homogeneous linear difference equation

$$a_0u_n + a_1u_{n-1} + au_{n-2} = 0 \ldots\ldots(3)$$

Let the solution of (3) be of the form $u_n = ck^n (c \neq 0)$. Substituting it in (3), we obtain

$$a_0\ ck^n + a_1\,ck^{n-1} + a_2\,ck^{n-1} = 0.$$

This implies $a_0k^2 + a_1k + a_2 = 0$,

Which is called the auxiliary equation.

**i.** If the auxiliary equation has two distinct real roots, $m_1$ and $m_2$ then, $c_1\, {m_1}^n + c_2\, {m_2}^n$is the general solution of (3), $c_1$ and $c_2$ are arbitrary constants.
If the roots of the auxiliary equation are real and equal, $m_1$= $m_2 = m,$ then, $(c_1 + c_2 n)\, m^n$is the general solution of (3), $c_1$ and $c_2$ are arbitrary constants.

**ii.** If the auxiliary equation has imaginary roots (which occur in conjugate pairs), $\alpha + i\beta$ and $\alpha - i\beta$, then $r^n(c_1 \cos n\theta + \ c_1 \sin n\theta)$ is the general solution of (3), $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$, $c_1$ and $c_2$ are arbitrary constants.

## *5.5 EXAMPLES*

**Example 5.5.1.** Obtain the difference equation by eliminating the arbitrary constants from $u_n = A2^n + B(-3)^n$.

**Solution:** Given, $u_n = A2^n + B(-3)^n$.
This implies that $u_{n+1} = A2^{n+1} + B(-3)^{n+1}$.
$\Rightarrow u_{n+2} = A2^{n+2} + B(-3)^{n+2}$.
Therefore, $u_{n+1} = 2A2^n - 3B(-3)^n$.
$u_{n+2} = 4A2^n + 9B(-3)^n$.

Solving, we get
$A = \frac{3u_{n+1}+u_{n+2}}{102^n}$ and $B = \frac{u_{n+2}-2u_{n+1}}{15(-3)^n}$.
Hence, the required difference equation is,

$$u_n = \frac{3u_{n+1} + u_{n+2}}{10} + \frac{u_{n+2} - 2u_{n+1}}{15},$$

$$u_{n+2} + u_{n+1} - 6u_n = 0.$$

**Example 5.5.2.** Find $u_n$ if $u_1 = 1$ and $u_{n+2} + 16u_n = 0$.

**Solution:** Let $u_n = ck^n (c \neq 0)$ be a solution of
$u_{n+2} + 16u_n = 0,$
Then the required auxiliary equation is
$k^2 + 16 = 0 \Rightarrow k = \pm 4i.$

The general solution is

$$u_n = c_1(4i)^n + c_2(-4i)^n,$$
$$= 4^n\left[c_1 e^{in\pi/2} + c_2 e^{-in\pi/2}\right],$$
$$u_n = 4^n[A_1 cos(n\pi/2) + A_2 sin(n\pi/2)],$$

Where $A_1$ and $A_2$ are arbitrary constants.

Now, $u_0 = 0$ and $u_1 = 1$ implies $A_1 = 0$ and $A_2 = \frac{1}{4}$.

Therefore,

$u_n = 4^{n-1}[sin(n\pi/2)]$ is the required solution.

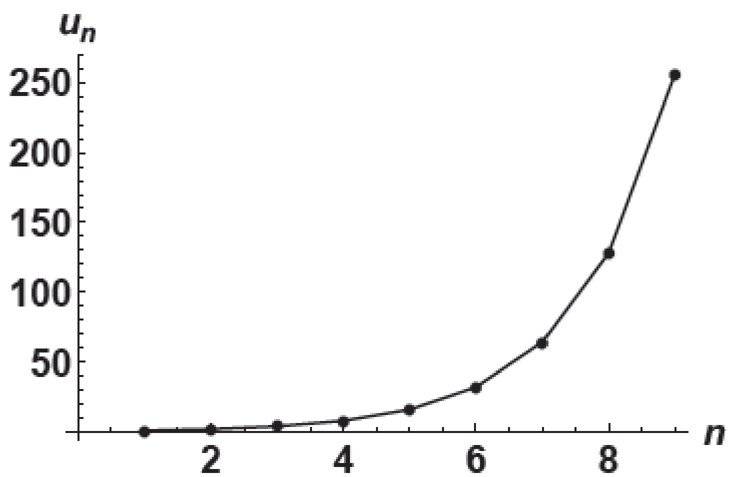


***Fig 5.5.1***
$k > 1$

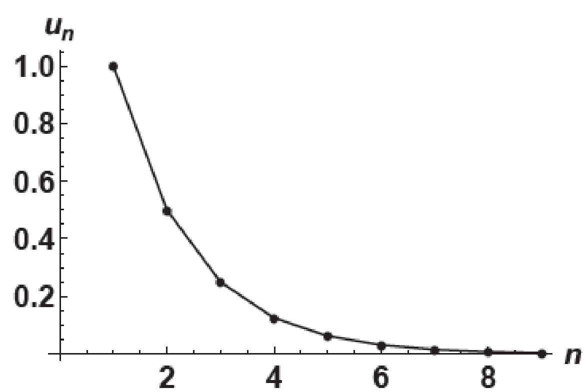


***Fig 5.5.2***
*0*$< k < 1$

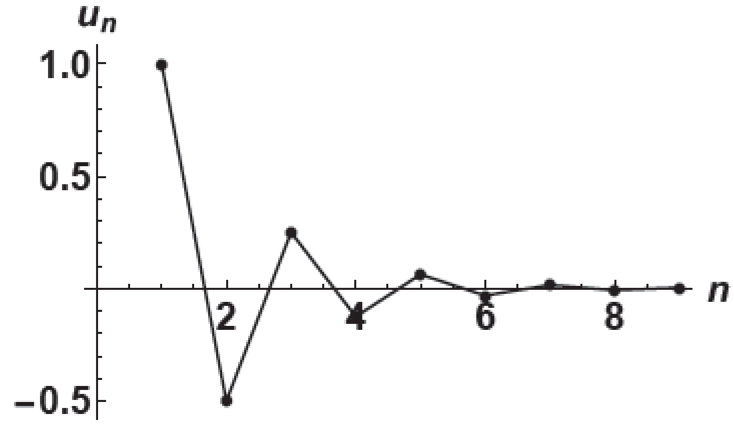


***Fig 5.5.3***
$-1 < k < 0$

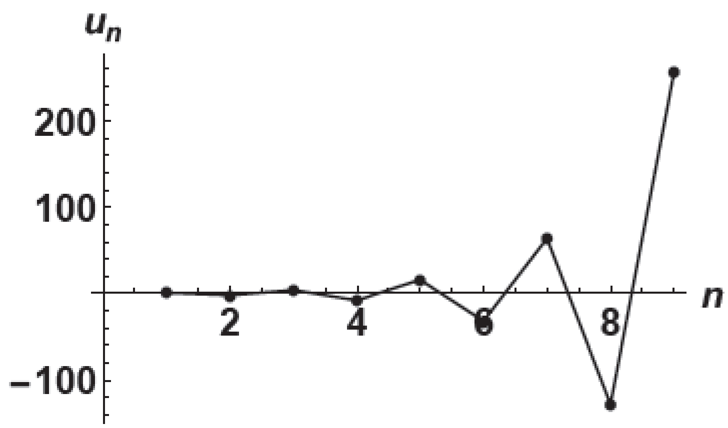


***Fig 5.5.4***
$k < -1$

The solutions of homogeneous linear difference equations with Constant coefficients are composed of linear combinations of the basic expressions of the form $u_n = ck^n$. The qualitative behavior of the basic solution will depend on the real values of $k$ namely, on the four possible ranges:

$$k \geq 1, k \leq -1, 0 < k < 1, -1 < k < 0$$

For $k > 1$, the solution $u_n = ck^n$ becomes unbounded as $n$ increases (*Fig 5.5.1.)* For $0 < k < 1$, $k^n$ goes to zero as $n$ increases, hence $u_n$ decreases (*Fig 5.5.2)* for $-1 < k < 0$ $k^n$ oscillates between positive and negative values, with diminishing magnitude to zero (*Fig 5.5.3)* and for $k < -1$, $k^n$ oscillates between positive and negative values with increasing magnitude (*Fig 5.5.4).*

The marginal points $k = 1, k = 0$ and $k = -1$ correspond to constant solution $(u_n = c)$, zero solution $(u_n = 0)$ and an oscillatory solution between $-c$ and $+c$ respectively. *Fig 5.5.1.* illustrates different behaviors of the solution for different ranges of $k$.

**Example 5.5.3.** Solve $x_{n+1} = \frac{x_n}{4} + y_n, y_{n+1} = 3\frac{x_n}{16} - \frac{y_n}{4}$.

**Solution:** $x_{n+1} = \frac{x_n}{4} + y_n \Rightarrow y_n = x_{n+1} - \frac{x_n}{4}$

$$\Longrightarrow y_{n+1} = x_{n+2} - \frac{x_{n+1}}{4}$$

Eliminating $y_{n+1}$ from both the equations, we get,

$$4x_{n+2} - x_n = 0.$$

Let $x_n = ck^n (c \neq 0)$ be a solution of $4x_{n+2} - x_n = 0$.
The required auxiliary equation is

$$4k^2 - 1 = 0 \Rightarrow k = \pm\frac{1}{2}.$$

The general solution is $x_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{2}\right)^n$,
where $c_1$ and $c_2$ are arbitrary constants. Similarly, it can be shown

$$y_n = d_1\left(\frac{1}{2}\right)^n + d_2\left(\frac{-1}{2}\right)^n$$

where $d_1$ and $d_2$ are arbitrary constants.

## *5.6 DIFFERENCE EQUATIONS: EQUILIBRIA AND STABLITY*

### *5.6.1 LINEAR DIFFERENCE EQUATIONS*

We consider an autonomous linear discrete equation of the form:

$$u_n = au_{n-1} + b(a \neq 1).$$

By equilibrium point (or fixed points or steady-state solutions), it is meant that there is no change from generation $(n-1)$ to generation $n$. If $u^*$ be the equilibrium solution of the model, then

$$u_n = u_{n-1} = u^* \Rightarrow au^* + b = u^* \Rightarrow \frac{b}{1-a}.$$

The equilibrium point $u^*$ is said to be stable if all the solutions of

$u_n = au_{n-1} + b$ approach $u^* = \frac{b}{1-a}$ as $n \to \infty$ ( as $n$ becomes large). The equilibrium point $u^*$ is unstable if all solutions (if exists) diverges from $u^*$ to $\pm\infty$.

The stability of the equilibrium point $u^*$ depends on $a$. The fixed point (equilibrium point) $u^*$of the autonomous discrete equation $u_n = au_{n-1} + b$ is

**i.** stable if $|a| < 1$.
**ii.** Unstable if $|a| > 1$ and
**iii.** If $a = \pm 1$, the case is ambiguous.

Note: If $a > 0$, the solutions converge monotonically to $u^*$ and if $a < 0$, the solutions converges to $u^*$ with oscillations.

## 5.6.2 *EXAMPLES*

**Example 5.6.2.1:** Find the equilibrium point of

$$x_{n+1} = a(x_n - 1)$$

For $a = \frac{4}{5}$ and determine its stability. Explain the dynamics when

$$a = -\frac{4}{5}, \frac{5}{4}, -\frac{5}{4}.$$

**Solution:**

**i.** If $x^*$ be the equilibrium point of

$x_{n+1} = \frac{4}{5}(x_n - 1)$,

then,

$$x_{n+1} = x_n = x^*,$$

This implies

$$x^* = \frac{4}{5}(x^* - 1),$$

This implies

$$x^* = -4.$$

The given equation of the form

$$x_{n+1} = ax_n + b,$$

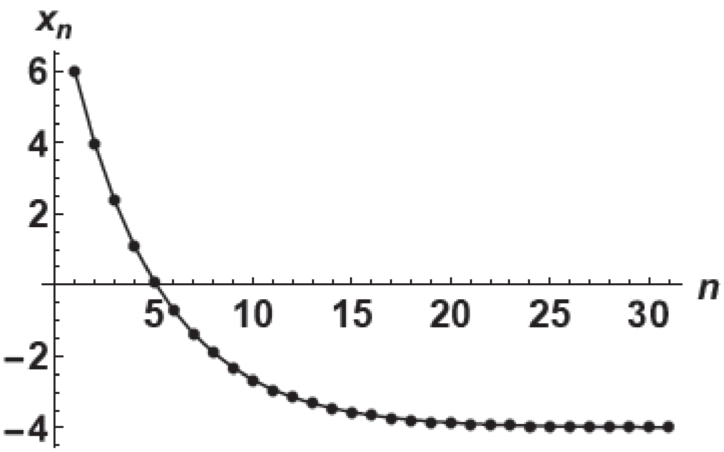


***Fig 5.6.2.1***

$$(a)\ a = \frac{4}{5} > 0, |a| < 1, b = \frac{-4}{5}$$

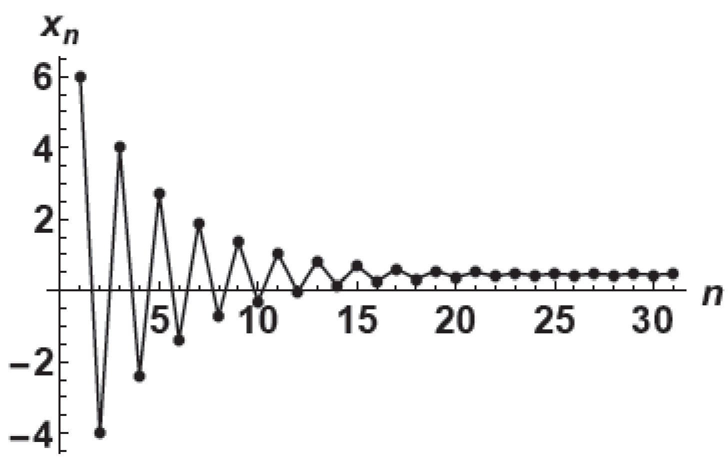


***Fig 5.6.2.2***

$$(a)\ a = \frac{-4}{5} < 0, |a| < 1, b = \frac{4}{5}$$

*The above figures show the plot of $x_{n+1} = ax_n + b$ for different values of $a$ and $b$ with $a_0 = 6$.The solution converges monotonically to $x^* = -4$ The solution converges with oscillations to $x^* = 0.444$.

where $|a| = \left|\frac{4}{5}\right| = \frac{4}{5} < 1$.

Therefore, the equilibrium point is stable.Since, $a > 0$, the solution with converge monotonically. ( ***Fig 5.6.2.1)***

**(ii)** For $a = \frac{-4}{5}$, the equilibrium point is given by

$$x^* = \frac{-4}{5}(x^* - 1).$$

This implies $x^* = \frac{4}{9} = 0.444$.

The given equation is of the form

$$x_{n+1} = ax_n + b,$$

where $|a| = \left|\frac{-4}{5}\right| = \frac{4}{5} < 1$.

Therefore, the equilibrium point is stable. Since, $a < 0$, the solution with converge to $x^*$ oscillations.( ***Fig 5.6.2.2)***

**(iii)** For $a = \frac{5}{4}$, the equilibrium point is given by

$$x^* = \frac{5}{4}(x^* - 1).$$

This implies $x^* = .5$.

The given equation is of the form

$$x_{n+1} = ax_n + b,$$

where $|a| = \left|\frac{5}{4}\right| = \frac{5}{4} > 1$.

Therefore, the equilibrium point is unstable.

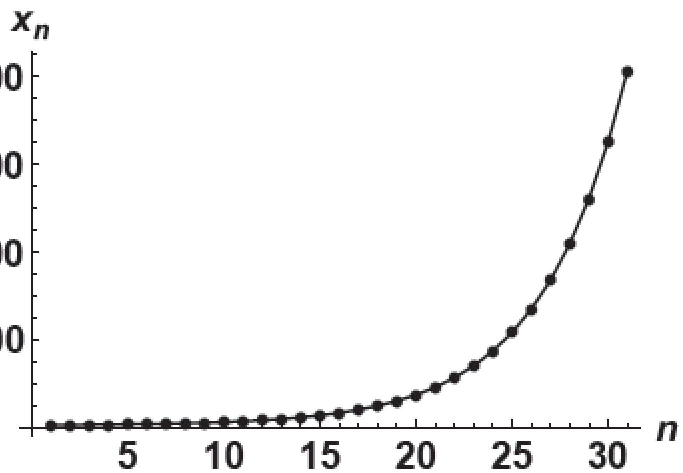


***Fig 5.6.2.3***

$$(a)\ a = \frac{5}{4} > 0, |a| > 1, b = \frac{-5}{4}$$

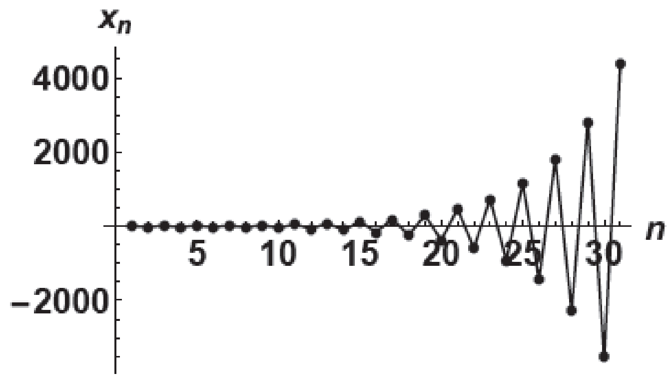


***Fig 5.6.2.4***

$$(a)\ a = \frac{-5}{4} < 0, |a| > 1, b = \frac{5}{4}.$$

*The above figures show the plot of $x_{n+1} = ax_n + b$ for different values of $a$ and $b$ with $a_0 = 6.$ The solution converges monotonically to $x^* = 5$ and The solution converges with oscillations to $x^* = 0.556$.

Since, $a < 0$, the solution diverges monotonically from $x^*$. ( ***Fig 5.6.2.3)***

**(iii)** For $a = -\frac{5}{4}$, the equilibrium point is given by

$$x^* = \frac{-5}{4}(x^* - 1).$$

This implies $x^* = \frac{5}{9} = 0.556$.

The given equation is of the form

$$x_{n+1} = ax_n + b,$$

where $|a| = \left|-\frac{5}{4}\right| = \frac{5}{4} > 1$.

Therefore, the equilibrium point is unstable. Since, $a < 0$, the solution diverges from $x^*$ with oscillation.( ***Fig 5.6.2.4)***

## *5.6.3 SYSTEM OF LINEAR DIFFERENCE EQUATIONS*

For a system of difference equations, it is possible to determine the stability of the system using eigenvalues. We consider the linear homogeneous system,

$$u_{n+1} = \alpha u_n + \beta v_n$$
$$v_{n+1} = \gamma u_n + \delta v_n$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

which can be expressed in the matrix form as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

This implies that

$$w_{n+1} = Aw_n,$$

where $w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.
Clearly $(0,0)$ is the equilibrium point of the homogeneous system.

## *5.6.4 THEOREMS*

**Theorem 5.6.4.1.** Let $\lambda_1$ and $\lambda_2$ be two real distinct eigenvalues of the coefficient matrix $A$ of the homogeneous linear system (6). Then, the equilibrium point $(0,0)$ is,

**i.** stable if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$,
**ii.** unstable if both $|\lambda_1| > 1$ and $|\lambda_2| > 1$,
**iii.** saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or if $|\lambda_1| > 1$ and $|\lambda_2| < 1$.

**Theorem 5.6.4.2.** Let $\lambda_1 = \lambda_2 = \lambda^*$ be real and equal eigenvalues of the coefficient matrix $A$ of the homogeneous linear system (6). Then, the equilibrium point $(0,0)$ is,

**i.** stable if both $|\lambda^*| < 1$
**ii.** unstable if both $|\lambda^*| > 1$ .

**Note:** $\lambda_1 = \lambda_2 = 1$ is a rare borderline case and will not be considered here.

**Theorem 5.6.4.2.** Let $a + ib$ and $a - ib$ be the complex conjugate eigenvalues of the coefficient matrix $A$ of a homogeneous linear system, then the equilibrium point $(0,0)$ is

**i.** stable focus or spiral if $|a \pm ib| < 1$,

**ii.** unstable focus or spiral or spiral if $|a \pm ib| > 1$.

Consider a non-homogeneous linear system of the form

$$w_{n+1} = Aw_n + b$$

where $w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $b = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$.

The equilibrium solution $w^* = (u^*, v^*)^T$ of the system is obtained by solving

$$u^* = \alpha u^* + \beta u^* + k_1 \text{ and } u^* = \gamma u^* + \delta v^* + k_2 \text{.......(6)}$$

This implies,

$$u^* = \frac{\beta k_2 - (\delta - 1)k_1}{(\alpha - 1)(\delta - 1) - \beta\gamma}, \qquad v^* = \frac{\gamma k_1 - (\alpha - 1)k_2}{(\alpha - 1)(\delta - 1) - \beta\gamma}$$

For stability, the same results hold as for the homogeneous system. This is due to the fact that the non-homogeneous system, with a unique equilibrium point, can be converted to a homogeneous system. The system (6) can be written as

$$w^* = A\,w^* + b \Rightarrow A\,w^* - \ w^* + b = 0 \text{(null matrix)}$$

$$z_{n+1} + \ w^* = A(\,z_n + w^*) + b$$

$$z_{n+1} \ \Rightarrow Az_n + A\,w^* - \ w^* + b$$

we get,

$$z_{n+1} \ \Rightarrow Az_n \text{ (since } A\,w^* - \ w^* + b = 0\text{).}$$

Which is a linear homogeneous system, whose stability has already been discussed.

## *5.6.5 EXAMPLES*

**Example 5.6.5.1.** Find the equilibrium point of the linear homogeneous system

$x_{n+1} \ \Rightarrow -x_n - 4y_n,$

$y_{n+1} \ \Rightarrow x_n - y_n,$

and check its stability.

**Solution:** If $(x^*, y^*)$ be the equilibrium solution, it is obtained by solving,

$$x^* = \ - \ x^* - \ 4y^* \Rightarrow \ x^* + 2y^* = 0,$$

$$y^* = \ \ x^* - \ y^* \Rightarrow \ x^* - 2y^* = 0.$$

Clearly, $(0, 0)$ is the only solution of the system as the coefficient matrix $\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$ is non-singular, that is, $\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4 \neq 0$.

The coefficient matrix of the linear homogeneous system is $\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$, whose eigenvalues are obtained by solving
$\begin{vmatrix} 1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = 0.$

This implies $\lambda = -1 \pm 2i$, and $|-1 \pm 2i| = \sqrt{1+4} = \sqrt{5} > 1$.
Hence the equilibrium point $(0,0)$ is unstable.

**Example 5.6.5.2.** Find the equilibrium point of the linear homogeneous system
$x_{n+1} \Rightarrow \alpha x_n + 0.12\, y_n,$
$y_{n+1} \Rightarrow 3x_n + \alpha\, y_n.$
Find all the real values of $\alpha$ for which the equilibrium point is stable.

**Solution:** Clearly $(0,0)$ is the equilibrium point
the coefficient matrix of the linear homogeneous system is
$\begin{pmatrix} \alpha & 0.12 \\ 3 & \alpha \end{pmatrix}$ whose eigenvalues are obtained by solving
$\begin{vmatrix} \alpha-\lambda & 0.12 \\ 3 & \alpha-\lambda \end{vmatrix} = 0.$

This implies $(\alpha - \lambda)^2 = 0.36,\ \lambda = \alpha \pm 0.6.$

The equilibrium point $(0,0)$ will be stable if
$|\alpha + 0.6| < 1$ and $|\alpha - 0.6| < 1$.
$|\alpha + 0.6| < 1$ this implies $-1 < \alpha + 0.6 < 1$.

This implies
$$-1.6 < \alpha < 0.4.$$
Similarly combining we obtain
$$-0.4 < \alpha < 0.4.$$
(common region of the two inequalities), which gives all the real values of $\alpha$ for which the equilibrium point is stable. ***(Fig 5.6.5.1)***

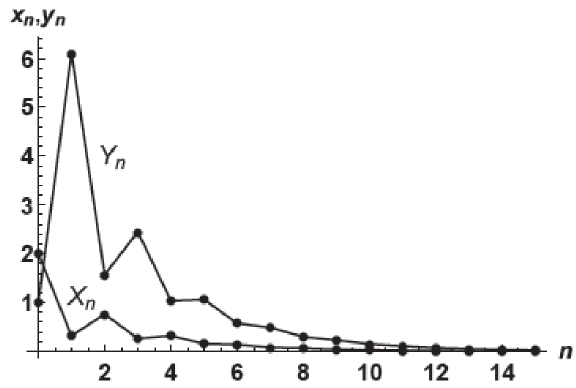


*Fig 5.6.5.1 Stable dynamics*

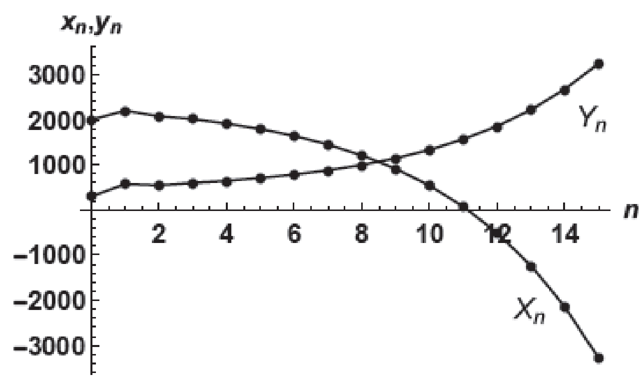


***Fig 5.6.5.2 Unstable dynamics(saddle)***

**Example 5.6.5.1.** Find the equilibrium point of the linear non-homogeneous system

$x_{n+1} \Rightarrow 0.75x_n - y_n + 1000,$

$y_{n+1} \Rightarrow -0.5x_n + 0.25\, y_n + 1500$

and check its stability.

**Solution:** If $(x^*, y^*)$ be the equilibrium solution of the given non- homogeneous system, it is obtained by solving

$x^* \Rightarrow 0.75x^* - y^* + 1000$. This implies $0.75x^* - y^* = 1000,$

$y^* \Rightarrow -0.5x^* + 0.25\, y^* + 1500$. This implies $0.5x^* + 0.75\, y^* = 1500.$

Solving, we get $(x^*, y^*) = (2400,\ 400)$ as the unique solution of the system

$$\begin{vmatrix} 0.75 & -1 \\ -0.5 & 0.25 \end{vmatrix}.$$

Whose eigenvalues are obtained by solving

$$\begin{vmatrix} 0.75-\lambda & -1 \\ -0.5 & 0.25-\lambda \end{vmatrix} = 0.$$

This implies $\lambda_1 = 1.25$ and $\lambda_1 = -0.25$.
Now $|\lambda_1| = 1.25 > 1$ and $|\lambda_2| = 0.25 < 1$.
Hence, the equilibrium point $(2400,\ 400)$ is a saddle.
***(Fig 5.6.5.2)***

## *5.6.6 NON-LINEAR DIFFERENCE EQUATIONS*

Non-linear difference equations are to be handled with special techniques and cannot be solved by simply setting $u_n = ck^n$. Here, we shall not discuss about the solutions of non-linear difference equations but focus on the qualitative behaviors, namely, equilibrium solution (fixed point or steady state), stability, cycles, bifurcations and chaos.

In the context of difference equations, $x^*$ is the steady-state solution (equilibrium solution) of the non-linear difference equation

$x_{n+1} = fx_n$ if $x_{n+1} = x_n = x^*$

That is, there is no change from generation $n$ to generation $(n+1)$.

By definition the steady-state solution is stable if for $\in > 0, \exists\ a\ \delta > 0$ such that $|x_0 - x^*| < \delta$ implies that for all $n > 0$, $|f^n(x_0) - x^*| < \in$. The steady – state solution is asymptotically stable if, in addition, $\lim_{n\to\infty} x_n = x^*$ holds.

After obtaining the equilibrium solution, we look into its stability, that is, given some value $x_n$ close to $x^*$, does $x_n$ tends towards $x^*$ or move away from it? To address this issue, we give a small perturbation to the system about the
steady state $x^*$. Mathematically, this means replacing $x_n$ by $x^*$ $+\in_n$, where $\in_n$ is small. Then,

$$\begin{aligned} x_{n+1} &= f(x_n), \\ \Rightarrow\quad x^* + \epsilon_{n+1} &= f(x^* + \epsilon_n), \\ &\approx f(x^*) + \epsilon_n f'(x^*) \quad \text{(by Taylor series expansion)}, \\ &= x^* + \epsilon_n f'(x^*). \end{aligned}$$

Since $x^*$ is the equilibrium solution, $x^* = f(x^*)$, which implies

$$\epsilon_{n+1} \approx \epsilon_n f'(x^*). \quad \ldots\ldots\ldots\ldots\ldots(7)$$

The solution of equation (7) will decrease
if $|f'(x^*)| < 1$,

and the solution of equation (7) will increase
if $|f'(x^*)| > 1$.

## *5.6.7 THEOREMS*

**Theorem 5.6.7.1.**

*The equilibrium solution $x^*$ of $x_{n+1} = f(x_n)$ is stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$. No definite conclusion if $|f'(x^*)| = 1$. Also, if $f'(x^*) < 0$, then the solution oscillates locally around $x^*$, but if $f'(x^*) > 0$ , they do not oscillate (f(x) must be differentiable at $x = x^*$).*

The equilibrium solution $(u^*, v^*)$ of a non-linear discrete system of the form $u_{n+1} = f(u_n, v_n)$ and $v_{n+1} = g(u_n, v_n)$ is obtained by solving

$$u^* = f(u^*, v^*) \text{ and } v^* = g(u^*, v^*).$$

Its stability analysis near the equilibrium point $(u^*, v^*)$ can be determined by linearizing the system about the equilibrium point.

**Theorem 5.6.7.2.**

*Let $(u^*, v^*)$ be an equilibrium solution of non-linear systems $u_{n+1} = f(u_n, v_n)$ and $v_{n+1} = g(u_n, v_n)$ and $A$ be the corresponding matrix of partial derivatives (also known as the Jacobian matrix) given by,*

$$A = \begin{pmatrix} f_u(u^*, v^*) & f_v(u^*, v^*) \\ g_u(u^*, v^*) & g_v(u^*, v^*) \end{pmatrix},$$

*then $(u^*, v^*)$ is stable if each eigenvalue of $A$ has modulus less than 1 and unstable if one of the eigenvalues of $A$ has modulus greater than 1*

## *5.6.8 EXAMPLE*

**Example 5.6.8.1**

Find the positive equilibrium point of the non-linear discrete equation

$$x_{n+1} = x_n\sqrt{2 - x_n}$$

and check its stability. Also, find its interval of existence and stability.

**Solution:** If $x^*$ be the equilibrium solution of the given non-linear discrete equation, it is obtained by solving

$$x^* = x^*\sqrt{2-x^*} \;\Rightarrow\; x^*(\sqrt{2-x^*}-1) = 0 \;\Rightarrow\; x^* = 0,\ 1.$$

Therefore, the positive equilibrium point is $x^* = 1$. Let

$$f(x) = x\sqrt{2-x} \;\Rightarrow\; f'(x^*) = \frac{2-2x}{\sqrt{2-x}}.$$

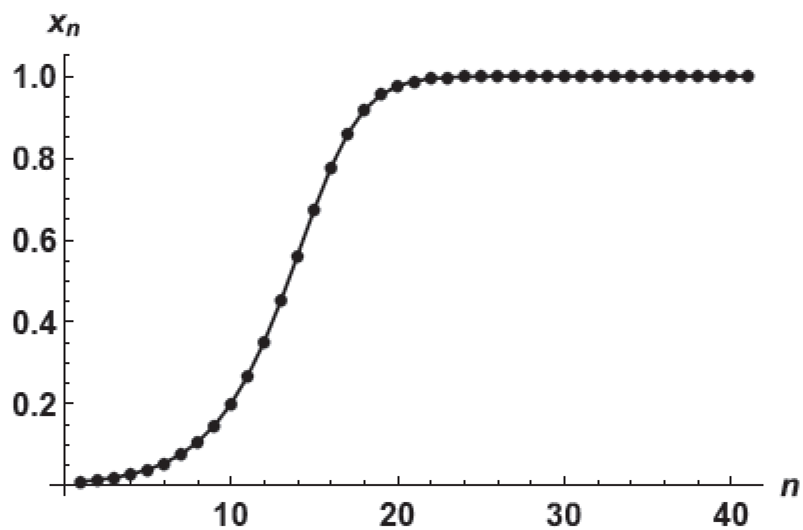


***Fig 5.6.8.1***

*The figure shows that the system approaches the stable equilibrium solution* $x^* = 1$.

Clearly, $|f'(x^* = 1)| = 0 < 1$, implying that the equilibrium point $x^* = 1$ is stable. The interval of the existence of the equilibrium point is,

$2 - x > 0 \Rightarrow\; x < 2$ and the interval of its stability is given by

$$\begin{aligned} |f'(x^*)| = \left|\frac{2-2x}{\sqrt{2-x}}\right| < 1 &\Rightarrow\; -1 < \frac{2-2x}{\sqrt{2-x}} < 1 \\ &\Rightarrow\; -\sqrt{2-x} < 2-2x \;\text{ and }\; 2-2x < \sqrt{2-x} \\ &\Rightarrow\; \sqrt{2-x} > 2x-2 \;\text{ and }\; 2-2x < \sqrt{2-x} \end{aligned}$$

Both the inequalities give $4x^2 - 7x + 2 < 0$, which implies

$$\frac{7-\sqrt{17}}{8} < x < \frac{7+\sqrt{17}}{8} \;\Rightarrow\; 0.36 < x < 1.39.$$

**Example 5.6.8.1.**

Find the equilibrium point of the non-linear discrete system

$$x_{n+1} = x_n + 2.5y_n - 0.1(x_n)^2 - 1,\ y_{n+1} = y_n + \frac{5}{x_n} - 1$$

and check its stability.

**Solution:** If $(x^*, y^*)$ be the equilibrium solution of the given non-linear discrete system, it is obtained by solving

$$\begin{aligned} x^* &= x^* + 2.5y^* - 0.1(x^*)^2 - 1, \\ y^* &= y^* + \frac{5}{x^*} - 1. \end{aligned}$$

Solving, we get $(x^*, y^*) = (5, 1.4)$ as the unique equilibrium solution of the system. The Jacobian matrix of the system is

$$\begin{pmatrix} 1 - 0.2x^* & 2.5 \\ -\dfrac{5}{(x^*)^2} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2.5 \\ -0.2 & 1 \end{pmatrix},$$

whose eigenvalues are obtained by solving

$$\begin{vmatrix} 0 - \lambda & 2.5 \\ -0.2 & 1 - \lambda \end{vmatrix} = 0 \ \Rightarrow\ \lambda = \frac{1 \pm i}{2},\ \text{where } |\lambda| = \frac{1}{\sqrt{2}} < 1.$$

Hence, the equilibrium point (5, 1.4) is stable.

## 5.7 SUMMARY

Present unit is a presentation of mathematical modelling through difference equations. In this unit Linear Difference Equation with Constant Coefficients, Solution of Homogeneous Equations, Difference Equations: Equilibria and Stability, Linear Difference Equations, System of Linear Difference Equations, Theorems, Non-Linear difference equations explained in a easy manner.

## 5.8 GLOSSARY

i. **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

ii. **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

iii. **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

iv. **Difference equation:** A difference equation is an equation involving differences. One can define a difference equation as a sequence of numbers that are generated recursively using a rule to the previous numbers in the sequence the difference equation

v. **Homogeneous Difference Equation and Non-Homogeneous Difference Equation:** The difference equation is homogeneous if $f(n) = 0$, otherwise it is non-homogeneous.

vi. **Order:** The order of the difference equation is the difference between the largest $(n)$ and smallest $(n-2)$ arguments appearing in the difference equation with unit interval.

vii. **Stable:** The equilibrium point $u^*$ is said to be stable if all the solutions of $u_n = au_{n-1} + b$ approach $u^* = \frac{b}{1-a}$ as $n \to \infty$ (as $n$ becomes large).

viii. **Unstable:** The equilibrium point $u^*$ is unstable if all solutions (if exists) diverges from $u^*$ to $\pm\infty$.

## CHECK YOUR PROGRESS

1. The recurrence relation $\frac{x_k}{x_{k-1}} = 2^k$ has order..................
2. The difference equation $Sinx_k - x_{k-1} = 5cosx_{k+2}$ has order.....
3. The difference equation $Sinx_k - x_{k-1} = 5cosx_{k+2}$ is linear difference equation. **True/False**
4. The difference equation $x_{k+1} - 2x_k + 2x_{k-1} = 2k - 1$ is non linear difference equation. **True/False**
5. The difference equation is homogeneous if ……………..., otherwise it is non-homogeneous.

6. The order of the difference equation is the difference between the …………………appearing in the difference equation with unit interval.
7. equilibrium point (or fixed points or steady-state solutions), it is meant that there is no change from ………………….. to …………………………..

## *5.9 REFRENCES*

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## *5.10 SUGGESTED READINGS*

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## *5.11 TERMINAL QUESTIONS*

Classify the following difference equations:

**TQ1**. $2y_{n+3} - ny_n = 0.$

**TQ2.** $y_n y_{n+3} - 2{y^3}_{n-2} = n^2.$

**TQ3.** $n^2 y_{n+5} + n!\, y_{n-4} = 5.$

**TQ4.** $y_{n+2} - 2y_{n+1} + 3y_n = 2n^2.$

## 5.12 ANSWERS

### TERMINAL QUESTIONS

**TQ1.** (i) Zero order (ii) Linear difference equation (iii) Homogeneous Equation

**TQ2.** (i) $4^{th}$ order (ii) Non- Linear difference equation (iii) Non - Homogeneous Equation

**TQ3.** (i) $9^{th}$ order (ii) Linear difference equation (iii) Non - Homogeneous Equation

**TQ4.** (i) zero order (ii) Linear difference equation (iii) Non - Homogeneous Equation

### CHECK YOUR PROGRESS

**CYP1**. 1.

**CYP2.** 3.

**CYP3.** False.

**CYP4.** False.

**CYP5.** $f(n) = 0$.

**CYP6.** largest $(n)$ and smallest $(n-2)$ arguments.

**CYP7.** generation $(n-1)$ generation $n$.

# UNIT 6: LINEAR MODELS

## CONTENTS:



## *6.1 INTRODUCTION*

In previous unit we have defined Linear Difference Equation with Constant Coefficients, Solution of Homogeneous Equations, Difference Equations: Equilibria and Stability, Linear Difference Equations System of Linear Difference Equations, Non-Linear Difference Equations.

## *6.2 OBJECTIVES*

After studying this unit, learner will be able to

**i.** Explain the Discrete Models
**ii.** Describe the Newton's Law of Cooling
**iii.** Analyze the Bank Account Problem
**iv.** Discuss the Drug Delivery Problem
**v.** Identify Harrod Model (Economic Model)
**vi.** Defined Arms Race Model

## 6.3 DISCRETE MODELS

In discrete models, the state variables change only at a countable number of points in time. These points in time are the ones at which the event occurs/changes in state.

Thus, indiscrete-time modelling, there is a state transition function which computes the state at the next time instant given the current state and input. In many situations, the changes are really discrete which occur at well defined time intervals. Moreover, in many cases, the data are usually discrete rather than continuous.

Hence, due to the limitations of the available data, we may be compelled to work with the discrete model, even though the underlying model is continuous.

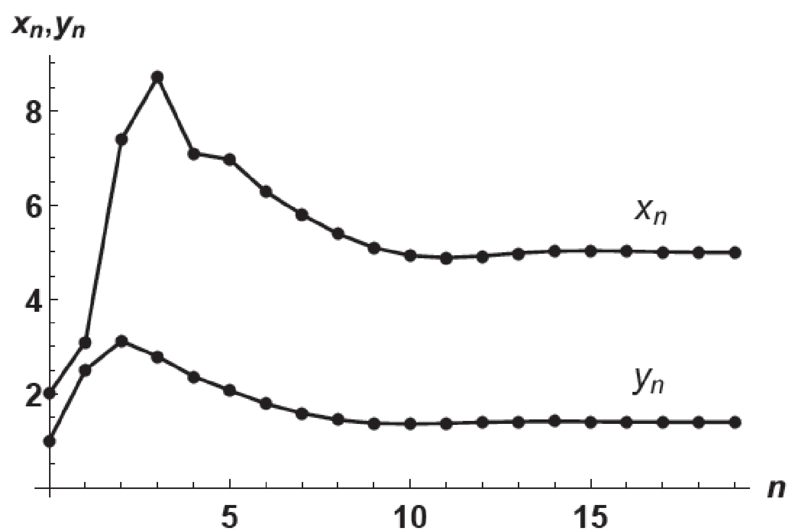



***Fig 6.3.1***

The figure shows that the system approaches the stable equilibrium solution $(x^*, y^*) = (5, 1.4)$.

**CHECK YOUR PROGRESS**

**1.** Define discrete models ……………………………………..
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………

## 6.4 LINEAR MODELS

### 6.4.1 NEWTON'S LAW OF COOLING

Suppose a cup of coffee, initially at a temperature of $190^{\circ}F\circ$ is placed in a room, which is held at a constant temperature of $70^{\circ}F$ . After 1 minute, the coffee has cooled to $180^{\circ}F$. If we need to find the temperature of the coffee after 15 minutes, we will use Newton's law of cooling, which states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (that is, the temperature of its surroundings). Mathematically, this means

$$t_{n+1} - t_n = k(S - t_n),$$

where $t_n$ is the temeperature of the coffee after $n$ minutes, $S$ is the temeprature of the room, and $k$ is the constant of proportionality.

We first make use of the information given about the change in the temperature of the coffee during the first minute to determine the value of the constant of proportionality $k$ . Thus,

$$\begin{aligned} t_1 - t_0 &= k(S - t_0), \\ \Rightarrow 180 - 190 &= k(70 - 190) \\ \Rightarrow k = \frac{1}{12}, \end{aligned}$$

$$\begin{aligned} \Rightarrow t_{n+1} - t_n &= \frac{1}{12}(70 - t_n) \\ t_{n+1} = \frac{11}{12}t_n + \frac{70}{12} \end{aligned}$$

This is of the form $u_n = au_{n-1} + b$.
Whose solution is given by,

$$\begin{aligned} t_{n+1} &= \left(\frac{11}{12}\right)^{n+1} 190 + 70\left[1 - \left(\frac{11}{12}\right)^{n+1}\right], \\ &= 70 + 120\left(\frac{11}{12}\right)^{n+1}, \quad \text{for } n = 0,1,2,\ldots . \end{aligned}$$

$$\text{For } n = 14,\ t_{15} = 70 + 120\left(\frac{11}{12}\right)^{15} = 102.54.$$

Hence, after 15 minutes, the coffee has cooled to $102.54^{\circ}F$.

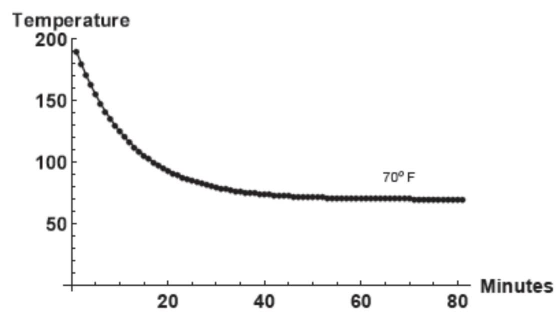
Since,

$$\lim_{x\to\infty}\left(\frac{11}{12}\right)^{n} = 0,$$

the temperature of the coffee will approach the equilibrium temperature of $70^{\circ}F$ (the room temperature) as $n$ increases.

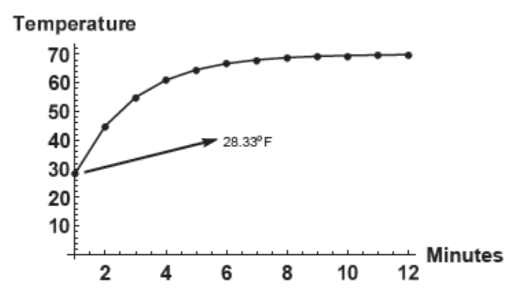
## *6.4.2 EXAMPLE*

**Example. 6.4.2.1** A soda–can is taken out from the refrigerator, and its Temperature is recorded after $1/2$ an hour. After await of another $1/2$ hour, The temperature is recorded again. If the two readings are $45^{\circ}F$ and $55^{\circ}F$ respectively, what is the temperature inside the refrigerator (assume the room temperature to be $70^{\circ}F$)?




(a) Coffee reaches room temperature.

***Fig. 6.4.2.1***




(b) Temperature inside refrigerator.

***Fig. 6.4.2.2***

Fig 6.4.2.1 show that the cup of coffee initially at $190^\circ F$ reaches the room temperature of $70^\circ F$ as $n$ increases the Fig 6.4.2.2 the temperature inside the refrigerator is $28.33^\circ F$.

**Solution:** Let $t_0$ be the temperature of the soda – can $1/2$ hour after it was removed the refrigerator (zero-time) and $t_1$ be the temperature after waiting $1/2$ hour more. Then, $t_{-1}$ will give the temperature when the soda – can was inside the refrigerator. Using Newton's law of cooling, we get

$$t_1 - t_0 = k(S - t_0),$$

$$\Rightarrow 55 - 45 = k(70 - 45)$$

$$\Rightarrow k = \frac{2}{5},$$

$$\Rightarrow t_{n+1} - t_n = \frac{2}{5}(70 - t_n)$$

$$\Rightarrow t_{n+1} = \frac{3}{5}t_n + 28.$$

This is of the form

$$u_n = au_{n-1} + b,$$

whose solution is given by,

$$t_{n+1} = \left(\frac{3}{5}\right)^{n+1} 45 + 28\left[\frac{1 - \left(\frac{3}{5}\right)^{n+1}}{1 - \left(\frac{3}{5}\right)}\right] = 70 - 25\left(\frac{3}{5}\right)^{n+1}.$$

Putting $n = -1, 0$, it can be easily checked

$$t_0 = 70 - 25\left(\frac{3}{5}\right)^0 = 45 \text{ and } t_1 = 70 - 25\left(\frac{3}{5}\right)^1 = 55.$$

Therefore, temperature inside the refrigerator is (putting $n = -2$)

$$t_{-1} = 70 - 25\left(\frac{3}{5}\right)^{-1} \simeq 28.3\ ^\circ\text{F}.$$

Fig. 6.4.2.2 clearly express the temperature inside the refrigerator is $28.33^\circ F$.

## *6.4.3 BANK ACCOUNT PROBLEM*

Suppose a savings account is opened that pays 4% interest Compounded yearly with an initial deposit of $10000, and a deposit of $5000 is made at the end of each year. For a savings account that is compounded yearly, the interest is added to the principal at the end of each year. If $a_n$ be the amount at the end of year n(n = 0, 1, 2, 3, ...),then

$$a_1 = a_0 + ra_0 = (1 + r)a_0,$$
$$a_2 = a_1 + ra_1 = (1 + r)a_1,$$

**………………………………………………………………………..**
**………………………………………………………………………..**

$$a_{n+1} = a_n + ra_n = (1 + r)a_n,$$

where r is the rate of interest. Now, if a deposit of $5000 is made at the end of each year, the n the dynamic model which describes this scenario is given by

$$a_{n+1} = (1 + r)a_n + 5000$$

$$= (1 + 0.04)a_n + 5000 = 1.04a_n + 5000.$$

Thus, the amount for three consecutive years will be

$$\begin{aligned} a_1 &= 1.04a_0 + 5000 = 1.04 \times 10000 + 5000 = 10400 + 5000 = 15400, \\ a_2 &= 1.04 \times 15400 + 5000 = 16016 + 5000 = 21016, \\ a_3 &= 1.04 \times 21016 + 5000 = 21856.64 + 5000 = 26856.64, \end{aligned}$$

and soon. Let us now consider a different scenario, where no deposits are made, but $2000 is withdrawn at the end of each year. We want to find out how much money be deposited initially, so that we never run out of cash. The model for this scenario is

$$a_{n+1} = 1.04a_n - 2000,$$

where we assume that the money is withdrawn after the interest from previous years has been added, and we are not penalized for withdrawing money each year. The equilibrium value is given by,

$$a^* = 1.04\ a^* - 2000 \ \Rightarrow a^* = \frac{2000}{0.04} = 50000.$$

Therefore, if the initial deposit in the account is $50000 and we withdraw $2000 each year, then the account will always have the same amount at the end of each year ***Fig. 6.4.3.1***

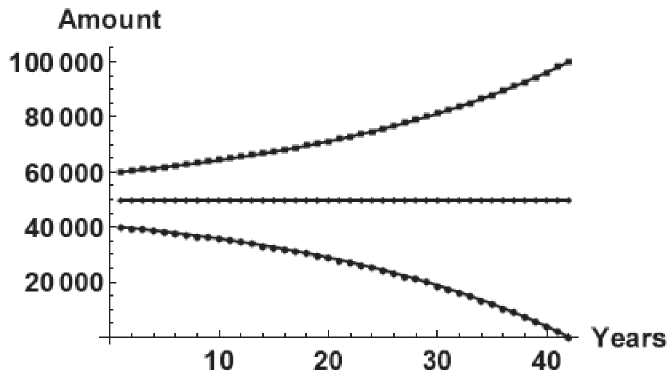
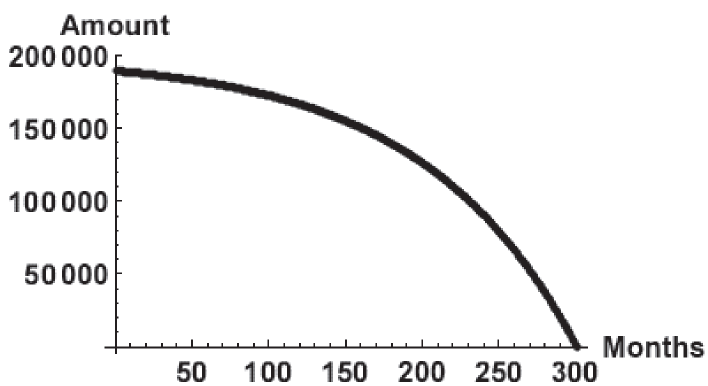



*Fig. 6.4.3.1*

(a) $a_0 > 50000, a_0 = 50000, a_0 < 50000.$

If $a_0 < 5000$, Fig 6.4.3.1 shows that if $a_0$ is less than50000, the amount in the account decreases to 0, and the amount grows without bound .




(b) House loan paid in 300 months.

*Fig. 6.4.3.2*

The figures show (a) different dynamics with different initial deposits $a_0$, (b) initial house loan amount $P_0$, = \$189894 is repaid in 300 months. If $a_0 > 5000$.Thus, the system approaches zero or increases without bound if $a_0 \neq 5000$, and therefore, this equilibrium value is unstable.

## *6.4.4 EXAMPLE*

**Example 6.4.4.1** A couple like to purchase their first house with their combined annual income of \$84000. They also have a savings of \$40000, which they want to use as a down payment. For the rest, they can take a loan from the bank with12% annual interest paid monthly for 25 years. However, the bank will not allow their monthly installment to exceed 1/4 of their monthly income. (a) What is the maximum budget, the couple can afford for the house under these conditions? (b) To afford a house costs \$290000; what would be their annual income?

**Solution.** Let $P_n$ be the amount of loan to be paid after $n$ months, then

$$P_{n+1} = P_n + rP_n - d,$$

where $r$ is the monthly rate of interest and $d$ is the monthly installment to be paid. This is of the form (5.4 b), whose solution is given by

$$P_{n+1} = P_0(1+r)^{n+1} - d\frac{(1+r)^{n+1}-1}{r},$$

where $P_0$ is the initial amount of loan

(a) The monthly income of the couple is:

$$\frac{96000}{12} = \$8000,$$

so their maximum repayment of the loan is:

$$\frac{8000}{4} = \$2000 = d$$

and

$$r = \frac{0.12}{12} = 0.01.$$

After 25 years, that is, 300 months, the loan amount will be zero, which implies $P_{300} = 0$ fig. 6.4.3.2. Therefore

$$P_{300} = P_0(1+0.01)^{299+1} - 2000\ \frac{(1+0.01)^{299+1}-1}{0.01} = 0,$$

$$\Rightarrow\ P_0 \times 19.79 = \frac{2000}{0.01}(19.79-1)\ \Rightarrow\ P_0$$

$P_0 = 0$ =189893.886 ≃ \$189894.

Therefore, the maximum budget that the couple can afford for the house is $189894+$40000=$229894.

(b) To afford a house which costs $290000, the couple has to take a load of (290000− 40000)=$250000 and their monthly instalment will be,

$$250000\ (1+0.01)^{299+1}-d\ \frac{(1+0.01)^{299+1}-1}{0.01}=0,$$
$$\Rightarrow d=\frac{250000\times 19.79\times 0.01}{18.79}\simeq \$2633.$$

Hence, the annual income of the couple is $2633 × 4 × 12=$126384.

## *6.4.5 DRUG DELIVERY PROBLEM*

Suppose a patient is given a drug to treat some infection. He/she is given the same dose of the medicine at equally spaced time intervals. The body metabolizes some of the drugs so that, after sometime, only a portion r of the original amount of the drug remains. After each dose, the amount of the Drug in the body is equal to the amount of the given dose b, plus the amount of the drug remnant from the previous dose. The dynamic model which describes this scenario is given by:

$$x_{n+1}=rx_n+b.$$

The equilibrium point is given by,

$$a^*=ra^*+b\ \Rightarrow\ a^*=\frac{b}{1-r}.$$

$$\begin{aligned}
\text{Let } x_0 &= 0 \text{ (no drug), then } x_1=b \text{ (first dose)}.\\
x_2 &= b \text{ (second dose)} + r\ b \text{ (amount remaining from previous dose)}.\\
x_3 &= b \text{ (third dose)} + r\ (b+rb) \text{ (amount remaining from previous dose)}.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_{n+1} &= b+r\ (b+rb+\ldots+r^{n-1}b)=b(1+r+r^2+\ldots+r^n)=b\frac{1-r^{n+1}}{1-r}.
\end{aligned}$$

$$\text{Since } r<1, x_{n+1}\ \longrightarrow\ \frac{b}{1-r},$$

the stable equilibrium point.

Suppose, the amount of drug in the patient's blood stream decreases at the rate of 80% per hour (this means 20% of the drug remains in the body). To sustain the drug to a certain level ,an injection is given at the end of each hour that increases the amount of drug in the blood stream by 0.2 unit. The dynamic model which describes this scenario is given by

$$a_{n+1} = a_n - 0.8a_n + 0.2$$
$$\Rightarrow \; a_{n+1} = 0.2a_n + 0.2,$$

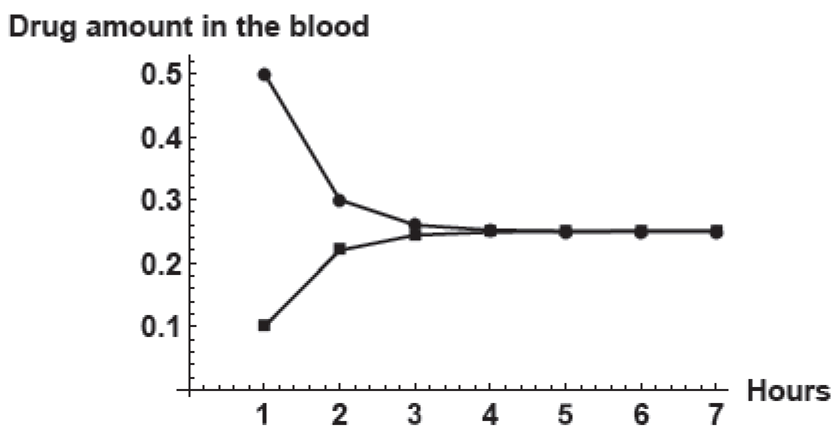
where $a_n$ is the amount of drug in the blood at the end of $n$ hours. The equilibrium solution of this model is given by,

$$a^* = 0.2a^* + 0.2 \;\Rightarrow\; a^* = 0.25.$$

The long-term behavior of the system will depend on the initial value $a_0$. The figure shows that no matter what is the value of $a_0$, the system always approaches the value of

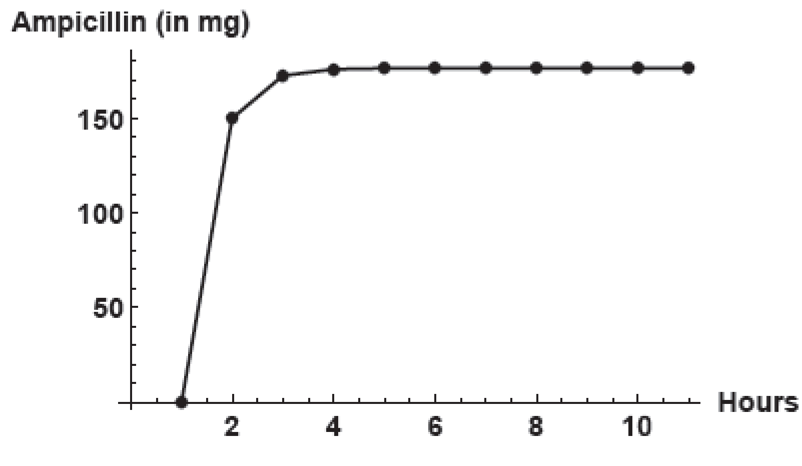
$$\frac{b}{1-r} = \frac{0.2}{1-0.2} = 0.25,$$

implying that 0.25 is a stable equilibrium point.




(a) Amount of drug.

***Fig 6.4.5.1***




*(b) Ampicillin in mg.*
*Fig 6.4.5.2*

The figures show (a) the amount of drug in a patient's blood stream always reaches the steady-state value 0.25, independent of the initial value $a_0$, implying a stable equilibrium, (b) ampicillin reaches steady-state value 176.47 mg.

## *6.4.6 EXAMPLE*

**Example 6.4.6.1.** A person with an ear infection takes 150mg ampicillin tablet once every 4 hours. About15% of the drug in the body at the start of a four – hour period is still there at the end of that period. What quantity of ampicillin is in the body (a) right after taking the third tablet? (b) at the steady – state level right after taking a tablet?(c) at the steady-state level right before taking a tablet?

**Solution**:
**(a)** The quantity of ampicillin right after taking the third tablet is

$$x_3 = b\,\frac{1-r^3}{1-r} = 150\,\frac{1-(0.15)^3}{1-0.15} = 175.87 \text{ mg.}$$

**(b)** The quantity of ampicillin at the steady-state level right after taking a tablet is

$$\frac{b}{1-r} = \frac{150}{1-0.15} = 176.47 \text{ mg.}$$

**(c)** The quantity of ampicillin at the steady-state level right before taking a tablet is

$$\frac{b}{1-r} - b = \frac{150}{1-0.15} - 150 = 26.47 \text{ mg.}$$

## 6.4.7 HARROD MODEL (ECONOMIC MODEL)

In J.W.Nevile. research paper the Harrod model which was developed in the 1930s, gives some Insight into the dynamics of economic growth. The model aims to determine an equilibrium growth rate for the economy. Let $G_n$ be the Gross Domestic Product (GDP) on national income, which is one of the primary indicators to determine a country's economy, and $S(n)$ and $I(n)$ be the savings and investment of the people. The Harrod model assumed that in a country people's savings depend on GDP or national income; that is, savings is a constant proportion of current income, which implies:

$$S_n = a\, G_n, \quad \text{..................(6.4.7.1)}$$

where $a > 0$ is a constant of proportionality.

Harrod further assumed that the investment made by the people depends on the difference between the GDP of the current year and the last year, that is,

$$I_n = b\,(G_n - G_{n-1})\,,\ b > a. \quad \text{........................(6.4.7.2)}$$

Finally, the Harrod model assumed that all the savings made by the people are invested, that is,

$$S_n = I_n. \quad \text{.........................(6.4.7.3)}$$

From (6.4.7.1), (6.4.7.2) and (6.4.7.3), we obtain,

$$b\,(G_n - G_{n-1}) = S_n = a\,G_n \;\Rightarrow\; G_n = \frac{b\,G_{n-1}}{b-a},$$

whose solution is

$$G_n = G(0)\left(\frac{b}{b-a}\right)^n$$

Thus, Harrod's model concludes that GDP or national income increases geometrically with time.

## *6.4.8 ARMS RACE MODEL*

We consider two countries engaged in an arms race. We assume that the two countries have similar economic strengths and the same level of distrust for each other. Let $T_n$ be the total amount of money spent by the two countries on arms after $n$ years. Let $g > 0$ measures the restraint of growth due to economic strength (or weakness) of the countries and $d > 0$ the level of distrust between the two countries. Both the countries also spent a constant amount (say $k$) of money for buying arms irrespective of involving in an arms race. Then, the dynamic discrete model for the total amount of money $T_n$ spent on arms by each country after $n$ years is given by:

$$T_n = (1-g)\,T_{n-1} + d\,T_{n-1} + k,$$

$$\Rightarrow T_n = (1-g+d)\,T_{n-1} + k,$$

$$T_n = (1-g+d)^n T_0 + k\left(\frac{1-(1-g+d)^n}{1-(1-g+d)}\right),$$

$$T_n = (1-g+d)^n T_0 + k\left(\frac{1-(1-g+d)^n}{g-d}\right).$$

The equilibrium solution is:

$$T_n = T_{n-1} = T^*,$$

$$\Rightarrow (1-g+d)T_{n-1} + k = T_{n-1},$$

$$\Rightarrow T^* = \frac{k}{g-d},\;\; (g > d).$$

Thus, as time increases, the total amount of money spent on arms reaches a steady state, and both the countries have a “stable” arms race.

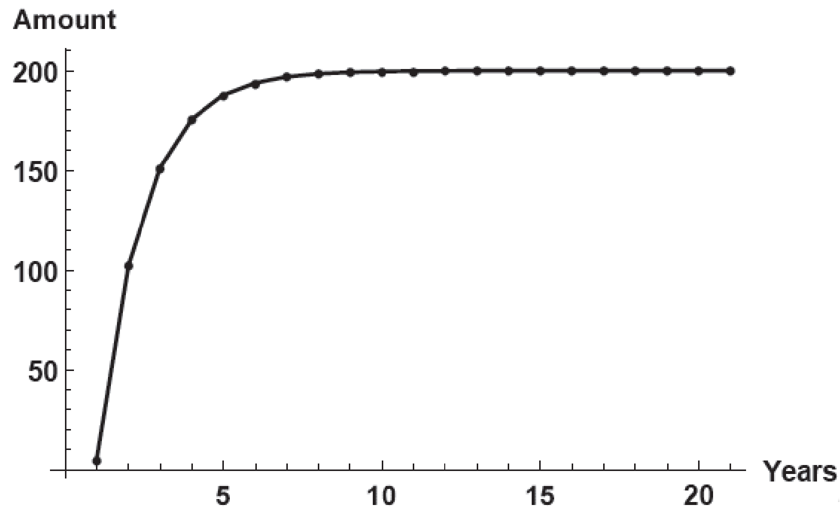



***Fig 6.4.8.1***

The amount of money spent by both the countries on arms reaches a steady-state value with increasing time. Parameter values g=0.6, d =0.1 and k=100.

## *6.4.9 LANCHESTER’S COMBAT MODEL*

F.W. Lanchester, a British Engineer, developed one of the first mathematical models for analyzing combats, whose greatest strength lies in its simplicity. The models helped in better planning, prediction of battles and their possible outcomes. Consider two adversaries, namely, A-team and B-team. Let $A_n$ and $B_n$ be the number of units of A-team and B-team, respectively, remaining in the battle after time $n$. By units, we will mean fighter planes, ships, tanks, soldiers, etc., depending on the context of the battle. It is assumed that the combat loss rate of both the teams is proportional to the size of their respective enemies. Under this assumption, the discrete dynamical system is

$$A_{n+1} = A_n - \beta B_n + r_1,$$
$$B_{n+1} = B_n - \alpha A_n + r_2,$$

where $\alpha > 0$ is the fighting effectiveness of $A$-team, $\beta > 0$ is the fighting effectiveness of $B$ -team, $r_1$ and $r_2$ are the respective numbers of reinforcements for A-team and B-team, respectively for each time step. The equilibrium solution is given by $(A^*, B^*) = \left(\frac{r_2}{\alpha}, \frac{r_1}{\beta}\right)$.

The Jacobian matrix of the system is

$$\begin{pmatrix} 1 & -\beta \\ -\alpha & 1 \end{pmatrix},$$

whose eigen values are obtained by solving

$$\begin{vmatrix} 1-\lambda & -\beta \\ -\alpha & 1-\lambda \end{vmatrix} = 0$$

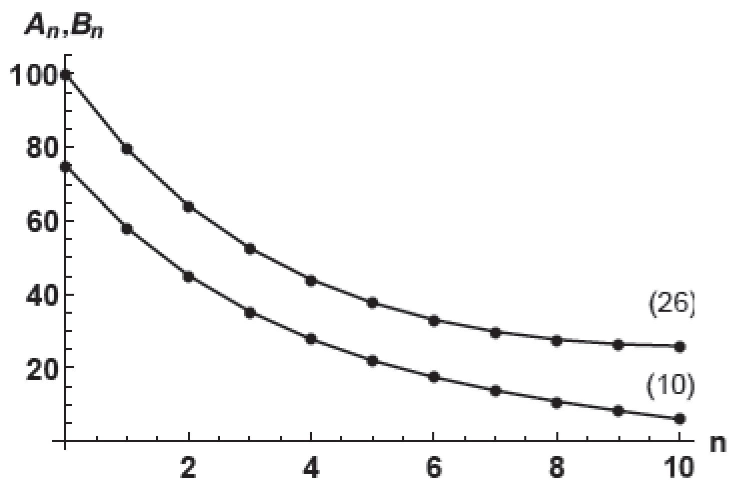
$$\Rightarrow \quad \lambda_{1,2} = 1 \pm \sqrt{\alpha\beta}.$$

Clearly,

$$|\lambda_1| = 1 + \sqrt{\alpha\beta} > 1,$$

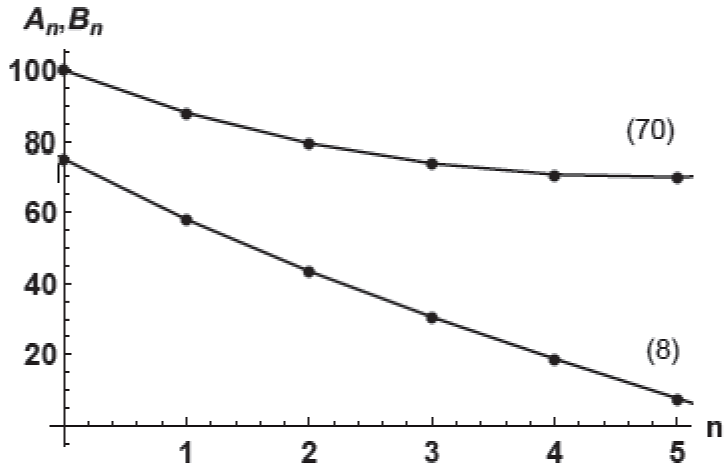
$$\text{and } |\lambda_2| = 1 - \sqrt{\alpha\beta} < 1.$$

This implies that the system is a saddle about the equilibrium point.




(a) $\alpha = 0.3, \beta = 0.2, r_1 = 3, r_2 = 2.$

***Fig 6.4.9.1***




(b) $\alpha = 0.2, \beta = 0.2, r_1 = 3, r_2 = 3.$

***Fig 6.4.9.2***

## *6.5. SUMMARY*

Present unit is a presentation of Discrete Models**,** Linear Models**,** Newton's Law of Cooling**,** Bank Account Problem**,** Drug Delivery Problem**,** Harrod Model (Economic Model)**,** Arms Race Model**,** Lanchester's Combat Model. The examples are also presented here.

## *6.6 GLOSSARY*

**i.** **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

**ii.** **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

**iii.** **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

iv. **Difference equation:** A difference equation is an equation involving differences. One can define a difference equation as a sequence of numbers that are generated recursively using a rule to the previous numbers in the sequence the difference equation

v. **Homogeneous Difference Equation and Non-Homogeneous Difference Equation:** The difference equation is homogeneous if $f(n) = 0$, otherwise it is non-homogeneous.

vi. **Order:** The order of the difference equation is the difference between the largest $(n)$ and smallest $(n-2)$ arguments appearing in the difference equation with unit interval.

vii. **Stable:** The equilibrium point $u^*$ is said to be stable if all the solutions of $u_n = au_{n-1} + b$ approach $u^* = \frac{b}{1-a}$ as $n \to \infty$ (as $n$ becomes large).

viii. **Unstable:** The equilibrium point $u^*$ is unstable if all solutions (if exists) diverges from $u^*$ to $\pm\infty$.

## CHECK YOUR PROGRESS

**CYP2.** Newton's law of cooling, which states that the rate of change of the temperature of an object is …………………..to the difference between its own temperature and the ambient temperature.

**CYP3.**Harrod model aims to determine an ……………..growth rate for the economy.

**CYP4.**………………….helped in better planning, prediction of battles and their possible outcomes

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## 6.9 TERMINAL QUESTIONS

**TQ1.** Let $t_n$ be the temperature in degrees centigrade and $n$ be the number of meters above the ground. The air cools by about $0.02^{\circ}C$ for each meter rise above the ground level.

- **i.** Formulate a discrete dynamical system to model this situation.
- **ii.** If the current temperature at ground level is $30^{\circ}C$, find the temperature $500$ m above the ground.
- **iii.** Find the height above the ground level at which the temperature is $0^{\circ}F$.

**TQ2.** A certain drug is effective in treating a disease if the concentration remains above 100 mg/L. The initial concentration is 640 mg/L. It is known from laboratory experiments that the drug decays at the rate of 20% of the amount present each hour.

**i.** Formulate a linear discrete system that models the concentration after each hour.
**ii.** Find graphically at what hour the concentration reaches 100 mg/L.
**iii.** Modify your model to include a maintenance dose administered every hour.
**iv.** Check graphically or otherwise to determine the maintenance doses that will keep the concentration above the minimum effective level of 100 mg/L and below the maximum safe level of 800 mg/L.
**v.** Working with the maintenance doses you found in
**vi.** Try varying the initial concentration. What do you observe about the tendency to stay within the necessary bounds, as well as the long-term tendency?

## *6.10 ANSWERS*

**TERMINAL QUESTIONS**

**TQ-1:**

**i.** $t_n = t_{n-1} - 0.02.$
**ii.** $t_n = t_0 - 0.02n.$
**iii.** 1.5 $km$

**TQ-2:**

**i.** $C_n = (0.8)^n \times 640.$
**ii.** After 9 hours the concentration reaches $100\frac{mg}{L}$.

**CHECK YOUR PROGRESS**

**CYP2.** Proportional
**CYP3.** Equilibrium
**CYP4**. Combat models

# UNIT 7: NON-LINEAR MODELS

## CONTENTS:



## *7.1 INTRODUCTION*

In previous unit we explained the concept of Discrete Models**,** Linear Models**,** Newton's Law of Cooling**,** Bank Account Problem**,** Drug Delivery Problem**,** Harrod Model (Economic Model)**,** Arms Race Model**,** Lanchester's Combat Model. In this unit we are describing the concepts of Non- Linear Models, Density-Dependent Growth Models, Richer's Model, The Learning Model, Dynamics of Alcohol: A Mathematical Model, Two Species Competition Model, 2-cycles, Stability of 2-cycles, 3-cycles.

## 7.2 OBJECTIVES

After studying this unit, learner will be able to

**i.** Explains the Density-Dependent Growth Models.

**ii.** Describe the Richer's Model and The Learning Model.

**iii.** Defined the Dynamics of Alcohol: Mathematical Model.

**iv.** Explained the Two Species Competition Model and 2-cycles, Stability of 2-cycles and 3-cycles.

## 7.3 NON-LINEAR MODELS

Density dependence is not considered by linear models, which assume that the same growth characteristics are applied to the population regardless of their sizes. In the natural world, linear growths are seldom seen (except for bacteria and viruses). Non – linear models or density-dependent models are quite successful in this regard. The non-linear models success fully capture the density dependence and their varying effects, which is reflected in the qualitative behavior of the solutions of the models.

## 7.3.1 DENSITY DEPENDENT GROWTH MODELS

### 7.3.1.1 RICHER'S MODEL

Richer's model is another example for a density-dependent model for the population of a species after $n$ generations and is given by

$$x_{n+1} = \alpha x_n e^{-\beta x_n} \quad (\alpha > 0, \beta > 0),$$

where $\alpha$ represents the maximal growth rate of the organism and $\beta$ is the inhibition of growth caused by over population.

If $x^*$ be the equilibrium solution of Richer's model, then

$$\alpha x_n e^{-\beta x^*} = x^* \;\Rightarrow\; x^*_{1,2} = 0,\; \frac{\ln(\alpha)}{\beta}.$$

Now, $f'(x) = \alpha e^{-\beta x} - \alpha\beta x e^{-\beta x}$.

Clearly, $|f'(x_1^* = 0)| = |\alpha| < 1$

$\Rightarrow 0 < \alpha < 1$(Since $\alpha > 0$). implying that the equilibrium point $x_1^* = 0$is stable if $\alpha \in (0,1)$.

The equilibrium point

$$x_2^* = \frac{\ln(\alpha)}{\beta}$$

is stable if,

$$|f'(x^* = x_2^*)| < 1 \;\Rightarrow\; |1 - \ln(\alpha)| < 1 \;\Rightarrow\; 1 < \alpha < e^2,$$

Implying that the equilibrium point

$$x^* = x_2^*$$

is stable,

$$\text{if } \alpha \in (1, e^2).$$

For $1 < \alpha < e$ $(\approx 2.7183)$, $f'(x^* = x_2^*) > 0$, hence, the solution of Richer's model is stable and monotonically approaches the equilibrium point $x_2^*$

For $e < \alpha < e^2$ $(\approx 7.389)$, $f'(x^* = x_2^*) < 0$, hence, the solution of Richer's model is stable and oscillates as it approaches the equilibrium $x_2^*$

For $\alpha > e^2$ $(\approx 7.389)$, the solution of Richer's model is unstable and oscillates as it grows away from the equilibrium point $x_2^*$

What is Richer's Method?

## *7.3.2.* THE LEARNING MODEL

When we learn a new topic, the following principle may apply. If the present amount learned is $L_n$, then $L_{n+1}$ equals $L_n$, minus the fraction $r$, of the $L_n$ forgotten, plus the new amount learned, which we assume is inversely proportional to the amount already learned $(\infty, 1/L_n)$. We also assume that the person learning the new topic cannot forget the whole part of the topic learned $(r \neq 1)$. Under the following assumptions, the model is given by

$$L_{n+1} = L_n - rL_n + \frac{k}{L_n} \quad (0 \le r < 1, k > 0),$$

where $k$ is the constant of proportionality. The steady-state solution is given by,

$$L^* = L^* - rL^* + \frac{k}{L^*} \;\Rightarrow\; L_n^* = \sqrt{\frac{k}{r}}.$$

$$\text{Let } f(L) = L - rL + \frac{k}{L},$$

$$\text{then } f'(L) = 1 - r - \frac{k}{L^2}.$$

For stability, we must have,

$$|f'(L^*)| < 1 \;\Rightarrow\; |1 - 2r| < 1 \;\Rightarrow\; 0 < r < 1.$$

Therefore, the system is stable if $0 < r < 1$ and unstable if $r > 1$.

What is learning model?

## *7.3.3 DYNAMICS OF ALCOHOL: A MATHEMATICAL MODEL*

Some teenagers and young adults have made drinking as an integral part of their social life. They believe that the quality of their social life is enhanced by moderate drinking without understanding the risks involved in the consumption of alcohol. About 45-55 percent of road traffic deaths in India occur under the influence of alcohol.

BAC or Blood alcohol content of 0.1 means1gram of alcohol is present in 100ml of blood. In India, the BAC legal limit is 0.03 percent per 100ml of blood or 30 mg of alcohol in 100ml of blood. In the United States, BAC legal limit is 0.08 and in most European countries, it is 0.05. A standard alcoholic drink, which is equivalent to 350 ml of beer ,will raise the BAC of a 68kg adult male to approximately 20-22mg of alcohol in 100ml of blood (count is different in a female). The alcohol consumed is absorbed into the body primarily through the lining of the stomach and that is there as on why a blood alcohol content (BAC) peak is obtained within 20 minutes of the last drink. The main content of alcohol is ethanol (a chemical compound) and the body deals with this chemical in the blood stream in two ways. The first is the metabolism of ethanol by liver enzymes and the second is the filtration by the kidneys. The average rate alcohol leaves the body is 15mg per 100ml per hour (a standard alcoholic drink for men). Thus, if a person 75mg of ethanol in his body at the beginning of an hour, then the body will eliminate 15mg in the next hour.

Let $a_n$ be the amount of alcohol in a person's body at the beginning of hour $n$ and let the person average eliminates about 15 percent of the alcohol from his/her body each hour, then the amount of alcohol eliminated during $(n-1)^{th}$ hour is $0.15a_{n-1}$ where $a_{n-1}$ is the amount of alcohol in a person's body at the beginning of hour $(n-1)$.. Therefore, the amount of alcohol in a person's body is modeled by the dynamical system

$$a_n = a_{n-1} - r\ a_{n-1} + A,$$

where $r$ is the fraction of alcohol filtrated by the kidneys during each time period and A is the amount of alcohol consumed in each time period (for alcohol $r = 0.15$).

Examples?.......
.....................

## 7.3.4 TWO SPECIES COMPETITION MODEL

In a certain forest, black bears and grizzly bears compete with each other for food. Suppose that in the absence of any competition or hunting, the black bear population will grow by $\alpha_1$ per year, while the grizzly bear population will grow by $\alpha_2$. Each year the competition between the two types of bears leads to the death of a certain number of each type of bear (due to fighting and food shortages). The number of black bears that die is equal to the product of the black and grizzly bear populations at a rate $\beta_1$. The number of grizzly bears that die is equal to the product of the black and grizzly populations at a rate $\beta_2$. Let $B_n$ and $G_n$ be the population of black bears and grizzly bears at year n, respectively. under the following assumptions, the model is given by

$$\begin{aligned} B_{n+1} &= B_n + \alpha_1\ B_n - \beta_1\ B_n\ G_n, \\ G_{n+1} &= G_n + \alpha_2\ G_n - \beta_2\ B_n\ G_n. \end{aligned}$$

The equilibrium points are obtained by solving,

$$B^* = B^* + \alpha_1\ B^* - \beta_1\ B^*\ G^*,$$

$$G^* = G^* + \alpha_2\ G^* - \beta_2\ B^*\ G^*,$$

This implies that,

$$(B^*, G^*) = (0,0) \text{ and } \left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}\right)$$

For stability, we calculate

$$\begin{pmatrix} \frac{\partial f}{\partial B} & \frac{\partial f}{\partial G} \\ \frac{\partial g}{\partial B} & \frac{\partial g}{\partial G} \end{pmatrix}_{(B^*,G^*)} = \begin{pmatrix} 1+\alpha_1-\beta_1\ G^* & -\beta_1 B^* \\ -\beta_2 G^* & 1+\alpha_2-\beta_2\ B^* \end{pmatrix},$$

where,

$$f(B,G) = B + \alpha_1\ B - \beta_1\ B\ G,\ g(B,G) = G + \alpha_2\ G - \beta_2\ B\ G.$$

At $(0,0)$, the matrix is $\begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha \end{pmatrix}$,

whose eigen values are

$1 + \alpha_1$

and

$1 + \alpha_2$.

Clearly, $|1+\alpha_1| > 1$, $|1+\alpha_2| > 1$, implying that the system is unstable at the origin. This means both the species will not become extinct simultaneously.

At $\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}\right)$, the matrix is $\begin{pmatrix} 1 & \frac{\beta_1\alpha_2}{\beta_2} \\ \frac{\beta_2\alpha_1}{\beta_1} & 1 \end{pmatrix}$, whose eigenvalues are $1 \pm \sqrt{\alpha_1\alpha_2}$. Clearly, $|1+\sqrt{\alpha_1\alpha_2}| > 1$, and $|1-\sqrt{\alpha_1\alpha_2}| < 1$, implying that the system is saddle at $\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}\right)$.

Since the model is unstable at both the equilibrium points, and we modify it by adding a self-competition term in both the bear species. We assume that each bear species also compete for food among each other. This will modify the model as

$$\begin{aligned} B_{n+1} &= B_n + \alpha_1\ B_n - \beta_1\ B_n\ G_n - \gamma_1\ B_n^2, \\ G_{n+1} &= G_n + \alpha_2\ G_n - \beta_2\ B_n\ G_n - \gamma_2\ G_n^2. \end{aligned}$$

The equilibrium points are obtained by solving

$$B^* = B^* + \alpha_1\ B^* - \beta_1\ B^*\ G^* - \gamma_1\ (B^*)^2,$$

$$G^* = G^* + \alpha_2\ G^* - \beta_2\ B^*\ G^* - \gamma_2\ (G^*)^2, \text{ and we get}$$

For stability, we obtain the matrix

$$A = \begin{pmatrix} 1+\alpha_1-\beta_1\ G^*-2\gamma_1\ B^* & -\beta_1 B^* \\ -\beta_2 G^* & 1+\alpha_2-\beta_2\ B^*-2\gamma_2\ G^* \end{pmatrix}, \text{ where}$$

$f(B,G) = B+\alpha_1\ B-\beta_1\ B\ G-\gamma_1\ B^2,\ g(B,G) = G+\alpha_2\ G-\beta_2\ B\ G-\gamma_2\ G^2.$

At (0, 0), the system is unstable,

At $\left(\frac{\alpha_1}{\gamma_1}, 0\right)$, $A = \begin{pmatrix} 1+\alpha_1 & -\frac{\alpha_1\beta_1}{\gamma_1} \\ 0 & 1+\alpha_2-\frac{\alpha_1\beta_1}{\gamma_1} \end{pmatrix}$, whose eigen values are

$$\lambda_1 = 1-\alpha_1,\ \lambda_2 = 1+\alpha_2-\frac{\alpha_1\beta_2}{\gamma_1}.$$

$\lambda_1 = 1-\alpha_1,\ \lambda_2 = 1+\alpha_2-\frac{\alpha_1\beta_2}{\gamma_1}$. The system will be stable if $|1-\alpha_1| < 1$ and $\left|1+\alpha_2-\frac{\alpha_1\beta_2}{\gamma_1}\right| < 1$.

At $\left(0, \frac{\alpha_2}{\gamma_2}\right)$, $A = \begin{pmatrix} 1+\alpha_1-\frac{\alpha_2\beta_1}{\gamma_2} & 0 \\ -\frac{\alpha_2\beta_2}{\gamma_2} & 1-\alpha_2 \end{pmatrix}$, whose eigen values are

$\lambda_1 = 1-\alpha_2,\ \lambda_2 = 1+\alpha_1-\frac{\alpha_2\beta_1}{\gamma_2}$. The system will be stable if $|1-\alpha_2| < 1$ and $\left|1+\alpha_1-\frac{\alpha_2\beta_1}{\gamma_2}\right| < 1$.

At $\left(\frac{\alpha_1\gamma_2-\alpha_2\beta_1}{\gamma_1\gamma_2-\beta_1\beta_2}, \frac{\alpha_2\gamma_1-\alpha_1\beta_2}{\gamma_1\gamma_2-\beta_1\beta_2}\right)$, the Jacobian matrix is given by

$$A = \begin{pmatrix} \frac{\beta_1(\beta_2-\alpha_2\gamma_1)-(1-\alpha_1)\gamma_1\gamma_2}{\beta_1\beta_2-\gamma_1\gamma_2} & \frac{\beta_1(\alpha_1\gamma_2-\alpha_2\beta_1)}{\beta_1\beta_2-\gamma_1\gamma_2} \\ \frac{\beta_2(\alpha_2\gamma_1-\alpha_1\beta_2)}{\beta_1\beta_2-\gamma_1\gamma_2} & \frac{\beta_1\beta_2-(\alpha_1\beta_2+(1-\alpha_2)\gamma_1)\gamma_2}{\beta_1\beta_2-\gamma_1\gamma_2} \end{pmatrix},$$

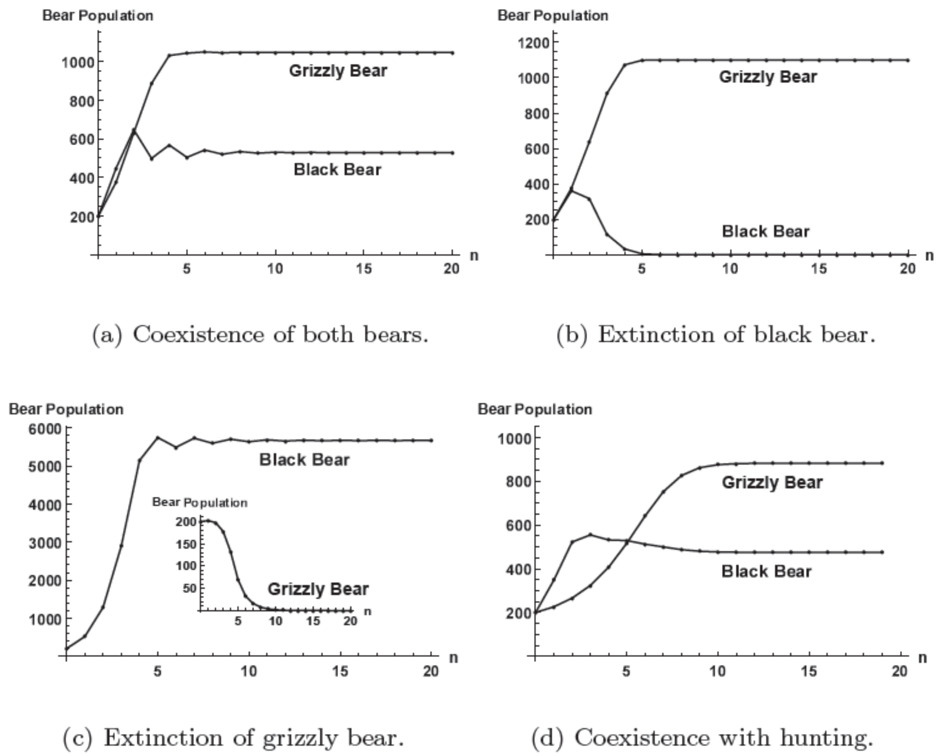



***Fig.7.3.4***

*The figure shows the dynamics of a competition model of black bear and grizzly bear over time. (a) Both bears coexist ($\alpha_1 = 1.9, \beta_1 = 0.0003, \gamma_1 = 0.003, \alpha_2 = 1.1, \beta_2 = 0.0001, \gamma_2 = 0.001$). (b) Black bear dies off ($\beta_1 = 0.0025$, rest same as (a)). (c) Grizzly bear goes to extinction ($\alpha_1 = 1.7, \beta_1 = 0.0003, \gamma_1 = 0.0003, \alpha_2 = 0.05, \beta_2 = 0.0001, \gamma_2 = 0.0001$). (d) Survival and coexistence of both the bears despite of hunting (same as (a)).*

Figure (a) shows that both the bears co-exist even though there is intra - specific and inter - specific competition among them. Reducing the inter-specific competition parameter $\beta_1$ to 0.0025, it is observed from figure (b) that the black bear goes to extinction in approximately 5 years. From figure (c) with some proper choice of parameters, the extinction of grizzly bears is seen.

*This model is a simplification of real-world ecological systems and makes assumptions that may not always hold true.*

## 7.3.5 2-CYCLES

Consider a non-linear difference equation $x_n = f(x_n)$. A pair of distinct points $x_1$ and $x_2$ is called a 2-cycle of $x_{n+1} = f(x_n)$ if,

$$f(x_1) = x_2 \text{ and } f(x_2) = x_1.$$

Each point is called a point of period 2 for $f(x)$.

## 7.3.6 STABLITY OF 2-CYCLES

Local stability of a 2-cycles of $x_n = f(x_{n+1})$ means each point of the Cycle is a stable fixed point of $f(f(x))$, otherwise the 2-cycle is unstable. The Condition for stability is given by

(Locally Stable)

$$|f'(x_1)f'(x_2)| < 1$$

(Unstable)

$$|f'(x_1)f'(x_2)| > 1$$

where $x_1$ and $x_2$ are two distinct points of 2-cycles of $x_{n+1} = f(x_n)$ .

## 7.3.7 3-CYCLES

Consider a non-linear difference equation

$$x_{n+1} = f(x_n)$$

A pair of three distinct points $x_1$, $x_2$ and $x_3$ is called a 3-cycle of $x_{n+1} = f(x_n)$ if,

$$f(x_1) = x_2, f(x_2) = x_3, f(x_3) = x_1.$$

Each point is called a point of period 3 for $f(x)$.

## 7.4. EXAMPLES

**Example 7.4.1.** Find a 2-cycle of

(i) $x_{n+1} = \dfrac{-3x_n}{x_n^2+1}$ (ii) $x_{n+1} = {x_n}^2 + x_n - 4$,

and determine its stability.

**Solution:**

(i) Let $f(x) = \dfrac{-3x}{x^2+1}$.

The equilibrium points are obtained by solving

$$x^* = \frac{-3x^*}{{x^*}^2+1},$$
$$\Rightarrow \; x^*\,(x^*+4) = 0 \;\Rightarrow\; x^* = 0, \pm 2i.$$

We next compute

$$f(f(x)) = \frac{3(x^2-1)}{(x^2+1)^2}.$$

Now, any point of period 2 for $f(x)$ is actually a fixed point of the composition $f(f(x))$ and we solve

$$f(f(x)) = x \;\Rightarrow\; \frac{9x(x^2+1)}{1+11x^2+x^4} = x,$$
$$\Rightarrow\; x(x^4+2x^2-8) = 0.$$

This equation also gives the fixed points of f(x). Hence, dividing by $x(x^2+4)$

$$x^2 - 2 = 0 \;\Rightarrow x = \pm\sqrt{2}.$$

Now, $f(\sqrt{2}) = -\sqrt{2}$ and $f(-\sqrt{2}) = \sqrt{2}$.

Therefore $\sqrt{2}$ and $-\sqrt{2}$ constitute a 2-cycle. Now,

$$f'(x) = 2x + 1\frac{3(x^2-1)}{(x^2+1)^2},$$

$$\Rightarrow \; f'(\sqrt{2}) = \frac{1}{3}, \; f'(-\sqrt{2}) = \frac{1}{3},$$

$$\text{and } |f'(\sqrt{2})f'(-\sqrt{2})| = \left|\frac{1}{9}\right| < 1,$$

This implies that 2-cycle is stable.(Fig.7.3.4(a)).

(ii) Let $f(x) = x^2 + x - 4$. We first obtain the fixed points of

$$x_{n+1} = {x_n}^2 + x_n - 4$$

by solving

$$x^* = {x^*}^2 + x^* - 4 \;\Rightarrow x^* = -2, 2.$$

We next compute

$$f(f(x)) = (x^2 + x - 4)^2 + (x^2 + x - 4) - 4 = x^4 + 2x^3 - 6x^2 - 7x + 8.$$

Now, any point of period 2 for $f(x)$ is actually a fixed point of the composition $f(f(x))$ and we solve

$$f(f(x)) = x \;\Rightarrow x^4 + 2x^3 - 6x^2 - 7x + 8 = x \;\Rightarrow x^4 + 2x^3 - 6x^2 - 8x + 8 = 0.$$

This equation also gives the fixed points of $f(x)$. Hence, dividing by $(x^2 - 4)$ we get,

$$x^2 + 2x - 2 = 0 \;\Rightarrow x = (-1+\sqrt{3}) \text{ and } (-1-\sqrt{3}). \text{ Now,}$$

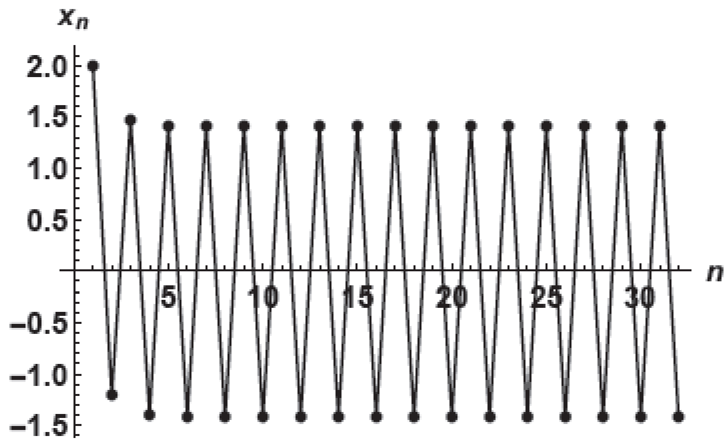
$$f(-1+\sqrt{3}) = (-1+\sqrt{3})^2 + (-1+\sqrt{3}) - 4 = (-1-\sqrt{3}) \text{ and } f(-1-\sqrt{3}) = (-1+\sqrt{3}).$$

Therefore $(-1+\sqrt{3})$ and $(-1-\sqrt{3})$ constitute a 2-cycle. Now,

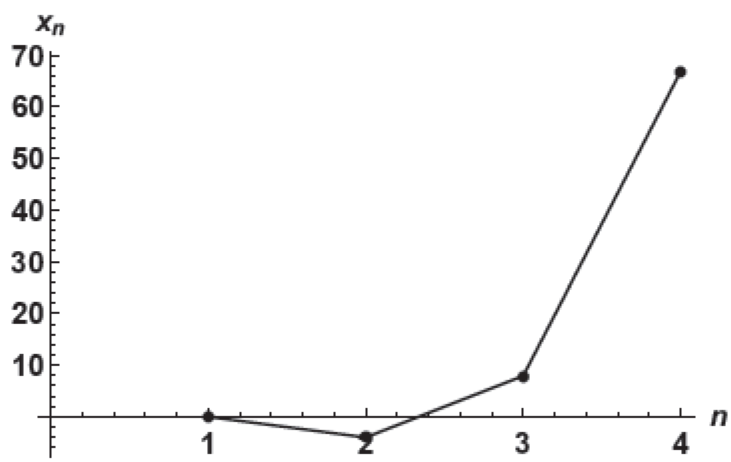
$$f'(x) = 2x + 1,$$

$$f'(-1+\sqrt{3}) = 2(-1+\sqrt{3}) + 1 = -1 + 2\sqrt{3},$$
$$f'(-1-\sqrt{3}) = 2(-1-\sqrt{3}) + 1 = -1 - 2\sqrt{3},$$
$$f'(-1+\sqrt{3})f'(-1-\sqrt{3}) = (-1+2\sqrt{3})(-1-2\sqrt{3}) = 1 - 12 = -11,$$
$$|f'(-1+\sqrt{3})f'(-1-\sqrt{3})| = |-11| = 11 > 1 \;\Rightarrow \text{the 2-cycle is unstable}$$

(Fig.7.3.4(b)).




(a) Stable 2-cycle.




(b) Unstable 2-cycle.

The figures show the dynamics of 2-cycles, (a) stable and (b)unstable.

**Example 7.4.2.**

For the system $x_{n+1} = 1-2|x_n|$, find the points in its 3-cycle,
find the points in its 3-cycle,
where $\frac{7}{9}$ is one such point. Also, determine its stability.

**Solution:** Let $f(x) = 1 - 2|x|$. Now,

$$f\left(\frac{7}{9}\right) = 1 - 2\left|\frac{7}{9}\right| = -\frac{5}{9},$$

$$f^2\left(\frac{7}{9}\right) = f\left(f\left(\frac{7}{9}\right)\right) = f\left(-\frac{5}{9}\right) = 1 - 2\left|-\frac{5}{9}\right| = -\frac{1}{9},$$

$$f^3\left(\frac{7}{9}\right) = 1 - f\left(f^2\left(\frac{7}{9}\right)\right) = 1 - f\left(-\frac{1}{9}\right) = 1 - 2\left|-\frac{1}{9}\right| = -\frac{7}{9}.$$

This implies $\frac{7}{9}, -\frac{5}{9}, -\frac{1}{9}$ is a 3-cycle of $x_{n+1} = 1 - 2|x_n|$.

Since, $f(x) = 1 - 2|x|$,

$$f'(x) = -2, \; x > 0,$$
$$= 2, \; x < 0.$$

Therefore,

$$f'(\frac{7}{9}) = -2, f'(-\frac{5}{9}) = 2, f'(-\frac{1}{9}) = 2,$$

and

$$\left|f'\left(\frac{7}{9}\right) f'\left(-\frac{5}{9}\right) f'\left(-\frac{1}{9}\right)\right| = |-8| = 8 > 1,$$

implying that the 3-cycle is unstable.

**Example 7.4.3.**
*Consider the price model $P_{n+1} = \frac{1}{P_n} + \frac{P_n}{2} - 1$. Find the two equilibrium points and determine their stability.*

**Solution**: The equilibrium points are given by

$$\frac{1}{P^*} + \frac{P^*}{2} - 1 = P^* \;\Rightarrow\; P^* = 0.732, -2.732.$$

$$\text{Let } f(P) = \frac{1}{P} + \frac{P}{2} - 1 \;\Rightarrow f'(P) = -\frac{1}{P^2} + \frac{1}{2}.$$

Now, $f'(0.732) = -1.36 \Rightarrow | f'(0.732) | > 1 \Rightarrow$ the system is unstable about $P^* = 0.732$, and $f'(-2.732) = 0.366 \Rightarrow | f'(-2.732) | < 1 \Rightarrow$ the system is stable about $P^* = -2.732$.

**Example 7.4.4.**

*Let $t_n$ be the temperature in degrees centigrade and $n$ be the number of meters The air cools by about 0.02 °C for each meter rise above the ground level.*

*(i) Formulate a discrete dynamical system to model this situation.*

*(ii) If the current temperature at ground level is 30 °C, find the temperature 500 m above the ground.*

*(iii) Find the height above the ground level at which the temperature is 0 °F.*

**Solution:** (i) The discrete dynamical model is given by

$$t_n = t_{n-1} - 0.02$$

(negative because temperature decreases (cools) as height increases), where $t_n$ is the temperature in degrees centigrade, $n$ meters above the ground.

(ii) The temperature 0.5 km. (500 m.) above the ground is given by

$$t_{500} = t_{499} - 0.02,$$
$$t_{499} = t_{498} - 0.02,$$
$$t_{498} = t_{497} - 0.02,$$
$$- - - - - - - - -$$
$$- - - - - - - - -$$
$$t_2 = t_1 - 0.02,$$
$$t_1 = t_0 - 0.02.$$

Adding, we get, $t_{500} = t_0 - 0.02 \times 500$

$= 30 - 0.02 \times 500 = 30 - 10 = 20\,°\text{C}.$

In general, $t_n = t_0 - 0.02\, n.$

(iii) If $n$ be the height above the ground at which the temperature is about 0 °C, then, $0 = 30 - 0.02 \times n \Rightarrow n = 1500$ m, that is, 1.5 km.

## 7.5 SUMMARY

This unit is a presentation of Non- Linear Models**,** Density-Dependent Growth Models, Richer's Model, The Learning Model Dynamics of Alcohol: A Mathematical Model, Two Species Competition Model, 2-cycles, Stability of 2-cycles, 3-cycles. We have define different models in brief and some examples are also present.

## 7.6 GLOSSARY

**i.** **Variables:** In mathematical modelling, variables are symbols that represent quantities that can change, such as time, distance, temperature, or population size. They are used to describe real quantitative situations by writing mathematical expressions in place of words. Variables can be independent or dependent.

**ii.** **Equations:** The equations in mathematical model contain variables, which are values to input into the equation, and parameters, which are constants whose value depends on the particular model and situation.

**iii.** **Constraints:** In mathematical modelling, constraints are the conditions that a solution to an optimization problem must satisfy. They represent restrictions or limitations on the variables used in equations that depict real-world scenarios. Constraints are essential to ensure that the mathematical model accurately reflects the situation.

**iv.** **Difference equation:** A difference equation is an equation involving differences. One can define a difference equation as a sequence of numbers that are generated recursively using a rule to the previous numbers in the sequence the difference equation

**v.** **Homogeneous Difference Equation and Non-Homogeneous Difference Equation:** The difference equation is homogeneous if $f(n) = 0$, otherwise it is non-homogeneous.

**vi.** **Order:** The order of the difference equation is the difference between the largest $(n)$ and smallest $(n-2)$ arguments appearing in the difference equation with unit interval.

vii. **Stable:** The equilibrium point $u^*$ is said to be stable if all the solutions of $u_n = au_{n-1} + b$ approach $u^* = \frac{b}{1-a}$ as $n \to \infty$ (as $n$ becomes large).

viii. **Unstable:** The equilibrium point $u^*$ is unstable if all solutions (if exists) diverges from $u^*$ to $\pm\infty$.

**CHECK YOUR PROGRESS**

i. Richer's model is another example for a ……… for the population of a species after $n$ generations.

ii. In a certain forest, black bears and grizzly bears compete with each other for food the model use here is ………………...

iii. Two Species Competition model is unstable at both the …………...

iv. Local stability of a 2-cycles of $x_n = f(x_{n+1})$ means each point of the Cycle is a …..

v. In 2-cycles each point is called a point of period……. for $f(x)$.

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## *7.9. TERMINAL QUESTIONS*

TQ1.

*A certain drug is effective in treating a disease if the concentration remains above* 100 *mg/L. The initial concentration is* 640 *mg/L. It is known from laboratory experiments that the drug decays at the rate of* 20% *of the amount present each hour.*

*(i) Formulate a linear discrete system that models the concentration after each hour.*
*(ii) Find graphically at what hour the concentration reaches* 100 *mg/L.*
*(iii) Modify your model to include a maintenance dose administered every hour.*
*(iv) Check graphically or otherwise to determine the maintenance doses that will keep the concentration above the minimum effective level of* 100 *mg/L and below the maximum safe level of* 800 *mg/L.*
*(v) Working with the maintenance doses you found in (iv), try varying the initial concentration. What do you observe about the tendency to stay within the necessary bounds, as well as the long-term tendency?*

## *7.10. ANSWERS*

**TERMINAL QUESTIONS**

**(i)**

$$C_n = (0.8)^n C_0 = (0.8)^n \times 640.$$

**(ii)** From the graph we can see that after 9 hours, the Concentration reaches100mg/L. Thus, doses must be provided before this time for recovery.

**CHECK YOUR PROGRESS**

- **i.** density-dependent model.
- **ii.** Two species competition model.
- **iii.** equilibrium points.
- **iv.** stable fixed point of $f(f(x))$.
- **v.** 2.

# UNIT 8: CONTINUOUS MODEL USING ORDINARY DIFFERENTIAL EQUATIONS

## CONTENTS:

## 8.1 INTRODUCTION

In previous unit we have defined Mathematical Modeling through Differential Equation, Linear Model Non-Linear Models. Now in this unit we defined continuous Models Using Ordinary Differential Equations, Steady State Solution, Stability, Linearization and Local stability Analysis, Lyapunov's Direct Method etc. we also discussed about Arms Race Models ,Epidemic modes and Combat Models .

## 8.2 OBJECTIVES

After studying this unit, learner will be able to

**i.** Explain the Continuous Model

**ii.** Describe the continuous change of a system over time or space using ODEs.

**iii.** Analyse the behaviour of a system with the help of ODEs.

**iv.** Identify the optimal parameters or control strategies to achieve the desired outcome.

## 8.3 INTRODUCTION TO CONTINUOUS MODELS

Continuous models are mathematical representations of systems or phenomena that can be described using continuous variables and functions. A continuous model consists of a dependent continuous variable, varying with some other independent continuous variables. We use a first -order ordinary differential equations to model a continuous system if we have some information or assumption about the rate of change of the dependent variable(s) with respect to the independent variables. There are some types of Continuous Models

**1.Differential Equations:** These models describe how a system changes over time or space using rates of change.
**2**. **Integral Equations:** These models describe the accumulation of a quantity over a given interval.
**3.Optimization Models:** These models describe the best solution to a problem given certain constraints.

## 8.4 STEADY STATE SOLUTION

We consider a system of n non linear, autonomous (does not explicitly depend on time, that is, time -invariant ) differential equations

$\frac{d\tilde{x}}{dt}$ = f($\tilde{x}$)

$where$

$\tilde{x}$ =$(x_1, x_2, \ldots\ldots\ldots\ldots x_n)^T and$ f($\tilde{x}$)= $f_1\tilde{x} f_2 \ \widetilde{(x)} \ ,\ldots\ldots\ldots f_n \ \ \widetilde{(x)}^T$

A steady -state solution or equilibrium solution or critical point is a constant solution that is the value of $\tilde{x}$does not change over time and is obtained by putting

$\frac{d\tilde{x}}{dt}$ = 0

**Note:** In order for the value of $\bar{x}$ to be the same (constant) over time, there must not be any change in$\bar{x}$ implying $\frac{d\tilde{x}}{dt}$ = 0. Therfore, the only value(s) of $\bar{x}$ for which this can happens is f($\bar{x}$) = 0 and so f($\tilde{x}$) =0 gives a steady state solution or an equlibrium solution or a fixed poinr.

In short if we consider a nonlinear time invariant system

$\frac{d\tilde{x}}{dt}$ =f($\tilde{x}$) , $\tilde{x}(t_0)$ = $\widetilde{x_0}$where f : R → $R$, a point $\tilde{x}_e$ is an equlibrium point or a steady state solution of the system if f ($\tilde{x}_e$) =0 .

## 8.5 STABILITY

In Layman's language, we say that an equlibrium point or a steady state solution $\tilde{x}_e$ is locally stable, if all solution that start near $\tilde{x}_e$ (that is, the initial condition are in the neighborhood of $\tilde{x}_e$ ) remain near $\tilde{x}_e$ for all the time. Furthermore , if all the solutions startring near $\tilde{x}_e$ approch $\tilde{x}_e$

t → ∞, we say that the equlibrium point or steady state solution $\tilde{x}_e$ is locally asymptotically stable .

Consider the dynamical system satisfying

$\frac{d\tilde{x}}{dt}$ =f($\tilde{x}$) , $\tilde{x}(t_0)$ = $\widetilde{x_0}$where f : R → $R$,

## 8.5.1 LINEARIZATION AND LOCAL STABILITY ANALYSIS

This method is known as Lyapunov's first method or reduced method, Where the stability analysis of a steady sate or equilibrium point is done by studding the stability of the corresponding linearized system in the neighborhood of the steady state. We consider the model of the form

$\frac{dy}{dx}$= f(x)

$Whose$ $l$ocal stability analysis we want to perform about the equilibrium point x* (obtained by putting f(x) = 0). We give a small perturbation to the system about the equilibrium point x*. Mathematically this means we put x = X+ x* into the above equation and get

## 8.5.2 LYAPUNOV'S DIRECT METHOD

Lyapunov's direct method, also known as Lyapunov's second method, determines the stability of a system without explicitly integrating the differential equation

$$\frac{d\tilde{x}}{dt} = f(\tilde{x}), \tilde{x}(t_0) = \tilde{x_0}, \text{ where } f: \mathrm{R} \to \mathrm{R}.$$

The method is a generalization of the idea that if the potential energy has a relative minimum at the equilibrium point, then the equilibrium point is stable; otherwise, it is unstable. Russian mathematician Aleksandr Mikhailovich Lyapunov generalized this principle to obtain a method for studying the stability of the general autonomous system.

## 8.5.2 LYAPUNOV'S CONDITION FOR LOCAL STABILITY

Consider the autonomous system

$$\frac{d\tilde{x}}{dt} = f(\tilde{x}),\ \tilde{x}(t_0) = \tilde{x_0}, \mathrm{f}: \mathrm{R} \to \mathrm{R}, \qquad \ldots(1)$$

where having isolated critical point at the origin $\tilde{x} = 0$ and $f(\tilde{x})$ has continuous partial derivatives for all $\tilde{x}$. Let $V(\tilde{x})$ be a positive definite function in a neighborhood **S** of the origin $\tilde{x} = 0$ and the derivative $\dot{V}(\tilde{x})$ of $V(\tilde{x})$ with respect to the system (1) is negative semi-definite in the neighborhood S of the origin $\tilde{x} = \mathbf{0}$, then $V(\tilde{x})$ is called a Lyapunov function for the System. Mathematically, $V(\tilde{x})$ is called a Lyapunov function if

(i) $V(\tilde{x}) > 0$ in the nbd. of the origin $\tilde{x} = 0$,

(ii) $V(\mathbf{0}) = 0$ for all $\tilde{x} = \mathbf{0}$,

(iii) $\dot{V}(\tilde{x}) \leq 0$ in the nbd. of the origin $\tilde{x} = 0$,

(iv) $\dot{V}(\mathbf{0}) = 0$ for all $\tilde{x} = \mathbf{0}$.

If there exists a Lyapunov function $V(\tilde{x})$ for the system (1) in the nbd. of the origin $\tilde{x} = 0$, then the steady-state solution or the equilibrium point $\tilde{x} = \mathbf{0}$ is stable (locally).

If the derivative $\dot{V}(\tilde{x})$ of $V(\tilde{x})$ is negative semi-definite in the neighborhood **S** of the origin $\tilde{x} = \mathbf{0}$, then the steady state solution or the equilibrium point $\tilde{x} = \mathbf{0}$ is locally asymptotically stable (LAS).

**Note:**There is no general method for constructing a Lyapunov function, but if one can construct a Lyapunov function for the system (1), then one can directly obtain information about the steady state of the equilibrium point $\tilde{x} = \mathbf{0}$ and hence the name Lyapunov's direct method.

## *8.6 CONTINUOUS MODELS*

A continuous model is a mathematical representation of a system or phenomenon that assumes the variables and functions involved are continuous, it means they can take on any value within a given range or interval. There are some different types of models are formulated and discussed Below.

## 8.6 .1 CARBON DATING

Carbon dating is method developed by W. F. Libby at the university of Chicago in 1947, that can be used to accurately date archaeological samples to determine the ages of the plant (wood Fossil) or any material.

## 8.6 .2 Drug Distribution in the body

The study of the movement of drugs in the body is called pharmacokinetics. Thescience of pharmacokinetics uses mathematical equations and utilizes them to describe the movement of the drug through the body.

Now we study a simple problem in pharmacology, where we will be dealing with the dose-response relationship of a drug. In this problem, the drug present in the system follows certain laws. Let us assume that the rate of decrease of the concentration of the drug is directly proportional to the square of its amount present in the body and $C_0$ be the initial dose of the drug given to the patient at time $t = 0$. The mathematical model that captures these dynamics is given by

$$\frac{dC(t)}{dt} = -kC^2 \qquad (1)$$

where $k$ is a constant depending on the drug used and its value can be obtained from the experiment. Solving (1), we get

$$C(t) = \frac{C_0}{1 + C_0 kt}, \text{ where } C(0) = C_0$$

Let an equal dose of drug $C_0$ be given to the body at equal time intervals, $T$. Then, immediately after the second dose, the concentration of the drug inside the body is

$$C_1 = C_0 + \frac{C_0}{1 + C_0 kT}$$

Immediately after the third dose, the concentration of the drug inside the body is

$$C_2 = C_0 + \frac{C_1}{1 + C_1 kT}.$$

In a similar manner, we can conclude that

$$C_n = C_0 + \frac{C_{n-1}}{1 + C_{n-1}kT}, \qquad (2)$$

which is a non-linear difference equation. Now,

$$C_{n+1} - C_n = \frac{C_n - C_{n-1}}{(1 + kTC_n)(1 + kTC_{n-1})}$$

From (2), we conclude that $C_n > C_0$, which implies $C_{n+1} - C_n$ and $C_n - C_{n-1}$ have the same sign. Noting that $C_n$ is an increasing function of $n$, we attempt to find the limiting value of the concentration by taking limits on both sides of (2), that is,\

$$\lim_{t \to \infty} C_n = \lim_{t \to \infty} \left( C_0 + \frac{C_{n-1}}{1 + C_{n-1}kT} \right),$$

$$\lim_{t \to \infty} C_n = C_0 + \frac{\lim_{t \to \infty} C_{n-1}}{1 + kT \lim_{t \to \infty} C_{n-1}},$$

⇨ $C_\infty = C_0 + \frac{C_\infty}{1 + C_\infty kT}$ where $C_\infty = \lim_{t \to \infty} C_n = \lim_{t \to \infty} C_{n-1}$'

⇨ $kTC^2{}_\infty$ - kT$C_0 C_\infty$ - $C_0$ =0,

⇨ $C_\infty = \frac{C_0}{2} + \frac{C_0}{2}\sqrt{1 + \frac{4}{C_0 kT}}$ taking positive sign only).

⇨ $C_0 < C_n < C_\infty$ (Since $C_n$ is an increasing function), that is concentration is bounde.

## 8.6.3 GROWTH AND DECAY OF CURRENT IN L-R CIRCUTT

We consider an L-R circuit where $L$ is the inductance of the coil and $R$ is the resistance. The coil is connected to a battery of voltage $V$ through a key $K$ in the given figure (1)

In the **ON** position, the current flows through the coil. When the current $i(t)$ starts to flow, the negative lines of force move outward from the coil and an electromotive force (e.m.f.) will induce across L. According to the law of electromagnetic induction, this e.m.f. will oppose the voltage, as a result of

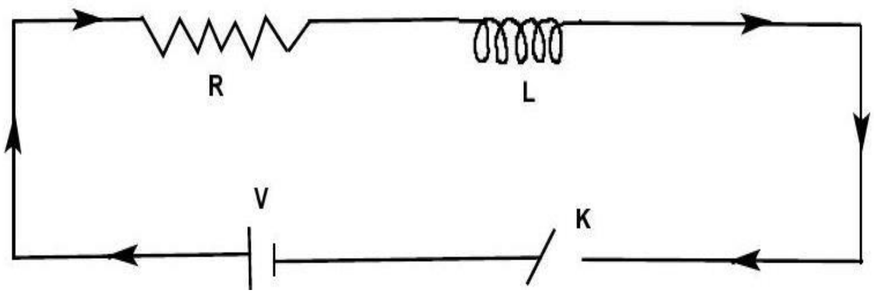
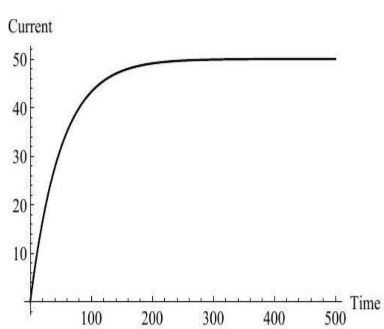



**Figure (1)**

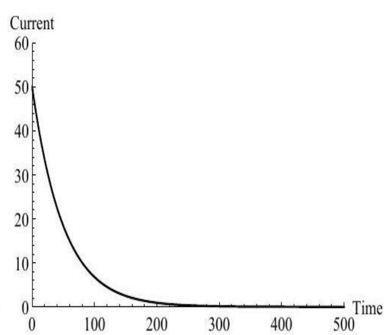
**FIGURE 1:** The inductance-resistance ( $L - R$ ) circuit, connected to a battery of voltage $V$ through a key $K$.
which there will be a voltage drop across $R$, which will also oppose the applied voltage. Let, at any time $t$, i be the current in the circuit increasing from 0 to a maximum value at a rate of increase $\frac{di}{dt}$. Now, the potential difference across the inductor is $V_1 = L\frac{di}{dt}$ and across the resistor is $V_2 = iR$. The differential equation modeling of this scenario is given by

$$V = L\frac{di}{dt} + iR \text{ (since } V = V_1 + V_2\text{)}$$
$$\Rightarrow \int_0^i \frac{di}{i - \frac{V}{R}} = -\frac{R}{L}\int_0^t dt \Rightarrow \log_e\left(i - \frac{i}{V/R}\right) = -\frac{R}{L}t$$
$$\Rightarrow i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

which shows that the current grows exponentially. As $t \to \infty, i \to \frac{V}{R} = I$ (say), a steady value (fig2).




(a) Current grows and reaches a steady value.




(b) Current decays to zero.

**Figure 2**

**FIGURE 2 :** Graphs showing the (a) growth and (b) decay of current, with $L = 50, R = 1$ and $V = 50$.

We now put the key in the OFF position. Initially, when the key was in the **ON** position, a steady current $I = \frac{V}{R}$ was flowing. With no current flowing in the circuit, the flux will reduce gradually, resulting in a voltage drop $iR$ across the resistance $R$ and the induced e.m.f. $L\frac{di}{dt}$ across the inductance $L$.

Now, since the key is OFF, the current becomes open, implying that the impressed voltage is zero.

The differential equation showing this scenario is given by

$$0 = L\frac{di}{dt} + iR(\text{ since } V = V_1 + V_2 = 0)$$
$$\Rightarrow \int_I^i \frac{di}{i} = -\int_0^t \frac{R}{L}dt(\text{ since at } t = 0, i = I)$$
$$\ln\left(\frac{i}{I}\right) = -\frac{R}{L}t \Rightarrow i(t) = \frac{V}{R}e^{-\frac{R}{L}t}$$

Thus, the current decays exponentially as time increases and ultimately goes to zero.

## 8.6.4 RECTILINEAR MOTION UNDER VARIABLE FORCE

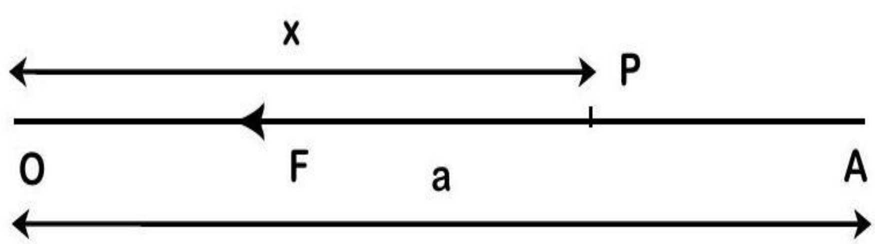



**Figure 3**

**FIGURE3 :**A particle moving in a straight line towards the origin O (fixed) and acted upon by a force $F$ directed towards $O$.

Let a particle move in a straight line and be acted upon by a force $F = \frac{m\mu}{x}, \mu(> 0)$ being the constant of proportionality, which is always directed towards a fixed point $O$. Here, $m$ is the mass of the particle and $P$ is the position of the particle at any time $t$ such that $OP = x$ (**fig. 3**). The equation of motion modeling the given scenario is given by

$$mv\frac{dv}{dx} = -m\frac{\mu}{x}$$

where $v\frac{dv}{dx}$ is the acceleration of the particle of mass $m$. Since the force is attractive, the sign of right-hand side is negative. Integrating, we get,

$$\frac{v^2}{2} = -\mu \ln x + \text{ constant}$$

Initially, let the particle start from rest (from A) at a distance $a$ from the fixed point $O$ (origin), then at time $t = 0, x = a, v = 0$,

$$\Rightarrow \text{ constant } = \mu \ln a \Rightarrow v^2 = 2\mu \ln\left(\frac{a}{x}\right).$$

$\Rightarrow v = \frac{dx}{dt} = -\sqrt{2\mu \ln\left(\frac{a}{x}\right)}$ (Negative sign as distance decreases with time).

$\Rightarrow \sqrt{2\mu}\int_0^T dt = -\int_a^0 \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}}$, where $T$ is the time taken by the particle

to reach the origin.

$$\sqrt{2\mu}T = -\int_0^\infty \frac{ae^{-y^2}(-2y)dy}{y}\left(\text{ Substitute } \ln\left(\frac{a}{x}\right) = y^2\right)$$

Therefore, $= 2a\int_0^\infty e^{-y^2}dy = a\int_0^\infty \frac{e^{-z}}{\sqrt{z}}dz(\text{ Put } z = y^2)$

$$= a\int_0^\infty e^{-z}z^{\frac{1}{2}-1}dz = a\Gamma\left(\frac{1}{2}\right) = a\sqrt{\pi} \Rightarrow T = a\sqrt{\frac{\pi}{2\mu}}$$

## *8.7 ARMS RACE MODELS*

The Arms Race model is a mathematical model used to describe the dynamics of an arms race between two or more countries. The model is based on the idea that each country's military expenditure is influenced by the military expenditure of its opponents.

We consider two neighboring countries $A$ and $B$ and let $x(t)$ and $y(t)$ be the expenditures on arms respectively by these two countries in some standardized monetary unit.

We construct a simple mathematical model by assuming the notion of mutual fear, that is, the more one country spends on arms, it encourages the other one to increase its expenditure on arms. Thus, we assume that each country spends on arms at a rate which is directly proportional to the existing expenditure of the other nation.

Mathematically, we can write

$$\frac{dx}{dt} = \alpha y, \frac{dy}{dt} = \beta x (\alpha, \beta > 0) \quad (1)$$

$$\Rightarrow \frac{d^2x}{dt^2} = \alpha \frac{dy}{dt} = \alpha\beta x$$

$$\Rightarrow x = A_1 e^{\sqrt{\alpha\beta}t} + A_2 e^{-\sqrt{\alpha\beta}t}$$

Similarly, we get, $y = B_1 e^{\sqrt{\alpha\beta}t} + B_2 e^{-\sqrt{\alpha\beta}t}$.

Thus, $x, y \to \infty$ as $t \to \infty$ and we conclude that both the countries $A$ and $B$ spend more and more money on arms with increasing time and no limits on the expenditure (fig.4). As the mathematical prediction of indefinitely large expenditure for both the countries is unrealistic, an improved model is desired.

In the modified model, other than the mutual fear, we also assume that the rate of change of one country's expenditure on arms will also be directly proportional to its own expenditure as the excessive expenditure on the arms puts the country's economy in the compromising position. Accordingly, the model (1) is modified as

$$\frac{dx}{dt} = \alpha y - \gamma x (\alpha, \beta, \gamma, \delta > 0) \quad (2)$$

Clearly, (0,0) is the only steady-state solution, provided $\gamma\delta - \alpha\beta \neq 0$. The characteristic equation is given by

$$\begin{vmatrix} -\gamma - \lambda & \gamma \\ \beta & -\delta - \lambda \end{vmatrix} = 0,$$

$$\Rightarrow \lambda^2 - (-\delta - \gamma)\lambda + \gamma\delta - \alpha\beta = 0$$

Hence, the system is stable if $\gamma\delta - \alpha\beta > 0 \Rightarrow \gamma\delta > \alpha\beta$. This implies if the product of the rates of depreciation $(\gamma\delta)$ on the expenditure of arms of both the countries $A$ and $B$ is greater than the product of rates of expenditure $(\alpha\beta)$ on arms of both the countries, the system will be stable and the countries will spend an allocated amount of money on arms, so that the economy of the country is not compromised.

A simple refinement of the model was made by Lewis F. Richardson (1881-1953), popularly known as the Richardson Arms Race model , where he assumed that the cause of the rate of increase of a country's armament, not only depend on mutual stimulation but also on the permanent underlying grievances of each country against the other.

$$\frac{dx}{dt} = \alpha y - \gamma x + r \quad (3)$$

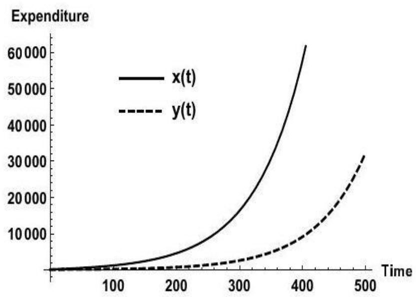
$$\frac{dy}{dt} = \beta x - \delta y + s \quad (4)$$

where $\alpha, \beta, \gamma, \delta$ are positive (as before) and $r, s$ are constants which will be negative or positive, depending on the fact whether the country has overcome the grievance or not. The unique steady-state solution is obtained by solving

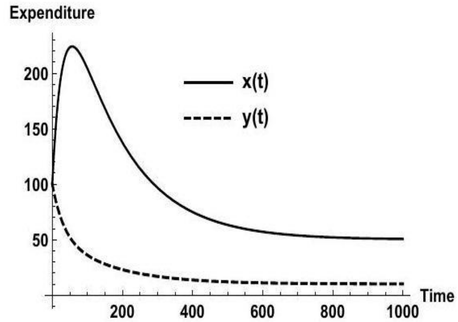
$$\alpha y^* - \gamma x^* + r = 0 \text{ and } \beta x^* - \delta y^* + s = 0$$

provided $\gamma\delta - \alpha\beta \neq 0$. Solving, we obtain

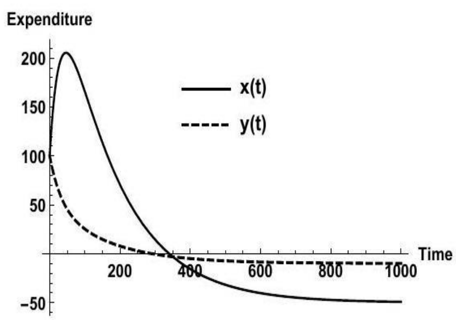
$$x^* = \frac{r\delta + s\alpha}{\gamma\delta - \alpha\beta} \text{ and } y^* = \frac{r\beta + s\gamma}{\gamma\delta - \alpha\beta}.$$



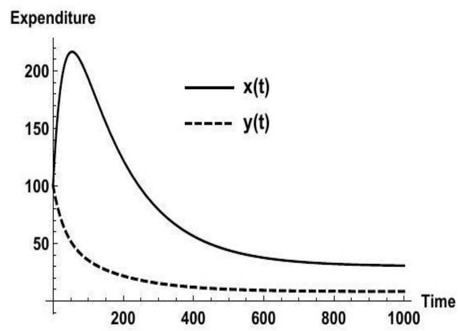

(a) $r = 0, \ s = 0.$




(b) $r = 0.1, \ s = 0.1.$




(c) $r = -0.1, \ s = -0.1.$




(d) $r = -0.1, \ s = 0.1.$

**Figure 4**

# *8.8 EPIDEMIC MODELS*

Mathematical epidemiology is the use of mathematical models to predict the course of infections disease and to compare the effects of different control strategies. In epidemic models, the population is divided into three main classes, namely, a susceptible class, denoted by S(t) (persons who are vulnerable to the disease or who can be easily infected by the disease), infected class denoted by I(t) (persons who already have the disease), and recovered class, denoted by R(t) (person who have recovered from the disease). One can defined more classes if the situation demands, for modification in the models.

**Susceptible – infective (SI) Model:** Let us suppose a population consist of (n+1) persons of which n persons are in susceptible group and only one is infected, so that S(t) + I(t) = n+1, S(0) = n, I(0) =1. A susceptible person gets infected when he comes in contact with an infected one. Mathematically we can say that the rate of increase of the infected class is proportional to the product of the susceptible and infected persons. Hence, the susceptible class also decreases at the same rate.

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI (\beta > 0).$$

Now, $\frac{dS}{dt} + \frac{dI}{dt} = 0 \Rightarrow S(t) + I(t) = \text{ constant.}$

$\Rightarrow \text{ constant } = S(0) + I(0) = n + 1 \Rightarrow S(t) + I(t) = n + 1$

$\Rightarrow \frac{dS}{dt} = -\beta S(n + 1 - S)$ and $\frac{dI}{dt} = \beta I(n + 1 - I)$

Integrating the first differential equation, we obtain

$$\int \frac{dS}{S(n + 1 - S)} = -\int \beta dt \Rightarrow \frac{1}{n + 1} \int \left[\frac{1}{n + 1 - S} + \frac{1}{S}\right] dS = -\int \beta dt$$

$\Rightarrow -\ln(n + 1 - S) + \ln(S) = -(n + 1)\beta t + A$ (constant).

At $t = 0, S(0) = n \Rightarrow \text{A} = \ln(n)$

$\Rightarrow \ln\left[\frac{S}{n(n + 1 - S)}\right] = -(n + 1)\beta t \Rightarrow \frac{S}{n(n + 1 - S)} = e^{-(n+1)\beta t}$

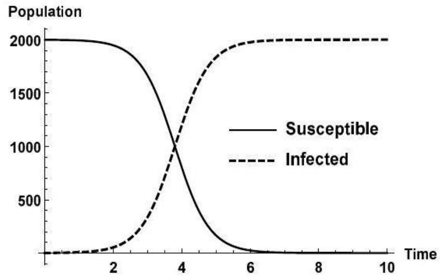
$\Rightarrow S(t) = \frac{n(n + 1)}{n + e^{(n+1)\beta t}}$. Therefore

$$I(t) = (n + 1) - S(t) = (n + 1) - \frac{n(n + 1)}{n + e^{(n+1)\beta t}} = \frac{n + 1}{1 + ne^{-(n+1)\beta t}}.$$

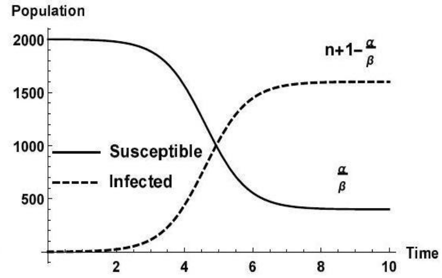
As $t \to \infty, S(t) \to 0$ and $I(t) \to (n + 1)$. Therefore, we conclude that as time increases, all the susceptible persons will become infected.

**Susceptible-Infective-Susceptible (SIS) Model:** A simple refinement of the previous model has been made and named as the SIS model, where it is assumed that the infected person has the ability to recover and move to the susceptible class at a rate $\alpha$ (say). Initially $S(0) = n, I(0) = 1$, then, we get the required model as

$$\frac{dS}{dt} = -\beta SI + \alpha I$$
$$\frac{dI}{dt} = \beta SI - \alpha I, (\beta, \alpha > 0)$$




(a) SI model.




(b) SIS model.

**Figure 5**

**FIGURE 5:** The figures show the dynamics of epidemic models. (a) SI model with $\beta = 0.001, S(0) = 2000, I(0) = 1$, (b) SIS model with $\beta = 0.001, \alpha = 0.4, S(0) = 2000, I(0) = 1$.

Now, $\frac{dS}{dt} + \frac{dI}{dt} = 0 \Rightarrow S(t) + I(t) = K$ (constant).

$\Rightarrow K = S(0) + I(0) = n + 1 \Rightarrow S(t) + I(t) = n + 1$

$$\frac{dS}{dt} = -[(n+1)\beta + \alpha]S + \beta S^2 + (n+1)\alpha$$

$$\frac{dI}{dt} = [(n+1)\beta - \alpha]I - \beta I^2 = cI - \beta I^2, \text{ where } c = (n+1)\beta - \alpha.$$

$$\Rightarrow \frac{dI}{I\left(1 - \frac{\beta}{c}I\right)} = cdt \Rightarrow \frac{\frac{c}{\beta}dI}{I\left(\frac{c}{\beta} - I\right)} = cdt \Rightarrow \left[\frac{1}{I} + \frac{1}{\frac{c}{\beta} - I}\right] = cdt$$

Integrating we obtain,

$\ln(I) - \ln\left(\frac{c}{\beta} - I\right) = ct + B$ (constant). Now, $I(0) = 1 \Rightarrow B = -\ln\left(\frac{c}{\beta} - 1\right)$.

$$\Rightarrow \ln(I) - \ln\left(\frac{c}{\beta} - I\right) + \ln\left(\frac{c}{\beta} - 1\right) = ct \Rightarrow \frac{I\left(\frac{c}{\beta} - 1\right)}{\frac{c}{\beta} - I} = e^{ct}$$

$$\Rightarrow I(t) = \frac{\frac{c}{\beta}}{1 + \left(\frac{c}{\beta} - 1\right)e^{-ct}} = \frac{(n+1) - \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{\alpha}{\beta} - 1\right)e^{-[(n+1)\beta - \alpha]t}}.$$

$$S(t) = n + 1 - I(t) = n + 1 - \frac{n + 1 - \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{\alpha}{\beta} - 1\right)e^{-[(n+1)\beta - \alpha]t}}$$

$$\Rightarrow S(t) = \frac{(n+1)\left(n + 1 - \frac{\alpha}{\beta} - 1\right)e^{-[(n+1)\beta - \alpha]t} + \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{\alpha}{\beta} - 1\right)e^{-[(n+1)\beta - \alpha]t}}.$$

AS $t \to \infty, S \to \frac{\alpha}{\beta}$ and $I \to n + 1 - \frac{\alpha}{\beta}$; provided$(n+1)\alpha - \beta > 0$. Hence,in this case, a fraction of susceptible persons will be there, which have not been infected or a fraction of infected persons have recovered and becomes susceptible again (fig. 5(b)).

**Susceptible-Infective-Recovered (SIR) Model:** This model was developed by Kermack and McKendrick and is given by the set of differential equations as follows

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \alpha I$$
$$\frac{dR}{dt} = \alpha I, (\beta, \alpha > 0)$$

It is assumed that the susceptibles become infected when they come in contact with one another $(\beta SI)$ and a fraction of the infected class $(\alpha I)$ recovers from the disease and moves to the recovered class. Now,

$$\frac{dS}{dR} = \frac{dS}{dt}\frac{dt}{dR} = \frac{-\beta SI}{\alpha I} = -\frac{\beta}{\alpha}S$$

$$\Rightarrow \frac{dS}{S} = -\frac{\beta}{\alpha}dR \Rightarrow \ln(S) = -\frac{\beta}{\alpha}R + \text{ constant.}$$

Initially, $S(0) = n, R(0) = 0 \Rightarrow$ constant $= \ln(n)$

$$\Rightarrow \ln(S) = -\frac{\beta}{\alpha}R + \ln(n) \Rightarrow S = ne^{-\frac{\beta}{\alpha}R}$$

Again,

$$\frac{dI}{dS} = \frac{dI}{dt}\frac{dt}{dS} = \frac{\beta SI - \alpha I}{-\beta SI} = -1 + \frac{\alpha}{\beta}\frac{1}{S}$$

$$\Rightarrow \int dI = -\int dS + \frac{\alpha}{\beta}\int \frac{dS}{S} \Rightarrow I(t) = -S + \frac{\alpha}{\beta}\ln(S) + \text{ constant.}$$

Initially at $t = 0, S(0) = n, I(0) = 1 \Rightarrow$ constant $= 1 + n - \frac{\alpha}{\beta}\ln(n)$.

$$\Rightarrow I(t) = -S + \frac{\alpha}{\beta}\ln(S) + n + 1 - \frac{\alpha}{\beta}\ln(n) = n + 1 - S + \frac{\alpha}{\beta}\ln\left(\frac{S}{n}\right).$$

Since $S = ne^{-\frac{\beta}{\alpha}R}$, we have

$$\frac{dR}{dt} = \alpha I = \alpha(n + 1 - S - R) = \alpha\left(n + 1 - ne^{-\frac{\beta}{\alpha}R} - R\right)$$

$$\Rightarrow \frac{dR}{dt} = \alpha\left[n + 1 - n\left(1 - \frac{\beta}{\alpha}R + \frac{\beta^2}{2\alpha^2}R^2\right) - R\right], \left(\text{Assuming } \frac{R}{\frac{\alpha}{\beta}} \text{ is small}\right),$$

$$\Rightarrow \frac{dR}{dt} = \alpha\left[1 - \frac{n\beta^2}{2\alpha^2}\left\{R^2 - \frac{2\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)R\right\}\right]$$

$$= \alpha\left[1 - \frac{n\beta^2}{2\alpha^2}\left\{R - \frac{\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)\right\}^2 + \frac{\alpha^2}{2n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)^2\right]$$

$$= \frac{n\beta^2}{2\alpha}\left[\frac{2\alpha^2}{n\beta^2} + \frac{\alpha^4}{n^2\beta^4}\left(\frac{n\beta}{\alpha} - 1\right)^2 - \left\{R - \frac{\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)\right\}^2\right],$$

$= \frac{n\beta^2}{2\alpha}[B^2 - (R - A)^2]$ where $A = \frac{\alpha^2}{n\beta^2}\left(\frac{n\beta}{\alpha} - 1\right)$, and

$B^2 = \frac{2\alpha^2}{n\beta^2} + \frac{\alpha^4}{n^2\beta^4}\left(\frac{n\beta}{\alpha} - 1\right)^2$. Integrating we get,

$$\int \frac{dR}{B^2 - (R - A)^2} = \int \frac{n\beta^2}{2\alpha}dt \Rightarrow \frac{1}{B}\tanh^{-1}\left(\frac{R - A}{B}\right) = \frac{n\beta^2}{2\alpha}t + \text{ constant.}$$

Initially at

$t = 0, R(0) = 0 \Rightarrow \text{constant} = \frac{1}{B}\tanh^{-1}\left(\frac{-A}{B}\right) = -\frac{1}{B}\tanh^{-1}\left(\frac{A}{B}\right)$.

$$\Rightarrow \frac{1}{B}\tanh^{-1}\left(\frac{R-A}{B}\right) = \frac{n\beta^2}{2\alpha}t - \frac{1}{B}\tanh^{-1}\left(\frac{A}{B}\right),$$

$$\Rightarrow \frac{R-A}{B} = \tan\text{h}\left[\frac{n\beta^2}{2\alpha}t - \frac{1}{B}\tanh^{-1}\left(\frac{A}{B}\right)\right]$$

$$\Rightarrow R(t) = A + B\tanh\left[B\frac{n\beta^2}{2\alpha}t - \frac{1}{B}\tanh^{-1}\left(\frac{A}{B}\right)\right].$$ Therefore,

$$S(t) = ne^{-\frac{\beta}{\alpha}\left[A+B\tanh\left\{B\frac{n\beta^2}{2\alpha}t - \frac{1}{B}\tanh^{-1}\left(\frac{A}{B}\right)\right\}\right]}$$ and

$$I(t) = n + 1 - S(t) - R(t).$$

The numerical solution of the model shows that both susceptible and infected goes to zero and there is a full recovery (fig.6(a)).

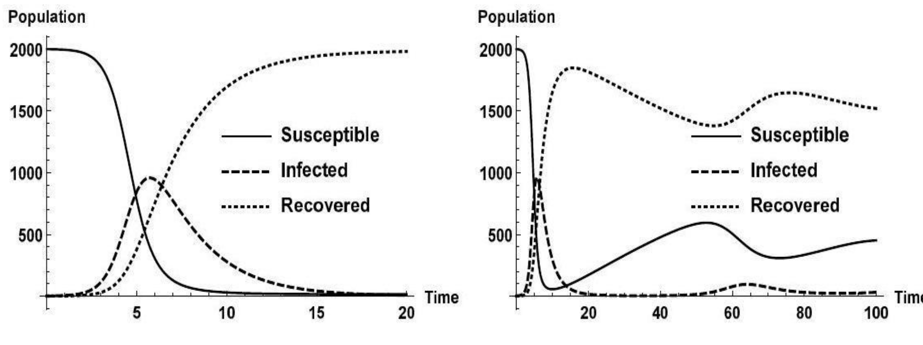
Susceptible-Infective-Removed-Susceptible (SIRS) Model: A refinement of the SIR model can be made by assuming that the recovered person becomes susceptible again due to loss of immunity at a rate proportional to the population in recovery class $R$, with proportionality constant $\gamma$. The following differential equations describe the model,

$$\frac{dS}{dt} = -\beta SI + \gamma R$$

$$\frac{dI}{dt} = \beta SI - \alpha I$$

$$\frac{dR}{dt} = \alpha I - \gamma R, (\beta, \alpha, \gamma > 0)$$

Fig.6(b) shows the dynamics of SIRS model for $\beta = 0.001, \alpha = 0.4$ and $\gamma = 0.01$.




(a) SIR model. (b) SIRS model.

**Figure 6**

**FIGURE 6:** The figures show the dynamics of epidemic models.

(a) SIR model with $\beta = 0.001, \alpha = 0.4, S(0) = 2000, I(0) = 1, R(0) = 0$,(b) SIRS model with $\beta = 0.001, \alpha = 0.4, \gamma = 0.01, S(0) = 2000, I(0) = 1, R(0) = 0$.

We can use the SIRS model to capture the dynamics of COVID-19. The susceptible population becomes infected by COVID-19 at a rate $\beta$ (per-capita effective contact rate), which is the number of effective contacts made by a given individual per unit time. We are trying to minimize the value of $\beta$ by practicing social distancing. Once infected, the susceptible population ($\beta SI$) moves to the infected class. The infected class recovers from the virus by hard immunity of individual (since no vaccine is available) but have the chance to reinfection. Hence, from the infected class, $\alpha I$ has moved to the recovered class, and $\gamma R$ has moved to the susceptible class. We want the see the dynamics of the spread of COVID-19 with $\beta = 0.00002856, \alpha = 0.19819303, \gamma = 0.001$and initial condition $S(0) = 15000, I(0) = 1, R(0) = 0$. The figure shows that the initial spread is high, which then decays over time (fig. 7(a)). The susceptible as well as recovered class also show "ups and down" behavior before reaching a steady value (fig. 7 (b)).

## *8.9 COMBAT MODELS*

The Combat Model is a mathematical model used to describe the dynamics of combat between two or more opposing forces. The model is based on the idea that the outcome of combat depends on the relative strengths and weaknesses of the opposing forces.

The Combat Model can be formulated mathematically using a system of differential equations. Let's consider a simple model with two opposing forces, A and B.

Where x(t) = Number of units of force A at time t

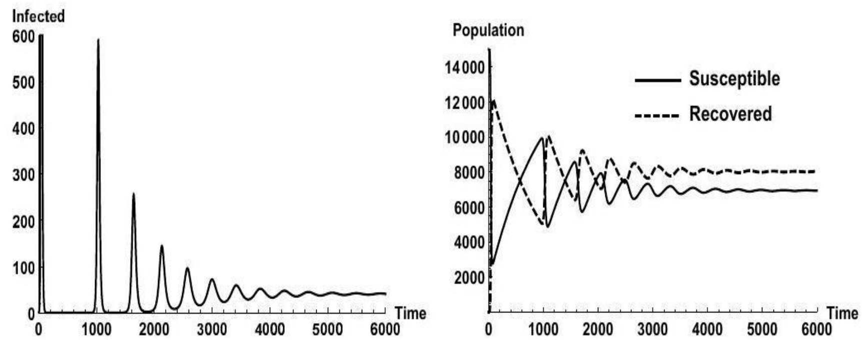
y(t) = Number of units of force B at time t

The system of differential equations can be written as:

dx/dt = -a * x * y

dy/dt = -b * x * y

where a and b are parameter that represent the effectiveness of each force in destroying the other.




(a) Infected class. (b) Susceptible and recovered.

**Figure 7**

**FIGURE 7:** The figures show the dynamics of the spread of COVID-19 along with the dynamics of susceptible and recovered classes.
were built on equations, very similar to the law of mass action:

$$\text{rate of change} = \text{rate in} - \text{rate out},$$

where rate in denotes the number of new troops supplied (reinforcement) and rate out denotes the number of troops not available to fight the battle, either due to attacks from the opposing army (combat loss) or other factors like desertion, sickness, etc. (operational loss).

## 8.9.1 CONVENTIONAL COMBAT MODELS

Conventional Combat Models are mathematical models used to describe the dynamics of conventional warfare between two or more opposing forces. These models are based on the principles of military operations research and are used to analyze and predict the outcomes of battles and campaigns.
Let $x(t)$ and $y(t)$ denote the number of troops or combatants for army A and army B, respectively. In conventional combat, the two opposite armies must directly interact with one another in the open, using conventional means (like knives, guns, etc.) with the exclusion of any chemical, biological and nuclear weapons. The following assumptions are made to formulate the model:

- The operational losses are neglected.
- There is no new supply of troops (no reinforcement).
- The combat loss rate of a conventional army A is proportional to the size of the opposing army B.

The system of differential equations describing the model is

$$\frac{dx}{dt} = -\alpha y, \frac{dy}{dt} = -\beta x$$

where $\alpha, \beta (> 0)$ are the fighting effectiveness coefficients of the armies B and A, respectively. Suppose, the initial number of troops for army A and army B are $x_0$ and $y_0$, respectively. Then, we have an initial value problem

$$\frac{dx}{dt} = -\alpha y, \frac{dy}{dt} = -\beta x, x(0) = x_0, y(0) = y_0,$$

which has a unique solution.

## 8.9.2 GUERRILA COMBAT MODEL

The Guerrilla Combat Model is a mathematical model used to describe the dynamics of guerrilla warfare, which is a type of asymmetric warfare characterized by small, mobile, and decentralized forces.Suppose $x(t)$ and $y(t)$ denote the number of troops or combatants for the army A and army B, respectively. In Guerrilla combat Model, troops are deployed in small groups that are hidden (covert). If the troops are large, they are more likely to get caught. However, if the opposing army is large, they are more likely to find the guerrilla troops. The following assumptions are made to formulate the model:

- The operational losses are neglected.
- There is no new supply of troops (no reinforcement).
- The combat loss rate of a guerrilla troop army is proportional to the product of the sizes of both the armies A and B .

The system of differential equations describing the model is

$$\frac{dx}{dt} = -\alpha xy, \frac{dy}{dt} = -\beta xy$$

with initial conditions

$$x(0) = x_0, y(0) = y_0,$$

where $\alpha, \beta (> 0)$ are fighting effectiveness coefficients of the forces B and A , respectively.

## 8.9.3 MIXED COMBAT MODELS

Suppose $x(t)$ and $y(t)$ denote the number of troops or combatants for a conventional army A and a guerrilla force B. The following assumptions are made to formulate the model:

- The operational losses are neglected.
- There is no new supply of troops (no reinforcement).
- The combat loss rate for the conventional army A is proportional to the size of the opposing guerrilla force B and the combat loss rate for guerrilla force B is proportional to the product of the sizes of both the armies A and B.

The system of differential equations describing the model is

$$\frac{dx}{dt} = -\alpha y, \frac{dy}{dt} = -\beta xy$$

with initial conditions

$$x(0) = x_0, y(0) = y_0,$$

where $\alpha, \beta (> 0)$ are fighting effectiveness coefficient of the forces B and A , respectively.

## *8.10 SUMMARY*

Present unit is a presentation of,

**Continuous Models:** Continuous models describe systems that change smoothly over time. This model is often use differential equations.

**Arms Race Models:** The Arms Race model is a mathematical model used to describe the dynamics of an arms race between two or more countries.

**Epidemic Models**: Mathematical epidemiology is the use of mathematical models to predict the course of infectious disease and to compare the effects of different control strategies.

**Combat Models:** The Combat Model is a mathematical model used to describe the dynamics of combat between two or more opposing forces. The model is based on the idea that the outcome of combat depends on the relative strengths and weaknesses of the opposing forces.

## *8.11 GLOSSARY*

**Ordinary Differential Equation (ODE):** An equation that involves an unknown function and its derivatives.

**Derivative:** A measure of how a function changes as its input changes.

**Differential:** An infinitesimal change in a variable.

**Integral:** The reverse operation of differentiation.

**Solution**: A function that satisfies the ODE.

**Separation of Variables**: A method for solving ODEs by separating the variables and integrating.

**Integrating Factor:** A method for solving ODEs by multiplying the equation by an integrating factor.

**Undetermined Coefficients:** A method for solving ODEs by assuming a solution in the form of a polynomial or trigonometric function.

**CHECK YOUR PROGRESS**

**CYP1 ..** A continuous model consists of a ………..variable, varying with some other independent continuous variables.

**CYP2.**The study of the movement of drugs in the body is called ………..

**CYP3.**The ……….. helped in better planning, prediction of battles and their possible outcomes.

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## *8.14* TERMINA*L QUESTIONS*

1. What is continuous modelling approach?
2. What is the difference between Arms Race Model and Epidemic Models.
3. A fossil is found that has 20% $C^{14}$ compared to the living sample. How old is the fossil, knowing that the $C^{14}$ half – life is 5730 years?
4. Model the population growth of a city using the logistic equation, and solve for the population at time t.
5. What is an Ordinary Differential Equation (ODE), and how is it used to model continuous systems?

## *8.15 ANSWERS*

**TERMINAL QUESTIONS**

**TQ2.** 13305 years.

**CHECK YOUR PROGRESS**

**CYQ1.** Dependent continuous
**CYQ2**.Pharmacokinetics.
**CYQ3.** Combat Models

# COURSE NAME: MATHEMATICAL MODELLING

# COURSE CODE: MAT 610

## BLOCK-III

## MATHEMATICAL MODELLING - III

# UNIT 9: SPATIAL MODELS USING PARTIAL DIFFERENTIAL EQUATIONS

## CONTENTS:



## *9.1 INTRODUCTION*

In previous unit we have defined continuous Models Using Ordinary Differential Equations, Steady State Solution, Stability, Linearization and Local stability Analysis, Lyapunov's Direct Method etc. we also discussed about Arms Race Models, Epidemic modes and Combat Models and their examples. In this unit we discussed about spatial models using partial differential equations.
Spatial modeling is a mathematical approach used to describe and analyze phenomena that vary over space and time. It involves using mathematical equations to model the behavior of systems that exhibit spatial heterogeneity, such as population growth, disease spread, and environmental changes.PDEs are particularly useful for

spatial modeling because they can capture the complex interactions between spatially distributed variables.

## 9.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Describe and predict the evolution of phenomena that vary both spatially and temporally,
- **ii.** To predict the future behaviour of spatial-temporal systems, informing decision-making in fields like public health, environmental management, and urban planning.
- **iii.** To identify the most important factors influencing spatial-temporal dynamics, such as diffusion rates, reaction rates, or boundary conditions.

## 9.3 THE ADVANTAGE OF PARTIAL DIFFERENTIAL EQUATION MODELS

The advantage of PDE models is that they include derivatives of at least two independent variables, and hence, they can describe the dynamical behavior of the problem of interest in terms of two or more variables at the same time. For example, if we consider the flow of heat in a metal bar, it would be inappropriate NOT to model it with partial differential equations, to compute the temperature distribution with respect to time as well as space.

Let us consider a simple predator-prey model (Lotka-Volterra)

$$\frac{dP_1(t)}{dt} = \alpha P_1(t) - \beta P_1(t)P_2(t)$$
$$\frac{dP_2(t)}{dt} = -\gamma P_2(t) + \delta P_1(t)P_2(t)$$

where we have used one independent variable (namely, time) to study the dynamics of the system. But, one can consider the effect of movement of the prey and the predator by adding a diffusion term to the equations, thereby making it a PDE model as

$$\frac{\partial P_1(t,x,y)}{\partial t} = \alpha P_1(t,x,y) - \beta P_1(t,x,y)P_2(t,x,y) + D_1\nabla^2 P_1(t,x,y)$$
$$\frac{\partial P_2(t,x,y)}{\partial t} = -\gamma P_2(t,x,y) + \delta P_1(t,x,y)P_2(t,x,y) + D_2\nabla^2 P_2(t,x,y)$$

where $\nabla^2 \cong \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

This model is able to capture the spatial aspect of the model and can give a complete picture of the dynamics of the predator-prey system with respect to both time and space.

The general solution of the partial differential equation involves as many arbitrary functions as the order of the equation (order of the highest partial differential coefficient in the equation). Certain conditions are required in order to find these arbitrary functions. The standard notation for the space variables in applications are $x, y, z$, etc., and a solution may be required in some region $\Omega$ of space. In such a case, there will be some conditions to be satisfied on the boundary $\partial\Omega$, which are called boundary conditions (BCs). Similarly, in the case of the independent variables, one of them is generally taken as time (say, $t$ ), then there will be some initial conditions (ICs) to be satisfied. The conditions of partial differential equations are classified into two categories:

**(i) Initial value problem (IVP):** A partial differential equation with initial conditions, that is, dependent variable and an appropriate number of its derivatives are prescribed at the initial point of domain is called an initial value problem.

**(ii)Boundary value problem (BVP):** A partial differential with boundary conditions, that is, dependent variable and an appropriate number of its derivatives are prescribed at the boundary of domain is called a boundary value problem.

There are four broad categories of boundary conditions:
**(a)** Dirichlet boundary condition: On the boundary, the values of the dependent variable are specified.

**(b)** Newmann boundary condition: On the boundary, the normal derivative of the dependent variable is specified.
**(c)** Cauchy boundary condition: On the boundary, both the values of the dependent variable and its normal derivative are specified.
**(d)** Robin boundary condition: On the boundary, a linear combination of the dependent variable and its normal derivatives are specified.

A second-order partial differential equations of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

is elliptic, parabolic or hyperbolic when $B^2 - 4AC$ is respectively less than, equal to or greater than zero. This classification gives a better understanding of the choice of initial or boundary conditions. To have a unique, stable solution, each class of PDEs requires a different class of boundary conditions.

**(i)** Dirichlet or Neumann boundary conditions on a closed boundary surrounding the region of interest are the requirements to elliptic equations. Other boundary conditions are either insufficient to determine a unique solution, overly restrictive or lead to instabilities.

**(ii)** Cauchy boundary conditions on an open surface are the requirements to elliptic equations. Other boundary conditions are either too restrictive for a solution to exist, or insufficient to determine a unique solution.

**(iii)** Parabolic equations require Dirichlet or Neumann boundary conditions on an open surface. Other boundary conditions are too restrictive.

**Example 9.1.1** Initial value problem (IVP):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, 0 \le x \le l, t > 0$$

$$ICs: u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = x$$

**Example 9.1.2** Boundary value problem (BVP):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = 0, 0 \le x \le L, t > 0$$

**(i)** BCs: $u(0,t) = 0, u(L,t) = 0$ (Dirichlet boundary condition).
**(ii)** BCs: $\frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(L,t)}{\partial x} = 0$ (Neumann boundary condition).
**(iii)**$BCs: u(0,t) = 0, \frac{\partial u(L,t)}{\partial x} = 0$ (Cauchy boundary condition).
**(iv)**$BCs: u(0,t) = 1, \frac{\partial u(L,t)}{\partial x} + u(L,t) = 0$ (Robin boundary condition).

**Example 9.1.3** Initial boundary value problem (IBVP):

$$BCs: u(0,t) = 0\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = 0, 0 \le x \le l, t > 0, u(l,t) = 0$$

(Dirichlet boundary condition),$ICs: u(x,0) = x$.

## 9.4 HEAT FLOW THROUGH A SMALL THIN ROD (ONE DIMENSIONAL*)*

We consider a thin rod of length $L$, made of homogenous material (material properties are translational invariant). We assume that the rod is perfectly insulated along its length so that heat can flow only through its ends (fig.9.1).

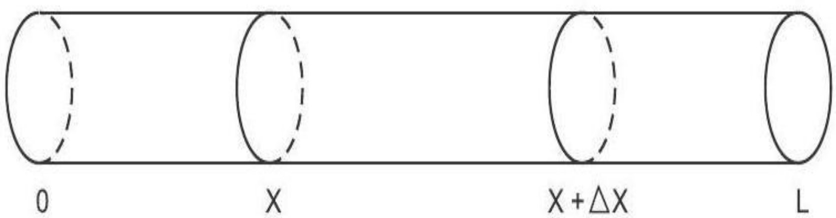



**FIGURE 9.1:** A thin homogenous rod of length L, perfectly insulated along its length.

Let $u(x,t)$ be the temperature of this homogenous thin rod at a distance $x$ at time $t$. We consider an infinitesimal piece from the rod with length $[x, x+\Delta x]$. Let $A$ be the cross-section of the rod, $\rho$ be the density of the material of the rod, then the infinitesimal volume is given by $\Delta V = A\Delta x$ and the corresponding infinitesimal mass is $\Delta m = \rho A \Delta x$. The amount of heat for the volume element is $Q = \sigma \Delta m u(x,t)$, where $\sigma$ is the specific heat of the material of the rod.

At time $t + \Delta t$, the amount of heat is

$$Q_1 = \sigma \Delta m u(x, t+\Delta t)$$

$$\text{Change in heat} = Q_1 - Q = \sigma \Delta m u(x,t+\Delta t) - \sigma \Delta m u(x,t)$$
$$= \sigma \rho A[u(x,t+\Delta t) - u(x,t)]\Delta x$$

Now, by the Fourier law of heat conduction, the heat flow is proportional to the temperature gradient, that is, $Q = -k\frac{\partial u}{\partial x} = -ku_x(x,t)$ (in one dimension), where $k$ is the thermal conductivity of the solid and the negative sign denotes that the heat flux vector is in the direction of decreasing temperature. Therefore, the change in heat must be equal to the heat flowing in at $x$, minus the heat flowing out at $x+\Delta x$, during the time interval $\Delta t$, that is,

$$\sigma\rho A[u(x,t+\Delta t)-u(x,t)]\Delta x=[-ku_x(x,t)-(-ku_x(x+\Delta x,t))]\Delta t,$$
$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=\left(\frac{k}{\sigma\rho A}\right)\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}$$

Taking $\Delta x,\Delta t\to 0$, we obtain $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$, which gives the required heat equation determining the heat flow through a small thin rod. $c^2=\frac{k}{\sigma\rho A}$ is called the diffusivity of the material of the rod.

We use the separation of variables to solve the heat equation, which can also be termed as a one-dimensional diffusion equation.

Let $u(x,t)=X(x)T(t)$ be a solution of $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$. Substituting, we get,

$$X(x)T'(t)=c^2X''(x)T(t)\Rightarrow\frac{X''}{X}=\frac{1}{c^2}\frac{T'}{T}=\mu\text{ (a constant)}$$

Both functions must be equal to some constant as one of them is a function of $x$ only, and the other is a function of $t$.

$$\Rightarrow X''=\mu X\text{ and }T'=\mu c^2T\qquad(9.1)$$

**Case I:** $\mu$ is positive ( $=\lambda^2$, say). From (9.1), we get $X(x)=A_1\cosh(\lambda x)+A_2\sinh(\lambda x)$ and $T(t)=A_3e^{\lambda^2c^2t}$. Then, the solution of the heat equation is

$$u(x,t)=[C_1\cosh(\lambda x)+C_2\sinh(\lambda x)]e^{\lambda^2c^2t},C_1=A_1A_2,C_2=A_2A_3.$$

**Case II**: $\mu=0$. From (9.1), we get, $X(x)=A_4x+A_5$ and $T(t)=A_6$. Then, the solution of the heat equation is

$$u(x,t)=C_3x+C_4.$$

**Case III:** $\mu$ is negative ( $=-\lambda^2$, say). From (9.1), we get $X(x)=A_7\cos(\lambda x)+A_8\sin(\lambda x)$ and $T(t)=A_9e^{-\lambda^2c^2t}$. Then, the solution of the heat equation is

$$u(x,t)=[C_5\cos(\lambda x)+C_6\sin(\lambda x)]e^{-\lambda^2c^2t}$$

Combining, we can write the general solution of the heat equation as

$$u(x,t) = \begin{cases} [C_1\cosh(\lambda x) + C_2\sinh(\lambda x)]e^{\lambda^2 c^2 t}, \mu = \lambda^2 > 0 \\ C_3 x + C_4, \mu = 0 \\ [C_5\cos(\lambda x) + C_6\sin(\lambda x)]e^{-\lambda^2 c^2 t}, \mu = -\lambda^2 < 0 \end{cases}$$

**Note: (i)** All three solutions are not consistent .

**(ii)** The first solution indicates $t \to \infty, u \to \infty$. So, it is reasonable to assume that $u(x,t)$ is bounded as $t \to \infty$, from a realistic physical Point of view.

**(iii)** The consistency of the third solution is always there; however, the second solution is consistent in some cases along with the first. (iv) If the boundary conditions are of Dirichlet's type, homogeneous as well as periodic, that is, $u(0,t) = 0, u(L,t) = 0$, third solution is the only solution is

**(v)** If the boundary conditions are of Dirichlet's type, but non-homogeneous or non-periodic, that is, $u(0,t) = \alpha, u(L,t) = \beta$, second and third solution are the general solution of the given problem.

**(vi)** Second and third solutions constitute the general solution of the given problem, in the case of Neumann and Robin boundary conditions.

**Example 9.2.1**$A$ uniform rod of length 20 cm with diffusivity $D$ of the material of the rod, whose sides are insulated, is kept at initial temperature $x$, when $0 \le x \le 10$ and $20 - x$, when $10 \le x \le 20$. Both ends of the rod are suddenly cooled at $0°\text{C}$ and are kept at that temperature.

**(i)** If $u(x,t)$ represents the temperature function at any point $x$ at time $t$, formulate a mathematical model of the given situation, stating clearly the boundary and initial conditions. (ii) Using the method of separation of variables, find the temperature function $u(x,t)$. Obtain the numerical solution of the problem for $D = 0.475$ and plot the graph.

Solution: (i) The mathematical model of the given situation represents an initial boundary value problem of heat conduction and is given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2};\ 0 \le x \le 20, t > 0 \tag{9.2}$$

Boundary Condition (BCs): $u(0,t) = 0 = u(20,t); t > 0$ (since both ends of the rod are cooled suddenly at $0°\text{C}$ ). Initial Condition (ICs): $u(x,0) = \begin{cases} x, & 0 \leq x \leq 10, \\ 20 - x, & 10 \leq x \leq 20. \end{cases}$

**(ii)** Let $u(x,t) = X(x)T(t)$ be a solution of $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$. Then,

$$\frac{X''}{X} = \frac{1}{D}\frac{T'}{T} = -\lambda^2 (\text{ separation constant }) \tag{9.3}$$

Since the boundary conditions are periodic and homogenous in $x$, the periodic solution of (9.2) exists if the separation constant is negative. One can also consider the other two cases, that is, the separation constant to be positive and zero but will arrive at the same conclusion. Basically, a negative separation constant gives a physically acceptable general solution. Solving (9.3) we get,

$$X(x) = A_1\cos(\lambda x) + A_2\sin(\lambda x) \text{ and } T(t) = A_3 e^{-\lambda^2 Dt}.$$

Therefore, the complete solution of (9.2) is given by $u(x,t) = [C_1\cos(\lambda x) + C_2\sin(\lambda x)]e^{-\lambda^2 Dt}$, where $C_1 = A_1A_2, C_2 = A_2A_3$.

Applying the boundary conditions $u(0,t) = 0 = u(20,t)$, we obtain

$$(C_1\cos 0 + C_2\sin 0)e^{-\lambda^2 Dt} = 0 \text{ and } [C_1\cos(20\lambda) + C_2\sin(20\lambda)]e^{-\lambda^2 Dt} = 0,$$

$\Rightarrow C_1 = 0$ and $C_2\sin(20\lambda) = 0, \Rightarrow \sin(20\lambda) = 0 \Rightarrow \lambda = \frac{n\pi}{20}, n$ being an integer (for non-trivial solution $C_2 \neq 0$).Therefore, the required solution is of the form

$$u(x,t) = C_2\sin\left(\frac{n\pi}{20}x\right)e^{\frac{-n^2\pi^2 D}{400}t}.$$

Noting that the heat conduction equation is linear, we use the principle of superposition to obtain its most general solution as

$$u(x,t) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{20}x\right)e^{\frac{-n^2\pi^2 D}{400}t}$$

Using the initial condition, we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{20}x\right)$$

which is a half-range Fourier series, where

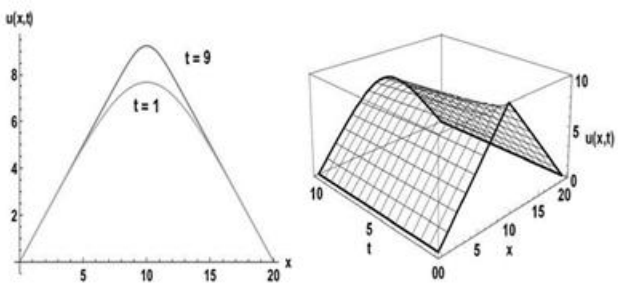
$$B_n = \frac{2}{20}\int_0^{20} u(x,0)\sin\left(\frac{n\pi}{20}x\right)dx$$

$$= \frac{2}{20}\int_0^{10} x\sin\left(\frac{n\pi}{20}x\right)dx + \frac{2}{20}\int_{10}^{20}(20-x)\sin\left(\frac{n\pi}{20}x\right)dx$$

$$= \frac{80}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1,2,3,\ldots \\ \frac{80}{(2m-1)^2\pi^2}, \quad \text{if } n = 2m-1 \text{ and } m = 1,2,3\ldots \end{cases}$$

Hence, the temperature function is given by (fig. 9.2)

$$u(x,t) = \sum_{n=1}^{\infty}\frac{80}{(2m-1)^2\pi^2}\sin\left(\frac{(2m-1)\pi x}{20}\right)e^{\frac{-(2m-1)^2\pi^2 D}{400}t}.$$




**(a) Solution of u(x,t) at t = 1,9 (b) Three dimensional graph of u(x,t)**

**FIGURE 9.2:** The figures show the dynamics of heat flows with Dirichlet boundary conditions for $D = 0.475$. (a) Heat flow between $x = 0$ and $x = 20$, the value of $u(x,t)$ is zero at the boundaries. (b) Three-dimensional visualization of one-dimensional heat flow, which is smooth over time, even though the initial condition is a piecewise function.

**Example 9.2.2** Consider a laterally insulated rod of length 100 cm with diffusivity $D$ of the material of the rod, whose ends are also insulated. The initial temperature is $x$, when $0 \le x \le 40$ and $100 -$x, when $40 \le x \le 100$

(i) If $u(x,t)$ represents the temperature function at any point $x$ at time $t$, formulate a mathematical model of the given situation, stating clearly the boundary and initial conditions.
(ii) Using the method of separation of variables, find the temperature

function $u(x,t)$. Obtain the numerical solution of the problem for $D=0.475$ and plot the graph.

**Solution: (i)** The mathematical model of the given situation represents an initial boundary value problem of heat conduction and is given by

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}; 0\leq x\leq 100, t>0 \qquad (9.4)$$

Boundary Condition (BCs): $\frac{\partial u(0,t)}{\partial x}=0, \frac{\partial u(100,t)}{\partial x}=0; t>0$ (both ends of the rod are insulated).
Initial Condition (ICs): $u(x,0)=\begin{cases}x, & 0\leq x\leq 40,\\ 100-x, & 40\leq x\leq 100.\end{cases}$

**(ii)** Let $u(x,t)=X(x)T(t)$ be a solution of $\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}$. Then,

$$\frac{X''}{X}=\frac{1}{D}\frac{T'}{T}=\mu\Rightarrow X''-\mu X=0 \text{ and } T'=\mu DT \qquad (9.5)$$

**Case I:** Let $\mu>0\ (=\lambda^2)$. From (4.5), we get $X(x)=A_1e^{\lambda x}+A_2e^{-\lambda x}\Rightarrow X'(x)=A_1\lambda e^{\lambda x}-A_2\lambda e^{-\lambda x}$ and $T(t)=A_3e^{\lambda^2 Dt}$. Using the given boundary conditions, we get $A_1=0, A_2=0$, which gives trivial solution $u(x,t)=0$. Hence, we reject $\mu=\lambda^2$.

**Case II:** Let $\mu=0$. Then, the solution of (4.5) is $X(x)=A_1x+A_2\Rightarrow X'(x)=A_1$ and $T(t)=$ constant $=A_3$ (say). Using the boundary conditions, we get $A_1=0$, then $X(x)=A_2$. Therefore, corresponding to $\mu=0$, a solution to the boundary value problem is given by

$$u(x,t)=A_2\times A_3=B_0(\text{ say }) \qquad (9.6)$$

**Case III:**$\mu<0(=-\lambda^2)$. From (4.5), we obtain

$$u(x,t)=[C_1\cos(\lambda x)+C_2\sin(\lambda x)]e^{-\lambda^2 Dt}$$
$$\Rightarrow\frac{\partial u}{\partial x}=[-C_1\lambda\sin(\lambda x)+C_2\lambda\cos(\lambda x)]e^{-\lambda^2 Dt}$$

Applying the boundary conditions $u_x(0,t)=0=u_x(100,t)$, we get $C_2=0$ and $-C_1\lambda\sin(100\lambda)=0\Rightarrow\sin(100\lambda)=0\Rightarrow\lambda=\frac{n\pi}{100}$, $n$being an integer (fornon-trivial solution $C_1\neq 0$ ).Therefore, the required solution is of the form $u(x,t)=C_1\cos\left(\frac{n\pi}{100}x\right)e^{\frac{-n^2\pi^2 D}{10000}t}$.

Noting that the heat conduction equation is linear, we use the principle of superposition to obtain

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{100}x\right) e^{\frac{-n^2\pi^2 D}{10000}t} \tag{9.7}$$

**Equations (9.6) and (9.7) c**onstitute a set of infinite solutions of (9.4). To obtain a solution, which will satisfy the initial condition, we consider a linear combination of these two solutions. Hence, the complete solution of **(9.4)** is of the form

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{100}x\right) e^{\frac{-n^2\pi^2 D}{10000}t}$$

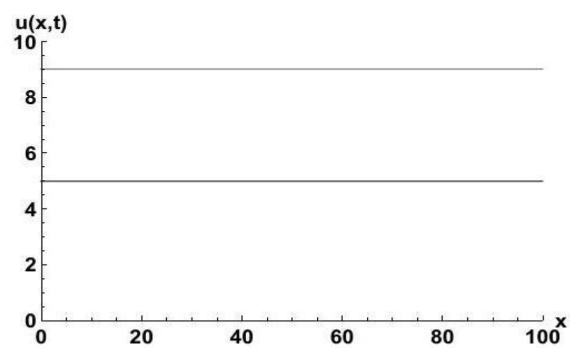
Using the initial condition, we get $u(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{100}x\right)$, which is a half-range Fourier series, where

$$B_0 = \frac{1}{100}\left[\int_0^{40} x dx + \int_{40}^{100} (100-x)dx\right] = 25, \text{ and}$$

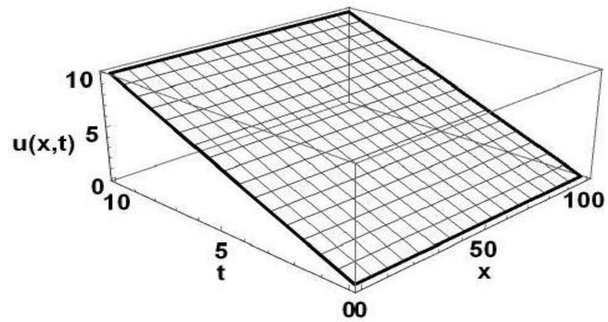
$$B_n = \frac{2}{100}\int_0^{100} u(x,0)\cos\left(\frac{n\pi}{100}x\right) dx = \frac{200}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right].$$

Hence, the temperature function is given by (fig. 4.3)

$$u(x,t) = 25 + \sum_{n=1}^{\infty} \frac{200}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right] \cos\left(\frac{n\pi x}{100}\right) e^{\frac{-n^2\pi^2 D}{10000}t}$$




(a) Solution of u(x,t) at t =5,9.




**(b)** Three-dimensional graph of $u(x,t)$.

**FIGURE 9.3**: The figures show the dynamics of heat flows with Neumann boundary conditions ( $D = 0.475$ ).

**(a)** Constant heat flow for $t = 5,9$ in $0 \leq x \leq 100$.

**(b)** Three-dimensional visualization of one-dimensional heat flow, shows constant increase.

## *9.5 TWO-DIMENSIONAL HEAT EQUATIONS (DIFFUSION EQUATION)*

Consider a thin rectangular plate made of some thermally conductive material, whose dimensions are $a \times b$. The plate is heated in some way and then insulated along its top and bottom. Our aim is to mathematically model the movement of thermal energy through the plate. Let $u(x,y,t)$ be the temperature of the plate at position $(x,y)$ at time $t$. It can be shown that under ideal assumptions (say, uniform density, uniform specific heat, perfect insulation, no internal heat sources etc.), $u(x,y,t)$ satisfies the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \text{ for } 0 \leq x \leq a, 0 \leq y \leq b \qquad (9.8)$$

Suppose, the four edges of the plate $x = 0, x = a, y = 0, y = b$ are kept at zero temperature, which imposes some sort of boundary conditions, namely,

$$u(0,y,t) = u(a,y,t) = 0, \qquad (9.9)$$
$$u(x,0,t) = u(x,b,t) = 0. \qquad (9.10)$$

The way the plate is heated initially is given by the initial condition

$$u(x, y, 0) = f(x, y), (x, y) \in \text{R}, \text{ where R} = [0, a] \times [0, b]. \qquad (9.11)$$

For a fixed $t$, the height of the surface $z = u(x, y, t)$ gives the temperature of the plate at time $t$ and position $(x, y)$. Our aim is to obtain a solution to the heat equation (9.8) subject to the boundary conditions (9.9), (9.10) and, initial condition (9.11). As before, we separate variables to produce simple solutions to (9.8), (9.9) and (9.10) and then use the principle of superposition to build up a solution that satisfies (9.11) as well.

Let $u(x, y, t) = X(x)Y(y)T(t)$ be a solution of (9.8). Substituting in (9.8), we obtain,

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T''}{kT}. \qquad (9.12)$$

Since $x, y, t$ are independent variables, (9.12) will hold if each term on each side is equal to same separation constant, that is,

$$\frac{X'}{X} = \mu_1, \frac{Y'}{Y} = \mu_2, \frac{T'}{kT} = \mu_1 + \mu_2$$

Let $\frac{X'}{X} = \mu_1 \Rightarrow X'' - \mu_1 X = 0$. Using boundary conditions (9.9) and (9.10) we get,

$$X(0)Y(y)T(t) = 0 \text{ and } X(a)Y(y)T(t) = 0$$

$Y(y) = 0$ or $T(t) = 0$ will lead to the trivial solution $u = 0$, hence $Y(y) \neq 0, T(t) \neq 0$ and this implies $X(0) = 0$ and $X(a) = 0$.

**Case I:** If $\mu_1 = 0, X(x) = A_1 X + A_2$. Now, $X(0) = 0 \Rightarrow A_2 = 0$ and $X(a) = 0 \Rightarrow A_1 = 0 \Rightarrow X(x) = 0$. This leads to $u = 0$, which does not satisfy **(9.11).** So, we reject $\mu_1 = 0$.

**Case II:** If $\mu_1 > 0$( say, $\lambda_1 \;\; ^2$ ), then $X(x) = A_1 e^{\lambda_1 x} + A_2 e^{-\lambda_1 x}$. Using $X(0) = 0, X(a) = 0$, we get $A_1 + A_2 = 0, A_1 e^{\lambda_1 a} + A_2 e^{-\lambda_1 a} = 0 \Rightarrow A_1 = 0, A_2 = 0$, which again leads to $u = 0$, and does not satisfy **(9.11).** So, we reject, $\mu_1 > 0$.

**Case III:** If $\mu_1 < 0$ (say, $-\lambda_1 \;\; ^2$ ), then $X(x) = A_1 \cos \lambda_1 x + A_2 \sin \lambda_1 x$. $X(0) = 0 \Rightarrow A_1 = 0$ and $X(a) = 0 \Rightarrow A_2 \sin(\lambda_1 a) = 0 \Rightarrow \sin(\lambda_1 a) = 0(A_2 \neq 0$, otherwise $A_2 = 0$ gives trivial solution, not satisfying **(9.11))**⇒ $\lambda_1 = \frac{n\pi}{a}, n = 1,2,3, \ldots.$ Hence, the non-zero solution is given by

$$X(x) = A_2 \sin\left(\frac{m\pi x}{a}\right), m = 1,2,3, \ldots$$

In a similar manner,

$$Y(y) = B_2 \sin\left(\frac{n\pi y}{b}\right), n = 1,2,3, \ldots$$

Now,

$$\frac{T'}{kT} = \mu_1 + \mu_2 = -\lambda_1^2 - \lambda_2^2 = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$\Rightarrow T' = -\lambda_{mn}^2 T, \text{ where } \lambda_{mn}^2 = k\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Solving we obtain,

$$T(t) = Ce^{-\lambda_{mn}^2 t}, m = 1,2, \ldots, n = 1,2, \ldots.$$

We use the principle of superposition to obtain its most general solution as

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\lambda_{mn}^2 t}$$

Putting $t = 0$ and using the initial condition (4.11), we get,

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

which is a double Fourier sine series, where

$$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dy dx.$$

**Example 9.3.1 The four edges of a thin rectangular plate of length a and breadth $b$ kept at zero temperature and the faces are perfectly insulated. If the initial temperature of the plate is $xy(\pi - x)(\pi - y)$, find the temperature at any point in the plate.**

**Solution:** Let $u(x,y,t)$ be the temperature of the plate at position ( $x,y$ ) at time $t$. The model in the form of the initial boundary value problem is given by

$$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \text{ for } 0\leq x\leq a, 0\leq y\leq b$$

Boundary Condition: u(0,y,t) = u (a,y,t) =0, u(x,0,t)= u(x,b,t)=0

Initial Condition:$u(x,y,0)=xy(\pi-x)(\pi-y)$.

Proceeding as in section (9.3), we obtain the solution as

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\lambda_{mn}^2 t}, \text{ where}$$

$$A_{mn}=\frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b}xy(\pi-x)(\pi-y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dydx$$

$$\text{and } \lambda_{mn}^2=k\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right), m=1,2,3,\ldots, n=1,2,3,\ldots.$$

Therefore,

$$A_{mn}=\frac{4}{ab}\int_{x=0}^{a}(\pi x-x^2)\sin\left(\frac{m\pi x}{a}\right)dx\times\int_{y=0}^{b}(\pi y-y^2)\sin\left(\frac{n\pi y}{b}\right)dy$$

$$=\frac{16a^2b^2}{m^3n^3\pi^6}[1-(-1)^m][1-(-1)^n],$$

$$=\begin{cases}0, & \text{when } m=2p \text{ or } n=2q(\text{ even }).\\ \dfrac{64a^3b^3}{m^3n^3\pi^6}, & \text{when } m=2p-1, n=2q-1(\text{ odd }).\end{cases}$$

Hence, the required solution is **(fig. 9.4)**

$$u(x,y,t)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left[\frac{64a^3b^3}{\pi^3(2p-1)^3(2q-1)^3}\times\right.$$
$$\left.\sin\frac{(2p-1)\pi x}{a}\sin\frac{(2q-1)\pi y}{b}e^{-k\pi^2\left(\frac{(2p-1)^2}{a^2}+\frac{(2q-1)^2}{b^2}\right)t}\right]$$

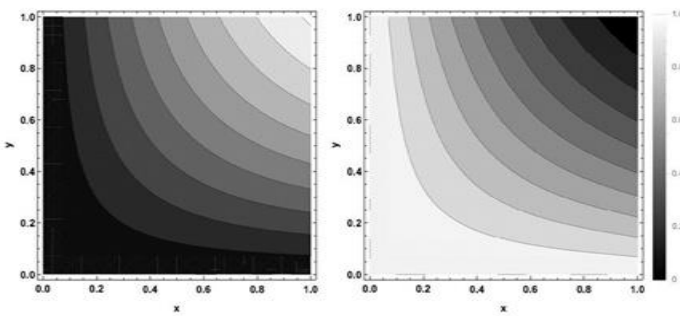


**FIGURE 9.4:** The figures show the dynamics of heat flows with Dirichlet boundary conditions and initial condition $u(x, y, 0) = xy(\pi - x)(\pi - y)$.

## *9.6 STEADY HEAT FLOW: LAPLACE EQUATION*

In heat flow problems, sometimes, one has to deal with inhomogeneous boundary conditions, which require the study of steady heat flow (that is, time-independent solutions of the heat equation). These are called steady-state solutions, and they satisfy $\frac{\partial u}{\partial t} = 0$. In one-dimensional case, the heat equation for steady state becomes $\frac{\partial^2 u(x)}{\partial x^2} = 0$, whose solutions are straight lines. In a two-dimensional case, the heat equation for steady state becomes $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$, which is the well-known Laplace equation. Solutions to the Laplace equation are called harmonic functions.

## *9.7 LAPLACE EQUATION WITH DIRICHLET'S CONDITIOIN*

To obtain the steady-state solutions to the Laplace equation

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad (9.13)$$

with Dirichlet's boundary condition:

$$u(0,y) = u(a,y) = 0, 0 \leq y \leq b$$
$$u(x,0) = 0, 0 \leq x \leq a \qquad (9.14c)$$
$$u(x,b) = f(x), 0 \leq x \leq a$$

we assume the solution of the form

$$u(x,y) = X(x)Y(y) \qquad (9.15)$$

Substituting we get,

$$\frac{X''}{X} = -\frac{Y''}{Y} = \mu(\text{ say }) \Rightarrow X'' - \mu X = 0 \qquad (9.16a)$$
$$Y'' + \mu Y = 0 \qquad (9.16b)$$

(9.14a) and (9.15) gives

$X(0)Y(0) = 0$ and $X(a)Y(y) = 0 \Rightarrow X(0) = 0$ and $X(a) = 0$
$[Y(y) \neq 0$ or else $u = 0$, which does not satisfy $(9.14c)]$

As before, $\mu = 0$ and $\mu > 0$ give trivial solution, which does not satisfy (9.14c), hence we reject both of them. For $\mu < 0 (= -\lambda^2)$, the solution of (9.16a) is

$$X(x) = A_1 \cos(\lambda x) + A_2 \sin(\lambda x)$$

Now, $X(0) = 0 \Rightarrow A_1 = 0$ and $X(a) = 0 \Rightarrow \lambda = \frac{n\pi}{a}, n = 1,2,3, \ldots.$
Hence, the non-zero solution of (9.16a) is given by

$$X(x) = A_2 \sin\left(\frac{n\pi x}{a}\right)$$

From (9.16b), we get

$$Y''(y) = \lambda^2 Y = \frac{n^2\pi^2}{a^2} Y \Rightarrow Y(y) = B_1 e^{\frac{n\pi}{a}y} + B_2 e^{-\frac{n\pi}{a}y}$$

(9.14b) and (9.15) give $u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 (X(x) \neq 0$ or else $u(0) = 0$, which does not satisfy (9.14c).

$$Y(0) = 0 \Rightarrow B_1 = -B_2 \Rightarrow Y(y) = 2B_1 \frac{e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}}{2}$$
$$= 2B_1 \sinh\left(\frac{n\pi y}{a}\right)$$

We use the principle of superposition to obtain the most general solution as

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \qquad (9.17)$$

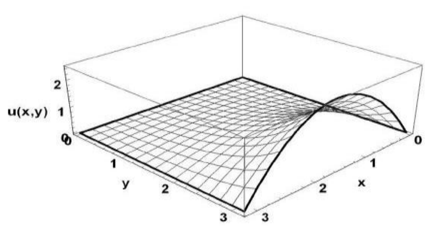
The general solution (9.17) satisfies the Laplace equation (9.13) inside the rectangle, as well as the three homogeneous boundary conditions on three of its sides (left, right and bottom). We now use the boundary condition on the top of the rectangle to determine the values of $A_n$, which requires

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)$$

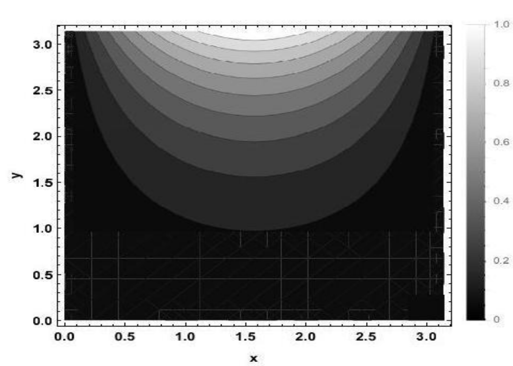
This is a half range Fourier sine series of $f(x)$ in $0 \le x \le a$, where

$$\Rightarrow A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx \qquad (9.18)$$

Therefore, the solution to the Laplace equation with Dirichlet's boundary conditions is (9.17), where $A_n$ is given by (9.18).




(a) Three-dimensional plot of steady heat flow.



(b) Contour plot of the flow.

**FIGURE 9.5:** The figures show a steady heat flow in a two-dimensional rectangular plate with Dirichlet boundary condition.

**Example 9.4.1 Find the steady-state temperature distribution in a rectangular plate of sides $a$ and $b$, insulated at the lateral surface and satisfying the boundary conditions**

$$\boldsymbol{u(0,y)=0, u(\pi,y)=0 \text{ for } 0\le y\le \pi}$$
$$\boldsymbol{u(x,0)=0, \text{ and } u(x,\pi)=x(\pi-x) \text{ for } 0\le x\le \pi}$$

**Solution:** We proceed as in section(9.4.1), and calculate

$$A_n=\frac{2}{\pi\sinh(n\pi)}\int_0^\pi x(\pi-x)\sin(\pi x)dx=\frac{2(2-2\cos(n\pi)-n\pi\sin(n\pi))}{n^3\pi\sinh(n\pi)}$$
$$=\frac{4}{\pi n^3}[1-(-1)^n]\cosh(n\pi)=\begin{cases}0, & \text{when } n=2m\\ \dfrac{8\cosh((2m-1)\pi)}{(2m-1)^3\pi}, & \text{when } n=2m-1.\end{cases}$$

Hence, the required steady temperature $u(x,y)$ is given by (fig. 9.5)
$u(x,y)=\sum_{m=1}^{\infty}\frac{8\cosh(2m-1)\pi}{\pi(2m-1)^3}\sin(2m-1)x\sin(2m-1)y.$

## 9.8 LAPLACE EQUATION WITH NEUMANN'S BOUNDARY CONDITION

To obtain the steady-state solution to the Laplace equation

$$\nabla^2 u\equiv\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

with Neumann's boundary conditions:

$$u_x(0,y)=0, u_x(a,y)=0, 0\le y\le b,$$
$$u_y(x,0)=0, u_y(x,b)=f(x) 0\le x\le a.$$

We assume the solution the solution of the form $u(x,y)=X(x)Y(y)$. Proceeding as in (9.4.1), we get $X(x)=A_1\cos\left(\frac{n\pi x}{a}\right)$ and $Y(y)=2B_1\cosh\left(\frac{n\pi y}{a}\right)$. Using the principle of superposition, the general solution can be written as

$$u(x,y) = \sum_{i=1}^{n} \frac{n\pi}{a} A_n \cos\left(\frac{n\pi x}{a}\right) \cosh\left(\frac{n\pi y}{a}\right).$$

$$u_y(x,b) = f(x) \Rightarrow f(x) = \sum_{i=1}^{n} \frac{n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right) A_n \cos\left(\frac{n\pi x}{a}\right), 0 \le x \le a$$

which is a Fourier Cosine series, where

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$$

## 9.9 VIBRATING STRING

We consider a homogenous flexible string of length $L$, which is stretched between two fixed points $(0,0)$ and $(L,0)$. Initially, the string is released from a position $u = f_1(x)$ with a velocity $u_t = f_2(x)$ parallel to the $y$-axis.

Mathematically, we can formulate the model as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, 0 \le x \le L, t > 0$$

Boundary Conditions (BCs): $u(0,t) = 0, u(L,t) = 0$.
Initial Conditions (ICs): $u(x,0) = f_1(x), \frac{\partial u(x,0)}{\partial t} = f_2(x)$.
We use separation of variables to solve the given wave equation. Let $u(x,t) = X(x)T(t)$ be a solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. Substituting, we obtain,

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \text{ constant}$$

Since the boundary conditions are periodic and homogenous in $x, X(x)$ must be periodic, which is only possible if the constant is negative $(-\lambda^2)$.

Solving $\frac{X''}{X} = -\lambda^2$ and $\frac{T''}{c^2 T} = -\lambda^2$, we obtain, $X(x) = A_1 \cos(\lambda x) + A_2 \sin(\lambda x)$ and $T(t) = A_3 \cos(c\lambda t) + A_4 \sin(c\lambda t)$.
Therefore, the general solution is given by

$$u(x,t) = [A_1 \cos \lambda x + A_2 \sin \lambda x][(A_3 \cos(c\lambda t) + A_4 \sin(c\lambda t)]. \quad (9.20)$$

Using the boundary conditions $u(0,t) = 0 = u(L,t)$ we obtain $A_1 = 0$ and $\sin(\lambda L) = 0 (A_2 \neq 0) \Rightarrow \lambda = \frac{n\pi}{L}, n$ being an integer. Therefore equation (9.20) becomes

$$u(x,t) = A_2 \sin\frac{n\pi x}{L}\left[A_3 \cos\left(\frac{cn\pi t}{L}\right) + A_4 \sin\left(\frac{cn\pi t}{L}\right)\right].$$

Noting that the wave equation is linear, we use the principle of superposition to obtain its most general solution as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right).$$

Using the initial condition we get,

$$u(x,0) = f_1(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \text{ and}$$

$$\frac{\partial u(x,0)}{\partial t} = f_2(x) = \sum_{n=1}^{\infty}\frac{cn\pi}{L} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Both are half-range Fourier sine series; therefore, we get,

$$A_n = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (9.21)$$

$$\frac{n\pi c}{L} B_n = \frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (9.21)$$

$$\Rightarrow B_n = \frac{2}{n\pi c}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (9.22)$$

Hence, the displacement of the vibrating string is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right),$$

where $A_n$ and $B_n$ are given by (9.21) and (9.22).

**Corollary 1:** If the homogenous flexible string of length $L$, stretched between two fixed points (0,0) and $(L,0)$, is initially released from rest from a position $u = f_1(x)$, then its initial velocity is zero, that is, $\frac{\partial u(x,0)}{\partial t} = 0$. The solution in that case will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right), \text{ where}$$
$$A_n = \frac{2}{L}\int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

**Corollary 2:** If the homogenous flexible string of length $L$, stretched between two fixed points (0,0) and $(L, 0)$, is initially released with velocity $f_2(x)$, and the initial deflection of the string is zero, then $u(x, 0) = 0$. The solution in that case will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right), \text{ where}$$
$$B_n = \frac{2}{cn\pi}\int_0^L f_2(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

**Example 9.5.1** A homogenous flexible string in a guitar is stretched between two fixed points (0,0) and $(L, 0)$, the length of the string being $L$ units. The string of the guitar is initially plucked from rest from a position $\mu x(L - x)$. Find the displacement $u(x, t)$ of the string of the guitar at time $t$.

**Solution:** Mathematically, we can formulate the model as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, 0 \le x \le L, t > 0$$

Boundary conditions (BCs): $u(0, t) = 0 = u(L, t)$.
Initial conditions (ICs): $u(x, 0) = \mu x(L - x), \frac{\partial u(x,0)}{\partial t} = 0$.
The solution is of the form

$$u(x,t) = (A_1 \cos \lambda x + A_2 \sin \lambda x)[A_3 \cos(c\lambda t) + A_4 \sin(c\lambda t)].$$

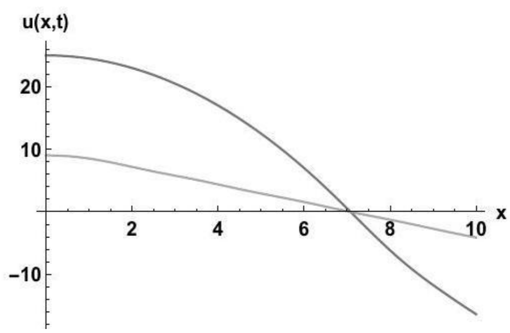
Applying boundary conditions, we get $A_1 = 0, \lambda = \frac{n\pi}{2} (A_2 \neq 0,)$, $n$ being an integer. Using the principle of superposition, the possible solution is $u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$.
Now, $\frac{\partial u(x,0)}{\partial t} = 0$ gives $B_n = 0 \Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$.
The initial condition gives $u(x, 0) = \mu x(L - x) =$

$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$,
which is a half-range Fourier sine series, where

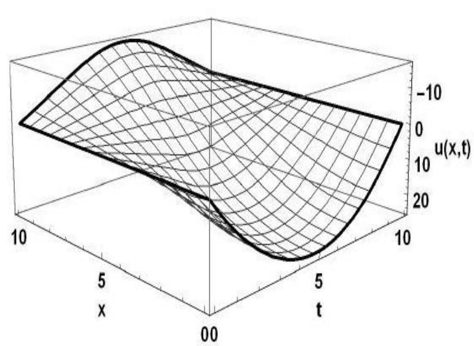
$$A_n = \frac{2}{L}\int_0^L \mu x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2\mu}{L}\left[2\left(\frac{L}{n\pi}\right)^3\{1-(-1)^n\}\right]$$
$$= \begin{cases} \frac{8\mu L^2}{n^3\pi^3}; & n = \text{ odd} \\ 0, & n = \text{ even} \end{cases}$$

Therefore, the required solution is (fig. 9.7)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8\mu L^2}{(2n-1)^3\pi^3}\cos\left\{\frac{(2n-1)c\pi t}{L}\right\}\sin\left\{\frac{(2n-1)\pi x}{L}\right\}$$




(a) Displacement $u(x,t)$ for $t = 5,9$.




(b) Three-dimensional view of $u(x,t)$.

**FIGURE 9.7:** The figures show the displacement of the homogeneous string in a guitar dynamic between (0,0) and (10,0).

## 9.9 WAVE EQUATION

The wave equation describes the propagation of oscillations and is represented by a linear second-order partial differential equation. Consider a homogeneous string of length $L$ is tied at both ends. We assume that the string offers no resistance due to bending; that is, it is thin and flexible; the tension in the string is much greater than the gravitational force, and hence, it can be neglected; the motion of the string takes place in the vertical plane only (fig. 9.6).

Let $\rho$ be the linear density of the string, and $P$ and $Q$ are two neighboring points on the string such that $\text{arc}PQ = \Delta s$. Let $T_1$ and $T_2$ be the tensions at points $P$ and $Q$, which make angles $\alpha$ and $\beta$, respectively with the $x$-axis and let $u(x,t)$ be the displacement of the string at time $t$ from its equilibrium state. Then, the equations of motion are

$$T_2\cos\beta - T_1\cos\alpha = 0 \text{ (along x-axis)} \qquad (9.19)$$

$$(\rho\Delta s)\frac{\partial^2 u}{\partial t^2} = T_2\sin\beta - T_1\sin\alpha \text{ (along y-axis)} \qquad (9.19)$$

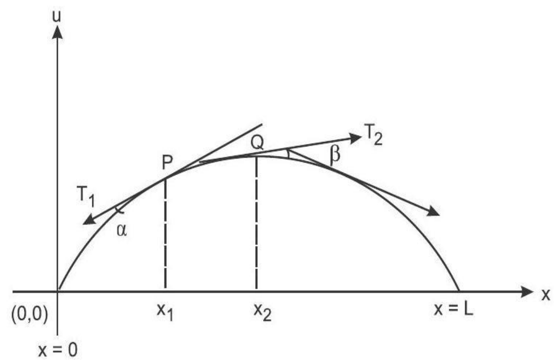



**FIGURE 9.6:** A homogeneous string of length L, tied at both ends such that the string offers no resistance due to bending.

From (9.19), we obtain, $T_1\cos\alpha = T_2\cos\beta = T$ (say), which implies

$$\frac{(\rho\Delta S)}{T}\frac{\partial^2 u}{\partial t^2} = \frac{T_2\sin\beta}{T} - \frac{T_1\sin\alpha}{T} = \frac{T_2\sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha}$$
$$= \tan\beta - \tan\alpha.$$

At the points $P$ and $Q$, the slopes of the string are given by $\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{x_1}$ and $\tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x_2}$. Therefore,

$\frac{(\rho\Delta S)}{T}\frac{\partial^2 u}{\partial t^2} = u_x(x_2, t) - u_x(x_1, t) \Rightarrow \frac{\rho}{T}\frac{\Delta S}{\Delta x}\frac{\partial^2 u}{\partial t^2} = \frac{u_x(x_1+\Delta x, t) - u_x(x_1, t)}{\Delta x}$.
As $\Delta x \to 0, \Delta s \to \Delta x$ and we get $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial u^2}$, where $c^2 = \frac{T}{\rho}$,
which is the one-dimensional wave equation, $c$ being the speed of the propagation of the wave.

## *9.10 CRIME MODEL*

Crime modeling with partial differential equations (PDEs) involves using mathematical equations to describe the spatial and temporal dynamics of crime patterns.
Crime occurs in both urban and rural environments. Some areas are reasonably safe while other are dangerous, demonstrating that crime is not uniformly distributed.This model can help:
**1. Identify crime hotspots:** PDE-based models can identify areas with high crime rates and predict future crime patterns.
**2. Understand crime diffusion**: PDE-based models can capture how crime spreads from one location to another, helping to identify underlying factors driving crime.
**3. Evaluate crime prevention strategies:** PDE-based models can simulate the impact of different crime prevention strategies, such as increased policing or community programs.

**Types of Crime Models using PDEs**
**1. Reaction-Diffusion Models:** These models describe how crime spreads from one location to another, taking into account factors like population density and policing.
**2. Wave Equation Models:** These models describe how crime patterns propagate through space and time, capturing the dynamics of crime waves.
**3. Nonlinear Diffusion Models:** These models capture the complex, nonlinear dynamics of crime patterns, including the impact of policing and community interventions.

## *9.17 SUMMARY*

**Spatial Modeling:** Spatial modeling involves analyzing and predicting phenomena that vary over space and time.

**Partial Differential Equations (PDEs):** PDEs are mathematical equations that describe the behavior of spatial-temporal systems, capturing complex interactions between variables.
**Crime Models:** Crime modeling with partial differential equations (PDEs) involves using mathematical equations to describe the spatial and temporal dynamics of crime patterns.

## *9.18 GLOSSARY*

**Partial Differential Equation (PDE):** A mathematical equation that describes the behavior of a system that changes over space and time.
**Dependent Variable**: A variable whose behaviour is being modelled, often denoted as u(x,t).
**Independent Variables:** Variables that describe the spatial and temporal coordinates, often denoted as x and t.
**Boundary Conditions**: Conditions that specify the behaviour of the dependent variable at the boundaries of the spatial domain.
**Initial Conditions:** Conditions that specify the initial state of the dependent variable.
**Spatial Domain**: The region in space where the PDE is defined.
**Spatial Grid:** A discrete representation of the spatial domain, used for numerical simulations.
**Spatial Resolution:** The level of detail at which the spatial domain is discretized.
**Spatial Autocorrelation**: The correlation between values of the dependent variable at different spatial locations.

**CHECK YOUR PROGRESS**

**CYQ1.** Spatial modeling is a mathematical approach used to describe and analyze phenomena that vary over …..and …...

**CYQ2.**The advantage of PDE models is that they include derivatives of at least……..

**CYQ3**…………capture the complex, nonlinear dynamics of crime patterns, including the impact of policing and community interventions.

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## 9.21 TERMINAL QUESTIONS

1. What is continuous modelling approach?
2. What is the difference between Arms Race Model and Epidemic Models.
3. A fossil is found that has 20% $C^{14}$ compared to the living sample. How old is the fossil, knowing that the $C^{14}$ half – life is 5730 years?
4. Model the population growth of a city using the logistic equation, and solve for the population at time t.
5. What is an Ordinary Differential Equation (ODE), and how is it used to model continuous systems?

## 9.22 ANSWERS

**TERMINAL QUESTIONS**

**TQ1.** What is meaning of PDE's Models ?

**TQ2.**A road of leanth l, whose side are insulated is kept, at uniform temperature $u_0$, both ends of the road are suddenly cooled at $0^0$c and are kept at that temperature. If u(x,t) represents the temperature function at any point x at time t,

(i) Formulate a mathematical model of the given solution using PDE's stating clearly the boundary and initial conditions.

(ii) Using the method of separation of variable, find the temperature function u(x,t).

**TQ3.** Find the traffic density $\rho(x,t)$. satisfying $\frac{\partial \rho}{\partial t}$ + x sin(t) $\frac{\partial \rho}{\partial x}$ = 0

With initial condition $\rho_0$(x) = 1 + $\frac{1}{x^2}$

## *ANSWERS*

**CYQ1.** Space and Time

**CYQ2.**Two independent variables

**CYQ3**. Nonlinear Diffusion Models

**TQ2. (ii)** u (x,t) = $\sum_{m=1}^{\infty} \frac{uu_0}{(2m-1)\pi}$ sin $\frac{(2m-1)\pi x}{L}$ $e^{\frac{-(2m-1)^2 \pi^2 c^2}{L}}$t.

**TQ3.**$\rho_0$(x,t)= 1+$\frac{1}{1+x^2 e^{-2+2\cos(t)}}$

# UNIT 10: MODELING WITH DELAY DIFFERENTIAL EQUATIONS

## CONTENTS:



## *10.1 INTRODUCTION*

In previous unit we have defined Mathematical Modeling through Differential Equation, Advantage of partial differential equation models, Laplace Equation with Dirichlet's conditions, Laplace Equation with Neumann's Boundary condition, wave equation and crime model. In this unit we discussed about different model with delay differential equations.

When the learner start reading this chapter the first question that comes to mind, what are Delay Differential Equations (DDE),. A delay differential equation (DDE) is a type of mathematical equation that describes the behavior of a system that changes over time, In layman's term, a DDE is a differential equation in which the derivatives of some unknown functions at the present time are dependent on the values of the functions at previous times. Let us consider a general DDE of the first order the form

$$\frac{dx(t)}{dt} = f\big(t, x(t), x(t-\tau_1), x(t-\tau_2), \dots, x(t-\tau_n)\big) \qquad (10.1)$$

where $\tau_i$ 's are positive constants (fixed discrete delays). When we solve an ODE (initial value problem), we only need to specify the initial values of the state variables. However, while solving a DDE, we have to look back to the earlier values of $x$ at every time step. Assuming that we start at time $t = 0$, we, therefore, need to specify an initial function, which gives the behavior of the system prior to time $t = 0$.

Consider a DDE with a single discrete delay, that is,

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau)) \qquad (10.2)$$

where $x(t-\tau) = \{x(\tau): \tau \le t\}$ gives the trajectory of the solution in the past. Here, the function $f$ is a functional operator from $\mathrm{R} \times \mathrm{R}^n \times C^1$ to R and x(t) $\in \mathrm{R}^n$. For this DDE, the initial function would be a function $x(t)$ defined on the interval $[-\tau, 0]$. The solution to (10.2) is a mapping from functions on the interval $[t-\tau, t]$ into the functions on the interval $[t, t+\tau]$. Thus, the solution of (10.2) can be defined as a sequence of functions $f_0(t), f_1(t), f_2(t), \dots$ defined over a set of adjacent time intervals of length $\tau$. The points $t = 0, \tau, 2\tau, \dots$, where the solution segments meet is called knots.

Now for solving Delay differential equations, let us consider a simple DDE of the form

$$\frac{dx}{dt} = -x(t-\tau), t > 0$$

Initial history: $\mathrm{x(t)} = 1, -\tau \le t \le 0$. Clearly, with $\tau = 0, x(t) = x(0)e^{-t}$. However, the presence of $\tau$ makes the situation a bit tricky. Hence, in the interval $0 \le t \le \tau$, we have

$$\frac{dx}{dt} = -x(t-\tau) = -1$$

$$\Rightarrow x(t) = x(0) + \int_0^t (-1)ds = 1 - t, 0 \le t \le \tau$$

In $\tau \le t \le 2\tau$, we get, $0 \le t - \tau \le \tau$ and so we have,

$$\frac{dx(t)}{dt} = -x(t-\tau) = -[1-(t-\tau)]$$

$$\Rightarrow x(t) = x(\tau) + \int_\tau^t [-\{1-(s-\tau)\}]ds$$

$$\Rightarrow x(t) = 1 - t + \frac{(t-\tau)^2}{2}, \tau \le t \le 2\tau$$

and so on. In general, it can be shown (use mathematical induction) that

$$x(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t - \overline{k-1}\tau]^k}{k!}, (n-1)\tau \le t \le n\tau, n \ge 1.$$

The above method is known as a procedure of steps.

## *10.2 OBJECTIVES*

After studying this unit, learner will be able to

1. Describe modeling Real-World Systems:
2. Understand complex dynamics, such as oscillations, instability, and bifurcations, that arise from delayed responses.
3. Analyze the stability of systems, identifying conditions under which the system will return to its equilibrium state.
4. Identify bifurcations, sudden changes in system behaviour, that occur as parameters are varied.

## 10.3 DIFFERENT MODELS WITH DELEY DIFFERENTIAL EQUATIONS

### 10.3.1 DELEY PROTEIN DEGRADATION

Let $P(t)$ be the concentration of proteins at any time $t$ in a system, then the production of proteins at any time is given by

$$\frac{dP(t)}{dt} = \alpha - \beta P(t) - \gamma P(t-\tau) \qquad (10.8)$$

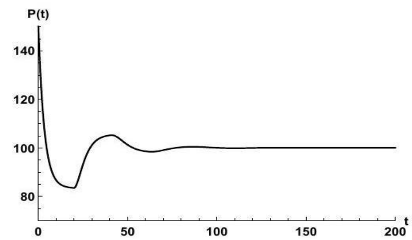
where $\alpha$ is the constant rate of protein production, $\beta$ is the rate of non-delayed protein degradation, and $\gamma$ is the rate of delayed protein degradation. The discrete time delay $\tau$ is due to the fact that the protein degradation machine degrades the protein after a time $\tau$ after initiation. The equilibrium points of **(10.8)** is given by

$$\alpha - \beta P^* - \gamma P^* = 0 \Rightarrow P^* = \frac{\alpha}{\beta + \gamma}$$

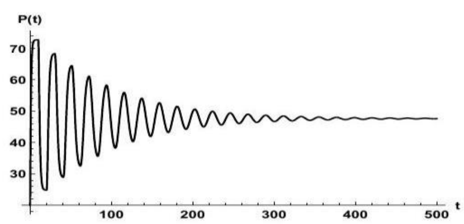
Using the transformation $P = P' + P^*$ in **(10.8),** we obtain

$$\frac{dP'(t)}{dt} = -\beta P'(t) - \gamma P'(t-\tau) \qquad (10.9)$$

Putting $P' = A_1 e^{\lambda t}$ in **(10.9),** we get the characteristic equation as $\lambda = -\beta - \gamma e^{-\lambda \tau}$. Comparing with **(10.2.1.1),** we get $A + B = -\beta - \gamma < 0$. Since, $-\gamma > -\beta$ (assumed), the system (10.8) is asymptotically stable about $P^* = \frac{\alpha}{\beta+\gamma}$. **Fig. 10.1** shows the degradation of protein for various parameter values,




(a)$\alpha = 40, \beta = 0.3, \gamma = 0.1, \tau = 20.$




(b) $\alpha = 100, \beta = 1.1, \gamma = 1, \tau = 10.$

**FIGURE 10.1:** The figures show delay-induced protein degradation.
**(a)** Protein concentration delay with initial history 150.

**(b)** Oscillatory behavior of protein degradation with initial history 20.

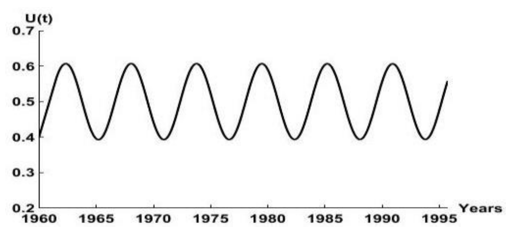
## 10.3.2 FOOTBALL TEAM PERFORMANCE MODEL

During the last 40 years R.B. Banks proposed a delay-induced mathematical model to analyze the performance of a National Football League (NFL) football team. The proposed model is

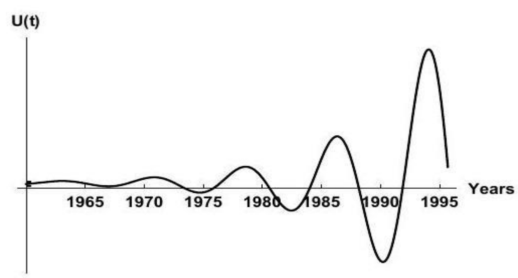
$$\frac{dU}{dt} = b\left[\frac{1}{2} - U(t-\tau)\right] \qquad (10.10)$$

where $U(t)$ is the fraction of games won by an NFL team during one season and it lies between 0 and 1, and $b$ is the growth rate. The computational formula for $U(t)$ is given by

$$U(t) = \frac{1\times \text{ no. of games won } + \frac{1}{2}\times \text{ no. of games tied } + 0\times \text{ no. of games lost}}{\text{Total no. of games}}.$$




**(a)** $\tau = 2$years.



**(b)** $\tau = 3$ years.

**FIGURE 10.2:** The performance of an NFL team from 1960 to 1992, with parameter values $b = 0.785, \tau = 2$ years and initial history 0.4. The performance of the team shows

**(a)** a periodic solution and
**(b)** an unstable situation, which matches with data .

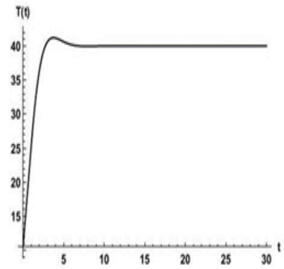
Basically, the proposed model says that at the present time, the rate of change of $U$ is proportional to the difference between $U = \frac{1}{2}$ (average values) and the values of $U$ at some previous time $t - \tau$. The equilibrium points of (10.10) is given by $b\left[\frac{1}{2} - U^*\right] = 0 \Rightarrow U^* = \frac{1}{2}$. Using the transformation $U = U' + U^*$ in (10.10), we obtain

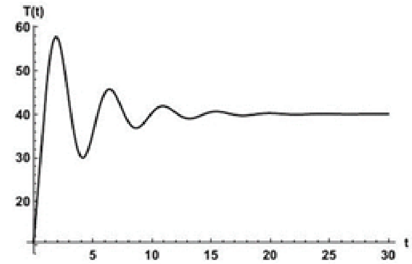
$$\frac{dU'(t)}{dt} = -bU'(t - \tau) \qquad (10.11)$$

Putting $U' = A_1 e^{\lambda t}$ in (5.11), we get the characteristic equation as $\lambda = -be^{-\lambda\tau}$. Comparing with (5.2.1.1), we get $A + B = -b < 0$ and $B < A$. Hence, the system (5.10) is asymptotically stable about $U^* = \frac{1}{2}$ for $0 < \tau < \tau^*$ and unstable for $\tau > \tau^*$concluded from his model that the time delay $\tau$ plays an important role in the ups and downs of the football team. It experiences a simple periodicity **(fig. 10.2(a))** and an unstable situation **(fig. $\mathbf{10.2}$( b) ).**

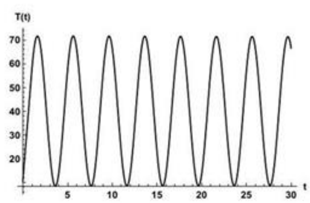
## 10.3.3 SHOWER PROBLEM

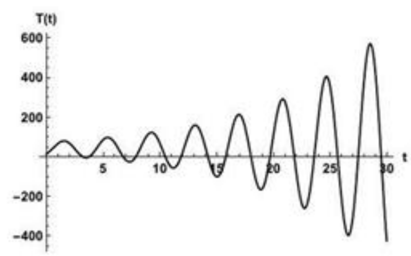
People enjoy showering, especially when they are able to control the water temperature. The dynamics of human behavior while taking a shower when the water temperature is not comfortable is quite interesting. A simple DDE model is proposed to capture such dynamics. We assume that the speed of water is constant (uniform flow) from the faucet to the shower head, which takes the time $\tau$ second (say)




$(a)$ $\alpha = 0.5$ (b) $\alpha = 1.1$




$(c)$ $\alpha = 1.5$ (d)= $\alpha = 1.8$

**FIGURE 10.3:** Varied temperatures of water for different values of $\alpha$. The parameter values are $T_d = 40°\text{C}, \tau = 1$ and initial history $=$ 0.5.

Let $T(t)$ denote the temperature of water at the faucet at time $t$, then the temperature evolution is given by

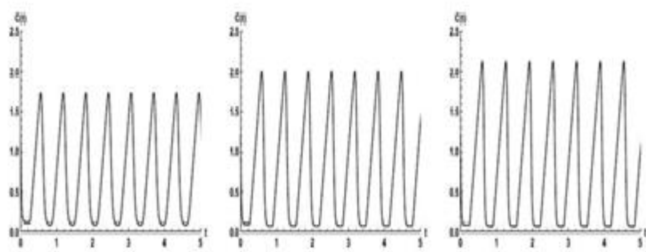
$$\frac{dT}{dt} = -\alpha[T(t-\tau) - T_d] \tag{10.12}$$

where $T_d$ is the desired temperature and $\alpha$ gives the measure of a person's reaction due to the wrong water temperature. One type of person might prefer a low value of $\alpha$, whereas another type of person would choose a higher value. The equilibrium point of (10.12) is given by $T^* - T_d = 0 \Rightarrow T^* = T_d$. Using the transformation $T = T' + T^*$ in (5.12), we obtain

$$\frac{dT'(t)}{dt} = -\alpha T'(t-\tau) \tag{10.13}$$

Putting $T' = A_1 e^{\lambda t}$ in (10.13), we get the characteristic equation as $\lambda = -\alpha e^{-\lambda\tau}$. Comparing with (10.2.1.1), we get $A + B = 0 + (-\alpha) = -\alpha < 0$ and $B < A$. Hence, the system (10.12) is asymptotically stable about $T^* = T_d$.

For $\alpha = 0.5$, the temperature of the water goes to $40°\text{C}$, which is comfortable to the body and the person remains calm **(fig. 10.3(a)).** For $\alpha = 1.1$, after initial fluctuation, the temperature of the water goes to $40°\text{C}$. A person may show initial discomfort with the start of the shower **(fig. 10.3(b)).** One person may prefer the value of $\alpha = 1.57$ while showering (a bathroom singer?), which shows cyclic behavior of water temperature **(fig. 10.3(c)).** For $\alpha = 1.8$, the temperature of the water is erratic and unpleasant while taking shower **(fig. 10.3(d)).**



n=3 n=6 n=9

**FIGURE 10.4**: The oscillatory behavior of carbon dioxide content for

**(a)** $n = 3$, **(b)** $n = 6$, **(c)** $n = 9$. The parameter values, are $\lambda = 6, \alpha = 1.0, V_{\max} = 80, \theta = 1, \tau = 0.25$ and initial history = 1.2.

## 10.3.4 BREATHING MODEL

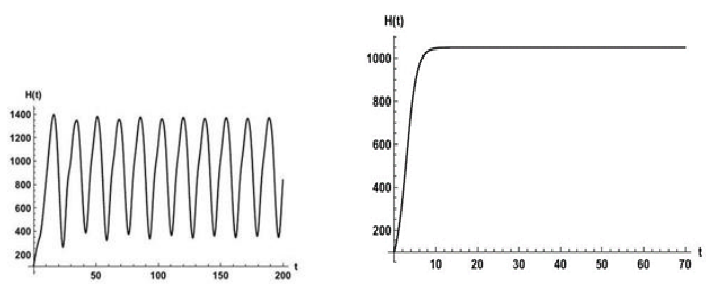
The breathing model is a mathematical representation of the respiratory system, which includes the lungs, airways, and breathing muscles. The model uses DDEs to describe the dynamics of breathing, taking into account the delays between the neural signals and the mechanical responses. The arterial carbon dioxide level controls our rate of breathing. A mathematical model was first developed by Mackey and Glass, where they assumed that carbon dioxide is produced at a constant rate $\lambda$ due to metabolic activity and its removal from the bloodstream is proportional to both the current carbon dioxide concentration and to ventilation. Ventilation, which is the volume of gas exchanged by the lungs per unit of time, is controlled by the carbon dioxide level in the blood. The process is complex and involves the detection of carbon dioxide levels by receptors in the brain stem. This carbon dioxide detection and its subsequent adjustment to ventilation is not an instantaneous process; there is a time lag due to the fact that the blood transport from the lungs to the heart and then back to the brain requires time. Thus, if C is the concentration of the carbon dioxide, then the rate of change of concentration of carbon dioxide due to breathing is given by

$$\frac{dC(t)}{dt} = \lambda - \alpha V_{\max} C(t) \dot{V}(t-\tau)$$

where $\dot{V}(t)$ is the rate of ventilation and is assumed to follow the Hill function, that is $\dot{V}(t) = \frac{(C(t))^n}{\theta^n + (C(t))^n}$; $V_{\max}$, $\theta, n,$ and $\alpha$ are constants. Thus, the rate of change of concentration of carbon dioxide is given by

$$\frac{dC(t)}{dt} = \lambda - \alpha V_{\max} C(t) \frac{(C(t-\tau))^n}{\theta^n + (C(t-\tau))^n}$$

**Fig. 10.4** shows the oscillatory solutions of the Mackey-Glass equation, representing the carbon dioxide content for $n = 3,6,9$. As $n$ increases, the amplitude of the oscillation also increases gradually.




(a) Oscillatory behaviour $\tau = 5$ (b) Logistic growth ($\tau = 0$)

**FIGURE 10.5: (a)** The oscillatory behavior of the adult houseflies. Parameter value $d_1 = 0.147, \beta = 1.81, k = 0.5107, M = 0.000226$, initial history 100.

**(b)** The adult houseflies follow a logistic growth for $\tau = 0$.

## 10.3.5 HOUSEFLY MODEL

The housefly model is a mathematical representation of the population dynamics of houseflies, which includes the delays between the different stages of their life cycle. The model uses DDEs to describe the dynamics of the housefly population, taking into account the delays between the egg, larval, and adult stages.

Taylor and Sokal proposed a model to describe the behavior of the adult housefly Musca domestic in laboratory conditions. To capture the dynamics of the housefly, the model is represented using as

$$\frac{dH}{dt} = -d_1 H(t) + \beta H(t-\tau)[k - \beta M H(t-\tau)].$$

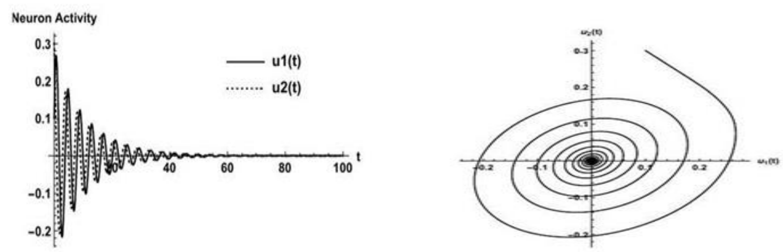
Here, $H(t)$ represents the number of adult houseflies at any time t, $d_1$ is the natural death of the houseflies, $\tau(> 0)$ is the discrete time delay, which is the time from laying eggs until their emerging from the pupal case (oviposition and eclosion of adults), $\beta$ is the number of eggs laid per adult, and assuming the number of eggs laid is proportional to the number of adults, the number of new eggs at time $t - \tau$ would be $\beta H(t-\tau)$. The term $k - \beta M H(t-\tau)$ gives the egg-to-adult survival rate, $k$ and $M$ being the maximum egg-adult survival rate and reduction in survival for each egg, respectively. **Fig. 10.5(a)** shows periodic solution as observed in the behavior of adult houseflies in laboratory conditions. Please note that for $\tau = 0$, the housefly population follows a logistic growth (**fig. 10.5(b)).**

## 10.3.6 TWO NEURON SYSTEM

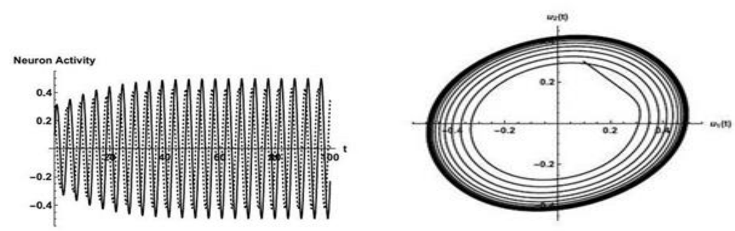
A two-neuron system of self-existing neurons is given by

$$\frac{du_1}{dt} = -u_1(t) + a_1 \tanh[u_2(t-\tau_{21})], \frac{du_2}{dt} = -u_2(t) + a_2 \tanh[u_1(t-\tau_{12})]$$

where $u_1(t)$ and $u_2(t)$ are the activities of the first and second neurons respectively, $\tau_{21}$ is the delay in signal transmission between the second neuron and the first neuron ( $\tau_{12}$ can be explained in a similar manner) and $a_1, a_2$ are the weights of the connection between the neurons. By taking $a_1 = 2, a_2 = -1.5, \tau_{21} = 0.2, \tau_{12} = 0.5$ such that $\tau_{21} + \tau_{12} < 0.8$, numerically it has been shown that the model is asymptotically stable about the origin (fig. 5.6(a), fig. 10.6(b)). For $\tau_{21} = 0.4, \tau_{12} = 0.6$ such that $\tau_{21} + \tau_{12} > 0.8$, a periodic solution bifurcates from the origin; that bifurcation is supercritical and the bifurcating periodic solution is orbitally asymptotically stable (fig. 10.6(c), fig. 10.6( d)). Interested readers may look into Ruan et al, to learn more about the analytical calculations and restrictions on $\tau_{21}$ and $\tau_{12}$.




(a) $\tau_{21}$= 0.2, $\tau_{12}$= 0. (b) Stable focus




**(c)** $\tau_{21}$= 0.4 $\tau_{21}$= 0.6 (d) Limit cycle

**FIGURE 10.6:** The figures show the activities of two self-exiting neurons. Parameter values: $a_1 = 2, a_2 = -1.5$ with initial history (0.1,0.3). (a) Asymptotically stable, (b) stable focus, (c) periodic solution, (d) limit cycle.

# 10.4 IMMUNOTHERAPY WITH INTERLEUKIN-2, A STUDY BASED ON MATHEMATICAL MODELING

## 10.4.1 BACKGROUND OF THE PROBLEM

The mechanism of establishment and destruction of cancer, one of the greatest killers in the world, is still a puzzle. Modern treatment involves surgery, chemotherapy, and radiotherapy, yet relapses occur.

Hence, the need for more successful treatment is clear. Developing schemes for immunotherapy or its combination with other therapy methods are the major focus at present and aim at reducing the tumor mass, heightening tumor immunogenicity, and removal of immunosuppression induced in an organism in the process of tumor growth. Recent progress has been achieved through immunotherapy, which refers to the use of cytokines (protein hormones that mediate both natural and specific immunities) usually together with adoptive cellular immunotherapy (ACI).

The main cytokine responsible for lymphocyte activation, growth, and differentiation is interleukin-2 (IL-2), which is mainly produced by T helper cells (CD4+ T-cells) and in relatively small quantities by cytotoxic T-lymphocytes (CD8+ T-cells). CD4 lymphocytes differentiate into T-Helper 1 and T-Helper 2 functional subjects due to the immune response. IL-2 acts in an autocrine manner on T-Helper 1 and also induces the growth of T-Helper 2 and CD8 lymphocytes in a paracrine manner. The T-lymphocytes themselves are stimulated by the tumor to induce further growth. Thus, the complete biological assumption of adoptive cellular immunotherapy is that the immune system is expanded in number artificially (ex vivo) in cell cultures by means of human recombinant interleukin-2. This can be done in two ways, either by (i) lymphokine-activated killer cell therapy, where the cells are obtained from the in vitro culturing of peripheral blood leukocytes removed from patients with a high concentration of IL-2, or (ii) tumor-infiltrating lymphocyte therapy (TIL), where the cells are obtained from lymphocytes recovered from the patient tumors, which are then incubated with high concentrations of IL-2 in vitro and are comprised of activated natural killer (NK) cells and cytotoxic T-lymphocyte (CTL) cells. The TIL is then returned into the bloodstream, along with IL-2, where they can bind to and destroy the tumor cells. It has been established clinically that immunotherapy with IL-2 has enhanced CTL activity at different stages of the tumor [133,134,148]. Also, there is evidence of the restoration of the defective NK cell activity as well as enhancement of polyclonal expansion of CD4+ and CD8+ T cells.

Kirschner and Panetta have studied the role of IL-2 in tumor dynamics, particularly long-term tumor recurrence and short-term oscillations, from mathematical perspective. The model proposed there deals with three populations, namely, the activated immune-system cells (commonly called effector cells), such as cytotoxic T cells, macrophages, and NK cells that are cytotoxic to the tumor cells, the tumor cells and the concentration of IL-2. The important parameters in their study are antigenicity of tumor (c), a treatment term that represents the external source of effector cells $(s_1)$, and a treatment term that represents an external input of IL-2 into the system $(s_2)$. Their results can be summarized as follows: (i) For non-treatment cases $(s_1 = 0, s_2 = 0)$, the immune system has not been able to clear the tumor for low-antigenic tumors, while for highly antigenic tumors, reduction to a small dormant tumor is the best-case scenario. (ii) The effect of adoptive cellular immunotherapy (ACI) $(s_1 > 0, s_2 = 0)$ alone can yield a tumor-free state for tumors of almost any antigenicity, provided the treatment concentration is above a given critical level. However, for tumors with small antigenicity, early treatment is needed while the tumor is small, so that the tumor can be controlled. (iii) Treatment with IL-2 alone $(s_1 = 0, s_2 > 0)$ shows that if IL-2 administration is low, there is no tumor-free state. However, if IL-2 input is high, the tumor can be cleared, but the immune system grows without bounds, causing problems such as capillary leak syndrome. (iv) Finally, the combined treatment with ACI and IL-2 $(s_1 > 0, s_2 > 0)$ gives the combined effects obtained from the monotherapy regime. For any antigenicity, there is a region of tumor clearance. These results indicate that treatment with ACI may be a better option either as a monotherapy or in conjunction with IL-2. Here I have proposed a modification of the model studied by Kirschner and Panetta by adding a discrete time delay which exists when activated T-cells produce IL-2.

## 10.4.2 THE MODEL

The proposed model is an extension of the Kirschner-Panetta ordinary differential equation model [83]

$$\frac{dE}{dt_1} = cT + \frac{p_1 E I_L}{g_1 + I_L} - \mu_2 E + s_1$$
$$\frac{dT}{dt_1} = r_2(1 - bT)T - \frac{aET}{g_2 + T}$$
$$\frac{dI_L}{dt_1} = \frac{p_2 ET}{g_3 + T} - \mu_3 I_L + s_2$$

to a DDE model with proper biological justifications and is given by

$$\frac{dE}{dt_1} = cT + \frac{p_1 E(t_1 - \tau) I_L(t_1 - \tau)}{g_1 + I_L(t_1 - \tau)} - \mu_2 E + s_1$$
$$\frac{dT}{dt_1} = r_2(1 - bT)T - \frac{aET}{g_2 + T}$$
$$\frac{dI_L}{dt_1} = \frac{p_2 ET}{g_3 + T} - \mu_3 I_L + s_2$$

Using the following scaling

$x = \frac{E}{E_0}, y = \frac{T}{T_0}, z = \frac{I_L}{I_{L_0}}, t = t_s t_1;\ \bar{c} = \frac{cT_0}{t_s E_0}, \overline{\rho_1} = \frac{p_1}{t_s}$,
$\overline{g_1} = \frac{g_1}{I_{L_0}}, \overline{\mu_2} = \frac{\mu_2}{t_s}, \overline{g_2} = \frac{g_2}{T_0}, \bar{b} = bT_0, \overline{r_2} = \frac{r_2}{t_s}, \bar{a} = \frac{aE_0}{t_s T_0}$,
$\overline{\mu_3} = \frac{\mu_3}{t_s}, \overline{p_2} = \frac{p_2 E_0}{t_s I_{L_0}}, \overline{g_3} = \frac{g_3}{T_0}, \overline{s_1} = \frac{s_1}{t_s E_0}, \overline{s_2} = \frac{s_2}{t_s I_{L_0}}$,
the given system is non-dimensional zed, given by (after dropping the overbar notation for convenience)

$$\frac{dx}{dt} = cy + \frac{p_1 x(t - \tau) z(t - \tau)}{g_1 + z(t - \tau)} - \mu_2 x + s_1$$
$$\frac{dz}{dt} = \frac{p_2 xy}{g_3 + y} - \mu_3 z + s_2$$

subject to the following initial conditions

$$\begin{aligned} x(\theta) = \psi_1(\theta), y(\theta) = \psi_2(\theta), z(\theta) = \psi_3(\theta) \\ \psi_1(\theta) \geq 0, \psi_2(\theta) \geq 0, \psi_3(\theta) \geq 0; \theta \in [-\tau, 0] \\ \psi_1(0) > 0, \psi_2(0) > 0, \psi_3(0) > 0 \end{aligned} \qquad (10.15)$$

where $C_+ = (\psi_1(\theta), \psi_2(\theta), \psi_3(\theta)) \in C([-\tau, 0], R^3_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $R^3_{+0}$, where $R^3_{+0}$ is defined as

$R^3_{+0} = ((x, y, z): x, y, z \geq 0)$ and $R^3_+$, the interior of $R^3_{+0}$ as
$R^3_+ = ((x, y, z): x, y, z > 0)$

In the system described by (5.14), $x(t), y(t)$, and $z(t)$, respectively, represent the effector cells, the tumor cells, and the concentration of IL-2 in the single site compartment. The first equation of the system (10.14) describes the rate of change for the effector cell population. The effector cells grow due to the direct presence of the tumor, given by the term $cy$, where $c$ is the antigenicity of the tumor. It is also stimulated by IL-2 that is produced by effector cells in an autocrine and paracrine manner (the term $\frac{p_1 xz}{g_1 + z}$, where $p_1$ is the rate at which effector cells grow, and $g_1$ is the half-saturation constant).

Clinical trials show that there are immune stimulation effects from treatment with IL-2, and there is a time lag between the production of IL-2 by activated T-cells and the effector cell stimulation from treatment with IL-2. Hence, a discrete time delay is being added to the second term of the first equation of the system (10.14), which modifies to $\frac{p_1 x(t-\tau) z(t-\tau)}{g_1 + z(t-\tau)}$, where $\mu_2 x$ gives the natural decay of the effector cells and $s_1$ is the treatment term that represents the external source of the effector cells such as ACI. A similar type of term was introduced by Galach in his model equation, where he

| Parameters | Values | Scales Values |
|---|---|---|
| $c$ (Antigenicity of tumor) | $0 \le c \le 0.05$ | $0 \le c \le 0.278$ |
| $p_1$ (Growth rate of effector cells) | 0.1245 | 0.69167 |
| $g_1$ (Half saturation constant) | $2 \times 10^7$ | 0.02 |
| $\mu_2$ (Natural decay rate of effector cells) | 0.03 | 0.1667 |
| $r_2$ (Growth rate of tumor cells) | 0.18 | 1 |
| $b$ (1/carrying capacity of tumor cells) | $1.0 \times 10^{-9}$ | 1 |
| $a$ (decay rate of tumor) | 1 | 5.5556 |
| $g_2$ (half saturation constant) | $1 \times 10^5$ | 0.0001 |
| $\mu_3$ (natural decay rate of IL-2) | 10 | 55.556 |
| $p_2$ (growth rate of IL-2) | 5 | 27.778 |
| $g_3$ (half saturation constant) | $1 \times 10^3$ | 0.000001 |

## TABLE 10.1 PARAMETRE VALUES USED FOR NUMERICAL RESULT

Assumed that the source of the effector cells is the term $x(t-\tau)y(t-\tau)$, as the immune system needs some time to develop a suitable response.

the second equation of the system (10.14) shows the rate of change of the tumor cells, which follows logistic growth (a type of limiting growth). due to tumor-effector cell interaction, there is a loss of tumor cells at the rate $a$ and which is modeled by michaelis menten kinetics to indicate the limited immune response to the tumor (the term $\frac{axy}{g_2+y}$, where $g_2$ is a half-saturation constant). the third equation of the system (10.14) gives the rate of change for the concentration of il-2. its source is the effector cells that are stimulated by interaction with the tumor and also has michaelis menten kinetics to account for the self-limiting production of il-2 (the term $\frac{p_2xy}{g_3+y}$, where $p_2$ is the rate of production of il-2 and $g_3$ is a half-saturation constant), $\mu_3 z$ is the natural decay of the il-2 concentration and $s_2$ is a treatment term that represents an external input of il-2 into the system.

proper scaling is needed as the system is numerically stiff, and numerical routines used to solve these equations will fail without scaling or inappropriate scaling (in this case, a proper choice of scaling is $E_0 = T_0 = I_{L_0} = 1/b$ and $t_s = r_2$ . the parameter values have been obtained from [83], which is put in tabular form (table 5.1). the units of the parameters are in $day^{-1}$, except for $g_1, g_2, g_3$, and $b$, which are in volumes.

the aim of this problem is to study this modified model and to explore any changes in the dynamics of the system that may occur when a discrete time delay has been added to the system and to compare with the results obtained by kirschner and panetta in .

## 10.4.3 POSITIVITY OF THE SOLUTION

The system of equations is now put in a vector form by setting

$$X = \mathrm{col}(M, N, Z) \in R^3_{+0},$$

$$F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \end{pmatrix} = \begin{pmatrix} cy + \dfrac{p_1 x(t-\tau) z(t-\tau)}{g_1 + z(t-\tau)} - \mu_2 x + s_1 \\ r_2(1 - by)y - \dfrac{axy}{g_2 + y} \\ \dfrac{p_2 xy}{g_3 + y} - \mu_3 z + s_2 \end{pmatrix},$$

where $F: C_+ \to R^3_{+0}$ and $F \in C^\infty(R^3_{+0})$. Then system (5.14) becomes

$$\dot{X} = F(X_t) \tag{10.16}$$

where $\cdot \equiv d/dt$ and with $X_t(\theta) = X(t+\theta), \theta \in [-\tau, 0]$ . It is easy to check in equation (10.16) that whenever we choose $X(\theta) \in C_+$such that $X_i = 0$, then we obtain $F_i(X)|_{X_i(t)=0, X_t \in C_+} \geq 0, i = 1,2,3$. Due to the lemma, any solution of equation (10.16) with $X(\theta) \in C_+$, say, $X(t) = X(t, X(0))$, is such that $X(t) \in R^3_{+0}$ for all $t > 0$.

## 10.5 SUMMARY

**Delay Differential Equations:** DDEs are mathematical equations that describe the behavior of a system that changes over time, with a delay or lag in the response.
**Delay**: The response of the system to changes in the input is delayed by a certain amount of time, τ.
**Nonlocal:** The behavior of the system at time t depends on the state of the system at previous times, specifically at time t-τ.

## 10.6 GLOSSARY

**Delay Differential Equation (DDE):** A mathematical equation that describes the behavior of a system that changes over time, with a delay or lag in the response.
**Delay:** The time lag between the input and output of a system, denoted by τ.
**State Variable:** A variable that describes the state of the system at a given time, denoted by y(t).
**Derivative:** A measure of the rate of change of the state variable with respect to time, denoted by dy/dt.
**Analytical Method:** A method for solving DDEs using mathematical techniques, such as Laplace transforms and Fourier analysis.

**Numerical Method:** A method for solving DDEs using numerical algorithms, such as Euler's method and Runge-Kutta methods.

**CHECK YOUR PROGRESS**

1. What is the primary objective of delay differential equation?
   **(a)** Non linearity
   **(b)** Time varying coefficient
   **(c)** Deley response
   **(d)** Stochasticity
2. Which of the following is common application of delay differential equation?
   **(a)** Image processing
   **(b)** Population Dynamics
   **(c)** Signal processing
   **(d)** Machine learning
3. What is the purpose of the delay term in delay differential equation?
   **(a)** To introduce nonlinearity
   **(b)** To model time-varying coefficients
   **(c)** To capture delayed responses
   **(d)** To add stochasticity
4. Which numerical method is commonly used to solve DDEs?
   **(a)** Euler's method
   **(b)** Runge-Kutta method
   **(c)** Finite difference method
   **(d)** All of the above

## 10.7 REFRENCES

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## 10.9 TERMINA*L QUESTIONS*

1. What is the difference between a Delay Differential Equation (DDE) and an Ordinary Differential Equation (ODE)?
2. How do DDEs model real-world systems that exhibit delayed responses?
3. What are the advantages and disadvantages of using numerical methods to solve DDEs?
4. Discuss the role of delay differential equations in modeling population dynamics. Provide examples of how DDEs can be used to study the spread of diseases or the growth of populations.
5. A population of rabbits is growing in a forest, with a delay of 2 months between the birth of new rabbits and their ability to reproduce. Model this system using a DDE and discuss the implications of the delay on the population dynamics.

## 10.10 ANSWERS

**CYQ1:** (c)
**CYQ2**: (b)
**CYQ3:** (c )
**CYQ:4** (d)

# UNIT 11: MODELING WITH STOCHASTIC DIFFERENTIAL EQUATIONS

## CONTENTS:

## *11.1 INTRODUCTION*

In previous unit we have defined Modeling with delay Differential Equations, Breathing Model, Two Neuron System, House fly Model etc. In this unit we have discussed Modeling with Stochastic Differential Equation. Stochastic Differential Equations (SDEs) are mathematical equations that describe the behavior of systems that are subject to random fluctuations or uncertainty. They are used to model systems that exhibit randomness or noise, such as stock prices, population dynamics, or chemical reactions. These SDEs are used to model systems with continuous-time stochastic processes and used to model systems with continuous-time stochastic processes, but with a different interpretation of the stochastic integral.

## *11.2 OBJECTIVES*

After studying this unit, learner will be able to

1. To describe Stochastic Differential Equation.
2. Analyze the dynamic behaviour of systems over time, taking into account the interactions between the system's components and the random fluctuations with the help of SDE's.

## 11.3 INTRODUCTION OF STOCHASTIC DIFFERENTIAL EQUATION

Before starting this unit we discussed some terminology and definition

### 11.3.1 RANDOM EXPERIMENT

Whenever we perform an experiment under nearly identical conditions, we expect to obtain results that are essentially the same. However, there are experiments in which the results will not be essentially the same, even though the conditions may be nearly identical. For example, if we throw two coins simultaneously, the results are TT, TH, HT or HH. We form the set of all possible outcomes as $\{TT, HT, TH, HH\}$. Each time we perform this experiment, the outcome is uncertain, although it will be one of the elements of the set $\{TT, HT, TH, HH\}$. Such an experiment is called a random experiment, where the result depends on chance.

## 11.3.2 OUTCOME

The results of the random experiment are known as the outcome. For example, in the random experiment of throwing two coins simultaneously, there are four possible outcomes, namely, TT, TH, HT or HH.

## 11.3.3 EVENT

Any phenomenon that occurs in a random experiment is called an event. An event can be elementary or composite. An elementary event corresponds to a single possible outcome, whereas a composite event corresponds to more than a single possible outcome. For example, When a dice is thrown, the event "multiple of 2 " is composite because it can be decomposed into elementary events 2,4,6.

## 11.3.4 SAMPLE SPACE

A sample space is a collection of all possible outcomes of a random experiment. In the random experiment of throwing two coins simultaneously, the sample space $\mathrm{S} = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$.

## 11.3.5 EVENT SPACE

An event space $(\Sigma)$ contains all possible events for a given random experiment. Sometimes, event space is confused with sample space. Consider "Toss" of a coin. Two possible outcomes are either head or tail; hence the sample space is $\mathrm{S} = \{\mathrm{H}, \mathrm{T}\}$. However, event space is a little different. In a 'toss' of a coin, the possible events are

(i) $\{H\} \rightarrow$ flipping the coin and getting head

(ii) $\{T\} \rightarrow$ flipping the coin and getting tail

(iii) $\{H, T\} \rightarrow$ flipping the coin and getting either head or tail.

Then, event space $\Sigma = \{\{H\}, \{T\}, \{H, T\}\}$.

## 11.3.6 AXIOMATIC DEFINITION OF PROBABILITY

Let $E$ be a random experiment described by the event space $S$ and $A$ be any event connected with $E$. Then the probability of event $A$, denoted by $P(A)$, is a real number that satisfies the following axioms

(Kolmogorov’s axioms):

(a) $\mathrm{P(A)} \geq 0$

(b) $\mathrm{P(S)} = 1$ (probability of a certain event is 1)
(c) If $A_1, A_2, \ldots.$ be a finite or infinite sequence of pairwise mutually exclusive events (that is, $A_i \cap A_j = \emptyset, i \neq j, i, j = 1,2, \ldots$ ), then, $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots \ldots \ldots$

## 11.3.7 PROBABILITY FUNCTION

A probability function is a mapping, $P: \Sigma \to [0,1]$, that assigns probabilities to the events in Σ and satisfies Kolmogorov's axioms.

## 11.3.8 PROBABILITY SPACE

A three-tuple $(S, \Sigma, P)$ whose components are sample space $(S)$, event space (Σ) and probability function $(P)$ is called a probability space.

## 11.3.9 RANDOM VARIABLE

Let $S$ be a sample space of the random experiment $E$. A random variable (or a variate) is a function or mapping $X: S \to \mathbf{R}$ (set of real numbers), which assigns to each element $\omega \in S$, one and only one number $X(\omega) = a$. The set of all values which $X$ takes, that is, the range of the function $X$, is called the spectrum of the random variable $X$, which is a set of real numbers $B = \{a: a = X(\omega), \omega \in S\}$. A random variable is called a discrete random variable if it takes on a finite or countably infinite number of values and it is called continuous random variable if it takes on a noncount ably infinite number of values.
Consider two "tosses" of an unbiased coin. Then, the sample space is

$$S = \{\omega_1 = (HH), \omega_2 = (HT), \omega_3 = (TH), \omega_4 = (TT)\}$$

We now define a function or a mapping $X: S \to \mathbf{R}$ such that $X(\omega_i) = \lambda_i$, where $\lambda_i$ is the number of heads, $i = 1,2,3,4, \ldots.$ Then, $X(\omega_1) = 2, X(\omega_2) = 1, X(\omega_3) = 1, X(\omega_4) = 0$. Here, $X$ is a random variable defined in the domain $S$. The spectrum of $X$ is $\{0,1,2\}$. Since, the spectrum is countable, $X$ is a discrete random variable. Consider the height of a group of high school students which lies between 155 cm and 185 cm. Here, $S = \{\omega: 155 < \omega < 185\}$. Since, $X$ takes any positive real values between 155 and $185, X$ is a continuous random variable.

## 11.3.10 SIGMA MEASURE

A collection $\Sigma$ of subsets of S is called $\sigma$-algebra if
**(i)** $\phi \in \Sigma$,
**(ii)** $A \in \Sigma \Rightarrow A^c \in \Sigma$
**(iii)** If $A_1, A_2, A_3, \ldots$ is a countable collection of subsets in $\Sigma$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.
$\sigma$ - algebra is closed under countable union and countable intersection. Then 2 -tuple $(S, \Sigma)$ is called a measurable space

## 11.3.11 MEASURE

Let $(S, \Sigma)$ be a measurable space. A measure on $(S, \Sigma)$ is a function $\mu: \Sigma \to [0, \infty)$ such that
**(i)** $\mu(\phi) = 0$
**(ii)** If $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\Sigma$, then the measure of the union (of countably infinite disjoint sets) is equal to the sum of measures of individual sets, that is, $\mu(\bigcup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty}\mu(A_i)$. The triplet $(S, \Sigma, \mu)$ is called a measure space. $\mu$ is said to be a finite measure if $\mu(S) < \infty$, otherwise $\mu$ is called infinite.

## 11.3.12 PROBABILITY MEASURE

A probability measure is a function $P: \Sigma \to [0,1]$ such that
(i) $P(\phi) = 0$
(ii) $P(S) = 1$
(iii) if $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\Sigma$, then $P(\bigcup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty}P(A_i)$. The triplet $(S, \Sigma, P)$ is called a probability space.

## 11.3.13 MEAN AND VARIANCE

The mean of the random variable X is defined as

$$\mu = E(X) = \sum_{-\infty}^{\infty} x_i f_i, \text{ for a discrete distribution}$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f_X(x) dx, \text{ for a continuous distribution.}$$

The mean gives a rough position of the bulk of the distribution, and hence called the measure of location.

The variance of the random variable $X$ is defined as

$$\sigma^2 = \text{Var}(X) = E[(X-\mu)^2] = E[X^2 + \mu^2 - 2\mu X]$$
$$= E(X^2) - [E(X)]^2.$$

The variance is a characteristic which describes how widely the probability masses are spread about the mean.

## 11.3.14 INDEPENDENT RANDOM VARIABLES

The cumulative distribution function (CDF) of a random variable $X$ is defined as

$$F_X(x) = P(-\infty < X \le x) \text{ for all } x \in \mathbf{R}$$

The joint distribution function of two random variables $X$ and $Y$ is defined by

$$F_{XY}(x,y) = P(-\infty < X \le x, -\infty < Y \le y), \text{ for all } x,y \in \mathbf{R}$$

where the event $(-\infty < X \le x, -\infty < Y \le y)$ means the joint occurrence of the two events $-\infty < X \le x$ and $-\infty < Y \le y$. If the events $(-\infty < X \le x)$ and $(-\infty < Y \le y)$ are the independent for all $x, y$, then

$$P(-\infty < X \le x, -\infty < Y \le y) = P(-\infty < X \le x)P(-\infty < Y \le y)$$
$$\Rightarrow F_{XY}(x,y) = F_X(x)F_Y(y)$$

Thus, the necessary and sufficient condition for the independence of the random variable $X$ and $Y$ is that their joint distribution function $F_{XY}(x,y)$ can be written as the product of the marginal distribution functions.

## 11.3.15 GAUSSIAN DISTRIBUTION (NORMAL DISTRIBUTION)

A random variable X is said to be normally distributed with mean $\mu$ and variance $\sigma^2$, if its probability density function (pdf) is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$$

Usually, the Gaussian or Normal distribution of $X$ is represented by

$$X \sim N(\mu, \sigma^2)$$

The probability density function (pdf) of the Gaussian or Normal distribution is a bell-shaped curve, symmetric about the mean $\mu$ (attains its maximum value $\frac{1}{\sqrt{2\pi}\sigma}$ there) and is completely characterized by the two parameters, namely, $\mu$ (mean) and $\sigma^2$ (variance). The mean $\mu$ is the centroid of the pdf, and in this case, it is also the point at which the pdf is maximum. The variance $\sigma^2$ gives the measure of the dispersion of the random variable around the mean. If $\mu = 0, \sigma = 1$, then the random variable $X$ is said to follow Standard Normal Distribution.

## 11.3.16 CHARACTERSTIC FUNCTION

The characteristic function of the random variable $X$ is a complex-valued function of a real variable $t$ and is defined as

$$\phi_X(t) = E\left(e^{itx}\right) = E[\cos(tX) + i\sin(tX)]$$

where $i = \sqrt{-1}$ and $t$ is a real number. The characteristic function satisfies the following properties:

**(i)** $\phi_X(0) = 1$ and $|\phi_X(t)| \leq 1, \forall t \in \mathbf{R}$.

**(ii)** $If\ Y = aX + b$, then $\phi_Y(t) = e^{ibt}\phi_X(at)$.

**(iii)** If $X$ and $Y$ are independent random variables and $Z = X + Y$, then, $\phi_Z(t) = \phi_X(t)\phi_Y(t)$.

## 11.3.17 CHARACTERSTIC FUNCTION OF GAUSSIAN DISTRIBUTION

Let $X \sim N(\mu, \sigma^2)$, then probability density function of $X$ is

$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$. Now,

$$\phi_X(t) = E(e^{itX}) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{itx} f_X(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}+itx}dx$$

$$= e^{i\mu t-\frac{\sigma^2 t^2}{2}} \times \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu-it)^2}{2\sigma^2}}dx = e^{i\mu t-\frac{\sigma^2 t^2}{2}} \times 1$$

$$\Rightarrow \phi_X(t) = e^{i\mu t-\frac{\sigma^2 t^2}{2}}$$

## 11.3.18 INVERSION THEORUM

Let $X$ be a continuous random variable, having probability density function $f_X(x)$, then the corresponding characteristic function is given by

$$\psi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx$$

Also, the probability density function $f_X(x)$ can be obtained from the characteristic function as

$$f_X(x) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{+T} e^{-itx}\psi_X(t)dt$$

at every point where $f_X(x)$ is differentiable. Now,

$$f_X(x) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T} e^{-itx} e^{i\mu t-\frac{\sigma^2 t^2}{2}}dt$$

$$= \frac{1}{2\pi} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \lim_{T\to\infty}\int_{-T}^{+T} e^{-\frac{\sigma^2}{2}\left(t+i\frac{x-\mu}{\sigma^2}\right)^2}dt = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

## 11.3.19 CONVERGENCE OF RANDOM VARIABLES AND LIMIT THEOREMS

Convergence in Probability: A sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ converges to a random variable $X$ in probability, denoted by $X_n \xrightarrow{p} X$, if for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\{|X_n - X| < \epsilon\} = 1 \text{ or } \lim_{n\to\infty} P\{|X_n - X| \geq \epsilon\} = 0.$$

**Almost Sure Convergence**: Consider a sequence of random variables $X_1, X_2, \dots \dots, X_n, \dots$, all defined on the same sample space $S$. For every $\omega \in S$, we obtain sample sequence $X_1(\omega), X_2(\omega), \dots, X_n(\omega), \dots.$ A sequence of random variables $X_1, X_2, \dots., X_n, \dots.$ converges to a random variable $X$ almost surely (also known as with probability 1), denoted by $X_n \overset{\text{a.s.}}{\to} X$, if

$$P\left\{\omega: \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = 1$$

or equivalently, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} P\{|X_n - X| < \varepsilon \text{ for every } n \geq m\} = 1$$

**Convergence in Mean Square:** A sequence of random variables $X_1, X_2, \dots., X_n, \dots.$ converges to a random variable $X$ in mean square (m.s.), denoted by $X_n \overset{\text{m.s.}}{\to} X$, if

$$\lim_{n\to\infty} E[(X_n - X)^2] = 0$$

Converge in Distribution: A sequence of random variables $X_1, X_2, \dots.., X_n, \dots.$ converges in distribution to a random variable $X$, denoted by $X_n \overset{\text{d}}{\to} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for every $x$ at which $F_X(x)$ is continuous.

**Notes:**

**(i)** Convergence in probability implies convergence in distribution but the converse is not true. However, convergence in distribution implies convergence in probability when the limiting random variable $X$ is a constant.
**(ii)** Almost surely convergence implies convergence in probability and hence implies convergence in distribution **(fig. 11.1).**
**(iii)** Convergence in mean square implies convergence in probability and hence implies convergence in distribution **(fig. 11.1).**
**(iv)** Convergence in probability does not necessarily implies almost surely convergence or mean square convergence. Convergence in probability is weaker than both almost surely convergence and convergence in mean square.
**(v)** Almost sure convergence does not imply convergence in mean square. Also, convergence in mean square does not imply almost sure

convergence.

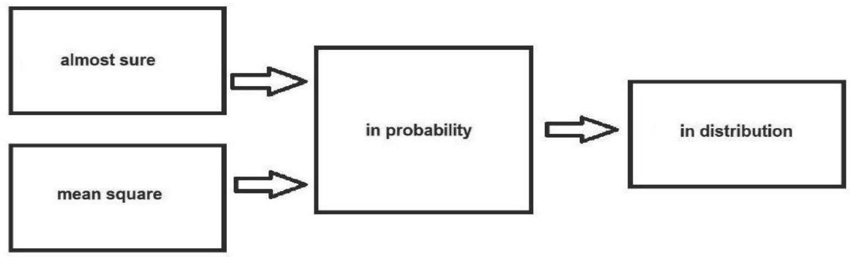



**FIGURE 11.1:** The figure shows the relationship between different types of convergences.

**Weak Law of Large Numbers (WLLN):** Let $X_1, X_2, \ldots\ldots, X_n, \ldots.$ be independent and identically distributed random variables (each r.v. has the same probability distribution as the others and all are mutually independent) with finite variance. Then, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\bar{X} - \mu| \geq \varepsilon) = 0, \text{ where } \bar{X} = \frac{1}{n}\sum_{1}^{n} X_i$$

Thus, the weak law of large numbers says that the probability of the difference between the sample mean and the true mean by a fixed number $\epsilon(> 0)$ become smaller and smaller, and converges to zweo as n goes to infinity.

**Central Limit Theorem:** Let $X_1, X_2, \ldots., X_n, \ldots$ be independent and identically distributed random variables with $E(X_i) = \mu$ (finite) and $\text{Var}(X_i) = \sigma^2$ (finite). Then, the random variable

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{(X_1 + X_2 + \cdots + X_n) - n\mu}{\sigma\sqrt{n}}$$

converges in distribution to the standard normal variable as $n$ goes to infinity, that is,

$$\lim_{n\to\infty} P(Z_n \leq x) = \Phi(x), \text{ for all } x \in \mathbb{R}$$

where $\Phi(x)$ is the standard normal distribution.

Statistical methods like testing of hypothesis and construction of confidence intervals are used in data analysis and these methods assume that the population is normally distributed. According to central limit theorem, we can assume the sampling distribution of an unknown or non-normal distribution as normal. This is the significance of central limit theorem.

## 11.3.20 STOCHASTIC PROCESS

A stochastic process is a collection of random variables $\{X_t, t \in T\}$, defined on some probability space $(S, \Sigma, P)$. We call the values of $X_t$ as state space denoted by $\Omega$. The index set $T$ from where $t$ takes its value is called a parameter set or a time set. A stochastic process may be discrete or continuous according to whether the index set $T$ is discrete or continuous. The range (possible values) of the random variables in a stochastic process is called the state space of the process.

**Example 11.3.1** $A$ stochastic process $\{X_n: n = 0,1,2,3, \cdots\}$ with discrete index set $\{0,1,2,3, \cdots\}$ is a discrete time stochastic process.

**Example 11.3.2** $A$ stochastic process $\{X_t: t \geq 0\}$ with continuous index set $\{t: t \geq 0\}$ is a continuous time stochastic process.

**Example 11.4.3** $\{X_n: n = 0,1,2,3, \cdots\}$, where the state space of $X_n$ is $\{1,2\}$, which represents whether an electronic component is acceptable or defective, and time $n$ corresponds to the number of components produced.

**Example 11.5.4** $\{X_t: t \geq 0\}$, where the state space of $X_t$ is $\{0,1,2, \cdots\}$, which represents the number of cars parked in the parking 1 to $t$ in front of a movie theater and $t$ corresponds to hours.

A filtration $\{\Sigma_t\}_{t\geq 0}$ is a family of sub-sigma algebras of some sigma-algebra $\Sigma$ with the property that $s < t$, then $\Sigma_s \subset \Sigma_t$. Thus, a stochastic process $\{X_t: 0 \leq t < \infty\}$ is adapted to $\{\Sigma_t\}_{t\geq 0}$ means that for any $t, X_t$ is $\Sigma_t$ is measurable.

## 11.3.21 MARKOV PROCESS

A Markov process is a stochastic process with the following properties:
**(i)** It has a finite number of possible outcomes or states.

**(ii)** The outcome at any stage depends only on the outcome of the previous stage.
**(iii)** Over time, the probabilities are constant.
Mathematically, a Markov Process is a sequence of random variables $X_1, X_2, X_3, \ldots$ such that

$$P(X_n = x_n / X_{n-1} = x_{n-1}, \ldots, X_1 = x_1) = P(X_n = x_n / X_{n-1} = x_{n-1})$$

Let the initial state of the system be denoted by $x_0$. Then, there is a matrix A, which gives the state of the system after one iteration by the vector $Ax_0$. Thus, we obtain a chain of state vectors, namely, $x_0, Ax_0, A^2x_0, \ldots$, where the state of the system after $n$ iterations is given by $A^n x_0$. Such a chain is called a Markov Chain and the matrix A is called a transition matrix.

**Example 11.1.5** A metro ride in a city was studied. After analyzing several years of data, it was found that 25% of the people who regularly ride on the metro in a given year do not prefer the metro rides in the next year. It was also observed 31% of the people who did not ride on the metro regularly in that year began to ride the metro regularly the next year.

In a given year, 8000 people ride the metro and 9000 do not ride the metro. Of the persons who currently ride the metro, 75% of them will continue to do so and of the persons who do not ride the metro, 31% will start doing so.

In order to find the distribution of metro riders/metro non-riders in the next year, we first obtain the number of people who will ride the metro next year. Therefore, the number of persons who will ride the metro next year $= b_1 = 0.75 \times 8000 + 0.31 \times 9000 = 8790$. Similarly, the number of persons who will not ride the metro next year $= b_2 = 0.25 \times 8000 + 0.69 \times 9000 = 8210$.

This can be expressed in matrix notation as $Ax = b$ where

$$A = \begin{bmatrix} 0.75 & 0.31 \\ 0.25 & 0.69 \end{bmatrix}, x = \begin{bmatrix} 8000 \\ 9000 \end{bmatrix}, \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8790 \\ 8210 \end{bmatrix}.$$

After two years, we use the same matrix $A$, but $x$ is replaced by $b$ and the
distribution becomes $Ab = A^2x$. Thus,

$$A^2x = \begin{bmatrix} 0.75 & 0.31 \\ 0.25 & 0.69 \end{bmatrix}^2 \begin{bmatrix} 8000 \\ 9000 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.4464 \\ 0.36 & 0.5536 \end{bmatrix} \begin{bmatrix} 8000 \\ 9000 \end{bmatrix}$$

$$= \begin{bmatrix} 9137.6 \\ 7862.4 \end{bmatrix}$$

After 2 years, the number of persons who will ride the metro is 9138 and the number of persons who will not ride the metro is 7862. In general, the distribution is $A^n x$ after $n$ years.

## 11.3.22 GAUSSIAN PROCESS

A Gaussian Process is a stochastic process such that the joint distribution of every finite subset of random variables is a multivariate normal distribution. Thus, a random process $\{X(t), t \in T\}$ is said to be a Gaussian (Normal) process if, for all $t_1, t_2, \ldots, t_n \in T$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normal.

## 11.3.23 WIENER PROCESS (BROWNIAN MOTION)

A Wiener process or a Brownian motion is a zero-mean continuous process with independent Gaussian increments (by independent increments we mean a process $X_t$, where for every sequence $t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent).

Mathematically, we can say that a one-dimensional standard Wiener process or Brownian motion $B(t): R_+ \longrightarrow R$ is a real-valued stochastic process on some probability space $(S, \Sigma, P)$ adapted to $\{\Sigma_t\}$ with the following properties:

**(i)** B (0) =0 (with probability 1).
**(ii)** B(t) is continuous for all t (with probability 1).
**(iii)** B(t) has independent increment.
**(iv)** $B(t) - B(s)$ has a Gaussian or Normal distribution with mean zero and variance $t - s$ for every $t > s \geq 0$. The density function of the random variable is given by

$$f(x; t, s) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}.$$

The consequences of this definition are

**(i)** The process {W(t)} is Gaussian.

**(ii)** $E[W(t)] = 0$ and $E[W(s)W(t)] = \min(s, t)$, for all $s, t \geq 0$ and in particular, $E[W^2(s)] = s$.

# 11.4 STOCHASTIC MODELS

In stochastic modeling, we take into account a certain degree of randomness or unpredictability. The million-dollar question is when to use deterministic models and when we really need to use stochastic ones. People argue that stochasticity put realism in models, and hence it should be added to make the model more realistic. However, I prefer that a stochastic model should be built when it is absolutely necessary and then stochasticity should be put in those parts of the model that are absolutely necessary to be stochastic, and then control the rest to improve the understanding of the model.

## 11.4.1 STOCHASTIC LOGISTIC GROWTH

The famous logistic growth model for a single species is given by (in the deterministic case)

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), x(0) = x_0 \qquad (\mathbf{11.4})$$

where $r$ is the intrinsic growth rate and $k$ is the carrying capacity. Clearly, $x = 0$ and $x = k$ are the two points of equilibria. It can be easily shown that the solution of (11.4) is

$$x(t) = \frac{e^{rt}x_0}{(e^{rt} - 1)\frac{x_0}{k} + 1}.$$

Suppose the logistic growth model for a single species is now subjected to the environment stochasticity or randomness $\eta(t)$, which is a Gaussian white noise with a time-varying intensity $\sigma^2(t)$. Then $\eta(t)dt = \sigma dW$, where $W(t)$ is a Wiener process and $\sigma$ is the intensity of the noise. The stochastic version of the model is given by

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{k}\right)dt + \sigma dW$$

It can be shown that the logistic model is stochastically stable if $\sigma^2 < \frac{2r}{k}$, $t \geq 0$ . Fig. 11.2 numerically confirms the result and shows that the equilibrium point $x^* = k$ is stochastically stable.

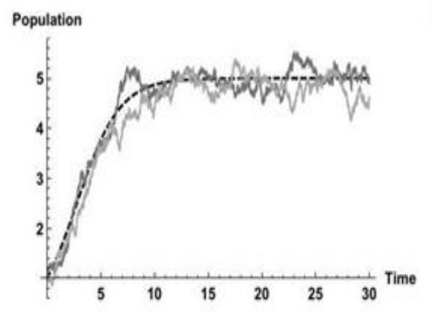
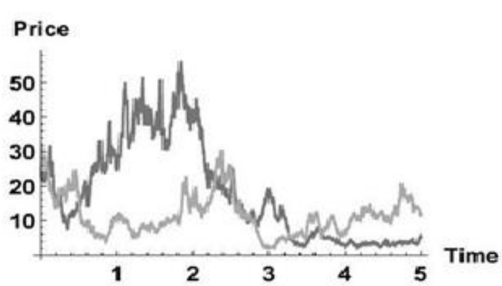


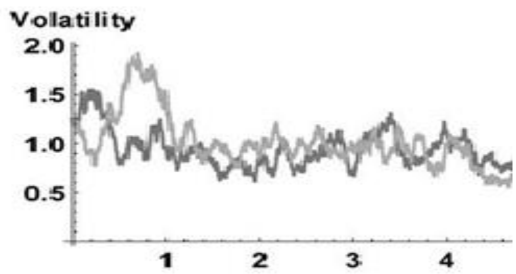

**FIGURE 11.2:** Effect of stochasticity on the logistic growth model. For $r = 0.5, k = 5.0, \sigma = 0.3$ and initial condition $x(0) = 1.0$, the steady-state solution $x^* = 5$ is stochastically stable.

## 11.4.2 TWO SPECIES STOCHASTIC MODEL

Lotka and Volterra proposed a two species competition model, which was later studied by gauss empirically. The proposed model is give below




(a) Price of the asset




(b) Volality of the asset

**FIGURE 11.3:** Heston model showing the price of the asset and its corresponding volatility. The parameter values are $\mu = 0.2, k = 2, \theta = 1, \sigma = 0.5, \rho = -0.35$ with initial condition: (25,1.25).

$$\frac{dN_1(t)}{dt} = r_1 N_1(t)\left(\frac{K_1 - N_1(t) - \alpha_{12}N_2(t)}{K_1}\right),$$
$$\frac{dN_2(t)}{dt} = r_2 N_2(t)\left(\frac{K_2 - N_2(t) - \alpha_{21}N_1(t)}{K_2}\right),$$

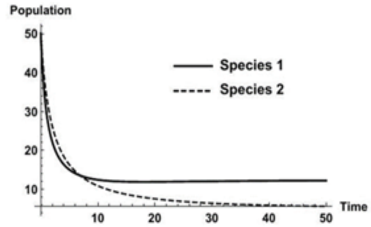
where $N_1(t)$ and $N_2(t)$ are densities of species 1 and species 2, respectively, at any time $t$. The meaning and interpretation of the positive parameters $r_1, K_1, \beta_{12}, r_2, K_2, \beta_{21}$ are left for the readers.

The model is now subjected to external noises and we obtain the stochastic two species competition model as

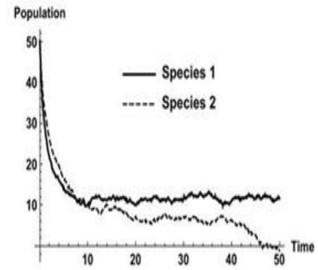
$$dN_1(t) = r_1 N_1(t)\left(\frac{K_1 - N_1(t) - \alpha_{12}N_2(t)}{K_1}\right)dt + \sigma_1 dW_1(t)$$
$$dN_2(t) = r_2 N_2(t)\left(\frac{K_2 - N_2(t) - \alpha_{21}N_1(t)}{K_2}\right)dt + \sigma_2 dW_2(t)$$

with initial conditions $N_1(0) = N_2(0) = 50$, where $W_1(t), W_2(t)$ are two independent Wiener processes and $\sigma_1, \sigma_2$ are the intensities of the noise.

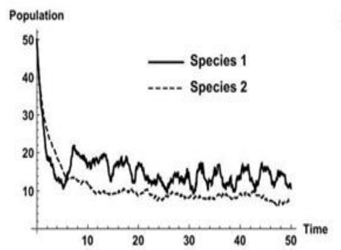
The model is solved numerically by taking $r_1 = 0.22, r_2 = 0.06, K_1 = 13, K_2 = 5.8, \alpha_{12} = 3.15, \alpha_{21} = 0.44$ and different values of the intensities $\sigma_1, \sigma_2$. Fig. 11.4(a) shows the deterministic model ( $\sigma_1 = 0, \sigma_2 = 0$ ), where, species 1 wins and species 2 dies off. With $\sigma_1 = 0.7, \sigma_2 = 0.7$, the stochastic model shows similar dynamics, that is, due to competition, species 2 goes to extinction **(fig. 11.4(b))**. However, it is observed that by manipulating the intensities of the noise, the dynamics of the model can be changed. Taking



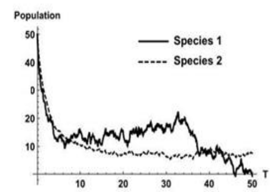

**(a)** $\sigma_1$ = 0 , $\sigma_2$ = 0




**(b)** $\sigma_1$ = 0.7 , $\sigma_2$ = 0.7




(***c*** ) $\sigma_1$ = 2.7, $\sigma_2$ = 0.7




**(d)** $\sigma_1$ = 3.7, $\sigma_2$ = 0.7

**FIGURE 11.4:** The figures show the dynamics of two species stochastic competing model with different values of the intensities of the noise. (a) Deterministic model: Species 1 wins, (b) stochastic model: Species 1 wins, (c) stochastic model: Coexistence of both the species, (d) stochastic model: Species 2 wins. $\sigma_1 = 2.7, \sigma_2 = 0.7$, we observe that neither species can contain the other and it is the case of stable coexistence (fig. 11.4(c)) of both the species. But, with $\sigma_1 = 3.7, \sigma_2 = 0.7$, species 2 dominates over species 1 and forces it to go to extinction **(fig. 11.4(d))**.

Therefore, from the behavior of the model, it is concluded that with proper intensities of the noise (which needs to be interpreted in terms of biology/ecology), different scenarios are obtained to give rich dynamics of the two species stochastic competition model.

**Note:** Please note that the stability analysis of the stochastic models discussed is actually research problems; hence they are left for the readers to research.

# 11.5 SUMMARY

**Stochastic Differential Equations (SDEs):** These equations are mathematical equations that describe the behavior of systems that are subject to random fluctuations or uncertainty.

**Randomness:** SDEs incorporate random fluctuations or noise, which are modeled using stochastic processes.
**Uncertainty:** SDEs account for uncertainty in the system's behavior, which can arise from various sources, such as measurement errors or inherent randomness.
**Dynamic behavior:** SDEs describe the dynamic behavior of systems over time, taking into account the interactions between the system's components and the random fluctuations.
**Analytical solutions:** These solutions involve finding closed-form expressions for the solution of the SDE.
**Numerical solutions:** These solutions involve approximating the solution of the SDE using numerical methods, such as Euler's method or Monte Carlo simulations.

## 11.6 GLOSSARY

**Stochastic Differential Equation (SDE):** A mathematical equation that describes the behaviour of a system that is subject to random fluctuations or uncertainty.
**Stochastic Process:** A mathematical representation of a system that exhibits randomness or uncertainty over time.
**Random Variable:** A variable that takes on random values, often denoted by ω.
**Probability Measure:** A mathematical function that assigns a probability to each possible outcome of a random experiment.
**Expectation:** A mathematical operation that calculates the average value of a random variable.

### CHECK YOUR PROGRESS

**1:** What is the primary characteristic of a Stochastic Differential Equation (SDE)?
**A)** Nonlinearity
**B)** Time-varying coefficients
**C)** Randomness
**D)** Determinism

**2:** Which of the following is a common application of SDEs?
**A)** Image processing
**B)** Population dynamics
**C)** Signal processing
**D)** Machine learning

**3:** What is the purpose of the stochastic integral in an SDE?
**A)** To introduce nonlinearity
**B)** To model time-varying coefficients
**C)** To capture random fluctuations
**D)** To add determinism

**4:** What is the purpose of the drift term in an SDE?
**A)** To introduce randomness
**B)** To model time-varying coefficients
**C)** To capture the deterministic component of the system
**D)** To add nonlinearity

**5:** Which of the following is a challenge in solving SDEs?
**A)** Nonlinearity
**B)** Time-varying coefficients
**C)** Randomness
**D)** All of the above

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## 11.8 SUGGESTED READINGS

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**2.** W.E. Boyce (1981) Case Studies in Mathematical Modelling, Boston, Pitman.
**3.** A. Friedman and W. Littman, (1994) Industrial Mathematics: A Course in Solving Real World Problems, Philadelphia, SIAM.

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## 11.9 TERMINAL QUESTIONS

1. Describe the main differences between a Stochastic Differential Equation (SDE) and an Ordinary Differential Equation (ODE).
2. What is the purpose of the stochastic integral in an SDE?
3. How do SDE model system with time varying coefficients?
4. Discuss the application of SDE in finance, including the modelling of stock prices and interest rates.

## 11.10ANSWARS

CYQ1. (C)

CYQ2: (B)

CYQ3. (C)

CYQ1. (C)

CYQ1. (D)

# UNIT 12: MATHEMATICAL MODELLING THROUGH GRAPHS

## CONTENTS:

## 12.1 *INTRODUCTION*

A graph is another type of mathematical model. We can determine the sale price of an item by locating its original price along the x-axis and then finding the corresponding y-value, or sale price, on the graph.

## 12.2 *OBJECTIVES*

After studying this unit, learner will be able to

i. Qualitative Relations in Applied Mathematics
ii. Balance Of Signed Graphs

## 12.3 *SITUATIONS THAT CAN BE MODELLED THROUGH GRAPHS*

### 12.3.1 QUALITATIVE RELATIONS IN APPLIED MATHEMATICS

It has been stated that "Applied Mathematics is nothing but solution of differential equations". This statement is wrong on many counts (*i*) Applied Mathematics also deals with solutions of difference, differential-difference, integral, integro-differential, functional and algebraic equations (*ii*) Applied Mathematics is equally concerned with inequations of all types (*iii*) Applied Mathematics is also concerned with mathematical modelling; mathematical modelling has to precede solution of equations (*iv*) Applied Mathematics also deals with situations which cannot be modelled in terms of equations or inequations; one such set of situations is concerned with qualitative relations. Mathematics deals with both quantitative and qualitative relationships.

Typical qualitative relations are: $y$ likes $x$, $y$ hates $x$, $y$ is superior to $x$, $y$ is subordinate to $x$, $y$ belongs to the same political party as $x$, set $y$ has a non-null intersection with set $x$; point $y$ is joined to point $x$ by a road, state $y$ can be transformed into state $x$, team $y$ has defeated team $x$, $y$ is the father of $x$, course $y$ is a prerequisite for course $x$, operation $y$ has to be done before operation $x$, species $y$ eats species $x$, $y$ and $x$ are connected by an airline, $y$ has a healthy influence on $x$, any increase of $y$ leads to a decrease in $x$, $y$ belongs to the same caste as $x$, $y$ and $x$ have different nationalities and so on. Such relationships are very conveniently represented by graphs where a graph consists of a set of vertices and edges joining some or all pairs of these vertices. To motivate the typical problem situations which can be modelled through graphs, we consider the first problem so historically modelled viz. the problem of the seven bridges of Königsberg.

### 12.3.2 THE SEVEN BRIDGES PROBLEM

There are four land masses *A*, *B*, *C*, *D* which are connected by seven bridges numbered 1 to 7 across a river (Figure 12.1). The problem is to start from any point in one of the land masses, cover each of the seven bridges once and once only and return to the starting point.

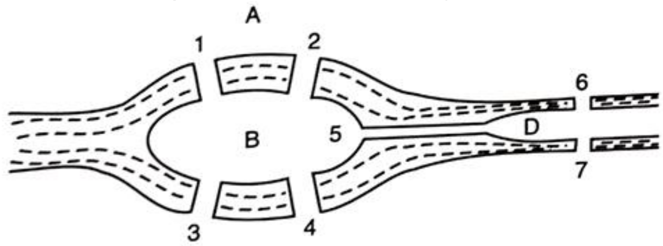


*Figure 12.1*

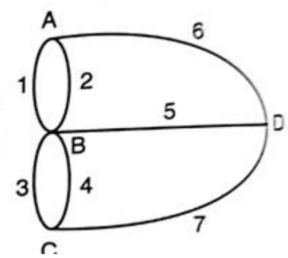


*Figure 12.2*

There are two ways of attacking this problem. One method is to try to solve the problem by walking over the bridges. Hundreds of people tried to do so in their evening walks and failed to find a path satisfying the conditions of the problem. A second method is to draw a scale map of the bridges on paper and try to find a path by using a pencil.

It is at this stage that concepts of mathematical modelling are useful. It is obvious that the sizes of the land masses are unimportant, the lengths of the bridges or even whether these are straight or curved are irrelevant. What is relevant information is that and *B* are connected by two bridges 1 and 2. *B* and *C* are connected by two bridges 3 and 4, and *D* are connected by one bridge number 5, 4 and *D* are connected by bridge number 6 and *C* and *D* are connected by bridge number 7. All these facts are represented by the graph with four vertices and seven edges in Figure 12.2. If we trace this graph in such a way that we start with any vertex and return to the same vertices and trace every edge once and once only without lifting the pencil from the paper, the problem can be solved. Again, the trial and error method cannot be satisfactorily used to show that no solution is possible. The number of edges meeting at a vertex is called the degree of that vertex. We note that the degrees of *A*, *B*, *C*, *D* are 3, 5, 3, and 3 respectively and each of these is an odd number. If we have to start from a vertex and return to it, we need in even number of edges at that vertex. Thus, it is easily seen that Konigsberg bridges problem cannot be solved.

This example also illustrates the power of mathematical modelling. We have not only disposed of the seven-bridges problem, but we have discovered a technique for solving many problems of the same type.

## 12.3.3 SOME TYPES OF GRAPHS

A graph is called *complete* if every pair of its vertices is joined by an edge (Figure 12.3(a)).

A graph is called a *directed graph* or a *digraph* if every edge is directed with an arrow. The edge joining A and B may be directed from *A* to *B* or from *B* to *A*. If an edge is left undirected in a digraph, it will be assumed to be directed both ways (Figure 12.3(b)).

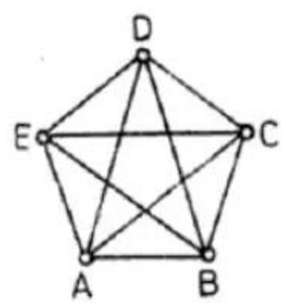

*Figure 12.3a*

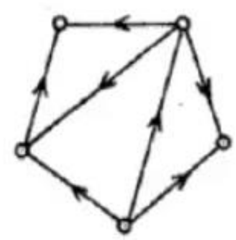
*Figure 12.3b*

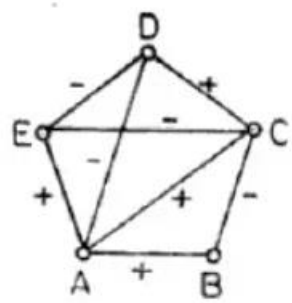

*Figure 12.3c*

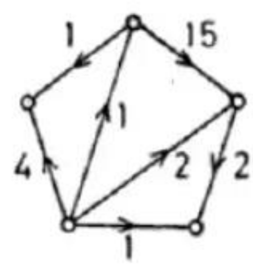

*Figure 12.3d*

A graph is called a *signed graph* if every edge has either a plus or minus sign associated with it (Figure 12.3(c)).

A digraph is called a *weighted digraph* if every directed edge has a weight (giving the importance of the edges) associated with it (Figure 12.3(d)). We may also have digraphs with positive and negative numbers associated with signs. These will be called *weighted signed digraphs*.

### 12.3.4 NATURE OF MODELS IN TERMS OF GRAPHS

In all the applications we shall consider, the length of the edge joining two vertices will not be relevant. It will not also be relevant whether the edge is straight or curved. The relevant facts would be: (a) which edges are joined; (b) which edges are directed and in which direction(s); (c) which edges have positive or negative signs associated with them; (d) which edges have weights associated with them and what these weights are.

## 12.4 *MATHEMATICAL MODELS IN TERMS OF DIRECTED GRAPHS*

### 12.4.1 REPRESENTING RESULTS OF TOURNAMENTS

The graph (Figure 12.4) shows that:

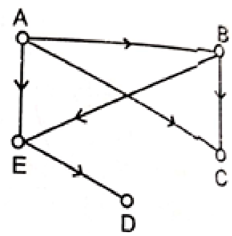


*Figure 12.4*

(i) Team *A* has defeated teams *B*, *C*, and *E*.
(if) Team *B* has defeated teams *C* and *E*.
(iii) Team *E* has defeated *D*.
(iv) Matches between *A* and *D*, *B* and *D*, *C* and *D*, and *C* and *E* have yet to be played.

### 12.4.2 ONE-WAY TRAFFIC PROBLEMS

The road map of a city can be represented by a directed graph. If only one-way traffic is allowed from point *a* to point *b*, we draw an edge directed from *a* to *b*. If traffic is allowed both ways, we can either draw two edges, one directed from *a* to *b* and the other directed from *b* to *a* or simply draw an undirected edge between *a* and *b*. The problem is to find whether we can introduce one-way traffic on some or all of the roads without preventing persons from going from any point of the city to any other point. In other words, we have to find when the edges of a graph can be given direction in such a way that there is a directed path from any vertex to every other, It is easily seen that one-way traffic on the road *DE* cannot be introduced without disconnecting the vertices of the graph (Figure 12.5).

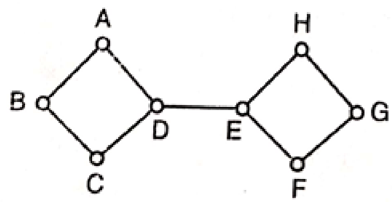

*Figure 12. 5(a)*

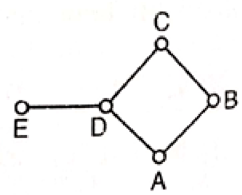

*Figure 12.5(b)*

In Figure 12.5(a), *DE* can be regarded as a bridge connecting two regions of the town. In Figure 12.5(b) *DE* can be regarded as a blind street on which a two-way traffic is necessary. Edges like *DE* are called separating edges, while other edges are called circuit edges. It is necessary that on separating edges, two-way traffic can also be permitted. It can also be shown that this is sufficient. In other words, the following theorem can be established:

If *G* is an undirected connected graph, then one can always direct the circuit edges of *G* and leave the separating edges undirected (or both way directed) so that there is a directed path from any given vertex to any other vertex.

### 12.4.3 GENETIC GRAPHS

In a genetic graph, we draw a directed edge from *A* to *B* to indicate that *B* is the child of *A*. In general, each vertex will have two incoming edges, one from the vertex representing the father and the other from the vertex representing the mother. If the father or mother is unknown, there may be less than two incoming edges. Thus, in a genetic graph, the local degree of incoming edges at each *A*, vertex must be less than or equal to two. This is a necessary condition for a directed graph to be a genetic graph, but it is not a sufficient condition.

Thus Figure 12.6 does not give a genetic graph even though the number of incoming edges at each vertex does not exceed two. Suppose $A_1$ is male, then $A_2$ must be female, since $A_1$, and $A_2$ have a child $B_1$. Then $A_3$ must be male,
since $A_2$, and $A_3$ have a child $B_2$.

*Figure 12.6*

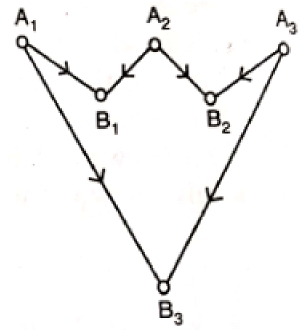


Figure 12.7

Now $A_1$, $A_3$ being both males cannot have a child $B_3$.

### 12.4.4 SENIOR SUBORDINATE RELATIONSHIPS

If a is senior to *b*, we write *aSb* and draw a directed edge from *a* to *b*. Thus, the organisational structure of a group may be represented by a graph like the following [Figure 12.7].

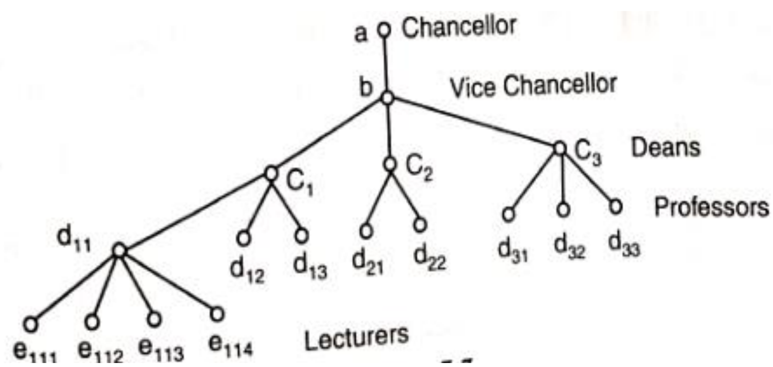


Figure 12.8

The relationship S satisfies the following properties:
(i) ~ *(aSa)* i.e. no one is his own senior.
(ii) *aSb* =~ (*bSa*) i.e. *a* is senior to *b* implies that *b* is not senior to *a.*
(iii) *aSb*, *bSc* = *aSc* i.e. if *a* is senior to *b* and *b* is senior to *c*, then *a* is senior to *c*.

The following theorem can easily be proved: "The necessary and the sufficient condition that the above three requirements hold is that the graph of an organization should be free of cycles".

We want now to develop a *measure for the status* of each person. The status *m(x)* of the individual should satisfy the following reasonable requirements:
(i) *m(x)* is always a whole number.

(ii) If *x* has no subordinate, *m(x)* = 0.
(iii) If, without otherwise changing the structure, we add a new individual subordinate to *x*, then *m(x)* increases.
(iv) If, without otherwise changing the structure, we move a subordinate of *a* to a lower level relative to *x*, then *m*(*x*) increases.
A measure satisfying all these criteria was proposed by Harary. We define the level of seniority of *x* over *y* as the length of the shortest path from *x* to *y*. To find the measure of status of *x*, we find $n_1$, the number of individuals who are one level below *x*, $n_2$, the number of individuals who are two levels below *x* and in general, we find $n_k$, the number of individuals who are *k* levels below *x*. Then the Harary measure *h(x)* is defined by

$$\mathrm{h(x)} = \sum_k kn_k \qquad (1)$$

It can be shown that among all the measures which satisfy the four requirements given above, the Harary measure is the least.

If however, we define the level of seniority of *x* over *y* as the length of the longest path from *x* to *y*, and then find H(x) = $\sum_k kn_k$ we get another measure which will be the largest among all measures satisfying the four requirements. For Figure 12.8, we get

*h*(*a*) = 1.2+4.2+2.3=16 *H*(*a*) = 1.1+3.2+2.3+2.4=21
*h*(*b*) = 1.3+2.4=11 *H*(*b*) = 2.1+2.2+2.3+1.4=16
*h*(*c*) = 1.2+1.2=4 *H*(*c*) = 1.1+1.2+1.3=6

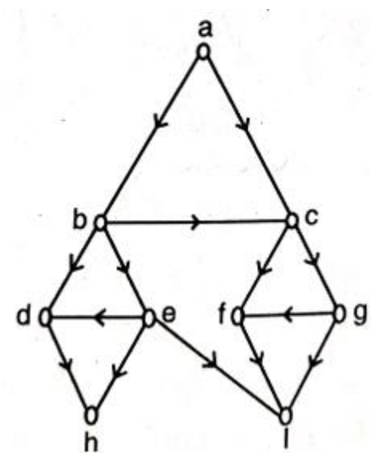

Figure 12.8

*h*(*d*) = 1.1 = 1 *H*(*d*) = 1.1
*h*(*e*) =1.3 = 3 *H*(*e*) = 1.2+2.1 = 4
*h*(*f*) =1.1 = 1 *H*(*f*) =1.1 = 1
*h*(*g*) =1.2 =2 *H*(*g*) = 1.2 = 2
*h*(*k*) = 0 *H*(*k*) = 0

$h(I) = 0$ $H(I) = 0$

### 12.4.5 FOOD WEBS

Here $aSb$ if $a$ eats $b$ and we draw a directed edge from $a$ to $b$. Here also ~($aSa$) and $aSb$ =~ ($bSa$). However, the transitive law need not hold. Thus consider the food web in Fig. 12.9. Here fox eats bird, bird eats grass, but fox does not eat grass.

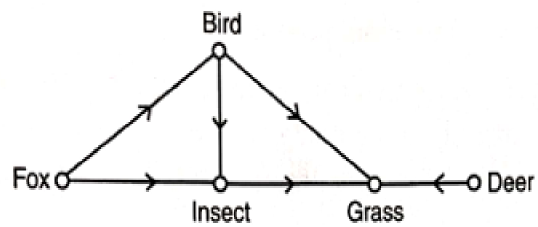


*Figure 12.9*

We can however calculate the measure of the status of each species in this food web by using Eqn. (1) $h$(bird)=2, $h$(fox)=4, $h$(insect)=1, $h$(grass)=0, $h$(deer) = 1.

### 12.4.6 COMMUNICATION NETWORKS

A directed graph can serve as a model for a communication network. Thus consider the network given in Figure 12.10. If an edge is directed from $a$ to $b$, it means that a can communicate with $b$. In the given network $e$ can communicate directly with $b$, but $b$ can communicate with $e$ only indirectly through $c$ and $d$. However, every individual can communicate with every other individual.

Our problem is to determine the importance of each individual in this network. The importance can be measured by the fraction of the messages on average that pass through him. In the absence of any other knowledge, we can assume that if an individual can send a message directly to $n$ individuals, he will send a message to any one of them with probability $1/n$. In the present example, the communication probability matrix is:

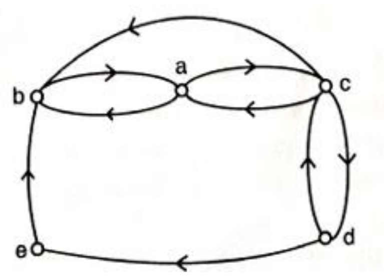


*Figure 12.10*

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \ldots\ldots\ldots\ldots (2)$$

No individual is to send a message to himself and so all diagonal elements are zero. Since all elements of the matrix are non-negative and the sum of elements of every row is unity, the matrix is a stochastic matrix and one of its eigenvalues is unity. The corresponding normalised eigenvector is [11/45, 13/45, 3/10, 1/10, 1/15]. In the long run, these fractions of messages will pass through *a*, *b*, *c*, *d*, *e* respectively. Thus, we can conclude that in this network, *c* is the most important person. If in a network, an individual cannot communicate with every other individual either directly or indirectly, the Markov chain is not ergodic and the process of finding the importance of each individual breaks down.

## 12.4.7 MATRICES ASSOCIATED WITH A DIRECTED GRAPH

For a directed graph with $n$ vertices, we define the $n \times n$ matrix A = ($a_{ij}$) by $a_{ij} = 1$ if there is an edge direct
ed from $i$ and $j$ and $a_{ij} = 0$ if there is no edge directed from $i$ to $j$. Thus, the matrix associated with the graph of Figure 12.11 is given by

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \ldots\ldots\ldots (3)$$

We note that:
(i) the diagonal elements of the matrix are all zero.
(ii) the number of non-zero elements are equal to the number of edges.
(iii) the number of non-zero elements in any row are equal to the local outward degree of the vertex corresponding to the row.
(iv) the number of non-zero elements

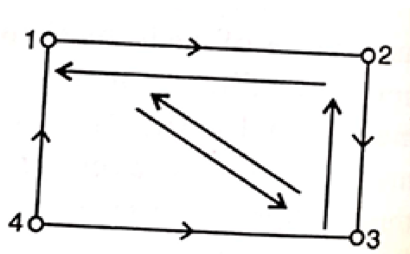

*Figure 12.11*

in a column are equal to the local inward degree of the vertex corresponding to the column.

Now

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array} = \left(a_{ij}^{(2)}\right) \qquad \ldots\ldots..(4)$$

The element $a_{ij}^{(2)}$ gives the number of 2-chains from *i* to *j*. Thus, from vertex 2 to vertex 1, there are two 2-chains via vertex 3 and vertex 4. We can generalize this result in the form of a theorem viz. "The element $a_{ij}^{(2)}$ of $A^2$ gives the number of 2-chains i.e. the number of paths with two-edges from vertex *i* to vertex *j*".

The theorem can be further generalised to "The element $a_{ij}^{(m)}$ of $A^m$ gives the number of *m*-chains i.e. the number of paths with *m* edges from vertex *i* to vertex *j*". It is also easily seen that "The *i*th diagonal element of $A^2$ gives the number of vertices with which *i* has a symmetric relationship".

From the matrix A of a graph, a symmetric matrix S can be generated by taking the elementwise product of A with its transpose so that in our case

$$S = A \times A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5)$$

*S* obviously is the matrix of the graph from which all unreciprocated connections have been eliminated. In the matrix *S* (as well as in $S^2$, $S^3$,....) the elements in the row and column corresponding to a vertex which has no symmetric relation with any other vertex are all zero.

## 12.4.8 APPLICATION OF DIRECTED GRAPHS TO DETECTION OF CLIQUES

A subset of persons in a socio-psychological group will be said to form a clique if (*i*) every member of this subset has a symmetrical relation with every other
member of this subset (*ii*) no other group member has a symmetric relation with all the members of the subset (otherwise it will be included in the clique) (*iii*) the subset has at least three members.

If other words a clique can be defined as a maximal completely connected subset of the original group, containing at least three persons. This subset should not be properly contained in any larger completely connected subset.

If the group consists of *n* persons, we can represent the group by *n* vertices of a graph. The structure is provided by persons knowing or being connected to other persons. If a person *i* knows *j*, we can draw a directed edge from *i* to j. If *i* knows *j* and *j* knows *i*, then we have a symmetrical relation between *i* and *j*.
With this interpretation, the graph of Figure 12 .11 shows that persons 1, 2, 3 form a clique. With very small groups, we can find cliques by carefully observing the corresponding graphs. For larger

groups analytical methods based on the following results are useful: ($i$) $i$ is a member of a clique if the $i$th diagonal element of $S^3$ is different from zero. ($ii$) If there is only one clique of $k$ members in the group, the corresponding k elements of $S^3$ will be
(k - 1) (k - 2)/2 and the rest of the diagonal elements will be zero. ($iii$) If there are only two cliques with $k$ and $m$ members respectively and there is no element common to these cliques, then k elements of $S^3$ will be (k-1)(k-2)/2, m elements of $S^3$ will be (m - 1)(m-2)/2 and the rest of the elements will be zero. ($iv$) If there are m disjoint cliques with $k_1$, $k_2$, ..., $k_m$ members, then the trace of $S^3$ is $\frac{1}{2}\sum_{i-1}^{m} k_i\,(k_i-1)(k_i-2)$. ($v$) A member is non-cliquical if only if the corresponding row and column of $S^2$ x $S$ consists entirely of zeros.

## 12.5 MATHEMATICAL MODELS IN TERMS OF SIGNED GRAPHS

### 12.5.1 BALANCE OF SIGNED GRAPHS

A signed (or an algebraic) graph is one in which every edge has a positive or negative sign associated with it. Thus, the four graphs of Figure 12.16 are signed graphs. Let the positive sign denote friendship and the negative sign denote enmity, then in graph *(i)* *A* is a friend of both *B* and *C,* and *B* and *C* are also friends. In graph (*ii*) *A* is a friend of *B* and *A* and *B* are both jointly enemies of *C*. In graph (*iii*), *A* is a friend of both *B* and *C*, but *B* and *C* are enemies. In graph (*iv*) *A* is an enemy of both *B* and *C*, but B and *C* are not friends.

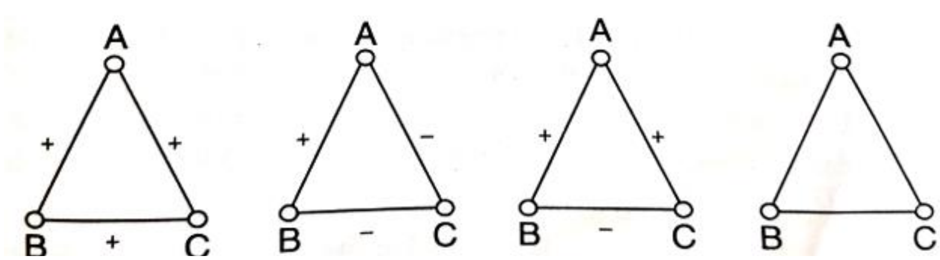


Figure 12.16

The first two graphs represent normal behavior and are said to be balanced, while the last two graphs represent unbalanced situations since if *A* is a friend of both *B* and *C* and *B* and *C* are enemies, this creates tension in the system and there is a similar tension when $B$ and $C$ have a common enemy $A$, but are not friends of each other.

We define the sign of a cycle as the product of the signs of component edges. We find that in the two balanced cases, this sign is positive and in the two unbalanced cases, this is negative.

We say that a cycle of length three or a triangle is balanced if and only if its sign is positive. A complete algebraic graph is defined to be a complete graph such that between any two edges of it, there is a positive or negative sign. A complete algebraic graph is said to be balanced if all its triangles are balanced. An alternative definition states that a complete algebraic graph is balanced if all its cycles are positive. It can be shown that the two definitions are equivalent.

A graph is locally balanced at a point $A$ if all the cycles passing through $A$ are balanced. If a graph is locally balanced at all points of the graph, it will obviously be balanced. A graph is defined to be $m$-balanced if all its cycles of length $m$ are positive. For an incomplete graph, it is preferable to define it to be balanced if all its cycles are positive. The definition in terms of triangle is not satisfactory, as there may be no triangles in the graph.

### 12.5.2 STRUCTURE THEOREM AND ITS IMPLICATIONS

*Theorem*: The following four conditions are equivalent:

*i.* The graph is balanced *i.e.*, every cycle in it is positive.

*ii.* All closed line-sequences in the graph are positive *i.e*. any sequence of edges starting from a given vertex and ending on it and possibly passing through the same vertex more than once is positive.

*iii.* Any two line-sequences between two vertices have the same sign.

*iv.* The set of all points of the graph can be partitioned into two disjoint sets such that every positive sign connects two points in the same set and every negative sign connects two points of different sets.

The last condition has an interesting interpretation with possibility of application. It states that if in a group of persons there are only two possible relationships *viz*. liking and disliking and if the algebraic graph representing these relationships is balanced, then the group will break up into two separate parties such that persons within a party like one another, but each person of one party dislikes every person of the other party. If a balanced situation is regarded as stable, this theorem can be interpreted to imply that a two-party political system is stable.

### 12.5.3 ANTIBALANCE AND DUOBALANCE OF A GRAPH

An algebraic graph is said to be anti-balanced if every cycle in it has an even number of positive edges. The concept can be obtained from that of a balanced graph by changing the signs of the edges. It will then be seen that an algebraic graph is anti-balanced if and only if its vertices can be separated into two disjoint classes, such that each negative edge joins two vertices of the same class and each positive edge joins persons from different classes.

A signed graph is said to be duo-balanced if it is both balanced and anti-balanced.

### 12.5.4 THE DEGREE OF UNBALANCE OF A GRAPH

For many purposes it is not enough to know that a situation is unbalanced. We may be interested in the degree of unbalance and the possibility of a balancing process which may enable one to pass from an unbalanced to a balanced graph.
The possibility is interesting as it can give an approach to group dynamics and demonstrate that methods of graph theory can be applied to dynamic situations
also.

Cartwright and Harary define the degree of balance of a group *G* to be the ratio of the positive cycles of *G* to the total number of cycles in *G*. This balance index obviously lies between 0 and 1. $G_1$ has six negative triangles *viz* (*abc*), (*ade*), (*bcd*), (*bce*), (*bde*), (*cde*) and has four positive triangles. $G_2$ has four negative triangles viz (*abc*), (*abd*), (*bce*) and (*bde*) and six positive triangles. The degree of balance of $G_1$ is therefore less than the degree of balance of $G_2$.

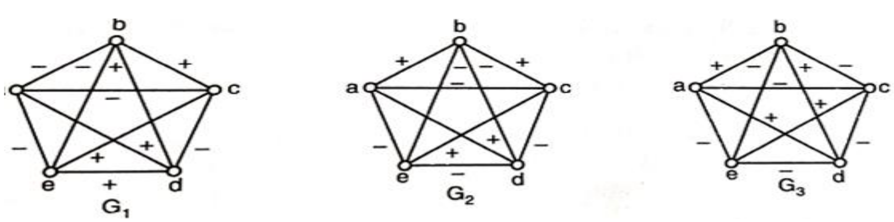


Figure 12.17

However, in order to get a balanced graph from $G_1$, we have to change the sign of only two edges *viz*. *bc* and *de* and similarly to make $G_2$, balanced we have to change the signs of two edges *viz bc* and *bd*. From this point of view both $G_1$, and $G_2$ are equally unbalanced.

Abelson and Rosenberg therefore gave an alternative definition. They defined the degree of unbalance of an algebraic graph as the number of the smallest set of edges of G whose change of sign produces a balanced graph.

The degree of an anti-balanced complete algebraic graph (*i.e*., of a graph all of whose triangles are negative) is given by [n(n -2) + k]/4 where k = 1.
if $n$ is odd and $k = 0$ if $n$ is even. It has been conjectured that the degree of unbalancing of every other complete algebraic graph is less than or equal to this value.

## *12.6 MATHEMATICAL MODELLING IN TERMS OF WEIGHTED GRAPHS*

### 12.6.1 COMMUNICATION NETWORKS WITH KNOWN PROBABILITIES OF COMMUNICATION

In the communication graph of Figure 12.19, we know that *a* can communicate with both *b* and *c* only and in the absence of any other knowledge, we assigned equal probabilities to *a*'s communicating with *b* or *c*.

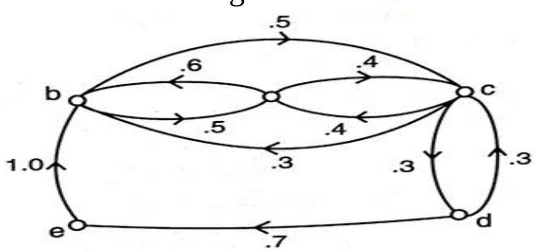


*Fig.12.18*

However, we may have priori knowledge that *a*'s chances of communicating with *b*
and *c* are in the ratio 3:2, then we assign probability .6 to *a*'s communicating with *b* and .4 to *a*'s communicating with *c*.

Similarly, we can associate a probability with every directed edge and we get the weighted digraph (Figure 12.19) with the associated matrix

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0 & .3 & 0 & 0.7 \\ 0 & 1.0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \ldots\ldots(6)$$

We note that the elements are all non-negative and the sum of the elements of every row is unity so that $B$ is a stochastic matrix and unity is one of its eigenvalues. The eigenvector corresponding to these eigenvalues will be different from the eigenvector found in Section 12.4.6 and so the relative importance of the individuals depends both on the directed edges as well as on the weights associated with the edges.

## 12.6.2 WEIGHTED DIGRAPHS AND MARKOV CHAINS

A Markovian system is characterised by a transition probability matrix. Thus, if the states of a system are represented by 1, 2. .... $n$ and $p_{ij}$ gives the probability of transition from the $i$th state to $i$th state, the system is characterised by the transition probability matrix (t.p.m)

$$T = \begin{bmatrix} p_{11} & p_{12} & \cdot\cdot & p_{1j} & \cdot\cdot & p_{1n} \\ p_{21} & p_{22} & \cdot\cdot & p_{2j} & \cdot\cdot & p_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{i1} & p_{i2} & \cdot\cdot & p_{ij} & \cdot\cdot & p_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n1} & p_{n2} & \cdot\cdot & p_{nj} & \cdot\cdot & p_{nn} \end{bmatrix} \quad (7)$$

Since$\sum_{i=1}^{n} p_{ij}$ represents the probability of the system going from $i$th state to any other state or of remaining in the same state. This sum must be equal to unity. Thus, the sum of elements of every row of a t.p.m is unity.

Consider a set of $N$ such Markov systems where $N$ is large and suppose at any instant $NP_1$, $NP_2$,…..$NP_n$ of these ($P_1$ + $P_2$+… + $P_n$ = 1) are in states 1, 2, 3. …, $n$ respectively. After one step, let the proportions in these states be denoted by $P'_1$, $P'_2$,…,$P'_n$, then

$$P_1' = P_1p_{11} + P_2p_{21} + P_3p_{31} + \cdots + P_nP_{n1}$$
$$P_2' = P_2p_{12} + P_2p_{22} + P_3p_{32} + \cdots + P_nP_{n2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (8)$$
$$P_n' = P_1p_{1n} + P_2p_{2n} + P_3p_{3n} + \cdots + P_nP_{nn}$$

Or

$$P' = PT \quad (9)$$

$$P'_1 = P_1p_{11} + P_2p_{21} + P_3p_{31} = \ldots + P_np_{n1}$$

$$P'_2 = P_2p_{12} + P_2p_{22} + P_3p_{32} = \ldots + P_np_{n2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$P'_n = P_1p_{1n} + P_2p_{2n} + P_3p_{3n} = \ldots + P_np_{nn}$$

*Or* $P' = PT$

where $P$ and $P'$ are row matrices representing the proportions of systems in various states before and after the step and T is the t.p.m.

We assume that the system has been in operation for a long time and the proportions $P_1$, $P_2$, ... ,$P_n$ have reached equilibrium values. In this case

$$P = PT \text{ or } P(I - T) = 0$$

where $I$ is the unit matrix. This represents a system of $n$ equations for determining the equilibrium values of $P_1$, $P_2$,..., $P_n$. If the equations are consistent, the determinant of the coefficient must vanish *i.e.* $|T - I| = 0$. This requires that unity must be an eigenvalue of $T$. However, this, as we have seen already is true. This shows that an equilibrium state is always possible for a Markov chain.

A Markovian system can be represented by a weighted directed graph.

Thus consider the Markovian system with the stochastic matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (11)$$

Its weighted digraph is given in Figure 12.19.

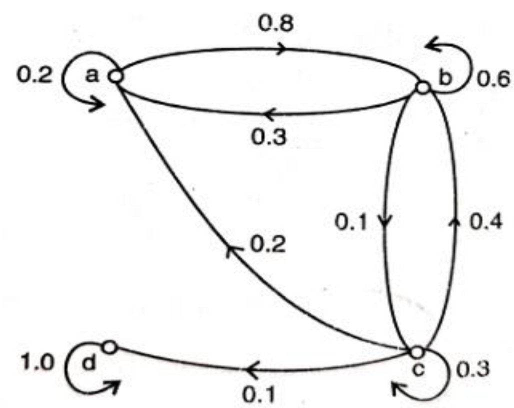


*Figure 12.19*

In this example *d* is an absorbing state or a state of equilibrium. Once a system reaches the state *d*, it stays there forever.

It is clear from Figure 12.19, that in whichever state, the system may start, it will ultimately end in state *d*. However, the number of steps that may be required to reach *d* depends on chance. Thus, starting from *c*, the number of steps to reach *d* may be 1, 2, 3, 4, ...; starting from *b* the number of steps to reach *d* may be 2, 3, 4, ... and starting from *a*, the number of steps may be 3, 4, 5, ... In each case, we can find the probability that the number of steps required in n and then we can find the expected number of steps to reach it.

Thus, for the matrix

$$\begin{array}{c} \\ a \\ b \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix} \end{array} \qquad (12)$$

*a* is an absorbing state. Starting from *b*, we can reach *a* in 1, 2, 3, ..., *n* steps with probabilities (1/3), (1/3) (2/3), (1/3) $(2/3)^2$ ,..., (1/3) $(2/3)^{n-1}$, ..., so that the expected number of steps is

$$\sum_{n=1}^{\infty} n\frac{1}{3}(\frac{2}{3})^{n-1} = 3 \qquad (13)$$

### 12.6.3 GENERAL COMMUNICATION NETWORKS

So, for we have considered communication networks in which the weight associated with a directed edge represents the probability of communication along that edge. We can however have more general networks *e.g.,*

*a)* for communication of messages where the directed edge represents the channel and the weight represents the capacity of the channel say in bits per second.
*b)* for communication of gas in pipelines where the weights are capacities, say in gallons per hour.
*c)* communication roads where the weights are the capacities in cars per hour.

An interesting problem is to find the maximum flow rate, of whatever is being communicated, from any vertex of the communication network to any other. Useful graph-theoretic algorithms for this have been developed by Elias. Feinstein and Shannon as well as by Ford and Fulkerson.

### 12.6.4 MORE GENERAL WEIGHTED DIAGRAMS

In the most general case, the weight associated with a directed edge can be positive or negative. Thus Figure 12.21 means that a unit change at vertex 1 at time $t$ causes changes of -2 units at vertex 2, of 2 units at vertex 4 and of 3 units at vertex 5 at time $t + 1$. Similarly, a change of 1 unit at vertex 2 causes a change of -3 units at 3 vertex, 4 units at vertex 4 and of 2 units at vertex 5 and so on.

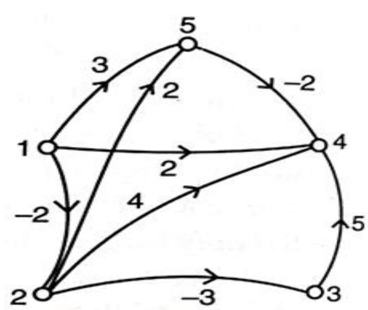


*Figure 12.20*

Given the values at all vertices at time $t$, we can find the values at time $t + 1$, $t + 2$, $t + 3$, .... The process of doing this systematically is known as the pulse rule.

These general weighted digraphs are useful for representing energy flows, monetary flows and changes in environmental conditions.

### 12.6.5 SIGNAL FLOW GRAPHS

The system of algebraic equations –

$$\begin{aligned} x_1 &= 4y_0 + 6x_2 - 2x_3 \\ x_2 &= 2y_0 - 2x_1 + 2x_3 \\ x_3 &= 2x_1 - 2x_2 \end{aligned} \qquad (14)$$

$x_1 = 4y_0 + 6x_2 - 2x_3$
$x_2 = 2y - 2x_1 + 2x_3$
$x_3 = 2x_1 - 2x_2$

can be represented by the weighted digraph in Figure 12.21. For solving for x, we successively eliminate x, and x2 to get the graphs in Figure 12.22 and finally we get $x_1 = 4y_0$

We can similarly represent the solution of any number of linear equations
graphically.

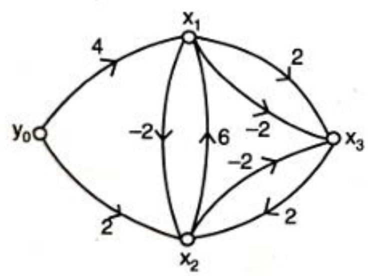

Figure 12.21

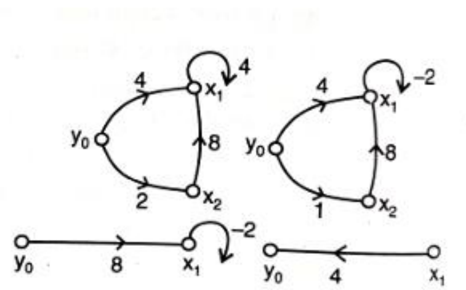

Figure 12.22

### 12.6.6 WEIGHTED BIPARTITE DIGRAPHS AND DIFFERENCE EQUATIONS

Consider the system of

difference equations

$$\begin{aligned} x_{t-1} &= a_{11}x_t + a_{12}\,y_t + a_{13}z_t \\ y_{t-1} &= a_{21}x_t + a_{22}\,y_t + a_{23}z_t \end{aligned} \qquad (15)$$

$$z_{t-1} = a_{31}x_t + a_{32}\,y_t + a_{33}z_t$$

This can be represented by a weighted bipartite digraph (Figure 7.24).
The weights can be positive or negative.

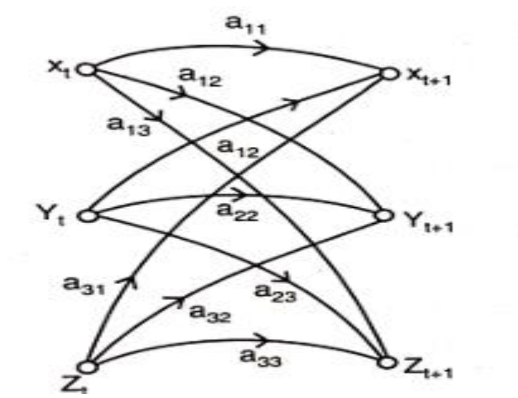


*Figure 12.23*

## 12.7 SUMMARY

A graph is called **complete** if every pair of its vertices is joined by an edge.
A graph is called a **signed graph** if every edge has either a plus or minus sign associated with it.
General weighted digraphs are useful for representing energy flows, monetary flows and changes in environmental conditions.

## 12.8 GLOSSARY

**Random Variable:** A variable that takes on random values, often denoted by ω.
**Probability Measure:** A mathematical function that assigns a probability to each possible outcome of a random experiment.
**Expectation:** A mathematical operation that calculates the average value of a random variable.

### CHECK YOUR PROGRESS

1. A graph is called ……. if every pair of its vertices is joined by an edge.
2. A graph is called a …… if every edge has either a plus or minus sign associated with it.

## *12.9* REFRENCES

**1.** J.N. Kapoor: (2009). Mathematical Modelling, New Age International (P) Limited Publishers.
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## *12.10 SUGGESTED READINGS*

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**2.** W.E. Boyce (1981) Case Studies in Mathematical Modelling, Boston, Pitman.
**3.** A. Friedman and W. Littman, (1994) Industrial Mathematics: A Course in Solving Real World Problems, Philadelphia, SIAM.
**4.** F.R. Giordano and M.D. Weir (1985). A First Course in Mathematical Modelling, Monterey, Brooks/Cole.

## *12.11 TERMINAL QUESTIONS*

**1.** Draw some antibalanced graphs and verify the structure theorem for them.
**2.** Show that a signed graph has an idealised party structure if and only if no circuit has exactly one – sign.
**3.** Define Mathematical Models in terms of directed graphs.
**4.** Define Mathematical Models in terms of Signed graphs.
**5.** Define Mathematical Models in terms of Weighted graphs.

## *12.12* ANSWARS

CYQ1. Complete CYQ2. Signed graph

# COURSE NAME: MATHEMATICAL MODELLING

# COURSE CODE: MAT 610

## BLOCK - IV

## MATHEMATICAL MODELLING - IV

# UNIT 13: MATHEMATICAL MODELLING THROUGH CALCULUS OF VARIATION

## CONTENTS:

## 13.1 *INTRODUCTION*

In calculus of variations the basic problem is to find a function y for which the functional I(y) is maximum or minimum. We call such functions as extremizing functions and the value of the functional at the extremizing function as extremum. The calculus of variation (or variational calculus) is a fold of mathematical analysis that uses variations, which are small changes in functions and functional to find maxima and minima of functional: Mapping from set of functions to the real numbers. Functional are often expressed as definite involving functions and their derivatives. Functions that maximize or minimize functional may be found using the Euler to langrage equation of the calculus of variations. A simple example of such a problem is to find the cause of shortest length connecting two points if there are no constraints, the solution is a straight line between the points. However, if the curve is constrained to lie on a surface in space, then the solution is less obvious, and possible many situations may exist. Such solutions are known ass geodesics. A related problem is posed be fermatas Principle: light follow the path of shortest optical length connecting two points, which depends upon the material of the medium. One corresponding concepts in mechanics is the principle of least or stationary action.
Differential dynamic programming is an optical control algorithm of the trajectory optimization class. The algorithm was introduced in 1966 by mayne and subsequently analyzed in Jocobson and mayne'book. The algorithm uses locally-quadratic models of the dynamics and cost functions, and displays quadratic convergence. It is closely related to Newton's Method. It can be used in the study of dynamic programming and other new mathematical formalisms in optical control problems, such as the determination of rocket trajectories, the correction of lunch error and inflight disturbances of spacecrafts and in the problems of optimal control found in economics, biology and the Social Sciences.

## 13.2 *OBJECTIVES*

After studying this unit, learner will be able to

**i.** Euler-Lagrange's equation.
**ii.** Maximum entropy distributions.
**iii.** Mathematical Modelling of Geometrical Problems through Calculus of Variations

## 13.3 *EULER – LAGRANGE EQUATION*

Consider

$$I = \int_a^b f\left(x, y, \frac{dy}{dx}\right) dx \qquad (1)$$

For every well-behaved function $y$ of $x$, we can find $I$ as a real number so that $I$ depends on what function $y$ is of $x$. The problem of calculus of variations is to find that function $y(x)$ for which $I$ is maximum or minimum. The answer is given by the solution of Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \qquad (2)$$

which is an ordinary differential equation of the second order. A proof of this result will be obtained in the next section by using dynamic programming.

$$\text{If } I = \iint f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dxdy \equiv \iint f(x, y, z, p, q)\, dxdy \qquad (3)$$

then $I$ is maximum or minimum when

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) = 0 \qquad (4)$$

## 13.4 *MAXIMUM – ENTROPY DISTRIBUTIONS*

**(a)** We want to find that probability distribution for a variate varying over the range $(-\infty, \infty)$ which has the maximum entropy out of all distributions having a given mean $m$ and a given variance $\sigma^2$.

Let $f(x)$ be the probability density function, then we have to maximize the entropy defined by

$$S = -\int_{-\infty}^{\infty} f(x) \ln f(x)\, dx \qquad (5)$$

subject to $\int_{-\infty}^{\infty} f(x)dx = 1, \int_{-\infty}^{\infty} xf(x)dx = m,$

$$\int_{-\infty}^{\infty} x^2 f(x)dx = \sigma^2 + m^2 \qquad (6)$$

We form the Lagrangian

$$L = \int_{-\infty}^{\infty} - f(x) \ln f(x) - \lambda \int_{-\infty}^{\infty} f(x)dx - \mu \int_{-\infty}^{\infty} xf(x)dx - v \int_{-\infty}^{\infty} x^2 f(x)dx \qquad (7)$$

Here the integrand contains only $x$ and $y(= f(x))$ and there is no $y'$ in it. As such (8) gives

$$-(1 + \ln f(x)) - \lambda - \mu x - v x^2 = 0 \qquad (8)$$

or

$$f(x) = Ae^{\mu x + v x^2} \qquad (9)$$

We use (11) to calculate $A, \mu, v$ to get

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-[\frac{1}{2}(x - \frac{m)^2}{\sigma^{2}]}} \qquad (10)$$

This shows that out of all distributions with a given mean $m$ and a given variance $\sigma^2$, the normal distribution $N(m, \sigma^2)$ has the maximum entropy.

Now mean and variance are the simplest moments and the maximum entropy distribution for which these moments have prescribed values is the normal distribution. This gives one reason for the importance of the normal distribution.

**(b)** We now want to find the distribution over the interval $[0, \infty)$ which has the maximum entropy, out of all those which have given arithmetic and geometric means.

Here we have to maximize

$$-\int_0^\infty f(x) \ln f(x)\, dx \qquad (11)$$

subject to

$$\int_0^\infty f(x) dx = 1, \int_0^\infty x f(x) dx = m, \int_0^\infty \ln x f(x)\, dx = \ln g \qquad (12)$$

Using Lagrange's method and (8) we get

$$f(x) = Ae^{-ax} x^{\gamma - 1} \qquad (13)$$

$A, a, \gamma$ are determined by using (2). Thus, gamma distribution has the maximum entropy out of all distributions which have given arithmetic and geometric means.

(c) We want to find the maximum entropy bivariate distribution when $x, y$ vary from $-\infty$ to $\infty$ and when means, variances and covariance are prescribed.

We have to maximize

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ln f(x,y)\,dxdy \qquad (14)$$

subject to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = m_1$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy = m_2, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)dxdy = \sigma_1^2 + m_1^2 \qquad (15)$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)dxdy = \sigma_2^2 + m_2^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \rho_{\sigma_1\sigma_2} + m_1 m_2$$

Forming the Lagrangian and using (4), we get

$$f(x,y) = Ae^{-a_1x-a_2y-b_1x^2-b_2y^2-cxy} \qquad (16)$$

Using (15) to find $a_1, a_2, b_1, b_2, c$, we get

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_1)^2}{\sigma_1^2} - \frac{2\rho(x-m_1)(y-m_2)}{\sigma_1\sigma_2} + \frac{(y-m_2)^2}{\sigma_2^2}\right)\right) \qquad (17)$$

which gives the density function for the bivariate normal distribution, so that out of all bivariate probability distributions for which $x, y$ vary from $-\infty$ to $\infty$ and which have given means, variances and covariance, the distribution with the maximum entropy is the bivariate normal distribution.

(d) We want to find the multivariate distribution for $x_1, x_2, \dots, x_n$ where

$$0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1, \dots, 0 \leqslant x_n \leqslant 1; x_1 + x_2 + \cdots + x_n = 1 \qquad (18)$$

for which $E(\ln x_1), \dots, E(\ln x_n)$ have prescribed values for which entropy is maximum.

Using the principle of maximum entropy, we get,

$$f(x_1, x_2, \dots, x_n) = \frac{T(m_1 + m_2 + \cdots + m_n)}{T(m_1)T(m_2)\dots, T(m_n)} x_1^{m_1-1} x_2^{m_2-1} \dots x_{n-1}^{m_{n-1}-1} (1 - x_1 - x_2 \dots - x_{n-1})^{m_{n-1}} \qquad (19)$$

which is Dirichlet distribution.

## 13.5 *MATHEMATICAL MODELLING OF GEOMETRICAL PROBLEMS THROUGH CALCULUS OF VARIATIONS*

**(a) Finding the path of the shortest distance between two points in a plane**

Here

$$I = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, f(x, y, y') = \sqrt{1 + y'^2} \tag{20}$$

(2) gives

$$\frac{d}{dx}(y') = 0, y' = \text{ const, } y = mx + c \tag{21}$$

Alternatively

$$I = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta, f(\theta, r, r') = \sqrt{r^2 + r'^2} \tag{22}$$

(2) gives

$$r\frac{d\theta}{dr} = \text{ const., } \tan\varphi = \text{ const., } \varphi = \text{ const.} \tag{23}$$

Thus the path of shortest distance between two points is a straight line.

**(b) Finding geodesics (paths of shortest distance) between two given points on the surface of a sphere**

Let

$$x = a\sin\theta\cos\varphi, y = a\sin\theta\sin\varphi, z = a\cos\theta \tag{24}$$

then

$$I = \int_{x_0,y_0,z_0}^{x_1,y_1,z_2} \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = a\int_{\varphi_2}^{\varphi_2} \sqrt{\sin^2\theta + \left(\frac{d\theta}{d\varphi}\right)^2}\, d\varphi \tag{25}$$

$$f = \sqrt{\sin^2\theta - \theta'^2} \qquad (26)$$

(2) gives

$$\frac{d\varphi}{d\theta} = \frac{\sin\alpha}{\sin\theta\sqrt{\sin^2\theta - \sin^2\alpha}} \qquad (27)$$

Integrating

$$\tan\alpha\cos\theta - \sin\theta\cos p\cos\beta + \sin\theta\sin p\sin\beta = 0 \qquad (28)$$

or

$$z\tan\alpha - x\cos\beta + y\sin\beta = 0; \qquad (29)$$

which is the equation of a plane passing through the center of the sphere. Hence a geodesic is a great circle arc passing through the two given points.

**(c) Finding the Minimal surface of revolution** i.e., finding the equation of a curve joining two given points in a plane, which when rotated about the x-axis gives a surface with minimum area.
The surface area is given by

$$S = 2\pi\int_a^b y\,ds = S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad (30)$$

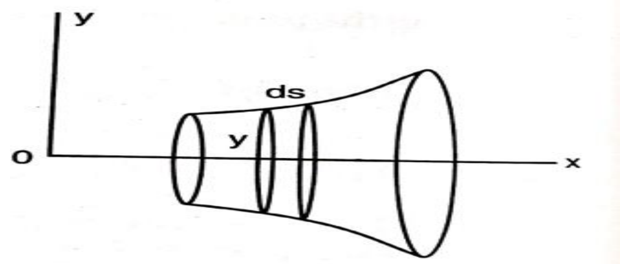


**Fig. 13.5.1**

$$f(x, y, y') = y\sqrt{1+y'^2} \qquad (31)$$

Equation (2) gives

$$y\sqrt{1+y'^2} = \text{constant} \qquad (32)$$

Integrating

$$y = c\cos h\left(\frac{x}{c}\right) \qquad (33)$$

**Fig.13.5.2**

Thus, the minimal surface of revolution is the catenoid obtained by rotating a catenary about its directrix. The soap film between two loops of circular wire is a practical example of a catenoid. As we go on increasing the distance between the loops, a stage comes when the film breaks down. This corresponds to the case when no catenoid is possible.

**(d) Determining a given plane closed curve with given perimeter enclosing maximum area (The isoperimetric curve)**

Using polar coordinates, we have to maximize

$$I = \frac{1}{2}\int_0^{2\pi} r^2 d\theta \tag{34}$$

$$\text{subject to} \quad \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \text{ constant} \tag{35}$$

Using Lagrange's method,

$$f = \frac{1}{2}r^2 - \lambda\sqrt{r^2 + r'^2} \tag{36}$$

(2) gives

$$\frac{1}{2}r^2 - \frac{\lambda r^2}{\sqrt{r^2 + r'^2}} = \text{ constant} \tag{37}$$

Differentiation with respect to θ, gives

$$\frac{r^2 + r'^2 - rr''}{(r^2 + r'^2)^{3/2}} = \frac{1}{\lambda} \tag{38}$$

but the LHS is the expression for the curvature of the curve. As such the required curve is a curve of constant curvature i.e. it is a circle.

The problem is supposed to have arisen from the gift of a king who was happy with a person and promised to give him all the land he could enclose by running round in a day. Since he could run a fixed distance, the perimeter of his path was fixed and as such the radius of the circle he should describe is known.

**(e) Finding the solid of revolution with a given surface area and maximum volume.**

If V is the volume and S is the surface area

$$V = \pi \int y^2 dx, \qquad S = 2\pi \int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{39}$$

$$f(x, y, y') = \pi y^2 - 2\lambda\pi y\sqrt{1 + y'^2} \tag{40}$$

(2) gives,

$$y^2 - \frac{2\lambda y}{\sqrt{1 + y'^2}} = \text{ constant} \tag{41}$$

Its integration for general values of the constant involves elliptic functions, but for the special case when the constant is taken as zero, (47) gives

$$y = 2\lambda\cos\psi \text{ so that } \sin\psi = \frac{dy}{ds} = -2\lambda\sin\psi\frac{d\psi}{ds} \tag{42}$$

or

$$\frac{d\psi}{ds} = -\frac{1}{2\lambda} = \text{constant}, \tag{43}$$

so that in this case the surface is obtained by rotating a circle and is thus a sphere.

## 13.6 *MATHEMATICAL MODELLING OF SITUATIONS IN MECHANICS THROUGH CALCULUS OF VARIATIONS*

**(a)** Finding the shape of a freely hanging uniform heavy string under gravity when the two ends of it are fixed

We minimize the potential energy V subject to the length of the string being fixed. As such we have to minimize

$$V = mg\int y\sqrt{1 + y'^2}dx \tag{44}$$

subject to

$$l = \int \sqrt{1 + y'^2}dx \tag{45}$$

Therefore

$$f = y\sqrt{1 + y'^2} - \lambda\sqrt{1 + y'^2} \tag{46}$$

(8) gives

$$\frac{dy}{dx} = \left(\frac{(y - \lambda)^2}{c^2} - 1\right)^{\frac{1}{2}} \tag{47}$$

Integrating

$$y - \lambda = \cos h\frac{x - a}{c}, \tag{48}$$

so that the required curve is a catenary.

**(b)** Finding the equation of the smooth vertical curve along which the time of descent under gravity between any two given points is minimum (Brachistochrone Problem)

Using the principle of conservation of energy, we get

$$\frac{1}{2}mv^2 - mgy = \text{ constant}$$

If the particle starts from rest when $y = 0$ (Fig. 13.6.1), we get

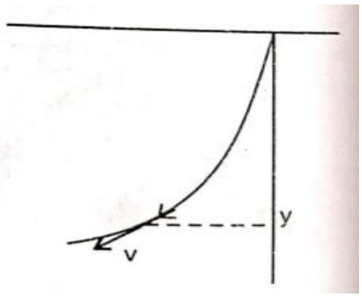


***Fig. 13.6.1***

$$t^2 = 2gy \text{ or } \frac{ds}{dt} - \sqrt{2gy} \tag{49}$$

$$\text{or } T = \int \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_a^b \frac{1}{\sqrt{y}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad (50)$$

so that

$$f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{y}} \quad (51)$$

(2) gives

$$y(1 + y'^2) = 2c \quad (52)$$

or

$$y = c(1 + \cos 2\psi) \quad (53)$$

Now

$$dx = \cot\psi\, dy = -4c \cos^2\psi d\psi \quad (54)$$

$$x = a - c(2\psi + \sin 2\psi) \quad (55)$$

Equations (53) and (55) give the parametric equations of a cycloid.
(c) Discussion of the shapes of vibrating strings and membranes.

These have already been discussed in section 6.4.

(d) Obtaining the equation of the free surface of a fluid rotating in a cylinder about its axis under gravity.

Consider the element of volume $2\pi x dx dz$ and mass $\rho 2\pi x dx dz$. Its potential energy is

$$\rho 2\pi x dx dz \left( gz - \frac{1}{2}\omega^2 x^2 + c \right) \quad (56)$$

so that the total potential energy of the fluid is

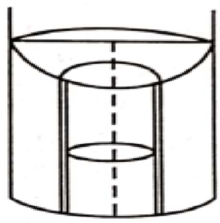

***Fig.13.4.2***

$$2\pi\rho\int_{x=0}^{a}\int_{z=0}^{y} x \left(gz - \frac{1}{2}\omega^2x^2 + c\right)dxdz = \pi\rho\int_{0}^{a}(gy^2 - \omega^2x^2y + 2cy)xdx \tag{57}$$

Since potential energy has to be minimum, we minimize (57). Here

$$f = (gy^2 - \omega^2x^2y + 2cy)x \tag{58}$$

(2) gives

$$2gy - \omega^2x^2 + 2c = 0 \tag{59}$$

which is a parabola. so that the free surface is a paraboloid of revolution.

**(e) Lagrange's equations of Motion**

Let $q_1, q_2, \ldots, q_n$ be 'generalised' coordinates in terms of which a dynamical system is described, then its kinetic energy T is a function of $q_1, q_2, \ldots, q_n;\ q_1', q_2', \ldots, q_n'$ and its potential energy V is a function of $q_1, q_2, \ldots, q_n$ only. According to the Hamiltonian principle, we then have to find an extreme value for

$$H = \int \left(T(q_1, q_2, \ldots, q_n; q_1', \ldots, q_n') - V(q_1 q_2, \ldots, q_n)\right)dt \tag{60}$$

Using an equation similar to (8) for $q_1, q_2, \ldots, q_n$, we get

$$\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} - \frac{d}{dt}\left(\frac{\theta T}{\partial q_i}\right) = 0, i = 1,2, \ldots, n \tag{61}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q'_i}, i = 1,2, \ldots, n \tag{62}$$

or

$$\frac{d}{d_i}\left(\frac{\partial L}{\partial q_2'}\right) - \frac{\partial L}{\partial q_1} = 0; L = T - V; i = 1,2, \ldots, n \tag{63}$$

Equations (61) or (62) or (63) are called Lagrange's equations of motion. These are n simultaneous ordinary differential equations of second order for determining $q_1, q_2, \ldots, q_n$ as functions of t.

- **Mathematical Modelling in Bioeconomic Through Calculus of Variations.**

Mathematical Bioeconomic is an interdisciplinary subject in which we use mathematical methods to optimize the economic profits from the utilization of renewable biological resources like forests and fisheries.

Let $x(t)$ be the fish population at time $t$ and let $h(t)$ be the rate at which it is harvested, then we get the equation

$$\frac{dx}{dt} = F(x) - h(t), \tag{64}$$

where $F(x)$ is the natural biological rate of growth. Let $c(x)$ be the cost of harvesting a unit of fish when the population size is $x(t)$ and let p be the selling price per unit fish so that the profit per unit fish is $(p - c(x))$ and the profit in time interval $(t, t + dt)$ is $[p - c(x)h(t)]dt$. If δ is the instantaneous discount rate, the present value of the total profit is

$$P = \int_0^{\infty} e^{-\delta t}(p - c(x))h(t)dt \tag{65}$$

If we know $h(t)$, we can use (64) to solve for $x(t)$ and then we can use (65) to determine P so that P depends on what function h is of $t$. We have to determine that function $h(t)$ for which P is maximum. Substituting for $h(t)$ from (64) in (65), we get

$$P = \int_0^{\infty} e^{-\delta t}(p - c(x))(F(x) - x')dt \tag{66}$$

so that

$$f(t, x, x') = e^{-\delta t}(p - c(x))(F(x) - x') \tag{67}$$

Using Euler-Lagrange eqn. (2),

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$e^{-\delta'}(-c'(x))(F(x) - x') + e^{-\delta t}\big(p - c(x)\big)\left(F'(x) - \frac{d}{dt}\big[e^{-\beta t}(c(x) - p)\big]\right. = 0 \tag{68}$$

Or

$$-c'(x)(F(x) - x') + \big(p - c(x)\big)F'(x) + \delta(c(x) - p) - c'(x)x' = 0$$

Or

$$-c'(x)F(x) + \big(p - c(x)\big)(F'(x) - \delta) = 0 \tag{69}$$

which determines a constant value $x^*$ for x and then (70) gives the rate of harvesting as constant and equal to $F(x^*)$.

If the initial population is less than $x^*$, we should do no harvesting till the population rises to $x^*$ and then begin harvesting at a constant rate $F(x^*)$. If the initial population is more than $x^*$, we should do harvesting at the maximum permissible rate till the population falls to $x^4$, and then begin doing harvesting at a constant rate $F(x^*)$.

- **Mathematical Modelling in Optics Through Calculus of Variations**

According to Fermat's principle of least time, light travel from a given point A to another point B in such a way as to take the least possible time. If $\mu(x, y)$ is the refractive index at the point $(x, y)$, then the velocity of light at the point is $c/\mu(x, y)$ and the time taken in going from A to B is

$$= \int_A^B \frac{ds}{c/\mu} = \int_A^B \mu(x, y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (70)$$

$$\therefore \qquad f(x, y, y') = \mu(x, y)\sqrt{1 + y'^2} \qquad (71)$$

(2) gives

$$\frac{\partial\mu}{\partial y}\sqrt{1 + y'^2} - \frac{d}{dx}\left[\mu\frac{y'}{\sqrt{1 + y'^2}}\right] = 0 \qquad (72)$$

$$\frac{\partial\mu}{\partial y} = \frac{d}{ds}(\mu \sin\psi) \qquad (73)$$

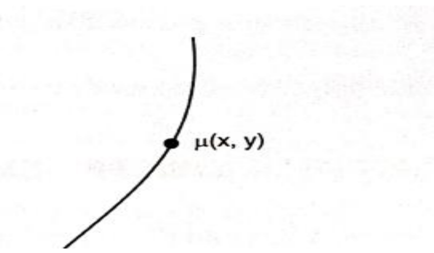


***Fig.13.4.3***

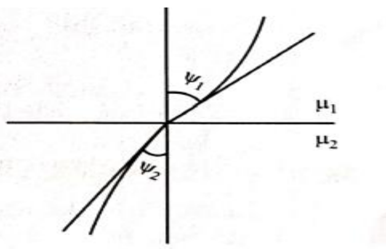


***Fig.13.4.4***

If y-axis separates two media of refractive indices $\mu_1$ and $\mu_2$, then $\frac{\partial\mu}{\partial y} = 0$ and so

$$\mu_1 \sin\phi_1 = \mu_2 \sin\psi_2 \qquad (74)$$

which is Snell's law of refraction.

## 13.7 SUMMARY

1. Euler-Lagrange's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0.$$

2. **Finding the Minimal surface of revolution** i.e., finding the equation of a curve joining two given points in a plane, which when rotated about the x-axis gives a surface with minimum area.The surface area is given by

$$S=2\pi\int_a^b yds = S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

## 13.8 GLOSSARY

**Random Variable:** A variable that takes on random values, often denoted by ω.
**Probability Measure:** A mathematical function that assigns a probability to each possible outcome of a random experiment.
**Expectation:** A mathematical operation that calculates the average value of a random variable.

**CHECK YOUR PROGRESS**

**1:** A graph is called ……. if every pair of its vertices is joined by an edge.
2. A graph is called a …… if every edge has either a plus or minus sign associated with it.

## 13.9 *REFRENCES*


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## 13.10 *SUGGESTED READINGS*

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**2.** W.E. Boyce (1981) Case Studies in Mathematical Modelling, Boston, Pitman.
**3.** A. Friedman and W. Littman, (1994) Industrial Mathematics: A Course in Solving Real World Problems, Philadelphia, SIAM.
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## 13.11 *TERMINAL QUESTIONS*

1. Show that the closed curve which encloses a given area and has minimum perimeter is a circle.
2. Show that the rectangle with given perimeter and enclosing maximum area is a square.
3. Define Maximum-Entropy Distributions.
4. Define Mathematical Modelling of Geometrical Problems through Calculus of Variations.

# UNIT 14: MATHEMATICAL MODELLING THROUGH DYNAMIC PROGRAMMING

## CONTENTS:

## 14.1 *INTRODUCTION*

In terms of mathematical optimization, dynamic programming usually refers to simplifying a decision by breaking it down into a sequence of decision steps over time. Dynamic programming is a useful mathematical technique for making a sequence of interrelated decisions. It provides a systematic procedure for determining the optimal combination of decisions. In contrast to linear programming, there does not exist a standard mathematical formulation of "the" dynamic programming problem. Rather, dynamic programming is a general type of approach to problem solving, and the particular equations used must be developed to fit each situation. Therefore, a certain degree of ingenuity and insight into the general structure of dynamic programming problems is required to recognize when and how a problem can be solved by dynamic programming procedures. These abilities can best be developed by an exposure to a wide variety of dynamic programming applications and a study of the characteristics that are common to all these situations. A large number of illustrative examples are presented for this purpose.

## 14.2 *OBJECTIVES*

After studying this unit, learner will be able to

- **i.** Euler-Lagrange's equation.
- **ii.** Maximum entropy distributions.
- **iii.** Applications of Dynamic programming

## 14.3 *MATHEMATICAL MODELLING THROUGH DYNAMIC PROGRAMMING*

Dynamic programming is an important technique for solving multi-stage optimization mathematical modelling problems. The main principle used is the principle of optimality discussed.

Quite often the problem of maximizing of a function of $n$ variables can be reduced to an $n$-stage decision problem where a stage corresponds to the choice of the optimizing values of a variable. Instead of dealing with one problem of maximizing a function of $n$ variables, we deal with $n$ problems of maximizing a function of one variable and we deal with these in a sequence. This leads to a considerable simplification of the problem, as we shall see in the examples below:

## 14.4 *TWO CLASSES OF OPTIMIZATION*

### (a) A Class of Maximization Problem

We have to allocate a total resource $c$ to $n$ activities so as to maximize the total output when the output from the $i$ th activity when an amount $x_i$ is allotted to it is $g_i(x_i)$ where $g_i(x_i)$ is a concave function of $x_i$ so that our problem is

$$\begin{array}{ll} \text{maximize} & g_1(x_1) + g_2(x_2) + \cdots + g_n(x_n) \\ \text{subject to} & x_1 + x_2 + \cdots + x_n = c; x_i \geq 0, i = 1,2, \dots n \end{array}$$

Let $f_n(c)$ be the maximum value, then the principle of optimality gives
Also

$$f_n(c) = \max_{0 \le x_n \le c} \left( g_n(x_n) + f_{n-1}(c - x_n) \right)$$

$$f_1(c) = g_1(c)$$

so that

$$f_2(c) = \max_{0 \le x_2 \le c} \left( g_2(x_2) + g_1(c - x_2) \right)$$

The function to be maximized is the sum of two concave function and its maximum arises when

$$g_2'(x_2) = g_1'(c - x_2)$$

Thus $x_2$ is known and therefore $f_2(c)$ is determined for all values of $c$. In particular if $g_1(x) = g_2(x) = g(x)$, then $g(x_1) + g(x_2)$ is maximum when $x_1 = x_2 = c/2$ and the maximum value is $2g(c/2)$. Similarly, if $g(x)$ is concave, then the maximum value of $g(x_1) + g(x_2) + \cdots + g(x_n)$ occurs when

$$x_1 = x_2 = \cdots = x_n = \frac{c}{n}$$

and the maximum value is $ng(c/n)$. For a general value of $n$, this result can be established by mathematical induction.

### Special Cases

**(i)** Since $\ln x$ is a concave function,
$\ln x_1 + \ln x_2 + \cdots + \ln x_n$ is maximum subject to $x_1 + x_2 + \cdots + x_n = c$, when

$x_1 = x_2 = \cdots x_n = c/n$ and the maximum value is $n\ln c/n$ and the maximum value of $x_1, x_2 \ldots x_n$ is $(c/n)n$.

**(ii)** Since $-x\ln x$ is a concave function,
$-(\sum_{i=1}^{n} p_i \ln p_i)$ is maximum subject to $\sum_{i=1}^{n} p_i = 1$ when $p_1 = p_2 = \cdots = p_n = 1/n$.

**(iii)** Since $(x^\alpha - x)/(1-\alpha)$ is concave function
$(\sum_{i=1}^{n} p_i^\alpha - 1)/(1-\alpha)$ is maximum subject to $\sum_{i=1}^{n} p_i = 1$, when $p_1 = p_2 = \ldots = p_n = 1/n$.

**(iv)** Since $-x\ln x + \frac{1}{a}(1+ax)\ln(1+ax)$ is a concave function,
$-\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}\sum_{i=1}^{n} (1+ap_i)\ln(1+ap_i)$ is maximum subject to $\sum_{i=1}^{n} p_i = 1$, when $p_1 = p_2 = \cdots = p_n = 1/n$.
All these results are of considerable importance in applications of information theory.

**(b) A Class of Minimization Problems**

If $h(x)$ is a convex function, the functional equation for obtaining the minimum value of $h(x_1) + h(x_2) + \cdots + h(x_n)$ subject to

$$x_1 + x_2 + \cdots + x_n = c, x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

is given by

$$f_n(c) = \min_{0 \leq x_n \leq c} \big(h(x_n) + f_{n-1}(c - x_n)\big)$$

and proceeding as before we find that the minimum value of

$$h(x_1) + h(x_2) + \cdots + h(x_n)$$

subject to (82) is $nh(c/n)$ and occurs when $c_1 = c_2 = \cdots = c_n = 1/n$
In the same way if $h_1(x_1), h_2(x_2), \ldots h_n(x)$ are all convex functions, the minimum value of $h(x_1) + h(x_2) + \cdots + h_n(x_n)$ subject to (82) occurs when

$$h_1'(x_1) + h_2'(x_2) + \cdots + h_n'(x_n).$$

## Special Cases

(i) Since $x\ln\frac{x}{y}$ is a convex function of $x$, the minimum value of $\sum_{i=1}^{n} x_i \ln\frac{x_i}{y_i}$ subject to $\sum_{i=1}^{n} x_i = c, \sum_{i=1}^{n} y_i = d$ occurs when

$$1 + \ln\frac{x_1}{y_1} = 1 + \ln\frac{x_2}{y_2} = \cdots = 1 + \ln\frac{x_n}{y_n}$$

or

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n} = \frac{c}{d}$$

and the minimum value of $\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i}$ is $c\ln\frac{c}{d}$. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ then the minimum value is zero.

**(ii)** Since $(x^{\alpha}y^{1-\alpha} - y)/(\alpha - 1)$ is a convex function of $x$, the quantity $\sum_{i=1}^{n} (x_i^{\alpha}y_i^{1-\alpha} - y)/(\alpha - 1)$ is minimum when (90) is satisfied and its minimum value is $((c/d)^{\alpha}c - d)/(\alpha - 1)$ and if $c =$ d, the minimum value is zero.

## 14.5 *SOME OTHER ALLOCATION PROBLEM*

**(a) A Cargo-loading Problem**

We consider a vessel whose maximum cargo capacity is $Z$ tons. Let $v_i$ and $w_i$ denote respectively the value end weight of the $i$ th item and let $x_i$ denote the

number of items of type $i$ chosen. The problem of determining the most valuable cargo consists in maximizing
subject to

$$L_n(X) = \sum_{i=1}^{n} x_i v_i$$

$$\sum_{i=1}^{n} x_i w_i \leq Z, x_i = 0,1,2, \ldots$$

Let $f_n(Z)$ denote the maximum value, then

$$f_1(Z) = \left[\frac{Z}{w_1}\right] v_1$$

where $[y]$ denotes the greatest integer less than or equal to $y$. The principle of optimality then gives

$$f_n(Z) = \max_{x_n}[x_n v_n + f_{n-1}(Z - x_n w_n)]$$

where the maximization with respect to $x_n$ is over the set of values

$$x_n = 0,1,2, \ldots \left[\frac{Z}{w_n}\right]$$

This is essentially a problem of linear integer programming which we have solved by using dynamic programming technique.

**(b) Reliability of Multicomponent Devices**

We consider an equipment containing $n$ components in series so that if one component fails, the whole equipment fails. For ensuring greater reliability of the equipment, we provide duplicate components in parallel at each stage. We assume that the units in each stage are supplied with switching circuits which have the property of shunting a new component into the circuit when an old one fails. We want to choose the number of components at each stage so that the probability of successful operation of the system is maximum subject to a given amount of money being available for duplicate components. Let $\varphi_j(m_j)$ denote the probability of successful operation of the system when $m_j$ components are used at the $j$ th stage. Let $c_j$ be the cost of a single component at the $j$ th stage so that we have the constraint

$$\sum_{j=1}^{n} m_j c_j \leq c$$

The reliability of the $n$-stage equipment i.e. the probability of its successful operation is given by

$$\prod_{j=1}^{n} \varphi_j(m_j)$$

Let its maximum value, which depends on $c$ and $n$ be denoted by $f_n(c)$, then by the principle of optimality

$$f_n(c) = \max_{m_n}[\varphi_n(m_n) f_{n-1}(c - c_n m_n)]$$

where $m_n$ can take value $0,1,2, \ldots [c/c_n]$. Also

$$f_1(c) = \varphi_1\left(\left[\frac{c}{m_1}\right]\right)$$

**(c) A Farmer's Problem**

A farmer starts with $q$ tons of wheat. He can sell a part, say $y$ tons for an amount $g(y)$ and he can sow the remaining $q - y$ tons and get a $(q - y)$ tons $(a \geq 1)$ out of it for further selling and sowing. It is required to find the optimum policy for him if he intends to remain in business for $n$ years. Let $f_n(q)$ be the maximum return on following an optimum policy, then by the principle of optimality
and

$$f_n(q) = \max_{0 \leq y \leq q}(g(y) + f_{n-1}(a(q - y)))$$
$$f_1(q) = \max_{0 \leq y \leq q}(g(y) + g(a(q - y)))$$

For an infinite stage process, applying the limiting process to (101), we get

$$f(q) = \max_{0\le y\le q}(g(y) + f(a(q-y)))$$

which is a functional equation to solve for $f(q)$.

**(d) A Purchase Problem**

An amount $x$ can be used to buy two equipments $A$ and $B$. If an amount $y$ is invested in type $A$, we get $g(y)$ hours of useful work in the course of a year and the equipment has a salvage value $ay(0 < a < 1)$. The remaining amount $x - y$ invested in equipment of type B gives $h(x-y)$ hours of useful work and has a salvage value $b(x-y)(0 < b < 1)$. If $f_n(x)$ is the number of useful hours on following an optimal policy, we get

$$f_n(x) = \max_{0\le y\le x}(g(y) + h(x-y) + f_{n-1}(ay + bx - by))$$
$$f_1(x) = \max_{0\le y\le x}(g(y) + h(x-y))$$

If the infinite-period optimal policy gives $f(x)$ as the number of useful hours of work, then taking the limit of (104) we get

$$f(x) = \max_{0\le y\le x}(g(y) + h(x-y) + f(ay - bx - by))$$

(e) Allocation Processes Involving Two Types of Resources

Suppose we have two types of resources in quantities $x$ and $y$ respectively. We have to allocate these resources to $n$ activities and if we allocate, $x_i, y_i$ to ith activity, the return is given by $g_i(x_i, y_i)$ so that the total return is

$$\sum_{i=1}^{n} g_i(x_i, y_i)$$

Let $f_n(x, y)$ be the maximum return for $n$ activities following an optimal policy, then the principle of optimality gives

$$f_n(x,y) = \max_{0\le x_n\le x}\max_{0\le y_n\le y}\big(g_n(x_n, y_n) + f_{n-1}(x_n - x_n, y - y_n)\big), n \ge 2$$

$$f_1(x,y) = g_1(x,y)$$

**<u>(f) Transportation Problem</u>**

We have $m$ origins $0_1, 0_2, \dots, 0_m$ where quantities $x_1, x_2, \dots, x_n$ of a certain commodity are available and these have to be supplied to $n$ destinations $D_1, D_2, \dots, D_n$ where quantities $y_1, y_2, \dots, y_n'$ are required. We further assume that

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j$$

The cost of transporting $x_{ij}$ commodities from $i$ th origin to $j$ th destination is $g_{ij}(x_{ij})$ so that we have to minimize

$$\sum_{j=1}^{n}\sum_{i=1}^{m} g_{ij}(x_{ij})$$

subject to

$$x_{ij} \geq 0, \sum_{j=1}^{n} x_{ij} = x_i, \sum_{i=1}^{m} x_{ij} = y_i, \sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j$$

Let $f_n(x_1, x_2, \ldots, x_m)$ denote the minimal cost obtained by following an optimal policy, then the principle of optimality gives

$$f_n(x_1, x_2, \ldots, x_m) = \min_{R_n}\big((g_{1n}(x_{1n}) + g_{2n}(x_{2n}) + \cdots + g_{mn}(x_{nm})\big)$$

where $R_n$ is the $m$-dimensional region determined by

$$0 \leq x_{in} \leq x_i, (i = 1,2, \ldots, m), \sum_{i=1}^{m} x_{in} = y_n$$

Instead of dealing with $mn$ independent variables $x_{ij}$ at one time, we have to minimize with respect to variations in $m$ variables at a time and the reduction in dimensionality is quite significant. Yet for $m > 2$, the problem of computation is still difficult. For $m = 2$ i.e., for the case of two origins, we get

$$f_n(x_1, x_2) = \min_{0 \leq x_{1n} \leq x_1} (g_{1n}(x_{1n}) + g_{2n}(y_2 - x_{1n})$$

which is more easily solvable.

## 14.6 *DYNAMIC PROGRAMMING AND CALCULUS OF VARIATIONS*

Let

$$I = \int_{x,y}^{x_0,y_0} F\left(x, y, \frac{dy}{dx}\right) dx$$

then the value of $I$ depends on what function $y$ is of $x$, the starting point $x, y$ and the final point $x_0, y_0$. If we choose different functions $y(x)$ and find the minimum value of $I$, this minimum value will depend on $x, y$ and $x_0, y_0$. If we keep $x_0, y_0$ fixed, the minimum value will depend on $x, y$ only. Let $f(x, y)$ be this minimum value.

To apply dynamic programming, we break up the interval $(x, x_0)$ into two parts $(x, x + \Delta x)$ and $(x + \Delta x, x_0)$. In the first interval, we choose an arbitrary slope $y'$, so that the contribution of the first interval to $I$ is

$$\int_x^{x+\Delta x} F(x, y, y')dx = F(x, y, y')\Delta x + 0(\Delta x)^2$$

The starting point for the second interval is $x + \Delta x, y + y'\Delta x$ and for this interval, we use the optimal policy to get

$$f(x + \Delta x, y + y'\Delta y) = f(x, y) + \Delta x \frac{\partial f}{\partial x} + y'\Delta x \frac{\partial f}{\partial y} + 0(\Delta x)^2$$

Applying the principle of optimality, we get

$$f(x, y) = \min_{y'} \left[\Delta x F(x, y, y') + f(x, y) + \Delta x \frac{\partial f}{\partial x} + y'\Delta x \frac{\partial f}{\partial y} + 0(\Delta x)^2\right]$$

Taking the limit as $\Delta x \to 0$

$$0 = \min_{y'} \left[F(x, y, y') + \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y}\right]$$

For the expression within brackets to be minimum

$$0 = \frac{\partial F}{\partial y'} + \frac{\partial f}{\partial y}$$

When we solve for $y'$ from Eqn. (121) and substitute in Eqn. (120) we get the minimum value of the expression as zero so that

$$0 = F(x, y, y') + \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y}$$

From Eqns. (121) and (122), we can determine

(i) $y$ as a function of $x$ and

(ii) $f(x, y)$ as a function of $x, y$.

Differentiating Eqn. (121) totally with respect to $x$, we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} y' = 0$$

Differentiating Eqn. (122) partially with respect to $y$, we get

$$F_y + F'_y \frac{\partial y'}{\partial y} + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} y' + \frac{\partial f}{\partial y}\frac{\partial y'}{\partial y} = 0$$

Eliminating $f(x, y)$, we get Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

For the more general case when there are several dependent variables $y_1, y_2, \dots, y_n$ i.e., where we have to minimize

$$I = \int F(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx$$

the equation corresponding to Eqn. (120) is

$$0 = \min_{y_1, y_2, \dots, y_n} \left[F + \frac{\partial f}{\partial x} + \sum_{j=1}^{n} y'_j \frac{\partial f}{\partial y_j}\right]$$

which gives the following two equations

$$\frac{\partial F}{\partial y'_i} + \frac{\partial F}{\partial y_i} = 0, i = 1,2, \dots, n$$

$$F + \frac{\partial f}{\partial x} + \sum_{j=1}^{n} y'_j \frac{\partial f}{\partial y_j} = 0$$

Eliminating $f_i$ we get Euler's equations

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'_1}\right) - \frac{\partial F}{\partial y_i} = 0, \mathrm{i} = 1,2, \dots, n$$

## 14.7 *SOME OTHER APPLICATIONS OF DYNAMIC PROGRAMMING*

**(a) A Defective Coin Search Problem**

We consider the problem of using an equal arms balance to detect the only heavy coin in a lot of $N$ coins of similar appearance. Let $f_N$ denote the maximum number of weightings required using an optimal policy. As each stage, we weigh one batch of $k$ coins against another and observe the result. Either the two sets of coins will balance or they will not. If the two sets balance, the heavy coin must be in the remaining $N-2k$ coins. If they do not balance, then we have already found the group of $k$ coins to which it belongs. Thus

$$f_N = 1 + \min_{0\le k\le N/2} \max[f_k, f_{N-2k}]$$

To minimize, we want $k$ and $N-2k$ to be as near as possible. Accordingly we take $k=[N/3]$ or $[N/3]+1$ depending on whether $N$ has the form $3m+1$ or $3m+2$.
(b) An Inventory Problem

At the beginning of each period, a businessman raises his stock to $y$. There is no time lag between his ordering and supplies being received. The cost of ordering an amount $z$ is $h(z)$. During a period, the probability that the demand lies between $s$ and $s+ds$ is $\varphi(s)ds$. If the demand
exceeds stocks, there is a penalty cost$p(z)$ associated with the shortage $z$. The businessman starts with a stock $x$ and wants to continue in business for $n$ periods. It is required to find $y$ so that his cost of ordering and stock-out is minimized.
In the first period, he has to spend $k(y-x)$ on ordering new stocks. If the demand lies between $s$ and $s+ds$, the expected stockout cost is $\int_y^\infty p(s-y)\varphi(s)ds$ since the cost will be there if $s\ge y$. Thus if $f_n(x)$ denotes the minimum cost for $n$ periods,

$$f_1(x) = \min\left[k(y-x) + \int_y^\infty p(s-y)\varphi(s)ds\right]$$

For writing the general recurrence relation, we note that at the end of the first period, the stock may be zero with probability $\int_y^\infty \varphi(s)ds$ or it may be $y-s$ if the demand has been for $s$ commodities in this period $(s\le y)$. The principle of optimality then gives

$$f_n(x) = \min_{y\ge x}[k(y-x) + \int_y^\infty p(s-y)\varphi(s)ds + f_{n-1}(0)\int_y^\infty \varphi(s)ds$$

(c) Optimal Exploitation of a Fishery Containing Many Interacting Species
Let $x_i(t)$ be the population of the $i$ th species at time $t$ and let $h_i(t)$ be its rate of harvesting at time $t$ so that

$$\frac{dx_i}{dt} = a_i x_i - h_i(t); i = 1,2,\dots,n$$

Let

$$h_i(t) = \alpha_i + \sum_{j=1}^{n} \beta_{ij}x_j + \gamma_i E, i = 1,2,\dots,n$$

where $E(t)$ is the effort per unit time. Let the cost of making an effort $E$ be $bE^2 - kE - m$, then the present value of the profit is

$$P = \int_0^\infty e^{-\delta t} \left\{ \sum_{i=1}^{n} p_i \left( \alpha_i + \sum_{j=1}^{n} \beta_{ij} x_j + \gamma_i E \right) - (bE^2 - kE - m) \right\}$$

where $p_i$ is the selling price per unit of the $i$ th species.
The maximum value of $P$ depends on the initial population sizes of the species. Let this maximum value be $f(R_1, R_2, \dots, R_n)$ where

$$X_i(0) = R_i, (i = 1,2, \dots, n)$$

We now split the integral in Eqn. (136) into two, over the ranges 0 to $\Delta$ and $\Delta$ to $\infty$, where $\Delta$ is small. We choose some arbitrary value for the initial effort $E$ and find the value of the first integral for this value of $E$ because $\Delta$ is small. From (136), if the maximum value is $f(R_1, R_2, \dots, R_n)$, then for the second integral, the maximum value is $f(R_1', R_2', \dots, R_n')$ when $R_1', R_2', \dots, R_n'$ are the population sizes at time $\Delta$ determined from (134), so that

$$R_1' = R_i + \Delta \left( a_i R_i - \alpha_i - \sum_{j=1}^{n} \beta_{ij} R_j - \gamma_i E \right)$$

We then find the sum of the first integral and the maximum value of the second integral. Both these depend on the choice of $E$. We now choose $E$ so as to maximize the sum. This gives the equation

$$f(R_1, R_2, \dots., R_n) = \max_E \left[ \Delta \left\{ \sum_{i=1}^{n} p_i (\alpha_i + \beta_{ij} R_j - \gamma_i E) - bE^2 - kE - m \right\} \right.$$
$$+ e^{-\delta\Delta} f \left( R_1 + \Delta \left( a_1 R_1 - \alpha_1 - \sum_{j=1}^{n} \beta_{1j} R_j - \gamma_1 E \right), \dots \right.$$
$$\left. \left. R_n + \Delta \left( a_n R_n - \alpha_n - \sum_{j=1}^{n} \beta_{nj} R_j - \gamma_n E \right) \right) \right]$$

Using Taylor's theorem expanding in power of $\Delta$, simplifying and proceeding to the limit as $\Delta \to 0$, we get

$$\delta f(R_1, R_2, \dots, R_n) = \max_E \left[ \sum_{i=1}^{n} p_i \left( \alpha_i + \sum_{j=1}^{n} \beta_{ij} R_j - \gamma_i E \right) - bE^2 - kE - m \right]$$

This gives the equations

$$\sum_{i=1}^{n} \left(p_i\gamma_i - \gamma_i \frac{\partial f}{\partial R_i}\right) - 2bE - k = 0$$

$$\delta f(R_1, R_2, \dots, R_n) = \sum_{i=1}^{n} p_i \left(\alpha_i + \sum_{j=1}^{n} \beta_{ij}R_j - m\right)$$

$$+\sum_{i=1}^{n} \left(a_iR_i - \alpha_i - \sum_{j=1}^{n} \beta_{ij}R_j\right)\frac{\partial f}{\partial R_i} + \frac{1}{4b}\left(\sum_{i=1}^{n} \gamma_i \left(p_i - \frac{\partial f}{\partial R_i}\right) - k\right)^2$$

Equation (142) gives a partial differential equation for determining $f$ as a function of $R_1, R_2, \dots, R_n$ and then Eqn. (141) determines $E(t)$.

**CHECK YOUR PROGRESS**

**1:** Which of the following is/are property/properties of a dynamic programming problem?

a ) Optimal substructure

b) Overlapping subproblems

c) Greedy approach

d) Both optimal substructure and overlapping subproblems

**2.** When dynamic programming is applied to a problem, it takes far less time as compared to other methods that don't take advantage of overlapping subproblems.

a) True

b) False

## 14.8 SUMMARY

**1. A Cargo-loading Problem**

We consider a vessel whose maximum cargo capacity is $Z$ tons. Let $v_i$ and $w_i$ denote respectively the value end weight of the $i$ th item and let $x_i$ denote the

number of items of type $i$ chosen. The problem of determining the most valuable cargo consists in maximizing
subject to

$$L_n(X) = \sum_{i=1}^{n} x_iv_i$$

$$\sum_{i=1}^{n} x_i w_i \leq Z, x_i = 0,1,2, \dots$$

Let $f_n(Z)$ denote the maximum value, then

$$f_1(Z) = \left[\frac{Z}{w_1}\right] v_1$$

where $[y]$ denotes the greatest integer less than or equal to $y$. The principle of optimality then gives

$$f_n(Z) = \max_{x_n} [x_n v_n + f_{n-1}(Z - x_n w_n)]$$

where the maximization with respect to $x_n$ is over the set of values

$$x_n = 0,1,2, \dots \left[\frac{Z}{w_n}\right]$$

This is essentially a problem of linear integer programming which we have solved by using dynamic programming technique.

**2.** Euler's equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) - \frac{\partial F}{\partial y_i} = 0, \mathrm{i} = 1,2, \dots, n$$

## 14.9 GLOSSARY

**Random Variable:** A variable that takes on random values, often denoted by ω.
**Probability Measure:** A mathematical function that assigns a probability to each possible outcome of a random experiment.
**Expectation:** A mathematical operation that calculates the average value of a random variable.

## 14.10 *REFRENCES*

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## *14.12 TERMINAL QUESTIONS*

**1.** Write Applications of Dynamic programming

**2.** Define Mathematical Modelling Through Dynamic Programming.

**3.** Define Two classes of optimization.

**4.** Write note on A Farmer's Problem.

## *14.13 ANSWERS*

CYQ 1. D

CYQ 2. A


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