Master of Science
(FOURTH SEMESTER)

MAT 609
THEORY OF RELATIVITY

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
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MAT 609

<u>COURSE INFORMATION</u>

The present self learning material **“Theory of Relativity”** has been designed for M.Sc. (Fourth Semester) learners of Uttarkhand Open University, Haldwani. This self learning material is writing for increase learner access to high-quality learning materials. This course provides a comprehensive introduction to the foundations and applications of relativity and tensor analysis across three major blocks. **Block I** begins with the **Special theory of Relativity**, covering the classical concepts leading to Einstein’s formulation, the **Lorentz transformation equations**, and their consequences in **relativistic mechanics**. It introduces the four-dimensional geometry of **Minkowski space** and explores key **applications** such as time dilation, length contraction, and relativistic dynamics. **Block II** focuses on **Tensor Analysis**, beginning with the concept of **tensors** and the **line element**, and proceeds to **geodesic equations**—which describe the paths of particles in curved spacetime—and the **curvature tensor**, crucial for understanding gravitational effects in curved geometry. **Block III** presents **General Relativity**, including the formulation of **Einstein’s field equations**, the exact **Schwarzschild solution** for spherically symmetric mass distributions, and their significance in **cosmology**, such as the expanding universe. The block concludes with an introduction to **relativistic electrodynamics**, which integrates electromagnetism into the relativistic framework. This structure equips students with both the theoretical tools and physical insights needed to study modern gravitational physics.

BLOCK I
SPECIAL RELATIVITY

UNIT 1:-Classical Theory of Relativity

CONTENTS:

1.1 *INTRODUCTION:-*

The Classical Theory of Relativity, formulated by Albert Einstein, revolutionized our understanding of space, time, and gravity. It consists of two key components: Special Relativity (1905) and General Relativity (1915). Special Relativity applies to observers in inertial frames and introduces the principle of relativity, stating that the laws of physics remain the same for all observers in uniform motion. It also establishes the constancy of the speed of light, leading to phenomena such as time dilation, length contraction, and mass-energy equivalence $E = mc^2$. General Relativity extends these ideas to accelerated frames and gravitational fields, describing gravity not as a force but as the curvature of space-time caused by mass and energy. This theory predicts effects such as gravitational time dilation, gravitational Lansing, black holes, and gravitational waves. The classical theory of relativity replaced Newton's concept of absolute space and time with a dynamic and relative framework, and it has been confirmed through numerous experiments, including light bending near the Sun, precise GPS calculations, and gravitational wave detections.

1.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To find the solutions of Inertial and Non Inertial Frame.
- To represent he Galilean Transformation
- To solved the Michelson and Morley Experiment

1.3 *THE NEWTONIAN FRAMEWORK OF SPACE AND TIME:-*

If we take into account the particle's path and velocity, we must always assume that there is a coordinate system (or frame of reference) in which the particle's position can be specified by a mean of measuring time that can determine the intervals of time at which the particle's position should be recorded, as well as some coordinates from instant to instant. The walls of a room or the position of the stars and the plumb line's direction might be thought of as examples of a coordinate system. The earth's rotational period can also be used to measure time. Such a frame of reference and the sources of time measurement allow for the verification of Newton's law or the law of mechanics, at least to a very good approximation. Newton's second law states that when a change occurs in the direction that the force acts, the rate of change of momentum is proportional to the net force impressed. i.e.

$$F = ma$$

where m is the mass of the body and a is acceleration. This law is valid in such frames of reference. However, this law does not hold in some frames of reference.

There are generally two types of reference systems:

1. Accelerated frame of reference
2. Unaccelerated frame of reference

1.4 *INERTIAL AND NON INERTIAL FRAME:-*

Inertial Frame

"The frames with respect to which an unaccelerated body appears unaccelerated are Inertial frame. In other words the frames which are at rest or in uniform translator motion relative to one other are inertial frames."

Let's look at a coordinate system that a moving body has co-ordinates (x, y, z) as well.Since the body is moving at a constant speed and is not being affected by any forces, the coordinates $x, y,$ and z are functions of time t.

$$\frac{d^2x}{dt^2} = 0, \frac{d^2y}{dt^2} = 0, \frac{d^2z}{dt^2} = 0,$$

$$\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w,$$

where u, v, w being velocity components in x, y, z directions respectively. This is Newton's first law of inertia.

We define this type of coordinate system as an inertial frame. Accordingly, "An inertial frame of reference is one in which Newton's first law is true." Or an inertial frame is an unaccelerated frame.

Non-Inertial Frames

"Non-Inertial frames" are the frames that make an unaccelerated body appear accelerated. Alternatively said, the accelerated frames are non inertial.

1.5 *GALILEAN TRANSFORMATION*:-

The outcome of research work of Galileo on the motion of the projectile led him to formulate Galilean Transformations. These are used to relates the motions which are observed by two observers in two different inertial frames. His two main results are as follows:

1. *The motion of a particle projected at any angle may be derived from the motion of the particle thrown vertically upward.*
2. *If a particle is thrown straight up from a cart which is moving with uniform speed, the observer on the cart may see the particle moving up and down, but the motion observed by an observer on*

the ground may be described by superimposing the motion of the cart into that of the projectile.

Let's look at two frames of reference, S and S', one at rest and the other traveling at a constant speed, v. Assume that the observers O and O' are located at the origins of S and S′, respectively. At any point P, they observe the identical event. Assume that the X′, Y′, and Z′ axes are parallel to one another, or that the two frames are parallel to one another. Let (x, y, z, t) and (x′, y′, z′, t′) be the coordinates of P with respect to origin O and O′, respectively.

Two frames have been chosen so that their origins overlap at time $t = 0\ (t' = 0)$.

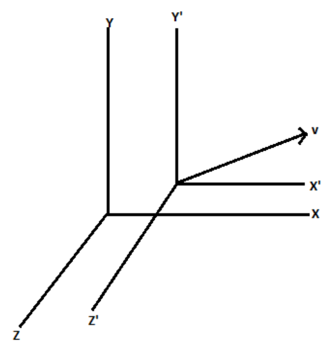


Fig.1.1

Case I: When the frame S′ have the velocity v only in X′ direction.

In that case, O' has velocity v only along the X axis (see figure 1.2). The two systems can be combined to each other by the following equations

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad \ldots (1)$$

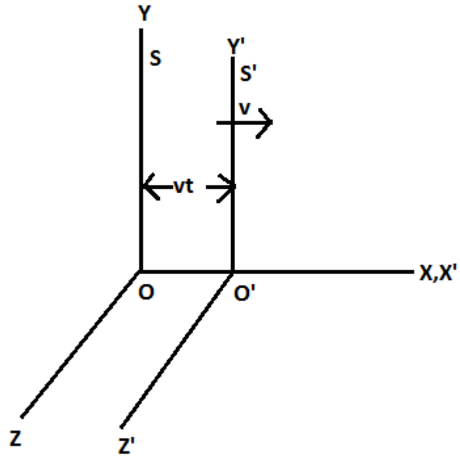


Fig.1.2

Case II: When the frame S′ have velocity v along any straight line in any direction such that $\boldsymbol{v = iv_x + jv_y + kv_z}$.

During a time t, the frame S' separated from S by tv_x, tv_y, tv_z and tv_z along the x, y, and z axes, respectively. Then, the following equations can be used to relate the two systems.

$$\left.\begin{aligned} x' &= x - tv_x \\ y' &= y - tv_y \\ z' &= z - tv_z \\ t' &= t \end{aligned}\right\} \qquad \ldots(2)$$

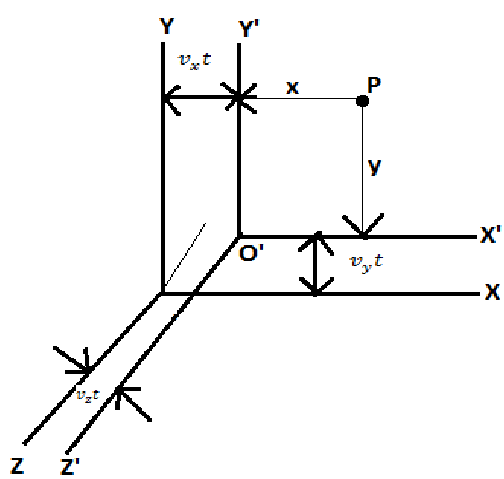


Fig.1.3

Transformations (1) and (2) are known as Galilean Transformations.

Case III: Galilean transformation in vector form:

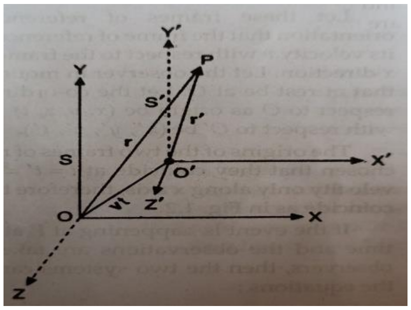


Fig.1.4

Let S and S' be two systems that are moving with a velocity v in relation to S. Two systems' origins initially coincide.

Suppose that, after time t, r and r' are the position vectors of any particle P with regard to origins O and O' of systems S and S', respectively. Then $\overrightarrow{OO'} = vt$.

Accordingly, using the law of triangles of vector addition, fig. 1.4

$$r = r' + vt$$

$$\Rightarrow \quad r' = r - vt \quad \dots(3)$$

$$also \quad t' = t \quad \dots(4)$$

These equations are known as Galilean transformations of space and time in vector form.

Equations (1), (2) and (3) are represent time dependent Galilean transformation since they are time dependent and were given by Galileo.

EXAMPLE1: Prove that the Galilean transformation of a position vector is expressed as $r = r_0 + r' + vt$, where v is the linear velocity of the frame O′ and r_0 is the position vector of origin O′ as measured by O at $t' = 0$.

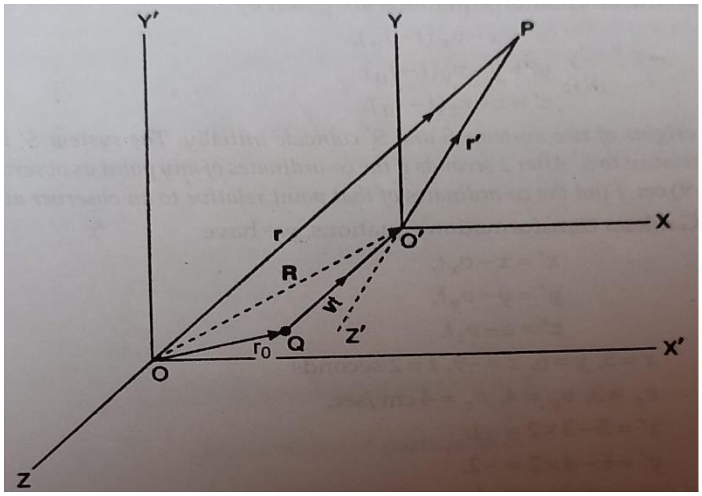


Fig.1.5

SOLUTION: Take two frames S and S', the latter moving with velocity v relative to former. Consider O and O' to be the observers of the event occurring at P, who are positioned at S and S', respectively. When r and r' represent the point P position vectors at any given time (fig. 1.5), we written

$$r = r' + R \qquad \text{... (1)}$$

where R is the position vector of observer O' relative to O after time t.

If $r_0(\overrightarrow{OQ})$ is the position vector of the observer O' relative to O at $t = 0$, then from Fig. 1.5, we get

$$R = \overrightarrow{OQ} + \overrightarrow{QO'}$$

$$= r_0 + vt \qquad \text{... (2)}$$

The distance traversed $\overrightarrow{QO'}$ by the observer O' in time t is vt.

Substituting the value of R from (2) in (1), we obtain

$$r = r' + r_0 + vt$$

EXAMPLE2: Let two systems S and S′ moving with velocity $v = iv_x + jv_y + kv_z$ relative to S. If the origins of the two systems coincide at $t = t' = t_0$, find the Galilean transformation equations.

SOLUTION: The system S′ is moving with respect to S at velocities v_x, v_y, and v_z along the X, Y, and Z axes in a positive direction, respectively. If two frames have the same origin at $t = t' = t_0$, then

The distance traversed by observer O′ in S′ relative to observer O in S at any instant t along axis of X = $v_x(t - t_0)$

The distance traversed by observer O′ relative to observer O at any instant t along Y axis = $v_y(t - t_0)$

The distance traversed by observer O′ relative to observer O at any instant t along Z axis = $v_z(t - t_0)$

Hence the Galleon transformation equations are

$$x' = x - v_x(t - t_0)$$

$$y' = y - v_y(t - t_0)$$

$$z' = z - v_z(t - t_0)$$

EXAMPLE3. The origin of two systems S and S′ coincide initially. The system S′ is moving with velocity $(3i + 4j + 6k)$cm/sec. relative to S. After 2 sec if the co-ordinates of any point as observed by an observer at the origin of S are (5, 6, -9) cm. Find the co-ordinates of the point relative to an observer at the origin of S′.

SOLUTION: we know that, the Galilean transformation equations are

$$x' = x - v_x t$$

$$y' = y - v_y t$$

$$z' = z - v_z t$$

Given $x = 5, y = 6, z = -9, t = 2sec, v_x = 3cm/sec, v_y = 4cm/sec, v_z = 4cm/sec$

$$\therefore x' = 5 - 3 \times 2 = -1$$

$$y' = 6 - 4 \times 2 = -2$$

$$z' = -9 - 6 \times 2 = -21$$

The co-ordinate of the point relative to an observer at the origin of S′ are $(-1, -2, -21)$cm.

1.6 *FICTITIOUS FORCE*:-

The presence of a mass particle in an accelerated frame relative to a stationary frame observer makes the frame non-inertial, and even when the particle is at rest, the acceleration of the frame gives the impression that a force is operating on it. The term "fictitious force" refers to this kind of force. As an illustration: Coriolis force

Example 4. Show that the length of the rod is invariant under Galilean transformation.

Solution: Let us suppose the co-ordinates of two point A and B in two inertial frame S and S′are $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and $(x_1', y_1', {z_1}'), (x_2', y_2', {z_2}')$ respectively.

If S′ is moving with velocity v relative to S along X′ axis, then according to Galilean transformation

$$\left.\begin{aligned} x_1' = x_1 - vt, y_1' = y_1, z_1' = z_1 \\ x_2' = x_2 - vt, y_2' = y_2, z_2' = z_2 \end{aligned}\right\} \qquad \dots(1)$$

The distance between the points A and B in the frame S′

$$= [({x_2}' - {x_1}')^2 + (y_2' - {y_1}')^2 + (z_2' - {z_1}')^2]^{\frac{1}{2}}$$

By using equation (1),

$$= [\{(x_2 - vt) - (x_1 - vt)\}^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}}$$

= distance between the poinsin the frame S

⇒the length of rod is invariant under Galilean transformation.

Theorem 1. Invariance of Newton's Law: To prove that Newtonian fundamental equations are invariant under Galilean transformation.

Proof: Using Newton's second rule of motion, we demonstrate this claim. A particle acted upon by a force F has an acceleration$\frac{d^2x}{dt^2}$, in the absolute system of coordinates, so that

$$F = m\frac{d^2x}{dt^2} \qquad \ldots(1)$$

In Galileon frame of reference, we get

$$x' = x - vt, y' = y, z' = z, t' = t$$

$$\Rightarrow \qquad \frac{dx'}{dt} = \frac{dx}{dt} - v, dt' = dt$$

$$\Rightarrow \qquad \frac{dx'}{dt'} = \frac{dx}{dt} - v$$

$$\Rightarrow \qquad \frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} \qquad \ldots(2)$$

Forces and masses are absolute quantities in Newtonian mechanics, thus that

$$m' = m, F' = F \qquad \ldots(3)$$

Substituting thevalues from (2) and (3) in (1)'

$$F' = m'\frac{d^2x\prime}{dt\prime^2} \qquad \ldots(4)$$

Newton's second rule of motion is invariant under Galilean transformation, according to a comparison of (1) and (4).

1.7 *ELECTRODYNAMICS*:-

The forces between two moving charges in classical mechanics are dependent on their distance from one another and are directed down the straight line that connects them. According to electrodynamics, the force between two moving charges is determined by their velocities and distance from one another. Furthermore, the active force's path does not connect the charges in a straight in a straight line.

This is how electrodynamics and classical mechanics vary from one another. Therefore, in the case of electrodynamics, the fundamental ideas of Newtonian or classical mechanics may not be applicable. Maxwell's basic equations of electrodynamics are the outcome of applying the principle of relativity to electrodynamics through experiments conducted in two distinct inertial frames. In line with Maxwell, *"Electrodynamics waves propagate in empty space with a uniform velocity $c = 3 \times 10^{10} cm/sec$. light waves are electromagnetic waves and the velocity of light in vacuum is independent to the state of motion of the source of light and is equal to the constant value, $c = 3 \times 10^{10} cm/sec$."*

Then, regardless of the motion of the light source, the velocity of light must have a constant value c with respect to all inertial frames. This is in contrast to the classical theory, which also shows in Galilean transformation that a system moving in the direction of the velocity of a particle has a lower velocity than a system at rest. Therefore, if the moving system is traveling with a constant velocity in the direction of light propagation, the velocity of light must be different in the two systems—one at rest and the other moving—and its value in the moving system must be lower than the stationary one.

As a result, the relativity principle and the constancy of the speed of light are incompatible with classical theory. Therefore, we must rethink our standard understanding of space and time if we embrace the relativity principle in the context of electromagnetism.

1.8 *FIZEAU'S EXPERIMENT*:-

The purpose of this experiment was to use ether to measure the earth's absolute velocity. Water served as Fizeau's medium within the block. He used two light beams in his experiment, one pointing in the direction of the water's velocity and the other in the opposite direction. The setup of the experiment is depicted in fig. 1.6.

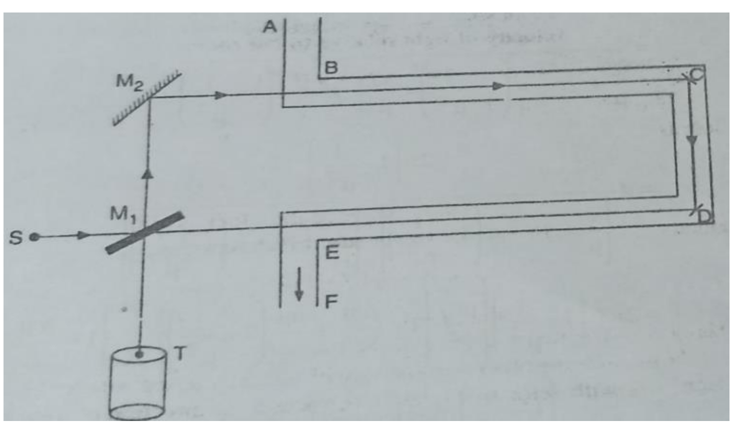


Fig.1.6

In the tube $ABCDEF$, the arrow at points A through F indicates the direction of the water flow, while S represents a light source. Following its emission from source S, the light beam strikes a semi-silvered mirror M_1, which is angled 45^0 degrees from the horizontal. A portion of the light beam falling in direction M_1 is transmitted in direction M_1D, while the other portion is reflected in direction M_1M_2.

The reflected beam follows the path $M_1M_2$is reflected towards M_2C, at M_2, is reflected again by mirrors C and D at C and D, falls on M_1 and then enters the telescope following reflection. As a result, the transmitted part follows path M_1EDCM_2M in the opposite direction of the water's velocity and subsequently enters the telescope, whereas the reflected part follows path $M_1M_2CDM_1$. Interference is a phenomena that results in interference fringes because the two beams enter the telescope at different times since they take different amounts of time to travel the same path.

The time difference between the two rays is:

$$= \frac{d}{\frac{c}{\mu} + fv - u\left(1 - \frac{1}{\mu^2}\right)} - \frac{d}{\frac{c}{\mu} + fv + u\left(1 - \frac{1}{\mu^2}\right)}$$

$$= d\,\frac{2u\left(1 - \frac{1}{\mu^2}\right)}{\left\{\frac{c}{\mu} + fv - u\left(1 - \frac{1}{\mu^2}\right)\right\}\left\{\frac{c}{\mu} + fv + u\left(1 - \frac{1}{\mu^2}\right)\right\}}$$

$$= 2ud.\left(1 - \frac{1}{\mu^2}\right)\left[\frac{\mu^2}{c^2}\left\{1 + \frac{fv\mu}{c} + \frac{\mu v}{c}\left(1 - \frac{1}{\mu^2}\right)\right\}^{-1}.\left\{1 + \frac{fv\mu}{c} + \frac{u\mu}{c}\left(1 - \frac{1}{\mu^2}\right)\right\}^{-1}\right]$$

$$= \frac{2ud\mu^2}{c^2}\left(1 - \frac{1}{\mu^2}\right)$$

(Neglecting higher order terms).

One oscillator's period T is divided by the equation to determine the phase difference. Let n be the frequency at which the light oscillates. Consequently, $nT = 1$.

Phase difference:

$$= \frac{2ud}{c^2T}(\mu^2 - 1) = \frac{2udn}{c^2}(\mu^2 - 1) \qquad \ldots (1)$$

Fizeau noticed that the interference pattern's fringes were shifting, and he discovered that this was just what equation (1) predicted. He was unsuccessful in determining the earth's velocity.

1.9 *MICHELSON AND MORLEY EXPERIMENT*:-

The Michelson and Morley experiment, conducted in 1887 by Albert A. Michelson and Edward W. Morley, is one of the most significant experiments in the history of physics. It was designed to detect the presence of a hypothetical medium known as the luminiferous aether, which was believed to be the substance through which light waves propagated analogous to how sound waves require air. Using an optical device called an interferometer, the experiment attempted to measure differences in the speed of light in perpendicular directions, assuming that the Earth's motion through the aether would cause a measurable "aether wind." According to classical physics, light moving with or against this wind would have different speeds, leading to observable shifts in interference patterns. However, the experiment produced a null result no significant change in the interference pattern was observed. This surprising outcome strongly suggested that the speed of light is constant in all directions; regardless of the motion of the source or the observer relative to the supposed aether.The implications of this result were profound. It undermined the ether theory and paved the way for Albert Einstein's theory of Special Relativity (1905), which postulated that the speed of light in a vacuum is constant for all inertial observers and that space and time are interwoven into a single continuum: space-time.

The experiment is set up so that a monochromatic light source S shines on a half-silvered plate P_1 that is angled $45\,^0$degrees from the light beam from S. The light beam is split into two halves by the half-silvered plate P_1; one is transmitted through it, and the other is reflected perpendicular to its initial direction. The reflected beam enters the telescope T after generally striking a flat mirror M_1 at A and reflecting back along its own path, passing via P_1. After being transmitted through P_1, the other beam travels through a plate P_2 that is parallel to it and has a thickness equal to P_1. It then travels along its own route and is generally reflected by a plane mirror M_2 at B. After going through P_2, the reflected beam hits P_1 and is reflected back to the telescope T. Both mirrors, M_1 and M_2, are at similar distances from P_1, such that ., $P_1B = P_1A$, and are highly polished to prevent double total internal reflection.

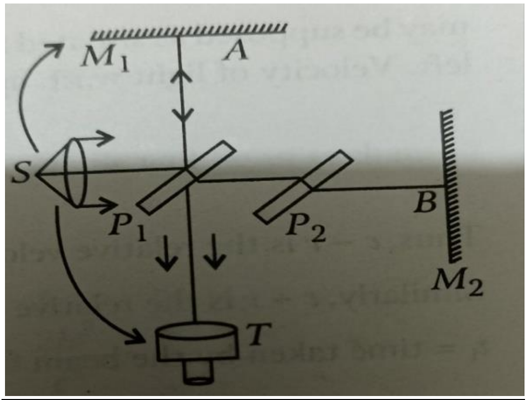


Fig.1.7

Interference fringes are created when the two beams that are reflected from the mirrors M_1 and M_2 enter the telescope. Since the reflected beam crosses plate P_2 twice, the plate P_2 serves just to offset the additional path it takes. Therefore, the plate P_2 is added in the path so that the two beams before entering the telescope can travel equal distances, allowing the transmitted beam to have to travel an equal additional distance.

Let's now assume that the entire apparatus is traveling toward the right at the earth's velocity, or along SP_1B ether, while remaining stationary. Because of the earth's motion, the beam is reflected by the mirror M_1 at A' rather than A, and again by M_2 at B' rather than B. During this period, the earth's motion causes the plate P_1 to shift to $P_1{}'$, which causes the two beams in the telescope T to collide. It is clear that in this instance, the transmitted and reflected beams' journey lengths are not equal.

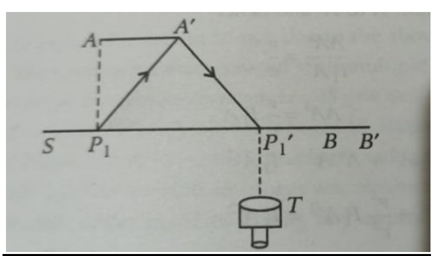


Fig.1.8

Suppose

$c =$ velocity of light w.r.t. ether.

v = velocity of earth w.r.t. ether, $P_1A = l$

$t_1 =$ time taken by the transmitted beam which travels the distance from P_1 $to\ B' and\ then\ B' to\ P_1{}'$

$t_2 =$ time taken by the reflected beam which travels from P_1 $to\ A' and\ then\ A' to\ P_1{}'$

Since the mirror M_1 at A and the mirror M_2 at B are placed at equal distance and therefore

$$P_1A = P_1B = l$$

To calculatet_1

The device is traveling to the right with a velocity of v. Consider the laboratory to be the norm. It implies that the device could be assumed to be located in the stream of ether traveling leftward at velocity v. velocity of light in relation to the device in the appropriate direction.

= velocity of light w.r.t. ether + velocity of ether w.r.t. apparatus

$$= c + (-v) = c - v$$

Thus, $c - v$ is the relative velocity of light along P_1B'.

Similarly, $c + v$ is the relative velocity of light along $B'^{P_1{}'}$

$t_1 =$ time taken by the beam from P_1 $to\ B' and\ B' to\ P_1{}'$

$$= \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2lc}{c^2}\left[1 - \frac{v^2}{c^2}\right]^{-1}$$

$$= \frac{2lc}{c^2}\left[1 + \frac{v^2}{c^2}\right] = \frac{2l}{c}\left[1 + \frac{v^2}{c^2}\right]$$

v being small in comparison to $c.\frac{v^3}{c^3}$ and higher power of v/c have been neglected.

To calculate t_2: for the path $P_1A'P_1{}'$, when the beam travels from $P_1 to\ A'$, the apparatus travels from A to A′ and hence

$$\frac{AA'}{P_1A'} = \frac{v}{c}$$

$$AA' = \frac{v}{c} P_1A'$$

$$\text{In } \Delta P_1AA', \ P_1A^2 + A'A^2 = P_1A'^2$$

$$l^2 + \frac{v^2}{c^2}P_1A'^2 = P_1A'^2$$

$$P_1A' = \frac{l}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} = l\left[1-\frac{v^2}{c^2}\right]^{-1/2} = l\left[1+\frac{v^2}{2c^2}\right]$$

$t_2 =$ time taken by the beam from $P_1 to\ A' and\ A' to\ P_1{}'$

= 2 time taken by the beam from $P_1 to\ A'$

$$= 2\frac{P_1A'}{c}. For\ time = \frac{distance}{velocity}$$

$$= \frac{2l}{c}\left[1+\frac{v^2}{2c^2}\right]$$

The difference in two timings results from the relative distance with respect to the apparatus being l in the travel towards b and backwards.

$$\Delta t = t_1 - t_2 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)$$

$$= \frac{2l}{c}\left[1+\frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right] = \frac{lv^2}{c^3}$$

The two systems will be in reverse positions when the device is turned $90°$ degrees and hence.

$$\overline{t_1} = \overline{t_2}, \overline{t_2} = t_1$$

$$\Delta\bar{t} = \overline{t_1} - \overline{t_2} = t_2 - t_1 = -(t_1 - t_2) = -\Delta t$$

$$\Delta\bar{t} - \Delta t = -\Delta t - \Delta t = -2\Delta t = -\frac{2lv^2}{c^3}$$

Let T be the time period, n the frequency.

Then $nT = 1$

$$phase\ difference = \frac{time\ difference}{T}$$

$$= \frac{2lv^2}{c^3} \cdot \frac{n}{1}$$

$$phase\ difference = -\frac{2nlv^2}{c^3}$$

Therefore, a shift in fringes could be anticipated as a result of the aforementioned discrepancy, but none was seen. In order to increase the distance, the experiment was performed using many mirrors; nonetheless, the outcome remained the same, i.e., all produced null results. $v = 0$ could be one reason, although Fresnel's law of drift contradicts this supposition. No change in fringes was seen when Trouton and Noble conducted the experiment again in 1904 using electromagnetic waves rather than visible light. All recent attempts to precisely avoid potential mistakes were unable to significantly alter the initial outcome.

1.10 *EXPLANATION OF NEGATIVE RESULTS*:-

1. Drag Theory: As started earlier, one possible explanation is $v = 0$ i.e. velocity of earth relative to ether is zero.

This gives $t_1 = t_2$. for

$$t_1 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right), t_2 = \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

In that case, the earth and the ether have no relative velocity. Stated differently, the ether is pulled with the earth's gravity at the same speed as the earth. There shouldn't be any aberration light, though, if this argument is adopted. The value of aberration and the lack of shift in fringes cannot be explained at the same time, even if the ether is thought to be partially pulled.

Lorentz and Fitzgerald Contraction Hypothesis: Fitzgerald proposed a theory in 1892 to account for the negative results of the Michelson-Morley experiment, which became known as the Lorentz Fitzgerald contraction hypothesis. Their hypothesis is "*All material bodies moving with velocity v are contracted in the direction of motion by a factor* $(1 - \beta^2)^{1/2} where\ \beta = \frac{v}{c}$ ".

According to this hypothesis, if l_0 is the length of a body at rest with regard to ether and l is its length when it is moving with velocity v with respect to ether, then$l = l_0(1-\beta^2)^{1/2}$.

Using this idea, they offered the following explanation for the negative results of the Michelson-Morley experiment:

$$t_1 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right), t_2 = \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)$$

According to Lorentz and Fitzgerald, then we get

$$t_1 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

$$= \frac{2l}{c}\left(1-\frac{v^2}{2c^2}\right)\left(1+\frac{v^2}{c^2}\right)$$

$$= \frac{2l}{c}\left(1+\frac{v^2}{c^2}-\frac{v^2}{2c^2}-\frac{v^4}{2c^4}\right) = \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right) = t_2$$

Neglecting the term $\frac{v^4}{c^4}$

$$\therefore t_1 = t_2.$$

Consequently, there is no phase difference between the transmitted and reflected beams because their response times are equal. As a result, no change in fringes is seen.

Example 5: In an experiment, the length of the arm of the interferometer was 11 meters, the wavelength of light 5.5×10^{-5} centimeters and the earth velocity 30 km/sec, calculate the amount of the fringe-shift.

Solution: given: $l = 1100cm, \lambda = 5.5 \times 10^{-5}cm, v = 30 \times 10^5 cm/sec, c = 3 \times 10^{10} cm/sec$

The required fringe-shift i.e., phase difference x_0 is given by

$$x_0 = \frac{2lnv^2}{c^3} = \frac{2lv^2}{c^3}.\frac{c}{\lambda} = \frac{2lv^2}{c^2\lambda}$$

$$x_0 = \frac{2 \times 1100 \times (3 \times 10^6)^2}{(3 \times 10^{10})^2 \times 5.5 \times 10^{-5}} = \frac{2}{5} = 0.4$$

SELF CHECK QUESTIONS

1. Frames for which law of inertia is valid are called
a) Inertial
b) Rotational
c) Non-inertial
d) None of these
2. the reference frame where fundamental laws of physics are invariant are called:
a) rotational
b) inertial frame
c) accelerated frame
d) frame attached to earth
3. the fundamental laws of physics are the same which are :
a) rotationary frame
b) inertial frame
c) accelerated frame
d) frames connected to earth
4. In Michelson-Morley experiment if the effective lenghth of path is 7 meter and wavelength of light is 700 $A°$, then fringe displacement is
a) 0.2
b) 0.1
c) 0.4
d) 0
5. Newton's 1^{st} law of motion holds good in
a) Inertial frame
b) Every frame
c) Non-inertial frame
d) None of these
6. Newton's 2^{nd} law of motion is invariant under
a) Galilean transformation
b) Lorentz transformation
c) Both of the above
d) None of these

1.11 *SUMMARY:-*

In this unit, we explored the Newtonian framework of space and time, where space and time are treated as absolute and independent entities. We distinguished between inertial frames (those moving with constant velocity) and non-inertial frames (accelerating frames), introducing the concept of fictitious forces that appear in non-inertial frames to explain apparent accelerations. The Galilean transformation was studied as the mathematical tool to relate the coordinates and velocities between different inertial frames in classical mechanics. We extended our study to classical electrodynamics, highlighting the inconsistencies it faced when subjected to Galilean transformations. Experimental efforts to resolve these issues, such as Fizeau's experiment (which tested the speed of light in moving media) and the Michelson-Morley experiment (which attempted to detect the aether), were discussed in detail. The null result of the Michelson-Morley experiment and its explanation marked a pivotal shift in physics, setting the stage for Einstein's theory of Special Relativity.

1.12 *GLOSSARY*:-

- **Relativity:** The principle that the laws of physics are the same in all inertial frames of reference.
- **Galilean Relativity:** A classical theory proposed by Galileo stating that mechanical laws are invariant under Galilean transformations between inertial frames.
- **Inertial Frame of Reference:** A frame of reference in which a body remains at rest or moves at constant velocity unless acted upon by a force.
- **Non-Inertial Frame of Reference:** A frame that is accelerating, where fictitious forces (like centrifugal force) must be introduced to apply Newton's laws.
- **Galilean Transformation:** Equations used to transform coordinates and time between two inertial frames in classical mechanics.
- **Absolute Time:** The concept in Newtonian mechanics that time flows uniformly for all observers, independent of their motion.
- **Absolute Space:** The idea that space exists independently and is the same for all observers.
- **Relative Motion:** The change in position of an object with respect to a particular frame of reference.

- **Velocity Addition Law:** In Galilean relativity, velocities add linearly ($e.g., u' = u + v$).
- **Fictitious Force:** A force that appears when observing motion from a non-inertial frame (e.g., Coriolis force or centrifugal force).
- **Michelson-Morley Experiment:** A famous experiment aimed at detecting the motion of Earth through the aether; it yielded a null result, challenging classical relativity.
- **Fizeau's Experiment:** An experiment measuring the speed of light in moving water, supporting the idea of partial aether drag.
- **Electrodynamics:** The study of electric and magnetic fields, particularly how they behave with moving charges; classical electrodynamics struggled under Galilean transformations.
- **Aether:** A hypothetical medium once thought to carry light waves through space.
- **Null Result:** An experimental result showing no expected effect; in relativity, often refers to the Michelson-Morley experiment's failure to detect the aether.

1.13 *REFERENCES*:-

- James J. Callahan (2019), "The Geometry of Spacetime: An Introduction to Special and General Relativity" (2nd Edition.
- Spencer A. Klein (2017), "Relativistic Mechanics and Electrodynamics".
- Ta-Pei Cheng (2015),"Relativity, Gravitation and Cosmology: A Basic Introduction" (2nd Edition).

1.14 *SUGGESTED READING*:-

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

1.15 *TERMINAL QUESTIONS:-*

(TQ-1) Discuss Michelson-Morley experiment and explain the outcome of this experiment.

(TQ-2) Explain Lorentz- Fitzgerald contraction idea. How was the idea used to account for the negative result of the Michelson-Morley experiment?

(TQ-3) In the Michelson-Morley experiment, the wavelength of the monochromatic light used in $5000A°$. What will be the expected fringe-shift on the basis of stationary ether hypothesis if the effective length of each path be 5 meter? Given velocity of earth =3$\times 10^4$ m/sec and $c = 3 \times 10^8 m/sec$.

(TQ-4) in Fizeau's experiment, the approximation values of the parameters were as follows

$$l = 1.5\, m, n = 1.33, \lambda = 5.3 \times 10^{-7} m, v_w = 7 m/sec$$

A shift of 0.23m fringes was observed from the case $v_w = 0$. Calculate the drag coefficient and compare it with the predicted value.

1.16 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

1. (a)
2. (b)
3. (b)
4. (a)
5. (a)
6. (a)

TERMINAL ANSWERS (TQ'S)

(TQ-3) $\frac{1}{5}m$

(TQ-4) d=0.4922

UNIT 2:- Lorentz Transformation Equations

CONTENTS:

2.1 *INTRODUCTION:-*

The Lorentz transformation is a set of linear equations that relate the space and time coordinates of events as measured in two different inertial frames of reference moving at a constant velocity relative to each other. It was developed by Hendrik Lorentz and later incorporated into Einstein's Special Theory of Relativity. The need for Lorentz transformation arose when the classical Galilean transformation failed to explain phenomena involving the speed of light, such as the null result of the Michelson-Morley experiment. According to Einstein's second postulate of special relativity, the speed of light is constant in all inertial frames, which contradicted the assumptions of classical mechanics. Lorentz transformations preserve the constancy of the speed of light and the form of the equations of electrodynamics (Maxwell's equations) across different inertial frames. They show that measurements of time, length, and simultaneity are relative, depending on the observer's frame of reference. This leads to important relativistic effects such as time dilation and length contraction.

2.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To explain relativistic concept of space and time.
- To explain the postulates of special theory of relativity.
- To derive Lorentz transformation equations.
- To understand the Doppler's Effect.

2.3 *THE RELATIVISTIC CONCEPT OF SPACE AND TIME:-*

Despite careful studies, Michelson-Morley, Trouton, Noble, and others were unable to ascertain the earth's velocity with respect to ether. Lorentz and Fitzgerald put out the following theory in an effort to explain the undesirable result: Every inertial frame requires the use of unique space coordinates (Fitzgerald hypothesis) and unique time coordinates (Lorentz hypothesis), which differ from the time and space coordinates in the absolute ether system. Thus, Lorentz and Fitzgerald suspected a new understanding of space and time in addition to the concept of an absolute ether frame. Einstein boldly asserted in 1905 that whereas motion via ether is a useless idea, motion relative to material entities has physical substance. This was done in response to the unfavorable findings of the Michelson-Morley and other tests that were carried out to ascertain the earth's velocity through ether, as well as the scientific tradition that prohibits making assumptions about things that are by definition impossible to observe. In other words, there is no absolute frame; all frames can be used to explain motion, but there may be circumstances in which a certain frame is more useful than others. He therefore rejected the notion that space is absolute. To clarify the two contradicting claims that follow:

1. The velocity of any motion varies depending on how observers move in relation to one another, according to classical mechanics.
2. Experimental investigations indicate that the motion of the frame of reference has no effect on the velocity of light.

Einstein concluded that the disagreement between them had to be caused by a flaw in the traditional theories of measuring time and space. By challenging our preconceived notions of simultaneity, he disproved the idea of absolute time.

His argument's nature can be viewed as follows:

The phrase "The two events X and Y take place simultaneously without reference of any co-ordinate system" may be meaningless, but let's check.

Let's examine a light signal that travels from point X to point Y in a straight path within a given inertial frame. Only if the clocks at X and Y are positioned correctly will the difference $(t_2 - t_1)$ obtained in this way provide the actual time it takes for light to travel from X to Y. This is presuming that a clock at X reads the emission time f and a clock at Y reads the arrival time t_2. The clocks at X and Y must be synced because this obviously requires that both clocks' hands be in the same place at the same time.

Now, how can we make sure that the two events occurring in two different places are occurring simultaneously? Will two events that happen at the same time in one frame also happen in any other frame? Examine two occurrences that occur in the inertial frame S at two fixed locations, X and Y. Since the velocity of light is c in all directions, it follows that for these occurrences to occur simultaneously with regard to system $S,$ the two light signals released from X and Y at the time of the events must meet in the center O of the line connecting X and Y .A same condition for simultaneity also applies to system $S',$ which has a constant velocity v compared to system S. Assume for the moment that the two events take place in relation to system S at the same time and that the line connecting X and Y runs parallel to the direction of system S' velocity. Next, examine two points X' and Y' in system S' that correspond to those points at the moment of the events. At that point, O will coincide with the center O' between X' and Y'. Like X' and Y', O' now moves with system S' at a velocity v with respect to system $S. O'$ will not coincide with O at the intersection of the light signals from X and Y. According to the aforementioned criterion, the two events are not synchronous with respect to system S' since the light signals do not meet in O'.

Simultaneity is therefore a relative rather than an absolute concept. Therefore, the idea of simultaneity between two events in separate spatial positions has an accurate meaning only when referring to a certain inertial system. In other words, each frame of reference has its own time. Therefore, unless we mention the reference system to which the time statement is referencing, expressing the time of an occurrence has no value. Since the absolute concept of simultaneity is excluded, the absolute concept of space is also ultimately excluded. Finding an object's end points at the same time is necessary to measure its length. The length measurements will be influenced by the frame of reference in the same manner as simultaneity is dependent on it. Therefore, rather from being absolute, the concept of length, or space, is relative. The experimental observation that observers moving relative to each other measure the speed of light at the same speed can be explained by Einstein's relativistic theories of space and time. Given these new concepts of space and time, a new class of transformation equations based on the invariant nature of the speed of light must be developed to replace the Galilean transformation equations.

The theory of relativity, which is separated into two sections and has a novel concept of space and time, applies to all optical and electromagnetic phenomena in addition to mechanical phenomena.

1. Special or restricted theory of relativity.
2. General theory of relativity.

According to the special theory of relativity, systems that move in uniform rectilinear motion relative to one another are called inertial systems. Thus, *"All systems of co-ordinates are equally suitable for description of physical phenomena."* When applied to accelerated systems that is, systems moving more quickly than one another the theory of relativity is known as the *"general theory of relativity."* The general theory of relativity provides a more sophisticated explanation of the laws of gravitation than Newton did, and it is relevant to them.

2.4 *POSTULATES OF SPECIAL THEORY OF RELATIVITY:-*

1. "*The natural laws must preserve their forms relative to all observers in a state of relative uniform motion.*"

According to this postulate, velocity is not absolute but relative. It is a fact drawn from the failure of Michelson and Morley experiment which was performed to determine velocity of earth through ether.

2. *"The velocity of light in vacuum is independent of the velocity of observer or the velocity of the source."*

According to Galilean transformation this postulate is not true. In fact, it is confirmed experimentally that the velocity of light calculated by any method is constant. The second postulate is important in the sense that it gives a clear distinction between classical theory and Einstein theory of relativity.

2.5 *LORENTZ TRANSFORMATION*:-

According to Einstein the theory of relativity is applicable to laws of optics. Thus for the constancy of velocity of light we have to introduce the new transformation equations which fulfill the following requirements:

1. The speed of light c must have the same value in every inertial frame.
2. The transformation must be linear and for low speed $v \ll c$ they should approach the Galilean transformations.
3. They should not be based on "absolute time and absolute space".

The above requirements were fulfilled by H. A. Lorentz by introducing transformation equations relating the observations of position and times made by two observers in two different inertial frames and are known as "Lorentz Transformation Equations".

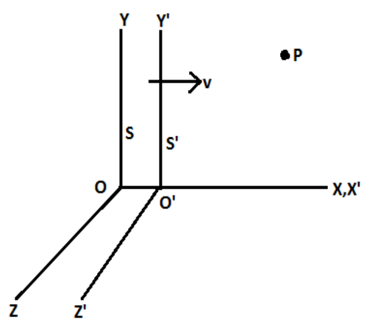


Fig.2.1

Assume that S and S' are two inertial frames of reference, and that S' is traveling relative to S with a constant velocity v. Assume that two observers Any event P from systems S and S' is observed by O and O'. Its coordinates are $(x.y, z, t)$ and(x', y', z', t') in S and S', respectively. When the origins of two frames coincide and both t and t' are zero, the event P a light signal is created.

Points which are at rest relative to S′ will move with velocity v relative to S in X-direction. In particular the point $x' = 0$ will move with velocity v in X-direction, i.e. $x' = 0$ will be identical with $x = vt$ so that

$$x' = \alpha(x - vt) \quad \dots (1)$$

Where α is some function of v.

Since the velocity of S′ is only along X-axis. Hence by symmetry

$$\left.\begin{matrix} y' = y \\ z' = z \end{matrix}\right\} \qquad \ldots(2)$$

An equation in t and t' must be created in order to complete the set of equations. Linearly, t' depends on $t, x, y,$ and z. Since clocks in S' would seem to conflict as seen from S, we suppose that t' does not depend on y and z for the zone of symmetry. Consequently, we have

$$t' = \beta t + \gamma x \qquad \ldots(3)$$

Where β and γ both are functions of v only. We are to determine the unknown α, β, γ.

The light pulse generated at $t = 0$ will expand into a growing sphere, and the wavefront radius will increase with speed c. Since (x, y, z, t) are the event's coordinates from the observer in system S at rest, the equation for a spherical surface whose radius increases with speed c is

$$x^2 + y^2 + z^2 = c^2t^2 \qquad \ldots(4)$$

Similarly the equation of spherical surface for observer O′ in system S′ is

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad \ldots(5)$$

From equation (1), (2) & (3) substituting value x', y', z' & t' in equation (5)

$$\alpha^2(x - vt)^2 + y^2 + z^2 = c^2(\beta t + x\gamma)^2$$

$$x^2(\alpha^2 - c^2\gamma^2) + y^2 + z^2 - 2xt(\alpha^2 v + c^2\beta\gamma) = (c^2\beta^2 - \alpha^2v^2)t^2$$

Equation (4) and the equation above both describe the same motion. Therefore, when we compare the coefficients of different terms, we get

$$\alpha^2 - c^2\gamma^2 = 1 \quad [6(i)]$$

$$\alpha^2 v + c^2\beta\gamma = 0 \quad [6(ii)]$$

$$c^2\beta^2 - \alpha^2v^2 = c^2 \quad [6(iii)]$$

$v \times [6(i)] - [6(ii)]$ gives

$$-c^2\gamma^2 v - c^2\beta\gamma = v$$

$$v(1 + c^2\gamma^2) + c^2\beta\gamma = 0 \qquad \dots(7)$$

Similarly $v^2 \times [6(i)] + [6(iii)]$ gives

$$-v^2c^2\gamma^2 + c^2\beta^2 = v^2 + c^2$$

$$-v^2(1 + c^2\gamma^2) + c^2\beta^2 = c^2 \quad \dots(8)$$

Similarly $v \times (7) + (8)$ gives

$$vc^2\beta\gamma + c^2\beta^2 = c^2$$

$$v\beta\gamma + \beta^2 = 1$$

$$\beta^2 - 1 = -v\beta\gamma \quad \dots(9)$$

Removing γ between (7) and (9)

$$v\left\{1 + c^2\left(\frac{\beta^2 - 1}{v\beta}\right)^2\right\} + c^2\left(\frac{1 - \beta^2}{v}\right) = 0$$

$$\frac{v[v^2\beta^2 + c^2(\beta^2 - 1)^2]}{v^2\beta^2} + \frac{[c^2(1 - \beta^2)]}{v} = 0$$

$$v^2\beta^2 + c^2(\beta^2 - 1)^2 + c^2\beta^2(1 - \beta^2) = 0$$

$$\beta^2[v^2 + c^2 - 2c^2] + c^2 = 0$$

$$\beta^2 = \frac{c^2}{c^2 - v^2}$$

Putting the value of β^2 in equation [6(iii)]

$$\frac{c^4}{c^2 - v^2} - \alpha^2 v^2 = c^2$$

$$\alpha^2 v^2 = \frac{c^4}{c^2 - v^2} - c^2 = \frac{v^2c^2}{c^2 - v^2}$$

$$\alpha^2 = \frac{c^2}{c^2 - v^2} = \beta^2$$

From equation [6(ii)]

$$\gamma = \frac{-\alpha^2 v}{c^2 \beta} = \frac{-\alpha v}{c^2} = \frac{-\beta v}{c^2}$$

$$\because \alpha = \beta$$

Also

$$\alpha = \beta = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$x' = \alpha(x - vt) = \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt)$$

$$t' = \beta t + \gamma x = \beta t - \beta \frac{vx}{c^2} = \beta\left(t - \frac{vx}{c^2}\right)$$

Thus Lorentz transformation equations are

$$x' = \beta(x - vt), y' = y, z' = z, t' = \beta\left(t - \frac{vx}{c^2}\right)$$

Where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note: If v is very small, then $(v/c) \to 0 \; so \; that \; \beta \to 1$. In this case Lorentz transformation equations becomes

$$x' = (x - vt), y' = y, z' = z, t' = t$$

These are the equations of Galilean transformation. So that Lorentz transformation reduce to Galilean transformation as $v \ll c$.

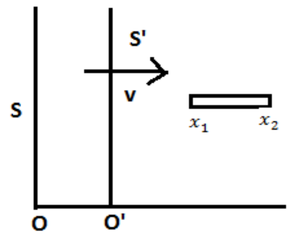
2.6 *CONSEQUENCE OF LORENTZ TRANSFORMATION:-*

(1) Lorentz and Fitzgerald Contraction (Length Contraction)

For a system S′ moving with velocity v in relation to a system S, the Lorentz transformation equations are obtained by

$$x' = \beta(x - vt), y' = y, z' = z, t' = \beta\left(t - \frac{vx}{c^2}\right)$$

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Assume that a rod of length l has been placed on the X-axis. If x_1& x_2 are the rod's end points, then

$$l = x_2 - x_1 \quad ...(1)$$

Since the measurement of the both ends are taken at the same time t, then

$$t_1 - t_2 = t$$

Assume that, according to an observer S' system, $x_1^{'} and\, x_2{'}$ are the same rod's locations along the X-axis at time t'. At time t', the rod's two end locations are simultaneously observed in order to

$$t_1' = t_2' = t'$$

$$l' = length\, of\, rod\, in\, S' system = x_2' - x_1{'}$$

By Lorentz transformation equations

$$x_1 = \beta(x_1' + vt_1'), x_2 = \beta(x_2' + vt_2')$$

Substituting the value of x_1& x_2 in equation (1)

$$l = \beta(x_2' + vt_2') - \beta(x_1' + vt_1')$$

$$l = \beta(x_2' - x_1') + v\beta(t_2' - t_1') = \beta l'$$

$$\because t_1' = t_2{'}$$

$$l' = \frac{l}{\beta} = l\left(1 - \frac{v^2}{c^2}\right)^{1/2} < l$$

$$\therefore l' < l$$

This illustrates how the ratio $(1-v^2/c^2)^{1/2}$ decreases the apparent length of a rigid body in the direction of motion.

2.7 *TIME DILATION OR APPARENT RETARDATION OF REST:-*

Examine two reference frames, $S\ and\ S'$. S' is traveling along the X-axis at a uniform velocity, v. Let $x = x_1$ be the location of a clock in system S. Let $t_1{}'$ be the time that an observer in S' measures in relation to the signal that this clock provides at time $t = t\ in\ S$. Afterward, by Lorentz transformation

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \quad ...(1)$$

$$where\ \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Let's say that at time t_2 in S, the clock gives another signal, and t_2' is the matching time in S'. Then,

$$t_2' = \beta\left(t_2 - \frac{vx_1}{c^2}\right) \quad ...(2)$$

Compose $\Delta t = t_2 - t_1, \Delta t' = t_2{}' - t_1'$

Equation (2)-(1) gives,

$$\Delta t' = \beta\left(t_2 - \frac{vx_1}{c^2}\right) - \beta\left(t_1 - \frac{vx_1}{c^2}\right)$$

$$\Delta t' = \beta\Delta t \quad ...(3)$$

$$\Delta t' = \Delta t\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \left(1+\frac{v^2}{2c^2}\right)\Delta t' > \Delta t$$

The following is equation (3)'s physical significance:

The interval $\Delta t'$, as it appears to the observer in motion, is lengthened, *i.e, the time is dilated and hence the name "time est dilation". It means that the time interval At appears to be dilated or miengthened by the factor ß to the moving observer.* Therefore, a clock moving in relation to an observer is observed to run more slowly than one that is at rest in relation to him, as per (3). Stated otherwise, a physical process with a finite duration will go significantly more slowly in a moving frame than it would in a stationary frame.

Let's do it the other way around and suppose that the clock is at $x_1{}'$. in system S'. When the clock gives a signal at t_1'in S', and t_1 is the observer's measurement of the corresponding time in S... As per the Lorentz inverse transformation,

$$t_1 = \beta\left(t_1{}' + \frac{vx_1}{c^2}\right)$$

$$t_2 = \beta\left(t_2{}' + \frac{vx_1{}'}{c^2}\right)$$

Then $\Delta t = t_2 - t_1 = \beta(t_2{}' - t_1{}') = \beta\Delta t' \quad ...(4)$

$$\Rightarrow \Delta t > \Delta t'$$

It states that, to an observer traveling with velocity v in relation to S′, the time interval $\Delta t'$ seems to be dilated by the factorβ. This is the same conclusion that was previously discussed.

Thus, based on the reasoning above, we may say, "A moving clock always appears to go slow". As a result, the clock at rest seems to be delayed by the factor $\sqrt{1-\frac{v^2}{c^2}}$ to the observer in motion. This means that: This appears to be clock retardation. From what has been done it follows:

Every clock appears to go at its fastest rate when it is at rest relative to the observer. If the clock moves w.r.t. the observer with velocity r, then it appears to go at its slowest rate by the factor.

$$\sqrt{1-\frac{v^2}{c^2}}$$

Thus, the issue deduction Clock Hypothesis or Clock Paradox The observer in S believes that the clock in S' is moving slowly, while from S' perspective, the clock in S' is moving quickly. As a result, when he returns to S', he discovers the exact opposite phenomenon.

2.8 *SIMULTANEITY*:-

Any two events are said to be simultaneous if they occur at the same time.

Let $S\ and\ S'$ be two frames of reference. S' is traveling along the X-axis with velocity v. Also let two events occur simultaneously in S at two distinct points $P_1(x_1, y_1, z_1, t_1) and\ P_2(x_2, y_2, z_2, t_2)$ so that

$$x_1 \neq x_2, t_1 \neq t_2$$

Since the events are simultaneous in S so that

$$t_1 = t_2$$

Assume that $t_1{}'$ and $t_2{}'$ are the times in S' that correspond to time $t_1\, and\, t_2$ in S. Through Lorentz transformations

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right), t_2{}' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$t_2' - t_1' = \beta\left(t_2 - \frac{vx_2}{c^2}\right) - \beta\left(t_1 - \frac{vx_1}{c^2}\right)$$

$$= \beta(t_2 - t_1) + \beta\frac{v}{c^2}(x_1 - x_2)$$

$$= \beta\frac{v}{c^2}(x_1 - x_2)\; for\; t_1 = t_2$$

However, since$x_1 \neq x_2$, the final statement reads $t_1{}' \neq t_2{}'$. Thus, in S', the two events occurrences are not happening at the same time.

Two events $(P_1\, and\, P_2)$ at two distinct locations For an observer S' traveling with velocity v relative to S along the X-axis, which are simultaneous for an observer at rest in S, are no longer simultaneous. It demonstrates that simultaneity is relative rather than absolute.

2.9 *LORENTZ TRANSFORMATION FOR A GROUP OR GROUP PROPERTY OF LORENTZ TRANSFORMATIONS:-*

Theorem 1: To prove that Lorentz transformations from a group.

Or

Show that the result of two successive Lorentz Transformations is itself a Lorentz Transformation.

Proof: Examine three frames of reference S, S', and $S"$ as shown in Fig. 2.3, S' has relative velocity v with respect to S along position X-axis and $S"$ has relative velocity v' with respect to S' along positive X-axis.

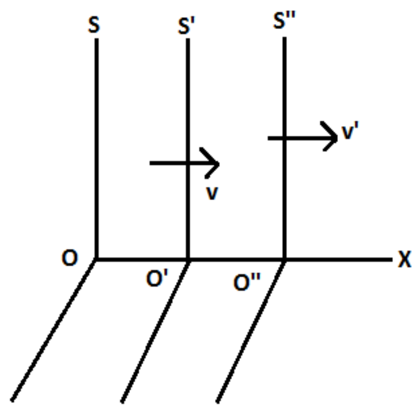


Fig. 2.3

By Lorentz transformation, the frames S and S′ can be related as

$$x' = \beta(x - vt), y' = y, z' = z, t' = \beta\left(t - \frac{vx}{c^2}\right) \quad \ldots (1)$$

$$where\ \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Similarly for the frames S′ and S″ can be related as

$$x\text{=}\beta'(x'\text{-}v't'), y''\text{=}y', z''\text{=}z', t''\text{=}\beta'\left(t' - \frac{v'x'}{c^2}\right) \quad \ldots (2)$$

$$where\ \beta' = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}}$$

Let us assume v″ is the resultant velocity of v and v′ then

$$v'' = \frac{v + v'}{1 + \frac{vv'}{c^2}} \quad \ldots (3)$$

Where v″ is the velocity of frame S″ relative to S. So that

$$\beta'' = \frac{1}{\sqrt{1 - \frac{v''^2}{c^2}}}$$

If we show that

$$x'' = \beta''(x - vt), y'' = y, z'' = z, t'' = \beta''\left(t - \frac{v''x}{c^2}\right) \quad \dots (4)$$

Then the required result will be proved.

$$\frac{1}{\beta''^2} = 1 - \frac{v''^2}{c^2} = 1 - \frac{1}{c^2}\left(\frac{v + v'}{1 + \frac{vv'}{c^2}}\right)^2$$

From equation (3)

$$= \frac{c^2\left(1 + \frac{vv'}{c^2}\right)^2 - (v + v')^2}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 + \frac{2vv'}{c^2} + \frac{v^2v'^2}{c^4}\right) - (v^2 + 2vv' + v'^2)}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 + 2vv' + \frac{v^2v'^2}{c^2} - v^2 - 2vv' - v'^2}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 + \frac{v^2v'^2}{c^4} - \frac{v^2}{c^2} - \frac{v'^2}{c^2}\right)}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$\beta'' = \left(1 + \frac{vv'}{c^2}\right) \Big/ \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \sqrt{\left(1 - \frac{v'^2}{c^2}\right)}$$

$$\beta'' = \beta\beta'\left(1 + \frac{vv'}{c^2}\right) \quad \dots (5)$$

From equation (1) & (2)

$$x" = \beta'(x' - v't') = \beta'\left[\beta(x - vt) - v'\beta\left(t - \frac{vx}{c^2}\right)\right]$$

$$= \beta'\beta\left[x\left(1 + \frac{vv'}{c^2}\right) - t(v + v')\right]$$

$$= \beta'\beta\left(1 + \frac{vv'}{c^2}\right)\left\{x - \frac{v + v'}{\left(1 + \frac{vv'}{c^2}\right)}.t\right\} = \beta"(x - v"t)$$

From equation (3) & (5)

Again from equation (2) & (1)

$$t" = \beta'\left(t' - \frac{v'x'}{c^2}\right) = \beta'\left[\beta\left(t - \frac{vx}{c^2}\right) - \frac{v'}{c^2}\beta(x - vt)\right]$$

$$= \beta'\beta\left[t\left(1 + \frac{vv'}{c^2}\right) - \frac{x}{c^2}(v + v')\right]$$

$$= \beta'\beta\left(1 + \frac{vv'}{c^2}\right)\left\{t - \frac{v + v'}{1 + \frac{vv'}{c^2}}.\frac{x}{c^2}\right\}$$

$$= \beta"\left(t - \frac{xv"}{c^2}\right)$$

From equation (3) & (5)

$$y" = y', y' = y \Longrightarrow y" = y$$

$$z" = z', z' = z \Longrightarrow z" = z$$

Thus, we have prove that

$$x" = \beta"(x - v\mathrm{t}), \mathrm{y}" = y, z" = z, t" = \beta"\left(t - \frac{v"x}{c^2}\right)$$

2.10 *ABERRATION (RELATIVISTIC TREATMENT):-*

The phenomenon of aberration was originally discovered by Bradley in 1927. *This phenomenon of light is very useful to determine the velocity of earth if the velocity of light is known.* The phenomenon of aberration results *"The speed of light is independent of the medium of transmission; but the direction of light rays depends on the motion of the source emitting light relative to the observer".*

The direction of a light ray emitted from a star is compared here with respect to the inertial frames S and S'. S' is traveling in the positive direction of the X-axis at a constant speed v in relation to S.
Given that the earth orbits the sun, we can presume that the system S is fixed in the sun and S' in the earth. A full light beam from star P is located in XY -plane or $X'Y'-$ plane. At any given time t, the observers at $O\ and\ O'$ observe that, the direction of this light ray makes angles $\alpha\ and\ \alpha'$ with the X-axis, respectively as shown in fig. 2.4. Then

$$u_x = c\ cos\alpha, u_y = -c\ sin\alpha, u_z = 0 \qquad ...(1)$$

$$u'_x = c\ cos\alpha', u'_y = -c \sin\alpha', u'_z = 0 \qquad ...(2)$$

By Lorentz Transformation

$$x' = \beta(x - vt), y' = y, z' = z, t' = \beta\left(t - \frac{vx}{c^2}\right)$$

$$where\ \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

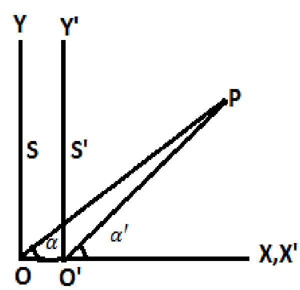


Fig. 2.4

$$\Longrightarrow dx' = \beta(dx - vdt), dy' = dy, dt' = \beta\left(dt - \frac{v}{c^2}dx\right)$$

$$\frac{dx'}{dt'} = \frac{\beta(dx - vdt)}{\beta\left(dt - \frac{v}{c^2}\right)dx}, \frac{dy'}{dt'} = \frac{dy}{\beta\left(dt - \frac{v}{c^2}dx\right)}$$

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, u'_y = \frac{u_y}{\beta\left(1 - \frac{v}{c^2}u_x\right)}$$

$$or\ \frac{u_y'}{u_x'} = -\frac{c\sin\alpha'}{c\cos\alpha'} = \frac{u_y}{\beta\left(1-\frac{v}{c^2}u_x\right)}.\frac{1-\frac{v}{c^2}u_x}{u_x - v}$$

$$-\tan\alpha' = \frac{u_y}{\beta(u_x - v)} = -\frac{c\sin\alpha}{\beta(c\cos\alpha - v)}$$

$$= -\frac{c\tan\alpha}{c\beta\left(1-\frac{v}{c\cos\alpha}\right)}$$

$$-\tan\alpha' = \frac{\tan\alpha}{\beta\left(1-\frac{v}{c}\sec\alpha\right)}$$

$$\tan\alpha = \frac{tan\alpha\left(1-\frac{v^2}{c^2}\right)^{1/2}}{1-\frac{v}{c}\sec\alpha}$$

This is known as the relativistic formula for aberration.

2.11 *DOPPLER'S EFFECT*:-

2.11.1 Non Relativistic Treatment

According to this phenomena, which most readers have probably read about in physics class, the pitch sound that an observer hears changes in two situations: first, when the source and observer are both stationary, and second, when they are moving relative to one another. The perceived frequency rises as the source and observer get closer to one another and falls as they get farther away. This phenomenon, known as Doppler's effect, happens with all types of wave motion, albeit it differs slightly from electromagnetic waves like light, which do not involve a medium, in the case of mechanical waves that involve a material medium. If the spectrum of light waves is viewed in a spectrometer, the motion of the source causes a shift in the spectral line's position from its initial position.

We start by looking at the scenario in which the source is traveling toward the observer at velocity u while the observer is at rest. Set the source's frequency to f so that the wave speed is c and the time period $T = 1/f$. The wave travels a distance cT in a time interval T during which it emits one cycle, yet the source waves in the same direction uT during the same period. Therefore, rather than $cT,$ the wave length, which is the distance between two successive peaks in the wave, is cT-ut. As a result, the matching frequency represented by $f_1{}'$ is provided by

$$f_1' = \frac{c}{T(c-u)} = \frac{cf}{c-u} = \frac{f}{1-c/u} \quad \dots(1)$$

This indicates a perceived rise in the frequency. We substitute - u for u in (1) to obtain the appropriate $f_1{}'$ if the source is moving away from the observer.
The apparent frequency in the second scenario, where the source is stationary but the observer is traveling v from the source, is because the wave speed in relation to the observer is c-v and rather than c.

$$f_2' = \frac{c-v}{\lambda} = \frac{c-v}{c/f} = \frac{c-v}{c} f = \left(1 - \frac{v}{c}\right) f \quad \dots(2)$$

Where $f = c/\lambda$ is the frequency in the stationary case.
Now when both the source of the observer is moving with velocities u and v along the same direction, the same result may be combined as follows:

The source's velocity causes the apparent wave length in the first scenario mentioned above to be

$$\lambda_1' = \frac{c}{f_1{}'} = \frac{c-u}{\lambda}$$

This indicates that the second wave maximum is $\lambda_1^{'}$ behind the first wave maximum when it reaches the observer. However, because of the observer's motion, the wave speed in relation to the observer is $c - v$, and as a result, the observer perceives the apparent frequency.

$$f' = \frac{c-v}{\lambda_1{}'} = \frac{c-v}{c-u} f$$

If the source and observer are moving in different directions, the signs of u and v will be adjusted appropriately.

$$if\ v > u\ then f' < f\ if\ v < u\ then\ f' > f$$

Therefore, the apparent frequency falls as the source and observer move apart, resulting in an increase in wave length; the opposite is true when the source and observer are moving closer together.

2.11.1.1 Experimental Evidence for Non-Relativistic Treatment

When light waves are viewed in a spectrometer, the mobility of the source causes a change in the spectral line's position from its initial position. There are two kinds of this spectral line shift.
A decrease in wave length is shown by several spectral lines shifting towards violet. However, in some situations, these lines change in the direction of red, signifying an increase in wave length. Doppler's effect provides the following explanation for the above:

The velocity of stars has a major impact in this change. The star is moving toward the earth (us) in the first scenario and away from the earth in the second. The shift is proportional to the distance of source from the earth. By (1),

$$f' = \left(\frac{c}{c-u}\right) f$$

$$\frac{1}{\lambda'} = \left(\frac{c}{c-u}\right)\frac{1}{\lambda}$$

For $v = n\lambda, i.e. c = f_1'\lambda', c = f_1\lambda$

$$\lambda' = \left(\frac{c-u}{c}\right)\lambda \qquad (for\ approach)$$

$$\lambda' = \left(\frac{c+u}{c}\right)\lambda \qquad (\text{for recession})$$

2.11.2 Relativistic Treatment

Examine two inertial frames, S and S', where S' is traveling along the X-axis at a velocity v relative to S. If f and T stand for frequency and time period respectively, in the S system, then f' and T' stand for the S' system. Using the standard time dilation formula,

$$T' = \frac{T}{\sqrt{1-\frac{v^2}{c^2}}}$$

But T= 1/f

$$\therefore T' = \frac{1}{f\sqrt{1-\frac{v^2}{c^2}}}$$

$$f' = \frac{c}{\lambda'} = \frac{c}{(c-v)T'}$$

$$= \frac{c}{c-v}.\frac{f\sqrt{1-v^2/c^2}}{1} = f\sqrt{\frac{c+v}{c-v}}$$

Finally we get

$$f' = \left(\frac{c+v}{c-v}\right)^{1/2} f$$

Solved Examples:

EXAMPLE 1: A particle with a mean proper life of 1μ second moves through the laboratory at $2.7 \times 10^{10} cm/sec$.

(1) What will be its life as measured by an observer in the laboratory?
(2) What will be the distance transversed by it before disintegrating?

(3) Find the distance transversed without taking relativity into account

SOLUTION: Given

$$\Delta t = 1\mu\, sec = 1 \times 10^{-6} sec, v = 2.7 \times 10^{10} cm/sec$$

(1) By the result of time dilation,

$$\Delta t' = \Delta t\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$\Delta t' = 1 \times 10^{-6} \Big/ \sqrt{\left(1 - \frac{(2.7 \times 10^{10})^2}{(3 \times 10^{10})^2}\right)}$$

$$\Delta t' = 10^{-6} \Big/ \sqrt{1 - \left(\frac{2.7}{3}\right)^2}$$

$$\Delta t' = 3 \times 10^{-6}/\sqrt{9 - (2.7)^2}$$

$$\Delta t' = 3 \times 10^{-6}/\sqrt{1.71} = 3 \times 10^{-6}/1.31 = 2.3 \times 10^{-6} sec$$

$$\Delta t' = 2.3\mu\, sec.$$

(2) Distance transversed by the particle:

$$= v.\Delta t' = 2.7 \times 10^{10} \times 2.3 \times 10^{-6} = 6.21 \times 10^4 = 621\, meter$$

(3) Distance transversed with relativistic effects

$$= v.\Delta t = 2.7 \times 10^{10} \times 1 \times 10^{-6} = 27 \times 10^4 cm = 270\, meter$$

EXAMPLE 2: A body has the dimensions represented by $6i + 7j$ meters in reference system S. How these dimensions will be represented in the system S′? If S is moving with velocity 0.6c along positive X axis i, j being unit vector along respective axis.

Solution: By Lorentz contraction

$$l' = l\sqrt{1 - \frac{v^2}{c^2}}$$

Given v=0.6c

$$l' = 6\sqrt{1 - \left(\frac{0.6c}{c}\right)^2} = 6\sqrt{0.64} = 6 \times 0.8$$

$$l' = 4.8$$

In S' system, the body's dimension along the X-axis is 4.8. However, since there is no motion along the y-axis in S', there is no contraction in the y-axis direction. Therefore, in S' system, the body will be represented by $4.8i + 7j$ meters.

EXAMPLE 3: The length of a rocket ship is 100 meters on the ground. When it is in fight its length observed on the ground is 99 meters, calculate its speed.

SOLUTION: By Lorentz transformation

$$l' = l\sqrt{1 - \frac{v^2}{c^2}}$$

$$99 = 100\sqrt{1 - \frac{v^2}{c^2}}$$

As $l' < l$

$$\left(\frac{99}{100}\right)^2 = 1 - \frac{v^2}{c^2} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{99^2}{100^2} = \frac{199}{10^4}$$

$$\Longrightarrow \frac{v}{c} = \frac{\sqrt{199}}{100} \Longrightarrow v = \frac{\sqrt{199}}{100} \times 3 \times 10^8$$

$$\Longrightarrow v = 42.3 \times 10^6 m/sec$$

EXAMPLE 4: A man in rocket ship is travelling with velocity 0.9c relative to an observer on the earth. He fires a proton in the direction of travel at a velocity of 0.9c relative to rocket ship. What is the velocity of proton relative to the observer on earth?

SOLUTION: We have

$$v = velocity\ of\ rocket\ ship\ relative\ to\ an\ observer\ on\ the\ earth = 0.9c$$

$$u' = velocity\ of\ proton\ relative\ to\ rocket\ ship = 0.9c$$

$$V = velocity\ of\ proton\ relative\ to\ an\ observer\ on\ earth$$

We know that

$$V = \frac{u' + v}{1 + u'\frac{v}{c^2}} = \frac{0.9c + 0.9c}{1 + \frac{0.9c \times 0.9c}{c^2}}$$

$$V = \frac{1.80c}{1.81} = 0..995c$$

EXAMPLE 5: If u and v are two velocities in the same direction and V their resultant velocity given by

$$\tanh^{-1}\frac{V}{c} = \tanh^{-1}\frac{u}{c} + \tanh^{-1}\frac{v}{c}$$

Then deduce the law of composition of velocities from this equation.

SOLUTION: Given that

$$\tanh^{-1}\frac{V}{c} = \tanh^{-1}\frac{u}{c} + \tanh^{-1}\frac{v}{c}$$

This equation is expressible as

$$\frac{1}{2}log\frac{c+V}{c-V} = \frac{1}{2}log\frac{c+u}{c-u} + \frac{1}{2}log\frac{c+v}{c-v}$$

$$log\frac{c+V}{c-V} = log\frac{c+u}{c-u}.\frac{c+v}{c-v}$$

$$\Rightarrow \frac{c+V}{c-V} = \frac{c+u}{c-u}.\frac{c+v}{c-v} = \frac{c^2+(u+v)c+uv}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{c+V}{c-V} - 1 = \frac{c^2+(u+v)c+uv}{c^2-(u+v)c+uv} - 1$$

$$\Rightarrow \frac{2V}{c-V} = \frac{2(u+v)c}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{c-V}{V} = \frac{c^2-(u+v)c+uv}{(u+v)c}$$

$$\Rightarrow \frac{c}{V} - 1 = \frac{c}{u+v} - 1 + \frac{uv}{(u+v)c}$$

$$\Rightarrow \frac{c}{V} = \frac{c}{u+v} + \frac{uv}{(u+v)c} = \frac{c^2+uv}{(u+v)c}$$

$$\Rightarrow \frac{V}{c} = \frac{(u+v)c}{c^2+uv}$$

$$\Rightarrow V = \frac{(u+v)c^2}{c^2\left(1+\frac{uv}{c^2}\right)}$$

$$\Rightarrow V = \frac{(u+v)}{\left(1+\frac{uv}{c^2}\right)}$$

This is required expression for V.

SELF CHECK QUESTIONS

1. The resultant of two velocities of light each of which is less than c is also
a) Less than c
b) Equal to c
c) Greater than c
d) None of these
2. Aberration of light stars is caused due to:
a) The travelling of light in the atmosphere
b) Elliptical orbit of the earth around the sun
c) The finite speed of light and the speed of earth in its orbit around the sun.
d) The scattering of light by the air particles.
3. The basic theory of field is governed by
a) Lorentz transformation
b) Laplace transformation
c) Legendre's transformation
d) Lagrange's formalism
4. Lorentz transformation reduce to Galilean transformation on if
a) $v = c$
b) $v \ll c$
c) $v \gg c$
d) None of these
5. The result of two successive Lorentz transformation is:
a) Galilean transformation
b) Lorentz transformation
c) Einstein transformation
d) None of these

2.12 *SUMMARY*:-

In this unit, we explored the relativistic concept of space and time, which replaces the classical notion of absolute space and time with a unified space-time framework. The unit began with the postulates of the Special Theory of Relativity proposed by Einstein, emphasizing that the laws of physics are the same in all inertial frames and that the speed of light in vacuum is constant for all observers, regardless of their motion. We then studied the Lorentz transformation, which mathematically relates the space and time coordinates between two inertial frames in relative motion, and ensures the invariance of the speed of light. The unit further examined the consequences of Lorentz transformation, including key relativistic effects such as:

- Time Dilation, where a moving clock appears to tick slower.
- Length Contraction, where objects in motion appear shortened along the direction of motion.
- Relativity of Simultaneity, which shows that simultaneous events in one frame may not be simultaneous in another.

We also studied the group properties of Lorentz transformations, establishing that they form a group under composition. Additionally, the unit covered relativistic optical phenomena such as aberration, which refers to the apparent shift in the direction of incoming light due to the motion of the observer, and the relativistic Doppler effect, which explains the frequency shift in light or sound due to the relative motion between source and observer.

2.13 GLOSSARY:-

- **Lorentz Transformation:** A set of equations that relate space and time coordinates between two inertial frames moving at constant velocity, ensuring the speed of light remains constant across all inertial frames.
- **Inertial Frame:** A reference frame in which a body remains at rest or moves with constant velocity unless acted upon by a force.
- **Relative Motion:** The motion of an object as observed from a particular frame of reference.
- **Space-Time Interval:** A quantity invariant under Lorentz transformations; it combines differences in space and time between two events.
- **Time Dilation:** A phenomenon where time appears to pass slower in a moving frame as observed from a stationary frame.
- **Length Contraction:** The shortening of an object's length in the direction of motion as observed from a stationary frame.
- **Simultaneity:** The concept that two events that are simultaneous in one frame may not be simultaneous in another due to relative motion.
- **Postulates of Special Relativity:**
  - The laws of physics are the same in all inertial frames.

   - The speed of light in a vacuum is constant in all inertial frames, regardless of the motion of the light source or observer.
- **Aberration of Light:** The apparent change in the direction of incoming light due to the motion of the observer.
- **Relativistic Doppler Effect:** The change in frequency or wavelength of light from a moving source, accounting for time dilation effects.
- **Group Property:** Lorentz transformations form a group, meaning they satisfy closure, associativity, identity, and inverse properties.

2.14 *REFRENCES:-*

- Ta-Pei Cheng (2015),"Relativity, Gravitation and Cosmology: A Basic Introduction" (2nd Edition).
- Spencer A. Klein (2017), "Relativistic Mechanics and Electrodynamics".
- James J. Callahan (2019), "The Geometry of Spacetime: An Introduction to Special and General Relativity" (2nd Edition.

2.15 *SUGGESTED READING:-*

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

2.16 *TERMINAL QUESTIONS:-*

(TQ-1) Explain the postulates of special theory of relativity.
(TQ-2) Explain the phenomenon of time dilation in special relativity.
(TQ-3) Obtain the law of transformation for the Lorentz contraction factor.
(TQ-4) Discuss the concept of Simultaneity in special theory.
(TQ-5) A rod has length 100cm when the rod is in a satellite moving with velocity 0.8c relative to laboratory, what is length of the rod as determined by an observer, (i) in the satellite and (ii) in the laboratory?

(TQ-6) Calculate the length of a rod moving with a velocity of 0.8c in a direction inclined at 60° to its own length. Proper length of the rod is given to be 100 cm.
(TQ-7) A man is in a car travelling at 30 miles/hour. He throws a ball in the direction of travel, at a velocity of 30 miles/hour relative to the car. What is the velocity of the ball relative to the ground?

2.17 *ANSWERS:-*

SELF CHECK ANSWERS

1. a
2. b
3. a
4. b
5. b

(TQ-5) (i) 100cm, (ii)60cm

(TQ-6) 91.6

(TQ-7) 59.999 miles/hour

UNIT 3:- Relativistic Mechanics

3.1 *INTRODUCTION:-*

Relativistic mechanics is the branch of physics that extends classical Newtonian mechanics to account for objects moving at or near the speed of light. It is based on Albert Einstein's theory of special relativity, which fundamentally changed our understanding of space, time, and motion. In relativistic mechanics, the assumptions of absolute time and space are replaced by the idea that measurements of time and distance depend on the relative motion between observers. One of the key insights is that the laws of physics are the same in all inertial frames, and the speed of light is constant for all observers, regardless of their relative motion. This leads to phenomena such as time dilation, length contraction, and the relativity of simultaneity, which have been experimentally verified. Additionally, the famous equation $E = mc^2$ emerges from relativistic mechanics, establishing the equivalence of mass and energy. This framework is essential not only in particle physics and astrophysics but also in technologies like GPS, where relativistic corrections are necessary for accurate functioning.

3.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To explain mass and momentum.
- To solve equivalence of mass and energy relation.
- To obtain the law of variation of mass with velocity.
- To understand the formulation of energy momentum.
- To solve the Binding energy.

3.3 *MASS AND MOMENTUM:-*

A moving particle's linear momentum (p) is described in classical mechanics as $p = mv$, where m is the mass and v is the velocity.
We assume in classical mechanics that

- A moving body's mass is equal to that of a stationary one.
- In the absence of external forces, a body's total momentum stays constant.
- The law of conservation of momentum refers to this. If we use Lorentz transformations to test assumption (1), it will not be true.

The law of conservation of momentum's Lorentz invariance suggests that a moving body's mass is not constant but rather varies with velocity, as we will see later.

3.4 *NEWTON'S LAW OF MOTION:-*

In classical mechanics, Newton has given three laws of motion namely.

1. A body at rest remains at rest and a body in motion continues with constant velocity in a straight line unless as external force is applied to it. Symbolically,

   $$F = 0 \Rightarrow a = 0$$

   Where F and a denote respectively net external force and acceleration.
2. If a force F acts on a body, then the momentum of the body will be changed so that rate of change of momentum is proportional to the

force and is in the direction of the force. Mathematically it is expressed as:

$$F = K\frac{dp}{dt}$$

Where K is constant of proportionality. We defined K s.t. $K = 1$ and dimension less.

$$\therefore \qquad F = \frac{dp}{dt}$$

In the non-relativistic limit the momentum is given by

$$F = m\frac{dv}{dt} = ma$$

It is non-relativistic form of Newton's second law.

3. Whenever two bodies intersect at, then forces $F_{1\to 2}$ on the second body exerted by the first body is equal and opposite to the force $F_{2\to 1}$ on the first body due the second. That is to say, action and reaction are equal and opposite.

$$F_{1\to 2} = -F_{2\to 1}$$

3.5 *MEASUREMENT OF DIFFERENT UNITS*:-

The three unit systems are C.G.S., F.P.S., and M.K.S. Keep in mind that the ergs is in C.G.S. and the Joule is in M.K.S.

1. In the formula $E = mc^2$,units of m, c and E are gram, cm/sec and ergs
2. $leV = 1$ electron Volt $= 1.6 \times 10^{-12}$ ergs
3. 1 Joule $= 10^7$ ergs
   $1eV = 1.6 \times 10^{-12} \times 10^{-7} = 1.6 \times 10^{-19}$Joule
   MeV =Million electron Volt, BeV = Billion electron Volt.
4. $1MeV = 10^6\ eV = 1.6 \times 10^{-12} \times 10^6$ergs
   $1BeV = 10^9\ eV = 10^9 \times 1.6 \times 10^{-12}$ ergs$= 1.6 \times 10^{-3}$ ergs
5. Rest mass of proton $= m_0 = 1.67 \times 10^{-24}$ gm.
6. Rest mass of electron $= m_0 = 9 \times 10^{-28}$ gram.
7. 1 Kilo watt hour $=$ 1 K.W.H. $= 3.6 \times 10^{12}$ ergs
8. 1gm $= 6 \times 10^{23}$a.m.u.
   a.m.u. =Atomic mass unit
9. 1 calorie $= 4.2 \times 10^7 ergs = 4.2$ joules

10. Distance from the earth to the sun is about 150×10^6 Km.

3.6 *VARIATION OF MASS WITH VELOCITY*:-

Examine two frames of reference, S and $S'.S$' is traveling along the X-axis at a constant rate, v. If m_1 is the mass of a particle traveling along the x-axis in system S at velocity u_1, then m_1' and $u_1{}'$ are the mass and velocity of the identical particle in system S', respectively.
Suppose

$$\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \beta_1 = \frac{1}{\sqrt{1-\frac{u_1{}^2}{c^2}}}, \beta_2 = \frac{1}{\sqrt{1-\frac{u_1{}'^2}{c^2}}} \quad \ldots (1)$$

By the formula of composition of velocities,

$$u_1 = \frac{u_1' + v}{1 + \frac{v}{c^2} u_1{}'}$$

$$u_1\left(1 + \frac{v}{c^2} u_1{}'\right) = u_1' + v$$

$$u_1' = \frac{u_1 - v}{1 - \frac{v}{c^2} u_1} \quad \ldots (2)$$

∴

$$\beta_1' u_1{}' = \frac{u_1 - v}{\sqrt{1-\frac{u_1{}'^2}{c^2}}\left(1 - \frac{v}{c^2}u_1\right)} \quad \ldots (3)$$

Now,

$$1 - \frac{u_1{}'^2}{c^2} = 1 - \frac{(u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2}$$

$$= \frac{c^2\left(1 - \frac{v}{c^2}u_1\right)^2 - (u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2}$$

$$\therefore c^2\left(1 - \frac{u_1{}'^2}{c^2}\right)\left(1 - \frac{v}{c^2}u_1\right)^2 = \left(1 - \frac{v}{c^2}u_1\right)^2 c^2 - (u_1 - v)^2$$

$$= c^2\left(1 + \frac{v^2}{c^4}u_1{}^2 - \frac{2u_1 v}{c^2}\right) - (u_1{}^2 + v^2 - 2u_1 v)$$

$$= c^2 + \frac{u_1{}^2 v^2}{c^2} - u_1{}^2 - v^2 = c^2\left[1 - \frac{u_1{}^2 + v^2}{c^2} + \frac{u_1{}^2 v^2}{c^4}\right]$$

Dividing by c^2

$$\left(1-\frac{u_1'^2}{c^2}\right)\left(1-\frac{v}{c^2}u_1\right)^2 = 1-\frac{u_1^2+v^2}{c^2}+\frac{u_1^2v^2}{c^4}$$

Taking square root both sides, we get

$$\sqrt{1-\frac{u_1'^2}{c^2}}\left(1-\frac{v}{c^2}u_1\right) = \left(1-\frac{u_1^2+v^2}{c^2}+\frac{u_1^2v^2}{c^4}\right)^{1/2}$$

Putting the above value in equation (3)

$$\beta_1'u_1' = \frac{u_1-v}{\left(1-\frac{u_1^2+v^2}{c^2}+\frac{u_1^2v^2}{c^4}\right)^{\frac{1}{2}}} \qquad \dots(4)$$

$$= \frac{u_1-v}{\left[\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u_1^2}{c^2}\right)\right]^{\frac{1}{2}}}$$

$$\beta_1'u_1' = \beta\beta_1(u_1-v)\ by\ (1)$$

$$\frac{\beta_1'u_1'}{\beta_1} = \beta(u_1-v) \qquad \dots(5)$$

Assuming that several of these particles are traveling along the X-axis and that their masses and momentum remain constant within the system S, we can now

$$\left.\begin{aligned}\sum m_1 &= constant\\ \sum m_1u_1 &= constant\end{aligned}\right\}$$

Since $\beta\ and\ v$ are same for every particle and therefore

$$\left.\begin{aligned}\sum m_1\beta v &= constant\\ \sum m_1u_1\beta &= constant\end{aligned}\right\} \qquad \dots(6)$$

Subtracting, we get

$$\sum m_1\beta(u_1-v) = constant$$

From equation (5)

$$\sum\left[m_1\frac{\beta_1'u_1'}{\beta_1}\right] = constant \qquad \dots(7)$$

Appling in S′, law of conservation of momentum

$$\sum m_1'u_1' = constant \qquad \dots(8)$$

Comparing equation (7) & (8)

$$\frac{m_1\beta_1'}{\beta_1} = m_1'$$

$$\frac{m_1}{\beta_1} = \frac{m_1'}{\beta_1'} = an\ absolute\ constant = m_0(say)$$

Then

$$m_1 = \frac{m_0}{\sqrt{1 - \frac{{u_1}^2}{c^2}}}, m' = \frac{m_0}{\sqrt{1 - \frac{{u_1'}^2}{c^2}}}$$

According to this, if a particle with mass m in relation to system S is traveling with velocity u in relation to system S, then

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

If $u = 0$, then the last given $m = m_0$

Hence is the mass of the body at rest. Hence, m_0 is also called rest mass or proper m_0 mass.

For it is the mass of the body measured, like proper length and proper time, in the inertial frame in which the body is at rest.

3.7 *EXPERIMENTAL VARIFICATION*:-

The mass velocity relation:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ... (1)$$

can be verified in the experiments measuring the mass or e/m of the final traveling electrons. We here offer the experiment by Guye and Lavanchy.

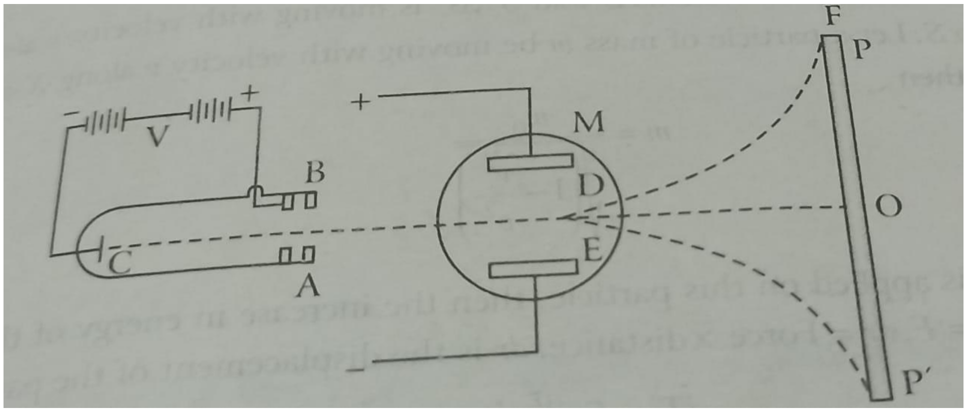


Fig.1

3.7.1 Experiment of Guye and Lavanchy

In 1915, Guye and Lavanchy used their "identical paths" method, which is seen in figure 1, to achieve the most accurate validation of equation (1). In a vacuum tube, a cathode-anode system C*A is set up and run at several thousand volts. The hole B in the anode collimates the cathode rays into a fine beam, which then travels in a straight line to impact a photographic plate Fat 0. By applying a potential difference between the plates D and E and the latter employing an electromagnet M, represented by a dotted circle, electric and magnetic fields are arranged in the course of the former's electron beam.

Two fields sequentially deflect the electron beam. The field strengths are set up so that the path taken by the electrons being studied matches the path taken by a reference beam of low-speed electrons. Assuming that the electric and magnetic field strengths X' and H' are such that the fast electrons (velocity v') experience the identical electric and magnetic deflectors as the slow electrons (velocity v) under the field strengths X and H, it can be shown that

$$\frac{v'}{v} = \frac{X'H}{XH'} and \ \frac{m'}{m} = \left(\frac{X'H}{XH'}\right)$$

Where m is the mass of the electron in the reference beam of low speed electrons and m' that of an electron in the beam of fast electrons under examination.

Guye and Lavanchy produced approximately 2000 determinations of for electrons with (m') / m velocity ranging from 26 to 48% that of light and showed that their results confirmed therewith to an accuracy of 1 part in 2000.

Theorem 1. Equivalence of mass and energy: To show that $E = mc^2$.

Proof: Let S and S' be two systems. S' is traveling along the x-axis with velocity v in relation to S. In the system S, suppose that a particle of mass m is traveling along the X-axis at velocity v. Then

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \dots (1)$$

The equation $dT = F.dr = force \times displacement$, where dr is the particle's displacement, gives the particle's increase in energy if a force F is applied to it.

$$dT = F.\frac{dr}{dt}.dt = F.vdt \qquad \dots (2)$$

Since force is defined as the rate of change in momentum so that

$$F = \frac{d}{dt}(mv) or\ Fdt = d(mv)$$

From equation (2)

$$dT = vd(mv) = vd\left[\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right]$$

$$= m_0 v \left[\sqrt{1 - \frac{v^2}{c^2}} + \frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\right] \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$dT = \frac{m_0 v dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \qquad \dots (3)$$

Taking differential of both sides in (1), we get

$$dm = \frac{m_0(-2v/c^2)dv}{(-2)\left(1 - \frac{v^2}{c^2}\right)^{3/2}} = \frac{m_0 v dv}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

$$c^2 dm = \frac{m_0 v dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

$$dT = c^2 dm \qquad \ldots (4)$$

From equation (3)

Assume the particle has a mass of m_0 and is initially at rest. Its mass becomes$m_0 + dm = m$, say, once force F is applied. The particle's total K.E. T is determined by

$$T = \int dT = \int_{m_0}^{m} c^2 dm = c^2(m - m_0)$$
$$T = mc^2 - m_0c^2$$
$$or\ T + m_0c^2 = mc^2$$
$$\Rightarrow E = K.E. of\ the\ moving\ particle + energy\ at\ rest$$
$$= T + m_0c^2$$

We obtain

$$E = mc^2$$

This formula is known as Einstein formula showing that the two fundamental conceptions of mass and energy are identical.

Here m_0c^2 is called interval or rest energy.

3.8 *TRANSFORMATION FORMULA FOR MASS*:-

Let's look at systems S and $S. S'$ is traveling along the x-axis at velocity $v.$ A body moving with velocity u and u' in S and S' has masses m and m' in S and S', respectively.

We have,

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}, m' = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} \qquad \ldots (1)$$

$$u^2 = u_x{}^2 + u_y{}^2 + u_z{}^2, u'^2 = u_x{}'^2 + u_y{}'^2 + u_z{}'^2$$

By the law of composition of velocities

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, u'_y = \frac{u_y\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{1 - \frac{v}{c^2}u_x}, u'_z = \frac{u_z\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{1 - \frac{v}{c^2}u_x}$$

From equation (1)

$$\frac{m}{m'} = \left(\frac{1 - \frac{u'^2}{c^2}}{1 - \frac{u^2}{c^2}}\right)^{\frac{1}{2}} \qquad \ldots (2)$$

$$1 - \frac{u'^2}{c^2} = 1 - \frac{1}{c^2}(u_x{}'^2 + u_y{}'^2 + u_z{}'^2)$$

$$= 1 - \left[(u_x - v)^2 + u_y{}^2\left(1 - \frac{v^2}{c^2}\right) + u_z{}^2\left(1 - \frac{v^2}{c^2}\right)\right].\frac{1}{c^2\left(1 - \frac{v}{c^2}u_x\right)^2}$$

$$= \frac{1}{c^2\left(1 - \frac{v}{c^2}u_x\right)^2}\left[c^2\left(1 - \frac{v}{c^2}u_x\right)^2 - \left\{(u_x - v)^2 + u_y{}^2\left(1 - \frac{v^2}{c^2}\right) + u_z{}^2\left(1 - \frac{v^2}{c^2}\right)\right\}\right]$$

Taking $\alpha^2 = 1/c^2\left(1 - \frac{v}{c^2}u_x\right)^2$, we get

$$1 - \frac{u'^2}{c^2} = \alpha^2\left[c^2\left(1 + \frac{v^2}{c^4}u_x{}^2 - \frac{2vu_x}{c^2}\right) - \left\{u^2 - 2xu_x - \frac{v^2}{c^2}(u_y{}^2 + u_z{}^2) + v^2\right\}\right]$$

$$= \alpha^2\left[c^2\left(1 - \frac{u^2}{c^2}\right) + \frac{v^2}{c^2}u^2 - v^2\right]$$

$$= \alpha^2\left[c^2\left(1 - \frac{u^2}{c^2}\right) + v^2\left(\frac{u^2}{c^2} - 1\right)\right]$$

$$= \alpha^2\left(1 - \frac{u^2}{c^2}\right)[c^2 - v^2]$$

$$1 - \frac{u'^2}{c^2} = \alpha^2 c^2\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$1 - \frac{u'^2}{c^2} = \frac{c^2\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{c^2\left(1 - \frac{v}{c^2}u_x\right)^2}$$

Taking square root

$$\left(1 - \frac{u'^2}{c^2}\right)^{\frac{1}{2}} = \frac{\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{v}{c^2}u_x\right)}$$

$$\left(\frac{1 - \frac{u'^2}{c^2}}{1 - \frac{u^2}{c^2}}\right)^{\frac{1}{2}} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{v}{c^2}u_x\right)}$$

$$\frac{m}{m'} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{v}{c^2} u_x\right)}$$

$$m' = \frac{m\left(1 - \frac{v}{c^2} u_x\right)}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

This is the transformation formula for mass.
If $u_x = 0$, then

$$m' = \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

3.9*TRANSFORMATION FORMULA FOR MOMENTUM AND ENERGY*:-

Consider the two systems, S and S'. S' is traveling along the X −axis with velocity v. Assume that a body in S and S' has masses m and m'. The body traveling with velocities $u(u_x, u_y, u_z)$ *and* $u'(u_x', u_y', u_z')$ is in S and S' respectively.
Next is the relationship,

$$m' = m\left(1 - \frac{v}{c^2} u_x\right)\beta$$

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2} u_x}, u_y{}' = u_y/\beta\left(1 - \frac{v}{c^2} u_x\right)$$

$$u_z{}' = u_z/\beta\left(1 - \frac{v}{c^2} u_x\right)$$

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The components of the momentum p are:

$$p_x = mu_x, p_y = mu_y, p_z = mu_z \ in\ S\ system.$$

$$p_x{}' = m'u_x{}', p_y{}' = m'u_y{}', p_z{}' = m'u_z{}' \ in\ S'\ system.$$

$$p_x' = m'u_x{}' = m\left(1 - \frac{v}{c^2} u_x\right)\beta \frac{u_x - v}{1 - \frac{v}{c^2} u_x} = (mu_x - mv)\beta$$

$$\because E = mc^2 \Rightarrow m = \frac{E}{c^2}$$

$$\therefore p'_x = (p_x - mv)\beta = \left(p_x - \frac{vE}{c^2}\right)\beta$$

$$p'_y = m'u'_y = m\left(1 - \frac{v}{c^2}u_x\right)\beta \frac{u_y}{1 - \frac{v}{c^2}u_x} = mu_y = p_y$$

$$p'_z = m'u'_z = m\left(1 - \frac{v}{c^2}u_x\right)\beta \frac{u_z}{1 - \frac{v}{c^2}u_x} = mu_z = p_z$$

$$E' = m'c^2 = m\left(1 - \frac{v}{c^2}u_x\right)\beta c^2 = \beta(mc^2 - mvu_x)$$

$$= \beta(E - vp_x)$$

$$[\because E = mc^2, p_x = mu_x]$$

Thus, we have shown that

$$p'_x = \left(p_x - \frac{vE}{c^2}\right)\beta, p'_y = p_y, p'_z = p_z, \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These are transformation equations for momentum. These transformation equations are exactly similar to Lorentz transformation equations if we replace x, y, z *by* $p_x, p_y, p_z, \frac{E}{c^2}$ respectively.

3.10*PARTICLE WITH REST MASS*:-

A particle with rest mass m_0 and momentum p has relativistic energy E, which may be found using

$$E^2 = p^2c^2 + m_0{}^2c^4$$

$$E = (p^2c^2 + m_0{}^2c^4)^{\frac{1}{2}} \quad \dots (1)$$

Where$m_0 = 0$, then equation (1) can be written as $E = pc$ ... (2)

But

$$p = \frac{vE}{c^2}$$

$$\therefore p^2 = \frac{v^2E^2}{c^4}$$

From equation (2)

$$\frac{E^2}{c^2} = p^2 = \frac{v^2E^2}{c^4}$$

$$\Rightarrow v^2 = c^2 \; or \; v = c$$

This proves that the particle of zero mass travels with the speed of light.

3.11 *BINDING ENERGY:-*

The binding energy of a nucleus is the amount of energy required to break its protons and neutrons apart across an infinite distance.

Therefore, we anticipate that the nucleus's mass will be less than that of the constituent nucleus by a factor of

$$\Delta m = \Delta E / c^2$$

Where ΔE is the nucleus' binding energy.

3.12*TRANSFORMATION FORMULA FOR FORCE:-*

Let's look at two systems. S and S' are traveling along the $X -$axis at a certain velocity. The masses of a body in S and S', with velocities a and u' in S and S', respectively, are denoted by m and m'. If a body with mass m and velocity an is subject to a force, then

$$F = rate\ of\ change\ of\ momentum = \frac{d}{dt}(mu) \quad \dots (1)$$

$$= u\frac{dm}{dt} + m\frac{du}{dt} = u\frac{dm}{dt} + m\dot{u}$$

$$F = iF_x + jF_y + kF_z, u = iu_x + ju_y + ku_z$$

This gives

$$\left.\begin{aligned} F_x &= u_x\frac{dm}{dt} + m\dot{u}_x \\ F_y &= u_y\frac{dm}{dt} + m\dot{u}_y \\ F_z &= u_z\frac{dm}{dt} + m\dot{u}_z \end{aligned}\right\} \quad \dots (2)$$

$$\frac{dm}{dt} = \frac{d}{dt}\left\{\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\right\} = \frac{u}{c^2}m_0\frac{du}{dt}\frac{1}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$$

$$= u\frac{du}{dt}.\frac{m}{c^2\left(1-\frac{u^2}{c^2}\right)} = \frac{m}{(c^2-u^2)}u\frac{du}{dt}$$

$$\frac{dm}{dt} = \frac{m}{(c^2 - u^2)} u \frac{du}{dt} \quad \ldots (3)$$

$$u^2 = {u_x}^2 + {u_y}^2 + {u_z}^2$$

Differentiating w.r.t. t, we get

$$u\dot{u} = u_x \dot{u_x} + u_y \dot{u_y} + u_z \dot{u_z}$$

Equation 3 now becomes

$$\frac{dm}{dt} = \frac{m(u_x \dot{u_x} + u_y \dot{u_y} + u_z \dot{u_z})}{(c^2 - u^2)}$$

Equation 2 now becomes

$$\left.\begin{aligned} F_x &= u_x \frac{m(u_x \dot{u_x} + u_y \dot{u_y} + u_z \dot{u_z})}{(c^2 - u^2)} + m\dot{u_x} \\ F_y &= u_y \frac{m(u_x \dot{u_x} + u_y \dot{u_y} + u_z \dot{u_z})}{(c^2 - u^2)} + m\dot{u_y} \\ F_z &= u_z \frac{m(u_x \dot{u_x} + u_y \dot{u_y} + u_z \dot{u_z})}{(c^2 - u^2)} + m\dot{u_z} \end{aligned}\right\} \quad \ldots (4)$$

By Lorentz transformation,

$$t' = \beta \left(t - \frac{vx}{c^2} \right)$$

$$\frac{dt'}{dt} = \beta \left(t - \frac{v}{c^2} u_x \right)$$

$$where\ \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the system S′, according to equation (1)

$$F_x' = \frac{d}{dt'} (m' u_x') = \frac{d}{dt} (m' u_x') \frac{dt}{dt'}$$

$$= \frac{d}{dt} \left[\frac{m\left(1 - \frac{v}{c^2} u_x\right)}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} . \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \right] . \frac{1}{\beta \left(1 - \frac{v}{c^2} u_x\right)}$$

$$= \frac{1}{\beta\left(1-\frac{v^2}{c^2}\right)^{1/2}} \frac{d}{dt}[m(u_x - v)].\frac{1}{\left(1-\frac{v}{c^2}u_x\right)}$$

$$= \frac{1}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{dm}{dt}(u_x - v) + m\dot{u}_x\right]$$

$$F_x' = \frac{1}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{m}{c^2-u^2}(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z)(u_x - v) + m\dot{u}_x\right] \quad \dots (5)$$

Observe that

$$F_x - \frac{\left(\frac{v}{c^2}\right)(u_yF_y + u_zF_z)}{\left(1-\frac{v}{c^2}u_x\right)}$$

From equation (4)

$$= \frac{m}{c^2-u^2}(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z)u_x + m\dot{u}_x - \frac{(v/c^2)}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{m}{c^2-u^2}({u_y}^2 + {u_z}^2)(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z) + m(u_y\dot{u}_y + u_z\dot{u}_z)\right]$$

$$= \frac{m}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{1}{(c^2-u^2)}.(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z)\left\{\left(1-\frac{v}{c^2}u_x\right)u_x - \frac{v}{c^2}({u_y}^2 + {u_z}^2)\right\} - (u_y\dot{u}_y + u_z\dot{u}_z)\frac{v}{c^2} + \left(1-\frac{v}{c^2}u_x\right)\dot{u}_x\right]$$

$$= \frac{m}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z)\left(u_x - \frac{v}{c^2}u^2\right)}{(c^2-u^2)} - \frac{v}{c^2}(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z) + {\dot{u}_x}^2\right]$$

$$= \frac{1}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{m}{c^2-u^2}(u_x\dot{u_x} + u_y\dot{u}_y + u_z\dot{u}_z)(u_x - v) + m\dot{u}_x\right] = F_x'$$

$$F_x' = F_x - \frac{\left(\frac{v}{c^2}\right)\left(u_y F_y + u_z F_z\right)}{\left(1 - \frac{v}{c^2}u_x\right)}$$

By virtue of (1)

$$F_y' = \frac{d}{dt'}\left(m'u_y{}'\right) = \frac{d}{dt}\left(m'u_y{}'\right)\frac{dt}{dt'}$$

$$= \frac{1}{\beta\left(1 - \frac{v}{c^2}u_x\right)}\frac{d}{dt}\left[\frac{m\left(1 - \frac{v}{c^2}u_x\right)}{\sqrt{1 - \frac{v^2}{c^2}}}.u_y\frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{v}{c^2}u_x\right)}\right]$$

$$= \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{v}{c^2}u_x\right)}\frac{d}{dt}\left(mu_y\right) = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{v}{c^2}u_x\right)}.F_y$$

Thus, we have shown that

$$F_x' = F_x - \frac{\left(\frac{v}{c^2}\right)\left(u_y F_y + u_z F_z\right)}{\left(1 - \frac{v}{c^2}u_x\right)}, F_y' = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{v}{c^2}u_x\right)}.F_y,$$

$$F_z' = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{v}{c^2}u_x\right)}.F_z$$

These are the required transformation formula for force acting on a body.

3.13*RELATIVISTICT RANSFORMATION FORMULA FOR DENSITY:-*

Let S and S' be two systems. Assume that S' is traveling with velocity relative to S in the r-axis' positive direction. Let

$$\beta = \sqrt{1 - \frac{v^2}{c^2}}$$

Case 1: Assume that a body in system S is at rest. Let $V_0 and\ V'$ represent the body's two systems' volumes. Then,

$$V' = V_0\beta$$

If $\rho\ and\ \rho'$ be the densities, then

$$\rho_0 = \frac{m_0}{V_0}, \rho' = \frac{m'}{V'}, m' = \frac{m_0}{\beta}$$

$$\Rightarrow\ \rho' = \frac{m_0}{\beta V_0 \beta} = \frac{\rho_0}{\beta^2}$$

$$\Rightarrow \rho' = \frac{\rho_0}{\left(1 - \frac{v^2}{c^2}\right)}$$

Case 2: A body moving with velocity $u(u_x, u_y, u_z)$ relative to S and $u'(u_x', u_y', u_z')$ related to S' is what we'll assume. Then

$$u = iu_x + ju_y + ku_z$$

$$u' = iu_x' + ju_y' + ku_z'$$

Let V represent the body's volume in system S and V' as determined by S'. Let $l_x, l_y and\ l_z$ be the lengths of the body's edges when it is at rest in system S, and let V_0 be the volume. Then,

$$V_0 = l_x l_y\ l_z$$

By Lorentz contraction, the length of edges in system S are

$$l_x\sqrt{1 - \frac{{u_x}^2}{c^2}}, l_y\sqrt{1 - \frac{{u_y}^2}{c^2}}, l_z\sqrt{1 - \frac{{u_z}^2}{c^2}}$$

respectively. Then

$$V = l_x l_y\ l_z\left[\left(1 - \frac{{u_x}^2}{c^2}\right)\left(1 - \frac{{u_y}^2}{c^2}\right)\left(1 - \frac{{u_z}^2}{c^2}\right)\right]^{1/2}$$

Let

$$A = \left[\left(1 - \frac{{u_x}^2}{c^2}\right)\left(1 - \frac{{u_y}^2}{c^2}\right)\left(1 - \frac{{u_z}^2}{c^2}\right)\right]^{\frac{1}{2}} \quad \dots(1)$$

Then,

$$V = V_0 A \quad \dots(2)$$

Mass of the body as observed from the system S is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$m_0 = rest\ mass\ of\ the\ body$$

$$\rho = \frac{m}{V} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} . \frac{1}{V_0 A}$$

From equation (2)

$$\rho = \frac{\rho_0}{A\sqrt{1 - \frac{u^2}{c^2}}} \quad \ldots (3)$$

$$where\ u^2 = {u_x}^2 + {u_y}^2 + {u_z}^2$$

To obtain the density expression for the system S',

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, u_y' = \frac{u_y \beta}{1 - \frac{v}{c^2}u_x}, u_z' = \frac{u_z \beta}{1 - \frac{v}{c^2}u_x}$$

$$m' = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

Lengths of edges in system S' are

$$l_x\sqrt{1 - \frac{{u_x}'^2}{c^2}}, l_y\sqrt{1 - \frac{{u_y}'^2}{c^2}}, l_z\sqrt{1 - \frac{{u_z}'^2}{c^2}}$$

We have

$$V' = l_x l_y\, l_z A'$$

$$V' = V_0 A' \quad \ldots (4)$$

$$\rho' = \frac{m'}{V'} = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} . \frac{1}{V_0 A'}$$

$$or\ \rho' = \frac{\rho_0}{A'\sqrt{1-\frac{u'^2}{c^2}}} \qquad \dots (5)$$

$$but\ \sqrt{1-\frac{u'^2}{c^2}} = \frac{\beta\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1-\frac{v}{c^2}u_x\right)}$$

Now from equation (5)

$$\rho' = \frac{\rho_0\left(1-\frac{v}{c^2}u_x\right)}{A'\beta\left(1-\frac{u^2}{c^2}\right)^{1/2}}$$

From equation (3), we get

$$\rho' = \frac{\rho A\sqrt{1-\frac{u^2}{c^2}}\left(1-\frac{v}{c^2}u_x\right)}{A'\beta\left(1-\frac{u^2}{c^2}\right)^{1/2}}$$

$$\rho' = \frac{\rho A\left(1-\frac{v}{c^2}u_x\right)}{A'\beta} \quad \dots (6)$$

Putting the value of A and A′, we get

$$\rho' = \frac{\rho\left(1-\frac{v}{c^2}u_x\right)\left[\left(1-\frac{{u_x}^2}{c^2}\right)\left(1-\frac{{u_y}^2}{c^2}\right)\left(1-\frac{{u_z}^2}{c^2}\right)\right]^{\frac{1}{2}}}{\sqrt{1-\frac{v^2}{c^2}}\left[\left(1-\frac{{u_x}'^2}{c^2}\right)\left(1-\frac{{u_y}'^2}{c^2}\right)\left(1-\frac{{u_z}'^2}{c^2}\right)\right]^{\frac{1}{2}}}$$

Solved Examples

Example 1: The rest mass of an electron is $9 \times 10^{-28} g$. what will be its mass if it were moving with 4/5th the speed of light?

Solution: The mass of an electron if it were moving with speed v is determined by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Where m_0 is the rest mass of the electron

Here

$$v = \frac{4}{5}c = 0.8c \;\&\; m_0 = 9 \times 10^{-28} g = 9 \times 10^{-31} kg$$

$$\therefore m = \frac{9 \times 10^{-31}}{\sqrt{1 - \left(\frac{0.8c}{c}\right)^2}}$$

$$= \frac{9 \times 10^{-31}}{\sqrt{1 - 0.64}} = \frac{9 \times 10^{-31}}{\sqrt{0.36}} = \frac{9 \times 10^{-31}}{0.6}$$

$$= 1.5 \times 10^{-30} kg$$

Example 2: How much electric energy could theoretically be obtained by annihilation of 1 gm of matter?

Solution: We have

$$\Delta E = \Delta m. c^2$$

$$= (1 \times 10^{-3} kg) \times (3 \times 10^{8} m/s)^2$$

$$= 9 \times 10^{13} joule$$

$$1\ electron\ volt = 1.62 \times 10^{-19} joule$$

Therefore, electrical energy obtained

$$= \frac{9 \times 10^{13}}{1.602 \times 10^{-19}} eV$$

$$= \frac{9 \times 10^{32}}{1.602} eV$$

$$= 5.618 \times 10^{32} eV$$

Example 3: Proton and neutron rest masses are 1.6725×10^{-24}gm and 1.6748×10^{-24} gm, respectively. The deuteron's measured mass is 3.3433×10^{-24} grams. Determine the binding energy.

Solution: We know that a nucleus of deuteron consists of one proton and one neutron.

$$\Delta m = (mass\ of\ proton + mass\ of\ neutron - mass\ of\ deuteron)$$

$$= (1.6725 + 1.6748) \times 10^{-24} - 3.3433 \times 10^{-24}$$

$$= 0.0040 \times 10^{-24} = 4 \times 10^{-27} gm$$

Binding energy ΔE is given by

$$\Delta E = \Delta m. c^2 = 4 \times 10^{-27} \times (3 \times 10^{10})^2 = 36 \times 10^{-7} erg$$

$$= \frac{36 \times 10^{-7}}{1.6 \times 10^{-12}} eV = \frac{36 \times 10^{-7}}{1.6 \times 10^{-12} \times 10^{6}} MeV$$

$$= \frac{3.6}{1.6} = 2.25 MeV$$

Example 4: A particle with rest mass $2 \times 10^{-24} kg$ is moving with speed $2.1 \times 10^{8} m/sec$. Calculate its moving mass.

Solution: Given

$$m_0 = 2 \times 10^{-24} kg, v = 2.1 \times 10^{8} m/sec, c = 3 \times 10^{8} m/sec$$

We have to calculate m.

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{2 \times 10^{-24}}{\sqrt{1 - \left(\frac{2.1 \times 10^{8}}{3 \times 10^{8}}\right)^2}}$$

$$= \frac{2 \times 10^{-24}}{0.714}$$

$$= 2.8 \times 10^{-24} kg$$

Example 5: Calculate rest mass of photon.

Solution: We know that for photon $v = c$, then

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$m_0 = m\sqrt{1-\frac{v^2}{c^2}} = m\sqrt{1-\frac{c^2}{c^2}}$$

$$m_0 = m \times 0 = 0$$

∴ rest mass of photon= 0

Example 6: From the relativistic concept of mass and energy show that the kinetic energy of the moving mass m with velocity v is $m_0v^2/2$ when $v \ll c$ where c being velocity of light.

Solution: We know that

$$E = K.E. of\ the\ moving\ particle + energy\ at\ rest$$

$$(\because E = mc^2, T = K.E. of\ the\ moving\ particle)$$

$$mc^2 = T + m_0c^2 \Rightarrow T = c^2(m - m_0)$$

$$T = c^2\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\right)$$

$$= m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right] = m_0c^2\left[1 + \frac{v^2}{2c^2} - 1\right]$$

Neglecting higher power as $v \ll c$

$$T = m_0c^2\frac{v^2}{2c^2} = \frac{1}{2}m_0v^2$$

SELF CHECK QUESTIONS

1. Prove that the relation between momentum and energy is
$$E^2 = p^2c^2 + {m_0}^2c^4$$
2. Rest mass of photon is
   a) $h/\lambda c$
   b) hv/c^2
   c) 0
   d) m
3. The variation of mass relation is given by

$a)\ m = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$b)\ m = m_0\sqrt{1-\frac{v^2}{c^2}}$

$c)\ m = \dfrac{m_0}{1-\frac{v^2}{c^2}}$

d) None of these

3.14 *SUMMARY*:-

In this unit, we studied the fundamental modifications required in classical mechanics to accurately describe physical phenomena at relativistic speeds. We began with a review of mass and momentum, followed by Newton's laws of motion and the measurement of physical quantities in different unit systems. A major focus was on the variation of mass with velocity, where we learned that mass increases with speed, as supported by experimental verifications. We then studied the transformation formulas for mass, momentum, and energy, which are essential for analyzing particle motion in different inertial frames. Special emphasis was given to the behavior of particles with zero rest mass, such as photons, and how they still carry energy and momentum. The concept of binding energy was introduced, illustrating the mass-energy relationship in nuclear processes. Additionally, we examined the relativistic transformation formula for force, showing how force components change under Lorentz transformations. Finally, we discussed the transformation of density in relativistic contexts, reinforcing the idea that even quantities like mass density are frame-dependent. Overall, this unit laid a solid foundation for understanding motion, forces, and energy in the realm of high velocities close to the speed of light.

3.15 *GLOSSARY*:-

- **Relativistic Mechanics:** The branch of physics that modifies classical mechanics to account for objects moving at speeds close to the speed of light, incorporating the principles of special relativity.

- **Special Relativity:** Einstein's theory that describes the physics of objects in inertial frames moving at constant high speeds, based on two postulates: the constancy of the speed of light and the invariance of physical laws in all inertial frames.
- **Lorentz Transformation:** A set of equations that relate space and time coordinates of events between two inertial frames moving at constant velocity relative to each other.
- **Rest Mass ($\boldsymbol{m_0}$):** The mass of an object as measured in its own rest frame; it is invariant (does not change with speed).
- **Binding Energy:** The energy required to separate the components of a bound system, such as a nucleus; it's equal to the mass defect multiplied by c^2.
- **Particle with Zero Rest Mass:** Particles like photons that have no rest mass but carry energy and momentum and always move at the speed of light.
- **Transformation of Force:** In relativistic mechanics, the components of force transform differently along and perpendicular to the direction of motion.
- **Relativistic Density:** The mass density of an object as observed from a moving frame, which changes due to length contraction.
- **Relativistic Mass ($\boldsymbol{m}$):** The effective mass of an object increases with its velocity according to
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$
- **Momentum (Relativistic):**Momentum in special relativity is given by
$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$
It grows without bound as velocity approaches the speed of light.
- **Energy-Mass Equivalence:** Expressed as $E = mc^2$, this famous relation shows that mass and energy are interchangeable.
- **Total Energy** ($\boldsymbol{E}$): The sum of rest energy and kinetic energy:
$$K.E. = E = m_0 c^2$$
It increases more steeply than in classical mechanics as speed increases.
- **Time Dilation:** The phenomenon where a moving clock appears to tick slower when observed from a stationary frame:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- **Length Contraction:** The phenomenon where a moving object appears shorter along the direction of motion:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

3.16REFERENCES:-

- James J. Callahan (2019), "The Geometry of Space-time: An Introduction to Special and General Relativity" (2nd Edition.
- Spencer A. Klein (2017), "Relativistic Mechanics and Electrodynamics".
- Ta-Pei Cheng (2015),"Relativity, Gravitation and Cosmology: A Basic Introduction" (2nd Edition).

3.17 *SUGGESTED READING:-*

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

3.18 *TERMINAL QUESTIONS:-*

(TQ-1) Derive and discuss $E = mc^2$

(TQ-2) The normal 12 volt car battery has the capacity to deliver 31 amperes for 20 minutes from full charge to discharge. It weighs 20 kg when fully charged. When it is not charged, how much less does it weigh?

(TQ-3) Calculate the velocity at which the mass of a particle becomes 8 times its rest mass.

(TQ-4) If the mass of a hydrogen atom is 1.00814 a.m.u., that of a neutron is 1.00898 a.m.u., and that of a helium atom is 4.00388 a.m.u., then determine the binding energy of one helium nucleus.

(TQ-5) Prove the formula

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

(TQ-6) Explain the formulation of energy- momentum vector in special relativity.

(TQ-7) Describe experimental verification of the variation of mass with velocity.

(TQ-8) A particle is moving with speed 0.6c. Calculate the ratio of rest mass to moving mass.

3.19 *ANSWERS:-*

SELF CHECK ANSWERS

1. c
2. a

TERMINAL ANSWERS

(TQ-2) 12.96×10^{-9}
(TQ-3) $0.992c$
(TQ-4) $28.687MeV$
(TQ-8) $4/5$

UNIT 4:-Minkowski Space

CONTENTS:

4.1 *INTRODUCTION:-*

Minkowski space is a fundamental concept in the theory of special relativity that combines space and time into a single four-dimensional continuum known as spacetime. Proposed by the German mathematician Hermann Minkowski in 1908, it provided a new geometric interpretation of Einstein's special theory of relativity. Unlike classical Newtonian mechanics, where space and time are treated as separate and absolute entities, Minkowski space treats them as interconnected dimensions. Each point in this space, called an "event," is described by four coordinates: three for space (x, y, z) and one for time t, often written as ct to ensure consistent units. This unification allows the laws of physics, especially the behavior of light and motion at high speeds, to be expressed more naturally and precisely. Minkowski space forms the mathematical foundation for analyzing relativistic effects and understanding the structure of spacetime in both special and general relativity.

4.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To explain Minkowski's four dimensional space- time continuums.
- To solve geometric interpretation of Lorentz transformation.
- To define space and time like interval.
- To define world points, world lines, light cone, proper time and four vector.
- To obtain relativistic equations of motion and Minkowski's equation of motion.

4.3*MINKOWSKI SPACE:-*

Minkowski argues that the external world is not Euclidean space of three dimensions, meaning that it is made up of events with coordinates of (x_1, x_2, x_3, x_4), where the first three $(x_1, x_2, x_3)$are space coordinates and the fourth one is time. In other words, the external world is not made up of points with coordinates of (x, y, z), where x, y, and z are real numbers. If anything happens in space, the location of the event in the four-dimensional continuum represents both the point where it happens and the moment it happens. The four directions are not interchangeable. Because a meter stick cannot be turned into a clock, an axis that measures distance in the X-direction can be rotated to measure distance in the Y and Z-direction. However, the same axis cannot be rotated to measure time interval. As a result, the time interval's direction is not unique. The space-time continuum is described as 3+1 dimensional instead of four dimensional to convey this distinction.

The interval between two events whose co-ordinates are

$(x, y, z, x_4) and\ (x + dx, y + dy, z + dz, x_4 + dx_4)$, is given by

$$ds^2 = dx^2 + dy^2 + dz^2 + {dx_4}^2 \qquad \dots (1)$$

Where the co-ordinates x_4 involves t. this interval must be independent of transformation from one system to another system.

We have seen that the expression

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

Is the Lorentz invariant. The invariant interval between two adjacent points must therefore have the following form

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \qquad \dots(2)$$

Comparing equation (1) and (2), we get

$$x_4 = ict, where\ i = \sqrt{(-1)}$$

Four elements (x, y, z, t) can be used to identify an event in Newtonian physics, where t is the time at which the event happens and x, y, and z are the rectangular Cartesian coordinates of the location. Since it is evident that an event requires four numbers to be identified, we say that the totality of all possible occurrences forms a four-dimensional continuum in Newtonian physics. We are unable to eliminate the hyphen and refer to space and time separately because this continuum is known as space-time. Since$x^1 = x, x^2 = y, x^3 = z, x^4 = t$, the coordinates of an event can thus also be obtained at (x^1, x^2, x^3, x^4).

4.4 *GEOMETRICAL INTERPRETATION OF LORENTZ TRANSFORMATION:-*

To prove that Lorentz transformation is simply a rotation in four dimensional spaces.

Assume $p = ict$. We are aware that under the Lorentz transformation $x^2 - c^2t^2 = x^2 + p^2$ is invariant. It indicates that there is no change in the distance between a point P(x, p) and the origin O. To create the new rectangular axes, Ox' and Op', rotate the rectangular axes Ox and Op via an angle of θ. Let P's coordinates be (x', p') with respect to the new axes. The relationships come next.

$$OP^2 = x^2 + p^2\ in\ system\ xp$$

$$OP^2 = x'^2 + p'^2\ in\ system\ x'p'$$

$$x' = xcos\theta + psin\theta \quad \dots(1)$$

$$p' = xcos(90° + \theta) + psin(90° + \theta)$$

$$p' = -xsin\theta + pcos\theta \quad \dots(2)$$

$$let\ \frac{v}{c} = \beta, tan\theta = i\beta, so\ that$$

$$sin\theta = \frac{i\beta}{\sqrt{(1-\beta^2)}}\ , cos\theta = \frac{1}{\sqrt{(1-\beta^2)}}$$

Putting the value of $sin\theta$ and $cos\theta$in equation (1), we get

$$x' = \frac{x}{\sqrt{(1-\beta^2)}} + \frac{pi\beta}{\sqrt{(1-\beta^2)}}$$

But we know that $p = ict$ &$\frac{v}{c} = \beta$,

$$x' = \frac{x - vt}{\sqrt{(1-\beta^2)}} \qquad \ldots (3)$$

Putting the value of $sin\theta$ and $cos\theta$ in equation (2), we get

$$p' = \frac{-xi\beta + p}{\sqrt{(1-\beta^2)}} \qquad \ldots (4)$$

$$-xi\beta + p = -xi\frac{v}{c} + ict = ic\left(t - \frac{vx}{c^2}\right)$$

Now equation (4) becomes

$$ict' = ic\frac{\left(t - \frac{vx}{c^2}\right)}{\sqrt{(1-\beta^2)}}$$

$$or\ t' = \frac{\left(t - \frac{vx}{c^2}\right)}{\sqrt{(1-\beta^2)}} \qquad \ldots (5)$$

The equation (3) & (5) represent Lorentz transformation. Thus, we have proved that Lorentz transformations are equivalent to rotation of axes in four dimensional space $(x, y, z) or\ (x, y, z, ict)$ through an hypothetical angle

$$\theta = \tan^{-1}(i\beta) = \tan^{-1}\left(\frac{iv}{c}\right)$$

4.5 *SPACE AND TIME LIKE INTERVAL:-*

Assume that two frame of references S and S′. S′ is moving with constant velocity v along X-axis. Then by Lorentz transformation

$$x' = \beta(x - vt), y' = y, z' = z, t' = \beta\left(t - \frac{vx}{c^2}\right) \quad \ldots (1)$$

$$where\ \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Assume two events whose coordinates are $(x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2)$ in S.

$$s_{12}{}^2 = -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2 \quad \ldots (2)$$

Similarly in system S′

$$s'_{12}{}^2 = -[(x_2{}' - x_1{}')^2 + (y_2{}' - y_1{}')^2 + (z_2{}' - z_1{}')^2] + c^2(t_2{}' - t_1{}')^2 \quad \ldots (3)$$

From equation (1)

$$s'_{12}{}^2 = -[\beta^2\{(x_2 - x_1) - v(t_2 - t_1)\}^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2\beta^2\left\{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right\}^2$$

$$= -\left[\beta^2(x_2 - x_1)^2\left(1 - \frac{v^2}{c^2}\right) + (y_2 - y_1)^2 + (z_2 - z_1)^2\right] + c^2\beta^2(t_2 - t_1)^2\left(1 - \frac{v^2}{c^2}\right) - 2v\beta^2(x_2 - x_1)(t_2 - t_1) + \frac{2v}{c^2}c^2\beta^2(x_2 - x_1) - v(t_2 - t_1)$$

$$but\ \beta^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

Hence

$$s'_{12}{}^2 = -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2 = s_{12}{}^2$$

$$\Rightarrow s'_{12}{}^2 = s_{12}{}^2$$

$$\Rightarrow s'_{12} = s_{12}$$

This proves that the interval s_{12} is Lorentz invariant.

Consequently, the following outcome is obtained.
The space-time, interval between two events is an invariant.

1. If$s_{12} = 0$, then the intervals s_{12},given by equation (2), is called singular. Also $s_{12} = 0$ given
   $$-[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2 = 0$$
   This suggest that
   $$-[dx^2 + dy^2 + dz^2] + c^2dt^2 = 0$$
   This equation is known as equation of null cone or light cone.
2. Let the two events occur at the same point in S′ and also let the first event occur after the second event so that
   $$x_2' = x_1', y_2' = y_1', z_2' = z_1', t_2' > t_1'$$
   Putting these values in equation (3)
   $$s'_{12}{}^2 = c^2(t_2' - t_1')^2 > 0$$
   $$s'_{12}{}^2 > 0\ \&\ s'_{12} > 0$$
   $$but\ s'_{12} = s_{12}\ .$$
   $$hence s_{12} > 0 \Rightarrow the\ interval\ s_{12} is\ real.$$

Real intervals are called time like intervals.
For s'_{12} contains only time component.
The condition that the intervals is time like interval is

$$c^2(t_2' - t_1')^2 > (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

If the interval is time like, then there exists a frame of reference for which the interval between two events is real.

3. Next, we assume that $t_1' = t_2'$ since the two events in S' happen simultaneously. 3 now turns into

$$s'_{12}{}^2 = -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] < 0$$

$$s'_{12}{}^2 < 0 \; or \; s_{12}{}^2 > 0$$

Or s_{12} is imaginary.
Imaginary intervals are called space like intervals. For s'_{12} contains only space co-ordinates. The condition that an interval is space like interval is

$$c^2(t_2' - t_1')^2 < (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Thus if the interval is space like, then there exists a frame of reference in which two events occur at the same time.
If the magnitude of a vector is real, it is said to be time-like. If its magnitude is imaginary, it is space-like. If its magnitude is zero, then it is null.

4.6*WORLD POINTS AND WORLD LINES*:-

Each particle corresponds to a certain line known as the world line, and the events in the four-dimensional space, or Minkowski space, are represented by points called world points. We just take into account one space axis, the X-axis, without sacrificing generality. A space-time diagram with a horizontal space axis and a vertical time axis that is orthogonal to one another can therefore be used to depict the coordinates (x, t) of an event. If we take ct (= m, say) rather than r, we can maintain the same dimensions of the coordinates. The formulae for the Lorentz transformation for and fare

$$x' = \frac{x - \beta m}{\sqrt{(1 - \beta^2)}} \qquad \dots(1)$$

$$m' = \frac{m - \beta x}{\sqrt{(1 - \beta^2)}} \qquad \dots(2)$$

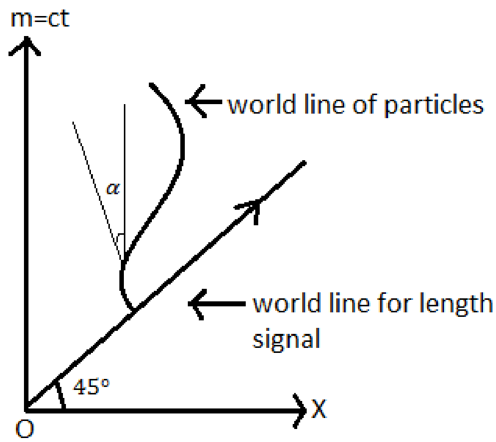


Fig. 4.1

$$x = \frac{x' - \beta m'}{\sqrt{(1-\beta^2)}}, m = \frac{m' - \beta x'}{\sqrt{(1-\beta^2)}} where \beta = v/c$$

In the frame S, we assume that the X-axis is horizontal and the m axis is vertical. A particle's trajectory in this frame will look like a world line, which is a curve whose points are determined by

$$tan\alpha = \frac{1}{c}\frac{dx}{dt} = \frac{v}{c}$$

Where α is the angle between the m-axis and the tangent. Additionally, for each material particle, $\alpha < 45° as\ v < c$. In such case, the world line for the light signal (v = c) is a straight line that forms a 45° angle with the m-axis.

The junction of two particles' world lines is represented by a collision. It is clear that an event and a space-time diagram at that event determine a material particle's world line. If the final velocity is the same as the initial velocity but different in direction, the collision is considered elastic.

4.7 *LIGHT CONE*:-

The quantity

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 \quad ...(1)$$

remains the invariant under Lorentz transformations. Here we take

$$x^1 = x, x^2 = y, x^3 = z, x^4 = ict = ct\sqrt{(-1)} \ ...(2)$$

Naturally, the square of the four-dimensional distance between the event $(x^i)$and the origin (0,0,0,0) equals the invariant from (1). The equation describes a surface made up of all the points with zero distance from the origin.

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad \dots(3)$$

We refer to this surface as the light cone. The propagation of a spherical light wave from the origin O(0, 0, 0) at $t = 0$ was described by equation (2). According to the inequalities, the light cone separates the $(3 + 1)$ space into two distinct and invariant domains, S_1 and S_2.

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 < 0$$

$$and\ s^2 = x^2 + y^2 + z^2 - c^2t^2 > 0$$

Respectively. We have the instance of simultaneity with respect to Lorentz transformations in the domain S_2. It is not possible to change two events in the domain S_1 in this way.

4.8 *PROPER TIME*:-

Working with the invariant $dT^2 = ds^2/c^2$ instead of ds^2 itself is sometimes more convenient. Thus, we designate it as the element of appropriate time and assign it a unique symbol, dT.

$$dT^2 = \frac{1}{c^2}ds^2 = -\frac{1}{c^2}(dx^2 + dy^2 + dz^2) + dt^2$$

$$dT = dt\left[1 - \frac{1}{c^2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}\right]^{1/2}$$

If the particle has a velocity u, then the last becomes

$$dT = dt\left(1 - \frac{u^2}{c^2}\right)^{1/2}$$

Integral of proper time along a world line

$$T = \int \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt$$

4.9 *ENERGY MOMENTUM FOUR VECTOR*:-

To describe how the energy-momentum vector is formulated in special relativity.

The definition of momentum p in classical mechanics is $p = mv$. Similarly, the definition of momentum p in relativistic terms is

$$p = \frac{m_0 v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \quad \ldots (1)$$

We know that

$$ds^2 = -(dx^2 + dy^2 + dz^2) + c^2 dt^2$$

$$\left(\frac{ds}{dt}\right)^2 = -\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] + c^2 = c^2 - v^2$$

$$\Rightarrow \frac{ds}{c} = dt\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

But we know that

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow \frac{ds}{c} = \frac{dt}{\beta} or \beta ds = cdt$$

Writing Cartesian equivalent of (1), we have

$$p_x = \frac{m_0 v_x}{\sqrt{1 - \frac{v^2}{c^2}}} = \beta m_0 v_x = \beta m_0 \frac{dx}{dt} = m_0 \frac{cdx}{ds}$$

From equation (2)

$or \qquad p_x = m_0 \frac{cdx}{ds}$

$$similarly \ p_y = m_0 \frac{cdy}{ds}, \ p_z = m_0 \frac{cdz}{ds}$$

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \beta m_0 c^2 = m_0 c^2 \frac{cdt}{ds}$$

From equation (2)

$$(E/c) = m_0 c \frac{cdt}{ds}$$

Write $x^1 = x, x^2 = y, x^3 = z, x^4 = ct$, we get

$$p_x = m_0 \frac{cdx^1}{ds}, \; p_y = m_0 \frac{cdx^2}{ds}, \; p_z = m_0 \frac{cdx^3}{ds}, \frac{E}{c} = m_0 \frac{cdx^4}{ds}$$

It is evident from this that the four quantities $\left(p_x, \; p_y, \; p_z, \frac{E}{c} \right)$ belong to the energy momentum four-vector, which we represent by p^μ i.e.

$$p^\mu = \left(p_x, \; p_y, \; p_z, \frac{E}{c} \right)$$

This is the significance of the fourth component of momentum.

4.10 *FOUR VECTOR (WORLD VECTOR)*:-

The ordinary vector analysis can be extended to four dimensions by introducing four-dimensional space (x, y, z, ict). We refer to these four-dimensional vectors as world vectors or four vectors. Below their emblems are bars that represent the world vectors.

$$\underset{\rightarrow}{A} = iA_1 + jA_2 + kA_3 + pA_4$$

$$\underset{\rightarrow}{B} = iB_1 + jB_2 + kB_3 + pB_4$$

If we write $\vec{u} = \frac{\overrightarrow{dr}}{dt} = iu_x + ju_y + ku_z$ in usual three dimensional velocity vector, then components of velocity four vector $\underset{\rightarrow}{u}$ are

$$u_1 = \beta u_x, u_2 = \beta u_y, u_3 = \beta u_z, u_1 = \beta(ic)$$

$$where \; \beta = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\underset{\rightarrow}{A} \; . \underset{\rightarrow}{B} = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$$

Now we can defined four velocity u^μ of a particle as

$$u^\mu = \frac{dx^\mu}{ds} = \dot{x}^\mu$$

Also we define four acceleration vector as

$$\dot{u}^\mu = \ddot{x}^\mu = \frac{du^\mu}{ds} = \frac{d}{ds}\left(\frac{dx^\mu}{ds}\right)$$

4.11 *RELATIVISTIC EQUATION OF MOTION:-*

The relativity theory adopts Newton's equations for motion and momentum, but with the distinction that a body of mass m traveling at velocity v fulfills the equation

$$m = \frac{m_0}{\sqrt{1-\beta^2}}$$

$$where\ \beta = \frac{v}{c}$$

Momentum p is defined as

$$p = mv = \frac{m_0 v}{\sqrt{1-\beta^2}}$$

Hence the components of momentum are

$$p_x = \frac{m_0\dot{x}}{\sqrt{1-\beta^2}}\ ,\ p_y = \frac{m_0\dot{y}}{\sqrt{1-\beta^2}},\ p_z = \frac{m_0\dot{z}}{\sqrt{1-\beta^2}}$$

$$where\ \beta^2 = \frac{v^2}{c^2} = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}$$

Here dots denotes differentiation w.r.t. time t. the equation of motion are

$$\frac{d\,p_x}{dt} = F_x, \frac{d\,p_y}{dt} = F_x, \frac{d\,p_z}{dt} = F_x$$

$$m_0\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{1-\beta^2}}\right] = F_x, m_0\frac{d}{dt}\left[\frac{\dot{y}}{\sqrt{1-\beta^2}}\right] = F_y, m_0\frac{d}{dt}\left[\frac{\dot{z}}{\sqrt{1-\beta^2}}\right] = F_z$$

4.12 *MINKOWSKI'S EQUATION OF MOTION:-*

The equation

$$\dot{p}^\mu = K^\mu i.e. \frac{d}{ds}\left(m_0 c\frac{dx^\mu}{ds}\right) = K^\mu$$

is referred to as Minkowski's equation of motion and the K^μ is four force. This K^μ is alo known as Minkowski's force.

To prove that Minkowski's equation reduces to the Newtonian form in the limit where $v/c \to 0$

Equation (1) is expressible as

$$\frac{d}{dt}\left(m_0\frac{dx^\mu}{dt}\frac{dt}{ds}\right)\frac{dt}{ds} = K^\mu$$

$$\frac{d}{dt}\left(m_0\frac{dt}{ds}\frac{dx^\mu}{dt}\right)\frac{dt}{ds} = K^\mu\frac{ds}{dt} \quad \dots (2)$$

$$ds^2 = -[(dx)^2 + (dy)^2 + (dz)^2] + c^2dt^2$$

$$\left(\frac{ds}{cdt}\right)^2 = 1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right].\frac{1}{c^2} = 1 - \frac{v^2}{c^2}$$

$$\frac{1}{c}.\frac{ds}{dt} = \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \quad \dots (3)$$

$$\therefore m_0c\frac{dt}{ds} = m_0c.\frac{1}{c\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = m \quad \dots (4)$$

(According the law of variation of mass with velocity)

From equation (2), (3) & (4)

$$\frac{d}{dt}\left(m\frac{dx^\mu}{dt}\right) = K^\mu.c\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

Taking limit as $v/c \to 0$, we get

$$\frac{d}{dt}\left(m\frac{dx^\mu}{dt}\right) = cK^\mu \quad for\ \mu = 1,2,3,4.$$

Hence in particular

$$\frac{d}{dt}\left(m\frac{dx^i}{dt}\right) = cK^i \quad for\ i = 1,2,3$$

This is Newton's form of equation of motion.

SELF CHECK QUESTIONS

1. What are the four coordinates used to describe an event in Minkowski space?
2. What is the formula for the spacetime interval between two events in Minkowski space?
3. What are the three types of intervals in Minkowski space, based on the sign of s^2?
4. Is the spacetime interval invariant under Lorentz transformations?
5. What transformation replaces Galilean transformations in Minkowski space?
6. Why is the time coordinate often written as ct in Minkowski space?

4.13 *SUMMARY*:-

In this unit we have studied the Minkowski space is a four-dimensional framework that merges the three dimensions of space with the dimension of time to form a unified concept called spacetime. Developed by Hermann Minkowski, it provides the geometric foundation of Einstein's special theory of relativity. In this space, events are represented by four coordinates (ct, x, y, z), where ct is the time component scaled by the speed of light. The key feature of Minkowski space is the spacetime interval, which remains invariant under Lorentz transformations, unlike distances in classical mechanics. This interval determines whether two events are causally connected and is classified as timelike, spacelike, or lightlike. The geometry of Minkowski space is pseudo-Euclidean, meaning it has one time dimension with a different sign in the metric compared to the three space dimensions. This structure allows for a natural explanation of relativistic phenomena such as time dilation, length contraction, and the constancy of the speed of light, making it essential for understanding modern physics.

4.14*GLOSSARY*:-

- **Minkowski Space:** A four-dimensional spacetime framework that combines three spatial dimensions and one time dimension, used to describe the structure of space and time in special relativity.
- **Spacetime:** The unified concept of space and time as a single four-dimensional continuum, where events are located using space and time coordinates.

- **Event:** A point in spacetime defined by four coordinates (ct,x,y,z), representing a specific place at a specific time.
- **Spacetime Interval (s^2):** An invariant quantity defined as $s^2 = x^2 + y^2 + z^2 - c^2t^2$ that remains the same in all inertial frames of reference.
- An interval with $s^2 < 0$<0, meaning two events can be connected by a signal moving slower than light; they are causally connected.
- **Spacelike Interval:** An interval with $s^2 > 0$, meaning two events cannot influence each other; they are outside each other's light cones.
- **Lightlike (Null) Interval:** An interval with $s^2 = 0$, representing the path of a light signal; the events lie on each other's light cone.
- **Worldline:**The path that an object traces in Minkowski spacetime, showing its position over time.
- **Light Cone:** A cone-shaped surface in spacetime representing all possible light paths from an event. It separates the past, future, and elsewhere (causally disconnected regions).
- **Proper Time (τ):**The time interval measured by a clock moving with the object; the actual experienced time between two events on a timelike path.
- **Metric Tensor ($\eta_{\mu\nu}$):** A matrix that defines the geometry of spacetime in special relativity. For Minkowski space, it typically has the form:

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1)$$

- **Causality:** The principle that a cause must precede its effect; preserved in Minkowski space by restricting causal influence to within the light cone.
- **Inertial Frame:** A reference frame in which an object not acted upon by a force moves in a straight line at constant speed.
- **Pseudo-Euclidean Geometry:** A geometry where the time component has a different sign than spatial components in the metric, as in Minkowski space.

4.15 *REFERENCES*:-

- Ashok Das (2011), Lectures on Gravitation, University of Rochester, USA ,Saha Institute of Nuclear Physics, India.
- Richard Feynman (2018), Feynman Lectures On Gravitation.

4.16 SUGGESTED *READING*:-

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

4.17 *TERMINAL QUESTIONS:-*

(TQ-1) Discuss Minkowski's four dimensional space- time continuums.

(TQ-2) Explain Minkowski's four dimensional formation bringing out the significance of the four components of momentum and the equations of motion.

(TQ-3) Explains the following terms in detail; light cone, world line, space like vector, time like vector.

(TQ-4) Derive the four vector equation of motion and discuss the physical significance of the force four-vector in terms of classical quantities**.**

(TQ-5) What is Minkowski space and how does it relate to special relativity?

(TQ-6) Explain the concept of the spacetime interval and its physical significance.

(TQ-7) How are Lorentz transformations derived from Minkowski space, and what is their role?

(TQ-8) Describe the geometry of Minkowski space and contrast it with Euclidean space.

(TQ-9) How does Minkowski spacetime help explain time dilation and length contraction?

(TQ-10) What is the significance of Minkowski space in general relativity?

4.18 *ANSWERS:-*

SELF CHECK ANSWERS

1. (ct, x, y, z)
2. $s^2 = -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]$
3. Timelike ($s^2 < 0$), Spacelike($s^2 > 0$), Null($s^2 = 0$).
4. The spacetime interval is invariant—it has the same value in all inertial frames.

5. Lorentz transformations replace Galilean transformations to preserve the spacetime interval.
6. Multiplying time t by c (the speed of light) makes the units of time and space the same (typically meters), simplifying the interval formula and unifying space and time.

UNIT 5:-Some Applications of Special Theory of Relativity

CONTENTS:

5.1 *INTRODUCTION:-*

The Special Theory of Relativity, formulated by Albert Einstein, is not only a profound theoretical advancement but also a practical framework with far-reaching applications in modern science and technology. At its core, the theory redefines fundamental notions of time, space, and energy, especially at speeds approaching that of light. While originally developed to resolve inconsistencies in classical mechanics and electromagnetism, special relativity now serves as the basis for interpreting high-speed phenomena in various domains. Its effects are no longer just theoretical curiosities—they manifest in everyday technologies such as satellite communication, particle accelerators, and even nuclear energy production. From ensuring the accuracy of GPS navigation to explaining the dynamics of cosmic rays, the theory's applications extend across both terrestrial and astronomical scales. Exploring these applications highlights the essential role of relativity in bridging abstract physics with real-world functionality.

5.2 *OBJECTIVES:-*

After studying this unit, the learner’s will be able to

- To explain Compton Effect and its importance.

- To discuss De-Broglie hypothesis of matter wave.

5.3 *COMPTON EFFECT:-*

The term “Compton effect” or “Compton scattering” refers to the elastic scattering of a photon from an electron. Compton asserts that when electrons in the scatter a high-frequency radiation beam, lower-frequency radiations are also produced. The Compton Effect is the name given to this observable shift in the frequency or wave length of the scattered high frequency radiations. Compton discovered that the direction the scattered beam travels, or the angle it makes with the indirect photon, affects this change. The simultaneous application of the theory of relativity provided an explanation for this effect on the fundamentals of quantum theory. According to the quantum theory's radiation principle,

I. Photons with energy $h\nu$, where h = Planck's constant and frequency, make up radiations.
II. The photons move at the speed of light c,

$$c = 3 \times 10^{10} cm/sec$$

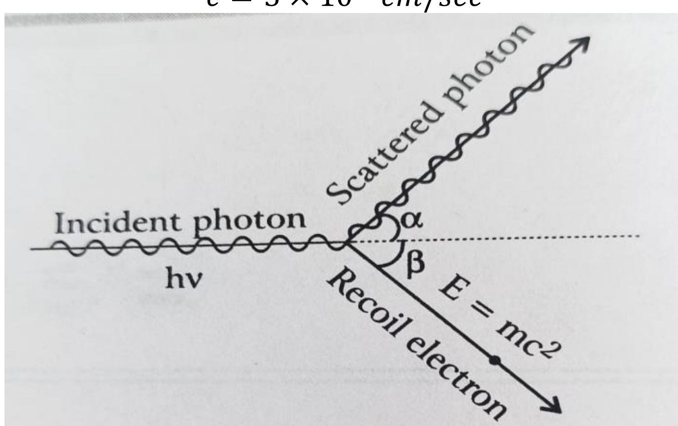


Fig.5.1

III. When the photons hit the scatterer's electrons, they follow all the rules of energy and momentum conservation.
IV. Some photons' K.E. is passed to electrons when they collide with them, resulting in scattered photons having a lower K.E. than the incident one.

To Derive the Formula for Scattering

Prior to the collision, let's assume that the electron is free and at rest. Figure 5.1 displays the dispersed photon and electron trajectories following a collision.

$$\because h\nu = energy\ of\ photon = mass.c^2$$

(By $E = mc^2$)

$$\therefore mass\ of\ photon = \frac{h\nu}{c^2}$$

$$momentum\ of\ photon = mass.velocity = mass.c = \frac{h\nu}{c^2}c = \frac{h\nu}{c}$$

Before Collision: $energy\ of\ photon = h\nu$

$$momentum\ of\ incident\ photon = \frac{h\nu}{c}$$

$energy\ of\ electron = m_0c^2$; m_0 is the mass of electron when it is at rest.

Momentum of electron=0 as electron is at rest before collision.

After Collision: Let after collision the scattered photon and recoil electron make angles $\alpha\ and\ \beta$ respectively with the direction of incident beam.

$energy\ of\ scattered\ photon = h\nu'$

$momentum\ of\ scattered\ photon = \frac{h\nu'}{c}$

$energy\ of\ electron = mc^2$

$momentum\ of\ recoil\ electron = mv$

Where m is the mass of the electron when it is moving with velocity v.

Calculations: By the principle of conservation of energy,

Energy before collision = Energy after collision

$$h\nu + m_0c^2 = h\nu' + mc^2 \quad ...(1)$$

By principle of conservation of momentum,

Momentum before collision = Momentum after collision along and perpendicular to the direction of incident photon

We get

$$\frac{h\nu}{c}cos0 + 0 = \frac{h\nu'}{c}cos\alpha + mvcos\beta$$

$$\& \ \frac{h\nu}{c}sin0 + 0 = \frac{h\nu'}{c}sin\alpha - mvsin\beta$$

$$\Rightarrow \frac{hv}{c} = \frac{hv'}{c} cos\alpha + mvcos\beta \;\&\; \frac{hv'}{c} sin\alpha - mvsin\beta = 0$$

$$\Rightarrow \qquad cmv\ cos\beta = h(v - v'cos\alpha) \qquad ...(2)$$

$$\&\ cmv\ sin\beta = hv'sin\alpha \qquad ...(3)$$

Squaring (2) and (3) and then adding,

$$m^2v^2c^2 = h^2[v^2 + v'^2 - 2vv'cos\alpha] \qquad ...(4)$$

Let p and E denote respectively momentum and energy of recoil electron. Then

$$p = mv, E = mc^2$$

$$equation\ (4) \Rightarrow \qquad p^2c^2 = h^2[v^2 + v'^2 - 2vv'cos\alpha]$$

$$equation\ (1) \Rightarrow \qquad E^2 = [h(v - v') + m_0c^2]^2$$

$$or \qquad E^2 = h^2(v^2 + v'^2 - 2vv') + {m_0}^2c^4 + 2m_0c^2h(v - v') \quad ...(5)$$

Subtracting equation (4) from (5), we get

$$E^2 - p^2c^2 = 2vv'h^2(cos\alpha - 1) + {m_0}^2c^4 + 2hm_0c^2(v - v')$$

But we know that

$$E^2 - p^2c^2 = {m_0}^2c^4$$

$$or \qquad {m_0}^2c^4 = 2vv'h^2(cos\alpha - 1) + {m_0}^2c^4 + 2hm_0c^2(v - v')$$

$$or \qquad 2vv'h^2(1 - cos\alpha) = 2hm_0c^2(v - v')$$

$$or \qquad \frac{h(1 - cos\alpha)}{m_0c^2} = \frac{(v - v')}{vv'}$$

$$or \qquad \frac{1}{v'} - \frac{1}{v} = \frac{h}{m_0c^2}(1 - cos\alpha) \qquad ...(6)$$

As α is acute so that $cos\alpha < 1. \frac{1}{v'} - \frac{1}{v} > 0$

$$or \qquad v - v' > 0\ or\ v > v'$$

But velocity = frequency. Wave length

$$\therefore \qquad c = v\lambda\ and\ c = v'\lambda'\ for\ photon$$

$\Rightarrow \qquad v = \frac{c}{\lambda}, v' = \frac{c}{\lambda'}$

Using this in equation (6), we get

$$\frac{\lambda'}{c} - \frac{\lambda}{c} = \frac{h}{m_0 c^2}(1 - cos\alpha)$$

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - cos\alpha) \qquad ... (7)$$

$\Rightarrow \lambda' - \lambda > 0 \Rightarrow \lambda' > \lambda$

The frequency change indicated by relation (6) implies that incoming radiation has a higher frequency than scattered radiation. The relation (7) indicates that the wave-length of incident radiation is smaller than the wave-length of dispersed radiation.
If $\alpha = \pi/2$, then $cos\alpha = 0$ so that (7) gives

$$\lambda' - \lambda = \frac{h}{m_0 c} \qquad ... (8)$$

$h = 6.62 \times 10^{-27} ergs\ sec, c = 3 \times 10^{10} cm/sec$

$electonic\ rest\ mass = m_0 = 9 \times 10^{-28} gram$

$$\frac{h}{m_0 c} = \frac{6.62 \times 10^{-27}}{9 \times 10^{-28} \times 3 \times 10^{10}} = \frac{66.2}{27 \times 10^{10}} = 2.4519 \times 10^{-10} cm$$
$$= 0.024519 Å$$

This quantity is called Compton wavelength and is denoted by λ_c.

$\lambda_c = 0.024519 Å$

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1 - cos\alpha) = 2\lambda_c sin^2(\alpha/2)$$

Since λ_c is finite in every inertial frame and hence it is impossible for a free electron to emit or absorb a photon.

5.4 *EXPERIMENT ON COMPTON EFFECT:-*

In 1905, De-Broglie and Geiger carried out an experiment to confirm Compton's **i**dea. In order to detect the photons and electrons produced by an X-ray beam scattering in hydrogen gas, two Geiger counters were placed opposite each other and perpendicular to the beam. While the other counter was sensitive to electrons exclusively, the platinum foil used to close one of the counters was sensitive to photons since it absorbs electrons, enabling only X-rays to enter the chamber. A secondary electron is created when a photon interacts with the gas inside the chamber. The photon counter actually reacts only to a secondary electron, not to a photon that strikes it directly.

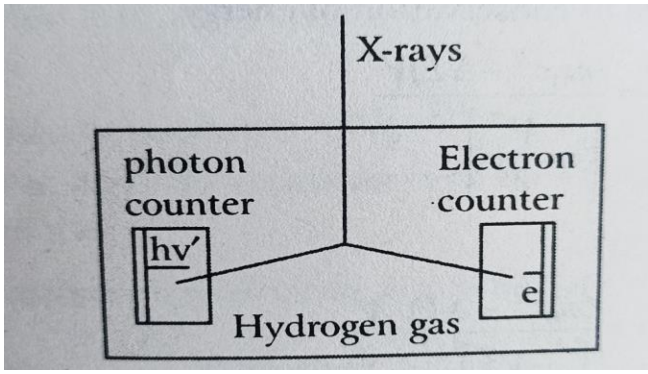


Fig.5.2

Compton theory states that an electron enters the electron counter for every photon entering the photon counter. About ten electrons were found in the electron counter for each scattered photon that was recorded in the photon counter. The reason is that no secondary electron is created by each photon that enters the chamber. The simultaneous detection of about 10% ionization in both counters can be attributed to chance and coincidence. The simultaneous emission of a scattered photon and a recoiling electron can be attributed to the observed coincidence, supporting the Compton theory as a two-particle action.

5.5 *DE-BROGLIE HYPOTHESIS OF MATTER WAVES*:-

De-Broglie Hypothesis: De-Broglie proposed that just like the dual nature of electromagnetic radiation, a material particle such as electron, proton etc., might have dual nature. He asserted, "A moving particle whatever its nature has wave properties associated with it". According to him: $\lambda = h/mv$ where λ is the wave length associated with the moving particle, m the mass of the particle, v its velocity and h is Planck's constant.

$$(h = 6.62 \times 10^{-34} joules/\ sec)$$

Derivation: In case of radiation, the momentum of photon:

$$p = \frac{hv}{c} = \frac{h}{\lambda} or\ \lambda = \frac{h}{p}$$

$as\ v = n\lambda\ gives\ c = v\lambda$

Similarly, the wavelength of the matter wave is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

The special theory of relativity can also be used to determine this value of the matter wave's wave length. Similar to a proton or electron, a material particle can be thought of as the standing wave system in the space region, then

$$\psi = \psi_0 e^{2\pi i v t} \qquad \dots (1)$$

Where v is the particle's frequency in the remaining frame, ψ_0 is the wave's amplitude at the position (x, y, z) at the moment &, and ψ is the quantity that varies periodically to produce matter waves. Using the Lorentz transformation of this wave function in the new frame of variables (x', y', z'), let the particle travel with velocity v in the positive direction of the X-axis., we have

$$\psi = \psi_0 exp\left\{2\pi i v \frac{\left(t' + \frac{vx'}{c^2}\right)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}\right\} \qquad \dots (2)$$

Now the standard equation of wave equation is

$$\psi = \psi_0 exp\left\{2\pi i v\left(t + \frac{x'}{u'}\right)\right\} \qquad \dots (3)$$

Where u' is the phase velocity of the wave in the new frame. From equation (2) & (3), we get

$$v' = \frac{v}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} and\ u' = \frac{c^2}{v} \qquad \dots (4)$$

Taking the mass of the particle m_0 in the position of rest, we get by Einstein's mass energy and quantum hypothesis

$$E = hv = m_0 c^2$$

$\Rightarrow$

$$v = \frac{m_0 c^2}{h}$$

Putting this value in equation (4), we get

$$v' = \frac{\frac{m_0c^2}{h}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} = \frac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} \cdot \frac{c^2}{h} = \frac{mc^2}{h}$$

$$as\ m = \frac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$$

Hence, the wave length of the matter particle is

$$\frac{u'}{v'} = \frac{\frac{c^2}{v}}{\frac{mc^2}{h}} = \frac{h}{mv}$$

Thus, the wave length of De-Broglie wave associated with a material particle is given by the expression

$$\lambda = \frac{h}{p}$$

A relation between the particle and wake aspects of the matter.

SOLVED EXAMPLE

EXAMPLE: 1. An excited nucleus of rest mass m_0 is at rest with respect to a chosen inertial frame. It goes over to the lower state whose energy is smaller by ΔE. As a result it emits a $\gamma - ray$ photon and undergoes a recoil. Show that frequency v of the $\gamma - ray$ photon is given by

$$v = \frac{\Delta E}{h}\left[1 - \frac{\Delta E}{2m_0c^2}\right]$$

SOLUTION: The mass m of the recoil nucleus in the inertial frame where its rest mass is m_0 is determined by taking v to be the frequency of the $\gamma - ray$ photon that is emitted and allowing the nucleus to recoil with velocity V.

$$m = \frac{m_0c^2 - \Delta E}{c^2\left(1-\frac{V^2}{c^2}\right)^{1/2}} \qquad \dots (1)$$

According to the principle of conservation of energy,

$$m_0c^2 = \frac{m_0c^2 - \Delta E}{c^2\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} + hv \qquad \ldots(2)$$

$p = mv \ gives \ p = mc \ for \ photon \ and \ so$

$$p = \frac{mc^2}{c} = \frac{E}{c} = \frac{hv}{c}$$

$$p = \frac{hv}{c} \qquad \ldots(3)$$

According to the principle of conservation of energy,

$$\frac{hv}{c} = \frac{(m_0c^2 - \Delta E)V}{c^2\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} \qquad \ldots(4)$$

It follows from (1) and (3)

$$(4) \Rightarrow \qquad hv = \frac{(m_0c^2 - \Delta E)}{\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} \cdot \frac{V}{c} \qquad \ldots(5)$$

$$(2) \Rightarrow \qquad m_0c^2 - hv = \frac{(m_0c^2 - \Delta E)}{\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} \qquad \ldots(6)$$

$$(5) \Rightarrow \qquad \frac{hvc}{V} = \frac{(m_0c^2 - \Delta E)}{\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} \qquad \ldots(5)$$

Equating (6) & (7)

$$m_0c^2 - hv = \frac{hvc}{V}$$

$$or \qquad \frac{m_0c^2 - hv}{hv} = \frac{c}{V}$$

$$or \qquad \frac{V}{c} = \frac{hv}{m_0c^2 - hv}$$

Putting the expression for V/c in (5), we get

$$hv = \frac{(m_0c^2 - \Delta E)}{\left[1 - \left(\frac{hv}{m_0c^2 - \ hv}\right)^2\right]^{\frac{1}{2}}}$$

$$1 = \frac{(m_0c^2 - \Delta E)}{[(m_0c^2 - \ hv)^2 - (hv)^2]^{1/2}}$$

$$(m_0c^2 - \Delta E)^2 = (m_0c^2 - 2hv)m_0c^2$$

$$(m_0c^2)^2 - 2m_0c^2\Delta E + (\Delta E)^2 = (m_0c^2)^2 - 2hvm_0c^2$$

$$\Delta E[\Delta E - 2m_0c^2] = -2hvm_0c^2$$

$$v = \frac{\Delta E}{h}\left[1 - \frac{\Delta E}{2m_0c^2}\right]$$

EXAMPLE2: Show that it is not possible for a photon to transfer all its energy to a free electron.

SOLUTION: Allow a photon with momentum p and energy E to transfer all of its energy to a free electron if at all possible. An electron with rest mass m_0 move with velocity v following the energy transfer.

We can regard the motion of an electron as a non-relativistic one so that

$$E = \frac{1}{2}m_0v^2 \qquad \dots(1)$$

(Since mass of electron compared to that of photon is infinitely large.)

$$\frac{E}{c} = p = m_0v$$

$$\frac{E}{c} = m_0v \qquad \dots(2)$$

Dividing (1) by (2), we get

$$c = \frac{v}{2} or\ v = 2c$$

This demonstrates that the electron travels at twice the speed of light. A contradiction. The maximum speed at which any particle in nature may travel is the speed of light.

SELF CHECK QUESTIONS

1. If λ and λ' are wavelengths at emission and reception respectively, if the condition of red sift is
a) $\lambda' = \lambda$
b) $\lambda' \leq \lambda$
c) $\lambda' > \lambda$
d) None of thes
2. What is time dilation and where is it observed in real-life applications?
3. How does the special theory of relativity apply to particle accelerators?
4. What role does length contraction play in high-speed travel?
5. How does special relativity help explain the decay of fast-moving muons?

5.6 *SUMMARY*:-

This unit focused on key discoveries that marked the transition from classical physics to quantum theory. We began by studying the Compton Effect, which provided strong evidence for the particle nature of light. The effect, observed during Compton scattering experiments, confirmed that photons carry momentum and that their interaction with electrons leads to a measurable shift in wavelength something that classical wave theory could not explain. Moving forward, we examined the groundbreaking De Broglie Hypothesis, which introduced the concept that all matter exhibits wave-like behavior. This idea extended the notion of wave-particle duality from light to matter, leading to the conclusion that particles such as electrons have a wavelength inversely proportional to their momentum. These topics not only deepen our understanding of the quantum world but also form the theoretical foundation for technologies like electron microscopes and quantum computing.

5.7GLOSSARY:-

- **Time Dilation:** The phenomenon where time appears to pass more slowly for an object moving at a high velocity relative to a stationary observer.

- **Length Contraction:** The shortening of an object in the direction of its motion as its speed approaches the speed of light, relative to an observer.
- **Relativistic Mass:** The concept that the mass of an object increases with its speed, becoming significantly larger as it approaches the speed of light.
- **Photon:** A quantum (particle) of electromagnetic radiation that has zero rest mass and travels at the speed of light.
- **Relativistic Momentum:** The momentum of a particle moving at relativistic speeds, given by $p = \gamma m v$, where γ is the Lorentz factor.
- **Global Positioning System (GPS):** A satellite-based navigation system that requires corrections from both special and general relativity to provide accurate location data.
- **Muons:** Elementary particles produced by cosmic rays in the upper atmosphere, whose extended life when traveling fast is evidence of time dilation.
- **Particle Accelerators:** Devices that accelerate particles to near-light speeds, where relativistic effects must be taken into account to predict motion and collisions.
- **Cosmic Rays:** High-energy particles from space that travel at nearly the speed of light, used in experiments confirming relativistic effects.
- **Simultaneity:** The concept that two events occurring at the same time in one frame may not occur simultaneously in another moving frame.
- **Twin Paradox:** A thought experiment that illustrates time dilation: a twin who travels at high speed and returns ages less than the one who stayed on Earth

5.8 *REFERENCES:-*

- **Rindler, W. (2018).** *Introduction to Special Relativity* (2nd ed.). Oxford University Press. A modern and accessible textbook covering both theory and applications, including GPS, relativistic kinematics, and particle physics.

5.9 *SUGGESTED READING:-*

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

5.10 *TERMINAL QUESTIONS:-*

(TQ-1) What is Compton Effect? Give its theory and importance.
(TQ-2) Using special theory of relativity derive De-Broglie hypothesis.
(TQ-3) Calculate De- Broglie wavelength of an electron whose kinetic energy is 50eV. $(m = 9.1 \times 10^{-28} gm)$

5.11 *ANSWERS:-*

SELF CHECK ANSWERS

1. c)
2. Time dilation is the effect where time appears to pass more slowly for an object moving at high speed relative to a stationary observer. It is observed in real-life in the accurate timing of GPS satellites, which must account for both special and general relativistic effects to provide precise location data.
3. In particle accelerators, particles move at speeds close to the speed of light. Due to relativistic effects, their mass increases with speed, and relativistic momentum and energy equations must be used to describe their behavior accurately.
4. Length contraction is the phenomenon where objects moving at relativistic speeds appear shortened in the direction of motion. Although not experienced at everyday speeds, it becomes significant in relativistic space travel or in cosmic ray interactions with Earth's atmosphere.
5. Muons created by cosmic rays in the upper atmosphere have a short lifetime. However, due to time dilation, they appear to live longer from the perspective of an Earth observer, allowing them to reach the ground before decaying.

TERMINAL ANSWERS

(TQ-3) $\lambda = 1.73\text{Å}$

BLOCK II
TENSOR ANALYSIS

UNIT 6:-Tensor and line Element I

CONTENTS:

6.1 *INTRODUCTION:-*

In this section, we are introduced to the fundamental concepts of tensors and the line element**,** which play a crucial role in the mathematical formulation of physics, especially in the theory of general relativity. A tensor is a generalization of scalars and vectors that remains invariant under coordinate transformations, allowing physical laws to be expressed in a form valid in all reference frames. The line element is an expression for the infinitesimal distance between two nearby points in space or space-time and is defined using the metric tensor, which encodes the geometric structure of the space. This forms the foundation for describing the curvature and geometry of space and time in both special and general relativity.

6.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To explain Summation Convention, Dummy suffix and Real suffix.
- To define Kronecker Delta.
- To explain four vectors and determinant.
- To discuss transformation of co-ordinate.

6.3 *SUMMATION CONVENTION:-*

The expression $a_1x^1 + a_2x^2 + \cdots + a_nx^n$ is represented by

$$\sum_{i=1}^{n} a_i x^i$$

Summation convention means drop the sigma sign and adopt the convention

$$\sum_{i=1}^{n} a_i x^i = a_i x^i$$

According to the summation convention, a suffix implies sum across a specified range if it appears twice in a phrase, once in the upper position and once in the lower position. We assume that the range is between 1 and 4 if it is not specified.

6.4*DUMMY SUFFIX:-*

The word *"dummy suffix"* refers to a suffix that appears twice in a term, once in the upper position and once in the lower position. For example i is dummy suffix in $a_i^\mu x^i$.

If we have

$$a_i^\mu x^i = a_1^\mu x^1 + \cdots + a_4^\mu x^4$$
$$a_j^\mu x^j = a_1^\mu x^1 + \cdots + a_4^\mu x^4$$

The last two equations prove that $a_i^\mu x^i = a_j^\mu x^j$. This shows that a dummy suffix can be replaced by another dummy suffix not used in that term. Also two or more than two dummy suffixes can be interchanged. For example

$$g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^j} = g_{\beta\alpha}\frac{\partial x^\beta}{\partial x'^i}\frac{\partial x^\alpha}{\partial x'^j}$$

6.5*REAL SUFFIX:-*

Real suffixes are those that are not repeated. For example μ is a real suffix in $a_i^\mu x^i$. A real suffix cannot be replaced by another real suffix. i.e.

$$a_i^\mu x^i \neq a_i^v x^i$$

6.6*KRONECKER DELTA*:-

It is denoted by δ_j^i and is defined as

$$\delta_j^i = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

Properties:

1. $\dfrac{\partial x^i}{\partial x^j} = \delta_j^i$

2. $\delta_j^i A^j = A^i$

3. $\delta_j^i = 4$

$for \; \delta_1^1 + \delta_2^2 + \delta_3^3 + \delta_4^4 = 1 + 1 + 1 + 1 += 4$

4. $\delta_j^i \delta_k^i = \delta_k^i$

6.7*DETERMINANT*:-

Consider the determinant

$$\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 \end{vmatrix} = a(say)$$

In this case, a_v^μ might be interpreted as this determinant's general element. The row and column to which the element a_v^μ belongs are indicated by the suffix $\mu \; and \; v$, which is also used to indicate this determinant$\left|a_v^\mu\right|$. A_μ^v stands for the element a_v^μ's cofactor. The determinant's symmetry or anti-symmetry is determined by

$$a_v^\mu = a_\mu^v \; or \; a_v^\mu = -a_\mu^v \; \forall \mu \; and \; v$$

We have

$$a_v^\mu A_\sigma^v = a_1^\mu A_\sigma^1 + a_2^\mu A_\sigma^2 + a_3^\mu A_\sigma^3 + a_4^\mu A_\sigma^4$$

By the well known property of determinant

$$a_v^\mu A_\sigma^v = 0 \; if \; \mu \neq \sigma$$

$$a_v^\mu A_\sigma^v = a \; if \; \mu = \sigma$$

These two results can be represented by a single equation

$$a_v^\mu A_\sigma^v = a\delta_\sigma^\mu$$

6.7.1. Differentiation of Determinant

Let the element a_v^μ be functions of independent variables $x, y, z, \ldots$ etc. So that

$$\frac{\partial a}{\partial x} = \begin{vmatrix} \frac{\partial a_1^1}{\partial x} & \frac{\partial a_2^1}{\partial x} & \cdots & \frac{\partial a_4^1}{\partial x} \\ a_1^2 & a_2^2 & \ldots & a_4^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^4 & a_2^4 & \ldots & a_4^4 \end{vmatrix} + \cdots + \begin{vmatrix} a_1^1 & \ldots & a_4^1 \\ a_1^2 & \ldots & a_4^2 \\ \ldots & \ldots & \ldots \\ \frac{\partial a_1^4}{\partial x} & \cdots & \frac{\partial a_4^4}{\partial x} \end{vmatrix}$$

First det. on R.H.S

$$\frac{\partial a_1^1}{\partial x}.A_1^1 + \frac{\partial a_2^1}{\partial x}.A_1^2 + \cdots + \frac{\partial a_4^1}{\partial x}A_1^4 = \frac{\partial a_v^1}{\partial x}.A_1^v$$

$$Similarly, last\ det. on\ R.H.S. = \frac{\partial a_v^1}{\partial x}.A_1^v$$

$$Finally, \qquad \frac{\partial a}{\partial x} = \frac{\partial a_v^1}{\partial x}.A_1^v + \cdots + \frac{\partial a_v^4}{\partial x}.A_4^v = \frac{\partial a_v^\mu}{\partial x}.A_\mu^v$$

$$Similarly, \qquad \frac{\partial a}{\partial y} = \frac{\partial a_v^\mu}{\partial y}.A_\mu^v$$

6.8 *FOUR VECTORS (WORLD VECTORS)*:-

The concept of four-dimensional space has been introduced to us. Ordinary vector analysis (three vectors) can now be extended to four dimensions, or four vectors. Four vectors or world vectors are these four-dimensional vectors.

Given that the coordinates in four dimensions are orthogonal,

$$i.i = j.j = k.k = p.p = 1$$

$$i.j = j.k = k.p = p.i = p.j = k.i = 0$$

The bars underneath their symbols represent the world vectors. Given two world vectors, A and B, then

$$\underline{A} = iA_1 + jA_2 + kA_3 + pA_4$$

And

$$\underline{B} = iB_1 + jB_2 + kB_3 + pB_4$$

The scalar product $\underline{A}.\underline{B}$ is defined as

$$\underline{A}.\underline{B} = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$$

$$or\ \underline{A}.\underline{B} = \overline{A}.\overline{B} + A_4B_4$$

Where $\overline{A}\ and\ \overline{B}$ are ordinary vectors.

$Thus,$ $\underline{A} = \underline{A}(A_1, A_2, A_3, iA_4)$

$Hence\ \underline{A}^2 = \underline{A}.\underline{A} = A_1^2 + A_2^2 + A_3^2 - A_4^2$

$or\ \underline{A}^2 = \overline{A}^2 - A_4^2$ where $\overline{A} = \overline{A}(A_1, A_2, A_3)$, is ordinary three dimensional vector.

The world vector **A** is said to be space-like $\mathbf{A}^2 \geq 0$ and time like if $\mathbf{A}^2 < 0$.

Corresponding to three dimensional operator ∇ we have four dimensional operator

$\boxtimes (D'alembertian\ operator) defined\ as$

$$\boxtimes = \mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z} + \mathrm{p}\frac{\partial}{\partial p}$$

$$Div\underline{A} = \boxtimes.\underline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_4}{\partial p}$$

$$Curl\underline{A} = \boxtimes \times \underline{A} = \begin{vmatrix} i & j & k & p \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ A_1 & A_1 & A_1 & A_1 \end{vmatrix}$$

6.9*TRANSFORMATION OF CO-ORDINATES:-*

We examine a transition from one coordinate system (x^1, x^2, x^3, x^4) to another (x'^1, x'^2, x'^3, x'^4) , where $x'^i = x'^i(x'^1, x'^2, x'^3, x'^4), i = 1,2,3,4.$

For co-ordinates x^i, the four functions x'^i are continuous differentiable with a single value. It is claimed that the four equations above define a transformation of coordinates. According to the equations, the differentials $(dx^1, dx^2, dx^3, dx^4)$are transformed.

$$dx'^1 = \frac{\partial x'^1}{\partial x^1} dx^1 + \frac{\partial x'^1}{\partial x^2} dx^2 + \frac{\partial x'^1}{\partial x^3} dx^3 + \frac{\partial x'^1}{\partial x^4} dx^4 = \frac{\partial x'^1}{\partial x^j} dx^j$$

Generalizing this, we get,

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j$$

This is the transformation law of co-ordinates.

SELF CHECK QUESTIONS

1. What is a tensor?
2. What is the role of the metric tensor in defining the line element?
3. What does the term "infinitesimal" in the line element mean?

6.10 *SUMMARY*:-

In this unit, we studied essential concepts that form the mathematical foundation for tensor analysis and special relativity. We began with the **Einstein Summation Convention**, which simplifies tensor notation by implying summation over repeated indices. The ideas of **dummy suffix** (repeated indices summed over) and **real suffix** (free indices that represent tensor components) were introduced to distinguish between variables in expressions. We explored the **Kronecker delta**, a special symbol used as the identity operator in tensor calculus. The concept of **determinants** was discussed in the context of coordinate transformations and matrix operations. We also studied **four-vectors** (or world vectors), which combine spatial and temporal components into a single object invariant under Lorentz transformations. Finally, we learned how **coordinate transformations** affect tensor components, preparing us for understanding more complex structures in relativistic physics.

6.11 *GLOSSARY*:-

- **Tensor:** A mathematical object that generalizes scalars and vectors, characterized by components that transform systematically under coordinate transformations. Tensors describe physical laws in a coordinate-independent way.
- **Line Element ds^2:** An expression representing the infinitesimal distance between two nearby points in space or space-time, typically written as $ds^2 = g_{\mu v} dx_\mu dx_v$.
- **Metric Tensor $g_{\mu v}$:** A symmetric tensor that defines the geometry of spacetime by specifying how distances and angles are measured. It appears in the line element.
- **Einstein Summation Convention:** A shorthand notation where repeated indices in a term are assumed to be summed over without explicitly writing the summation symbol.
- **Dummy Suffix (Index):** An index that appears twice in a term and is summed over. It does not appear in the final result and can be replaced by any other letter.
- **Real Suffix (Index):** A free index that appears only once in a term and indicates the specific component of a tensor. It must match on both sides of an equation.
- **Kronecker Delta$(\delta_{\mu v})$**: A symbol defined as 1 when μ=ν and 0 otherwise. It acts as the identity operator in tensor equations.
- **Determinant:** A scalar value calculated from a square matrix, used in transformations and to determine properties such as invertibility and volume scaling.
- **Four-Vector (World Vector):** A vector in four-dimensional spacetime, consisting of time and spatial components *(e.g.,* $x^\mu = (ct, x, y, z), x$ which transforms under Lorentz transformations.
- **Coordinate Transformation:** A rule that relates the coordinates in one frame to those in another. Tensors transform according to specific laws under such transformations.

6.12 *REFERENCES*:-

- Tevian Dray(2023), Differential Forms and the Geometry of General Relativity , CRC Press.
- Iva Stavrov (2020),Curvature of Space and Time, with an Introduction to Geometric Analysis, American Mathematical Society.

6.13 *SUGGESTED READING:-*

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

6.14 *TERMINAL QUESTIONS:-*

(TQ-1) Write short note on Kronecker Delta.

(TQ-2) Define four vectors.

(TQ-3) What do you understand by transformation of co-ordinates?

(TQ-4) What is meant by the term dummy suffix in tensor notation? Explain its role in tensor equations and how it relates to summation convention.

(TQ-5) How does the summation convention help in the efficient calculation of physical quantities in relativistic and tensor equations?

(TQ-6) Discuss the significance of free indices in tensor expressions, and give examples of how real indices are used in the formation of tensor components.

(TQ-7) Discuss the concept of the determinant in the context of tensors and their transformations. How is the determinant of the metric tensor important in general relativity?

(TQ-8) Explain how coordinate transformations work in tensor calculus. Discuss how the components of a tensor transform when changing from one coordinate system to another.

6.14 *ANSWERS:-*

SELF CHECK ANSWERS

1. A tensor is a mathematical object that generalizes scalars, vectors, and matrices. It can be described in terms of its components, which transform in a specific way under a change of coordinates. In essence, a tensor is a multi-dimensional array of quantities that obeys a set of transformation rules depending on the type (contravariant, covariant, mixed) and rank (order of the tensor).

2. The metric tensor $g_{\mu\nu}$ defines the geometry of the space or spacetime. It determines how distances and angles are measured. In the context of the line element, the metric tensor allows the calculation of the infinitesimal distance ds between two nearby points in a given coordinate system. It also dictates how vectors and tensors transform under coordinate changes.
3. The term "infinitesimal" refers to a very small quantity, approaching zero. In the context of the line element, it describes the infinitesimally small distance ds between two points that are arbitrarily close to each other in the manifold. This allows for the calculation of the distance between points in the limit as the separation between them tends to zero.

UNIT 7:-Tensor and Line Element II

CONTENTS:

7.1 *INTRODUCTION:-*

Tensor and Line Element II delves deeper into the mathematical framework used to describe curved spaces in differential geometry and general relativity. It focuses on key concepts like the Riemann curvature tensor, which measures the curvature of spacetime, and the Christoffel symbols, which describe how vectors change when parallel transported in curved spaces. The line element, expressed through the metric tensor, provides a way to calculate the infinitesimal distance between points in a curved manifold. Additionally, the covariant derivative, geodesics, and metric compatibility are explored to understand how objects move and interact in curved spacetime. These tools are essential for describing the geometry of space-time and are foundational in the study of general relativity and other areas of physics involving curved geometries.

7.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- T derives the mathematical expression for the Riemann tensor and understands its physical interpretation.
- To examine how the line element describes infinitesimal distances in both curved and flat spaces.

7.3 *TENSOR:-*

We will define scalar and vector first, followed by a tensor.

1. **Scalar:** A quantity that can be expressed by a single number is called a scalar. For example, body mass, and body temperature.
2. **Vector**. Any quantity that can be represented by three numbers in three dimensions is called a vector. For example, $u^1, u^2, and\, u^3$ three dimensions can be used to indicate the velocity$q$.
3. **Tensor:** A collection of numbers $A^{(i)}$ is said to be vector if it fulfills the transformation law

$$A'^{\mu} = A^{\alpha}\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \quad \dots (1)$$

Or if it satisfies the transformation law

$$A'_{\mu} = A_{\alpha}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \quad \dots (2)$$

It is referred to as a contra variant vector or contra variant tensor of rank one if it satisfies the first one, and as a covariant vector or covariant tensor of rank one if it satisfies the second.

The total number of real indices or suffixes for each component is known as the tensor's rank.

The suffix's upper place is set aside for indicating characters that are contra variants. The suffix's lower position is set aside to denote covariant character.

As an extension of (1) and (2), we express the

Contra variant tensor of rank p, i.e. tensor of the type $(p, 0)$…

$$A'^{\mu_1\mu_2\dots\mu_p} = A^{\alpha_1\alpha_2\dots\alpha_p}\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}}\dots\frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}}$$

Covariant tensor of rank q, i.e. tensor of the type $(0, q)$…

$$A'_{v_1 v_2 \dots v_q} = A_{\beta_1 \beta_2 \dots \beta_q} \frac{\partial x^{\beta_1}}{\partial x'^{v_1}} \frac{\partial x^{\beta_2}}{\partial x'^{v_2}} \cdots \frac{\partial x^{\beta_q}}{\partial x'^{v_q}}$$

Mixed tensor of rank $p + q$, i.e. tensor of the type (p, q)…

$$A'^{\mu_1 \mu_2 \dots \mu_p}_{v_1 v_2 \dots v_q} = A^{\alpha_1 \alpha_2 \dots \alpha_p}_{\beta_1 \beta_2 \dots \beta_q} \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial x'^{v_1}} \frac{\partial x^{\beta_2}}{\partial x'^{v_2}} \cdots \frac{\partial x^{\beta_q}}{\partial x'^{v_q}}$$

In $4\, dimensions\, (u\, dimensions)$, a tensor of rank m consists of $4^m (n^m)$ components. Therefore, scalars (tensor of rank zero) and vectors (tensor of rank one) are included in the general form of tensor.

Remark: if $A^{\alpha_1 \alpha_2 \dots \alpha_p}_{\beta_1 \beta_2 \dots \beta_q} = 0$, then evidently

$$A'^{\mu_1 \mu_2 \dots \mu_p}_{v_1 v_2 \dots v_q} = 0$$

This shows that if a tensor vanishes is one co-ordinate system then it vanishes in all co-ordinate systems.

7.4 *SYMMETRIC TENSOR:-*

If two contravariant or covariant indices may be interchanged without modifying the tensor, then the tensor is said to be symmetric with regard to these two indices. i.e.

$$\left.\begin{aligned} A_{\mu v} &= A_{v\mu} \\ A^{\mu v} &= A^{v\mu} \end{aligned}\right\} \quad \dots (1)$$

Claim 1: Symmetric property remains unchanged by tensor law of transformation. If we show that

$$A'_{\mu v} = A'_{v\mu}$$

The result will follow.

From equation (1)

$$A_{\alpha\beta} = A_{\beta\alpha} \quad \dots (2)$$

$$A'_{\mu v} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^v} = A_{\beta\alpha} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^v}$$

From equation (2)

$$= A_{\beta\alpha} \frac{\partial x^\beta}{\partial x'^v} \frac{\partial x^\alpha}{\partial x'^\mu} = A'_{v\mu}$$

$$or\ A'_{\mu v} = A'_{v\mu}$$

Claim 2: A symmetric tensor $A_{\mu v}$ has $\frac{4(4+1)}{2}$ independent components.

$A_{\mu v}$ has 4^2 components in 4 dimensions which are written as follows:

$$A_{11}\ A_{12}\ A_{13}\ A_{14}$$
$$A_{21}\ A_{22}\ A_{23}\ A_{24}$$

$$A_{31}\ A_{32}\ A_{33}\ A_{34}$$
$$A_{41}\ A_{42}\ A_{43}\ A_{44}$$

No. of components corresponding to a repeated suffix is 4.
No. of components corresponding to a distinct suffix is $4^2 - 4$.
Due to symmetric property this no. is reduced to $\frac{4^2-4}{2}$.
Total no. of independent components is

$$\frac{4^2-4}{2}+4=\frac{4^2-4+2\times 4}{2}=\frac{4(4+1)}{2}$$

Note: A tensor $A_{\mu v \sigma}$ is said to be symmetric in suffixes $\mu\ and\ v$ if

$$A_{\mu v \sigma} = A_{v \mu \sigma}$$

The total no. of independent component in this tensor has

$$\frac{n(n+1)}{2}.n=\frac{n^2}{2}(n+1)$$

7.5 *ANTI-SYMMETRIC TENSOR*:-

When two contravariant or covariant indices are switched, a tensor is said to be skew symmetric or anti-symmetric with regard to these two indices if each component changes in sign but not in magnitude. i.e.

$$\left.\begin{aligned} A_{\mu v} &= -A_{v\mu} \\ A^{\mu v} &= -A^{v\mu} \end{aligned}\right\} \quad \dots(1)$$

Claim 1: An anti-symmetric property remains unchanged by tensor law of transformation.

For this we have to show that

$$A'^{\mu v} = -A'^{v\mu} \quad \dots(2)$$

From equation (1)

$$A^{\alpha\beta} = -A^{\beta\alpha} \quad \dots(3)$$

$$A'^{\mu v} = A^{\alpha\beta}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{v}}{\partial x^{\beta}} = -A^{\beta\alpha}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{v}}{\partial x^{\beta}}$$

From equation (3)

$$= -A^{\beta\alpha}\frac{\partial x'^{v}}{\partial x^{\beta}}\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = -A'^{v\mu}$$

Claim 2: An anti-symmetric tensor $A^{\mu v}$ has $\frac{4(4-1)}{2}$ independent components.
$A^{\mu v}$ has 4^2 components in 4 dimensions which are written as follows:

$$A_{11}\ A_{12}\ A_{13}\ A_{14}$$
$$A_{21}\ A_{22}\ A_{23}\ A_{24}$$
$$A_{31}\ A_{32}\ A_{33}\ A_{34}$$

$$A_{41}\ A_{42}\ A_{43}\ A_{44}$$

Putting $\mu = v$ in equation (1)

$$A^{\mu\mu} = -A^{\mu\mu} \Rightarrow A^{\mu\mu} = 0$$

Hence no. of independent components corresponding to a repeated suffix is 0.

No. of independent components corresponding to a distinct suffix is $4^2 - 4$.

Due to anti-symmetric property this no reduce to $\frac{4^2-4}{2}$.

Total no. of independent components is

$$= \frac{4^2 - 4}{2} + 0 = \frac{4(4-1)}{2} = 6$$

Note:

1. A tensor $A_{\mu v \sigma}$ is said to be skew-symmetric in suffixes $\mu\ and\ v$ if
$$A_{\mu v \sigma} = -A_{v \mu \sigma}$$
The total no. of independent component in this tensor has
$$\frac{n(n-1)}{2}.n = \frac{n^2}{2}(n-1)$$
2. A tensor $A_{\mu v \sigma}$ is said to be skew-symmetric in suffixes $\mu, v\ and\ \sigma$ if
$$A_{\mu v \sigma} = -A_{v \mu \sigma}, A_{\mu v \sigma} = -A_{\mu \sigma v}, A_{\mu v \sigma} = -A_{\sigma v \mu}$$
The total no. of independent component in this tensor has
$$\binom{n}{3} = \frac{n}{6}(n-1)(n-2)$$

Theorem 1: To prove that tensor (mixed tensor) law of transformation posses group property.

Proof: Consider transformation of co-ordinate

$$x^{\mu} \to x'^{\mu} \to x''^{\mu}$$
$$(i) \to (ii) \to (iii)$$
$$A^{\mu}_{v} \quad A'^{\mu}_{v} \quad A''^{\mu}_{v}$$

In case of transformation $(i) \to (ii)$, we have

$$A'^{p}_{q} = A^{\alpha}_{\beta} \frac{\partial x'^{p}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{q}} \qquad \ldots(1)$$

In case of transformation $(ii) \to (iii)$

$$A''^{\mu}_{v} = A'^{p}_{q} \frac{\partial x''^{\mu}}{\partial x'^{p}} \frac{\partial x'^{q}}{\partial x''^{v}} = A^{\alpha}_{\beta} \frac{\partial x'^{p}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{q}} \frac{\partial x''^{\mu}}{\partial x'^{p}} \frac{\partial x'^{q}}{\partial x''^{v}}$$
$$= A^{\alpha}_{\beta} \frac{\partial x'^{p}}{\partial x^{\alpha}} \frac{\partial x''^{\mu}}{\partial x'^{p}} \frac{\partial x^{\beta}}{\partial x'^{q}} \frac{\partial x'^{q}}{\partial x''^{v}}$$

$$A''^{\mu}_{\ \ v} = A^{\alpha}_{\beta} \frac{\partial x''^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x''^{v}}$$

This implies that the same law of transformation is obtained if we directly change from $(i) \rightarrow (ii)$. The following is an expression for this property: Group property is present in the Tensor Law of Transformation.

7.6 *ADDITION OF TENSOR*:-

If two tensors are of the same rank and character, they can be added or subtracted.
Then sum or difference of two tensors is a tensor of the same rank and similar character. This is proved in the following theorem:

Theorem 2: To show that the sum of two tensors is a tensor of the same rank and similar character.
Proof: Let $A^{\sigma}_{\mu v}$ *and* $B^{\sigma}_{\mu v}$ be mixed tensors. Their sum is defined as

$$A^{\sigma}_{\mu v} + B^{\sigma}_{\mu v} = C^{\sigma}_{\mu v} \qquad \ldots (1)$$

If we show that $C^{\sigma}_{\mu v}$ is a mixed tensor of rank three, the result will follow:
From equation (1)

$$C^{\gamma}_{\alpha\beta} = A^{\gamma}_{\alpha\beta} + B^{\gamma}_{\alpha\beta} \qquad \ldots (2)$$

and

$$C'^{\sigma}_{\mu v} = A'^{\sigma}_{\mu v} + B'^{\sigma}_{\mu v}$$

$$= A^{\gamma}_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{v}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} + B^{\gamma}_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{v}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}}$$

$$= \left[\left(A^{\gamma}_{\alpha\beta} + B^{\gamma}_{\alpha\beta}\right) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{v}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}}\right]$$

$$C'^{\sigma}_{\mu v} = C^{\gamma}_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{v}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}}$$

This proves the required result.

7.7 *INNER PRODUCT OF TWO VECTORS*:-

Let B_{α} be a covariant vector and A^{α} be a contravariant vector. The inner product or scalar product of vectors A^{α} *and* B_{β} is the product $A^{\alpha}B_{\alpha}$. The outer or open product of A^{α} *and* B_{β} is the product $A^{\alpha}B_{\beta}$.

$$A'^{\mu}B'_{\mu} = A^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} B^{\beta} \frac{\partial x^{\beta}}{\partial x'^{\mu}}$$

$$= A^{\alpha}B^{\beta} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} = A^{\alpha}B^{\beta} \frac{\partial x^{\beta}}{\partial x^{\alpha}} = A^{\alpha}B^{\beta}\delta^{\beta}_{\alpha}$$

$$= A^{\alpha}B_{\alpha} = A^{\mu}B_{\mu}$$

or

$$A'^{\mu}B'_{\mu} = A^{\mu}B_{\mu}$$

This proves that $A^{\mu}B_{\mu}$ remains unchanged by tensor law of transformation and hence $A^{\mu}B_{\mu}$ is a scalar or invariant.

7.8 *MULTIPLICATION OF TENSORS*:-

A tensor whose rank is the sum of the ranks of the two tensors is called a product of two tensors. More broadly, if we multiply a tensor $A^{\mu_1\mu_2\ldots\mu_l}_{v_1v_2\ldots v_m}$ (which is covariant of order m and contravariant of order n) by tensor $B^{a_1a_2\ldots a_p}_{\beta_1\beta_2\ldots\beta_q}$ (which is covariant of order q and contravariant of order p). The result is a tensor that is covariant of order $m + q$ and contravariant of order$l + p$. This product is referred to as the outer product or the open product of two tensors.

Theorem 3: The product of two tensors is also a tensor.

Proof: Let A^{μ}_{v} *and* B_{σ} be any two tensors. Let

$$C^{\mu}_{v\sigma} = A^{\mu}_{v}B_{\sigma} \qquad \ldots(1)$$

If we show that $C^{\mu}_{v\sigma}$ is a tensor, the result will follow:
From equation (1)

$$C'^{\mu}_{v\sigma} = A'^{\mu}_{v}B'_{\sigma}$$

$$= A^{p}_{q}\frac{\partial x'^{\mu}}{\partial x^{p}}\frac{\partial x^{q}}{\partial x'^{v}}\,.B_{r}\frac{\partial x^{r}}{\partial x'^{\sigma}}$$

$$= A^{p}_{q}.B_{r}\frac{\partial x'^{\mu}}{\partial x^{p}}\frac{\partial x^{q}}{\partial x'^{v}}\frac{\partial x^{r}}{\partial x'^{\sigma}}$$

$$= C^{\mu}_{v\sigma}\frac{\partial x'^{\mu}}{\partial x^{p}}\frac{\partial x^{q}}{\partial x'^{v}}\frac{\partial x^{r}}{\partial x'^{\sigma}}$$

From this it follows that $C^{\mu}_{v\sigma}$ is a tensor.

7.9 *CONTRACTION*:-

When one contravariant and one covariant suffix are equivalent in a tensor, the process is referred to as contraction.
Let A^{pq}_{rst} be a tensor of rank five. Then by tensor law of transformation

$$A'^{pq}_{rst} = A^{\alpha\beta}_{ijk}\frac{\partial x'^{p}}{\partial x^{\alpha}}\frac{\partial x'^{q}}{\partial x^{\beta}}\frac{\partial x^{i}}{\partial x'^{r}}\frac{\partial x^{j}}{\partial x'^{s}}\frac{\partial x^{k}}{\partial x'^{t}}$$

Taking$t = p$, we get

$$A'^{pq}_{rsp} = A^{\alpha\beta}_{ijk} \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x'^q}{\partial x^\beta} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s} \frac{\partial x^k}{\partial x'^p}$$

But $$A^{\alpha\beta}_{ijk} \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^k}{\partial x'^p} = A^{\alpha\beta}_{ijk} \frac{\partial x^k}{\partial x^\alpha} = A^{\alpha\beta}_{ijk} \delta^k_\alpha = A^{\alpha\beta}_{ij\alpha}$$

Hence, $$A'^{pq}_{rsp} = A^{\alpha\beta}_{ij\alpha} \frac{\partial x'^q}{\partial x^\beta} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s} = tensor\ of\ rank\ 3$$

For R.H.S. contains three partial derivatives.

This shows that contraction reduces the rank of a tensor by two.

Theorem 4: Quotient law of tensors. A set of quantities, whose inner product with an arbitrary vector is a tensor, is itself a tensor.

Proof: Let $A^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m k}$ be a set of quantities whose inner product with an arbitrary vector u^k is a tensor of the type $B^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m}$.

To prove that $A^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m k}$ is a tensor.

By assumption $B^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m} = A^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m k} u^k$

From which, we get

$$B^{\alpha_1 \alpha_2 \dots \alpha_l}_{\beta_1 \beta_2 \dots \beta_m} = A^{\alpha_1 \alpha_2 \dots \alpha_l}_{\beta_1 \beta_2 \dots \beta_m a} u^a \qquad \dots (1)$$

and $$B'^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m} = A'^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m k} u'^k$$

$\because B'^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m}$ is a tensor.

$$\therefore B^{\alpha_1 \alpha_2 \dots \alpha_l}_{\beta_1 \beta_2 \dots \beta_m} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \dots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \dots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}} = A'^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m k} u'^k$$

Using equation (1), we get

$$A^{\alpha_1 \alpha_2 \dots \alpha_l}_{\beta_1 \beta_2 \dots \beta_m a} u^a \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \dots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \dots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}} - A'^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m k} u'^k = 0$$

Making use of the fact that u^k is a vector,

$$A^{\alpha_1\alpha_2\ldots\alpha_l}_{\beta_1\beta_2\ldots\beta_m a} u'^k \frac{\partial x^a}{\partial x'^k}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}} - A'^{i_1 i_2\ldots i_l}_{j_1 j_2\ldots j_m k} u'^k = 0$$

$$or\ u'^k\left[A^{\alpha_1\alpha_2\ldots\alpha_l}_{\beta_1\beta_2\ldots\beta_m a}\frac{\partial x^a}{\partial x'^k}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}} - A'^{i_1 i_2\ldots i_l}_{j_1 j_2\ldots j_m k}\right] = 0$$

Since u^k is an arbitrary vector and hence the expression within the bracket vanishes. Consequently

$$A'^{i_1 i_2\ldots i_l}_{j_1 j_2\ldots j_m k} = A^{\alpha_1\alpha_2\ldots\alpha_l}_{\beta_1\beta_2\ldots\beta_m a}\frac{\partial x^a}{\partial x'^k}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

This proves that $A^{i_1 i_2\ldots i_l}_{j_1 j_2\ldots j_m k}$ is a tensor.

7.10*RECIPROCAL SYMMETRIC TENSOR*:-

Theorem 5: If a_{ij} is a symmetric covariant tensor then conjugate tensor a^{ij} is also a tensor.

Proof: Let a_{ij} be a second rank covariant symmetric tensor. Consider the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a\ (say)$$

The cofactor of a_{ij} in this determinant is denoted by A^{ji}. We define

$$a^{ij} = \frac{A^{ji}}{a}$$

$a_{ij}\ is\ symmetric\ \Rightarrow\ |a_{ij}| = a\ is\ \ symmetric$

$\Rightarrow\ A^{ji}\ is\ symmetric$

$\Rightarrow\ a^{ij} = \dfrac{A^{ji}}{a}\ is\ symmetric$

$\Rightarrow\ a^{ij}\ is\ symmetric$

Let u^i be an arbitrary vector. Then $a_{ij}\,u^i$ is a tensor since the product of two tensors is a tensor. Let $B_j = a_{ij}\,u^i$

Now B_j is an arbitrary vector.

$$B_j\, a^{jk} = u^i a_{ij}\, a^{jk} = u^i a_{ij} \frac{A^{kj}}{a} = \frac{u^i}{a} a\delta_i^k = u^k = a\ tensor$$

$\therefore$ $$B_j\, a^{jk} = a\ tensor$$

This proves, by quotient law that a^{jk} is a tensor. i.e. a^{ij} is tensor.

The tensors $a_{ij}\ and\ a^{ij}$ are defined as reciprocal to each other. They are also called conjugate tensor.

Note:

1. $a_{ij}\, a^{jk} = \delta_i^k$
   $for\ a_{ij}\, a^{jk} = a_{ij}\ \frac{A^{kj}}{a} = \frac{a\delta_i^k}{a} = \delta_i^k$
2. $a_{ij}\, a^{ij} = 4$
   For $k = i$ the result (1) gives
   $a_{ij}\, a^{ji} = \delta_i^i = 4$
   $or\ a_{ij}\, a^{ij} = 4\ for\ \ a^{ij}\ is\ symmetric.$
3. These results are of vital importance for future study.

7.11 *RELATIVE TENSOR*:-

Let $A_{\mu v}$ be a tensor

$$if\ A'_{\mu v} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^v} \left|\frac{\partial x}{\partial x'}\right|^\omega$$

Then $A_{\mu v}$ is called relative tensor of weight ω.

A relative tensor of weight one is called tensor density; while if the weight is zero, the tensor is absolute.
A relative tensor of order one is called relative vector. Thus, if

$$A'_\mu = A_\alpha \frac{\partial x^\alpha}{\partial x'^\mu} \left|\frac{\partial x}{\partial x'}\right|^\omega$$

Then A_μ is called a relative vector of weight ω. A relative vector of weight one is called vector density, while if the weight is zero, the vector is absolute.
A relative tensor of rank zero is called relative scalar. Thus,

$$if\ a' = a\left|\frac{\partial x}{\partial x'}\right|^{\omega}$$

Then a is called relative scalar of weight ω. A relative scalar of weight one is called scalar density, while if the weight is zero, the scalar is absolute.

7.12 *RIEMANNIAN METRIC*:-

The term "line element" or "metric" refers to a formula that expresses the distance between adjacent points.
For example $ds^2 = dx^2 + dy^2 + dz^2$ is a line element. For it expresses the distance between adjacent (x, y, z) and $(x + dx, y + dy, z + dz)$. More generally for any curvilinear co-ordinates u, v, w,

$$ds^2 = adu^2 + bdv^2 + cdw^2 + 2fdvdw + 2gdwdu + 2hdudv$$

Where the coefficients $a, b, c, \ldots, h$ are functions of co-ordinates uvw.By defining the infinitesimal distance ds between neighboring points whose coordinates in any system are x^i and $x^i + dx^i$, Riemann expanded this concept to a space of n dimensions.

$$ds^2 = g_{ij}dx^i dx^j \ \ (i, j = 1,2, \ldots, n)$$

Where the coefficients g_{ij} are functions of co-ordinates x^i.

$ds^2 = g_{ij}dx^i dx^j$ is the quadratic differential form known as the Riemannian metric for n-dimensional space. A space that is defined by this metric is referred to as a Riemannian space of n dimensions. The term Riemannian geometry of n dimensions refers to the geometry based on this metric.
In general theory of relativity, the line element is given by

$$ds^2 = g_{\mu v}dx^{\mu}dx^{v}\ (\mu, v = 1,2,3,4).$$

The case of special theory of relativity corresponds to

$$g_{11} = g_{22} = g_{33} = 1, g_{11} = -c^2$$

$$g_{\mu v} = 0(\mu \neq v)$$

Note:

1. The determinant formed by the elements $g_{\mu v}$ is denoted by g and is always assumed to be non-zero. i.e.

$$Thus \qquad g \neq 0 \; and \; g = |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix}$$

2. We define

$$g_{\mu v} = \frac{cofactor\ of\ g_{\mu v}\ in\ this\ determinant}{g}$$

It can be shown that

$$g_{\mu v} g^{\mu v} = 4, g_{\mu v} g^{v\sigma} = g_{\mu}^{\sigma} = \delta_{\mu}^{\sigma}$$

This tensor $g^{\mu v}$ is called reciprocal tensor of $g_{\mu v}$. The tensors $g_{\mu v}\ and\ g^{\mu v}$ are called fundamental tensor.

Theorem 6: Fundamental tensor: to show that $g_{\mu v}$ is covariant symmetric tensor of second order.

Proof: Firstly we shall show that dx^{α} is a contravariant vector. Consider the transformation $x^{\mu} \rightarrow x'^{\mu}$.

$$evidently \qquad dx'^{\mu} = dx^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}}$$

If we write $dx^{\alpha} = A^{\alpha}$, then

$$A'^{\mu} = A^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}}$$

This confirms the tensor law of transformation. Hence, A^{α} is a contravariant vector. Secondly we shall show that $g_{\mu v}$ is a second rank covariant tensor. ds^2 is a invariant under any co-ordinate system. Then

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} : in\ x^i\ system$$

$$ds^2 = g'_{\mu v} dx'^{\mu} dx'^{v} : in\ x'^i\ system$$

From which, we get

$$g_{\alpha\beta} dx^{\alpha} dx^{\beta} = g'_{\mu v} dx'^{\mu} dx'^{v}$$

Since dx^{α} is a contravariant vector,

$$g_{\alpha\beta} dx^{\alpha} dx^{\beta} = g'_{\mu v} dx^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} dx^{\beta} \frac{\partial x'^{v}}{\partial x^{\beta}}$$

or $$\left(g_{\alpha\beta} - g'_{\mu v}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{v}}{\partial x^{\beta}}\right)dx^{\alpha}dx^{\beta} = 0$$

Since dx^{α} is arbitrary, the expression within the bracket vanishes.

∴ $$g_{\alpha\beta} = g'_{\mu v}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{v}}{\partial x^{\beta}}$$

This confirms the tensor law of transformation. $g_{\alpha\beta}$ is hence a covariant tensor of second rank. Finally, we demonstrate that $g_{\mu v}$ is symmetric. $g_{\mu v}$ can be expressed as

$$g_{\mu v} = A_{\mu v} + B_{\mu v}$$

$$where \quad A_{\mu v} = \frac{1}{2}\left(g_{\mu v} + g_{v\mu}\right) = symmetric\ tensor\ \&$$

$$B_{\mu v} = \frac{1}{2}\left(g_{\mu v} - g_{v\mu}\right) = anti - symmetric\ tensor$$

$$g_{\mu v}dx^{\mu}dx^{v} = \left(A_{\mu v} + B_{\mu v}\right)dx^{\mu}dx^{v}$$

$$or \quad \left(g_{\mu v} - A_{\mu v}\right)dx^{\mu}dx^{v} = B_{\mu v}dx^{\mu}dx^{v} \qquad \ldots(1)$$

$But\ B_{\mu v}dx^{\mu}dx^{v} = B_{v\mu}dx^{v}dx^{\mu}$, by interchanging dummy suffix $\mu\ and\ v$

$= -B_{\mu v}dx^{v}dx^{\mu}$. For $B_{\mu v}$ is anti-symmetric

$or\ 2B_{\mu v}dx^{\mu}dx^{v} = 0 \Rightarrow B_{\mu v}dx^{\mu}dx^{v} = 0$

$\Rightarrow \left(g_{\mu v} - A_{\mu v}\right)dx^{\mu}dx^{v} = 0$ From equation (1)

$\Rightarrow g_{\mu v} - A_{\mu v} = 0$

$\Rightarrow g_{\mu v} = A_{\mu v} = symmetric\ tensor$

$\Rightarrow g_{\mu v}\ is\ symmetric$

7.13 *ASSOCIATE TENSORS:-*

We define

$$A_{\mu} = g_{\mu\alpha}A^{\alpha} \qquad \ldots(1)$$

The tensor A_μ is called associate to A^μ. Also we say that the tensors A_μ and A^μ are associate to each other. We also define

$$A^\mu = g^{\mu\alpha} A_\alpha \qquad \dots (2)$$

This is called raising the subscript.

Multiplying equation (2) by $g_{\mu p}$, we get

$$g_{\mu p} A^\mu = g^{\mu\alpha}\, g_{\mu p} A_\alpha = \delta_p^\alpha A_\alpha = A_p$$

$$or\ A_p = g_{\mu p} A^\mu = g_{\alpha p} A^\alpha = g_{p\alpha} A^\alpha$$

$$or\ A_p = g_{p\alpha} A^\alpha\ or\ A_\mu = g_{\mu\alpha} A^\alpha$$

This is equation (1).

This is called lowering the superscript.

Thus, there are three processes:

1. Multiplication by $g^{\mu v}$ gives substitution with raising.
2. Multiplication by $g_{\mu v}$ gives substitution with lowering.
3. Multiplication by g_v^μ gives a simple substitution.

7.14 *MAGNITUDE OF A VECTOR*:-

The magnitude A of a vector A_α is defined as

$$A^2 = g^{\alpha\beta} A_\alpha A_\beta$$

$$obviously\ A^2 = g^{\alpha\beta} A_\alpha A_\beta = A^2 = A^\beta A_\beta = A^2 = A^\beta g_{\alpha\beta} A^\alpha = g_{\alpha\beta} A^\alpha A^\beta$$

$$or \qquad A^2 = g_{\alpha\beta} A^\alpha A^\beta$$

This shown that magnitude of contravariant component and covariant component of the same vectors are equal.

7.15 *ANGLE BETWEEN TWO VECTORS*:-

Let θ is the angle between any two vectors $A^\alpha\ and\ B^\alpha$, then we define

$$\cos\theta = \frac{g_{\alpha\beta}A^{\alpha}B^{\beta}}{\sqrt{(g_{\alpha\beta}A^{\alpha}A^{\beta})}\sqrt{(g_{\mu\nu}B^{\mu}B^{\nu})}}$$

SOLVED EXAMPLE

EXAMPLE1: Calculate the quantities g_{ij} for a V_3 where fundamental form in co-ordinates (u, v, w) is

$$a(du)^2 + b(dv)^2 + c(dw)^2 + 2fdvdw + 2gdwdu + 2hdudv$$

SOLUTION: Comparing $ds^2 = g_{ij}dx^i dx^j \ \ (i, j = 1,2,3)$

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}dx^1dx^2 + 2g_{23}dx^2dx^3 + 2g_{31}dx^3dx^1$$

$$with\ ds^2 = a(du)^2 + b(dv)^2 + c(dw)^2 + 2fdvdw + 2gdwdu + 2hdudv$$

$$we\ get\ x^1 = u, x^2 = v, x^3 = w, g_{11} = a, \quad g_{22} = b, \quad g_{33} = c, \quad g_{12} = g_{21} = h, g_{23} = g_{32} = f, g_{13} = g_{31} = g$$

$$|g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = g$$

This completes the problem.

Deduction: To find quantities g^{ij},

$$g^{ij} = \frac{cofactor\ of\ g_{ij}}{|g_{ij}|} = \frac{cofactor\ of\ g_{ij}}{g}$$

$$\therefore\ g^{11} = \frac{cofactor\ of\ g_{11}}{g} = \frac{bc - f^2}{g}, g^{12} = \frac{-(ch - gf)}{g}$$

Similarly we calculate the other g^{ij}

Form the determinant

$$|g^{ij}| = \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix} = \frac{1}{g}\begin{vmatrix} bc - f^2 & gf - ch & fh - bg \\ gf - ch & ac - g^2 & gh - af \\ fh - bg & gh - af & ab - h^2 \end{vmatrix}$$

EXAMPLE2: Prove that Kronecker delta is an invariant tensor.

SOLUTION: Consider two co-ordinate systems $x^i \; and \; x'^i$. By tensor law of transformation,

$$\delta'^i_j = \delta^\alpha_\beta . \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^j}$$

$$= \left(\delta^\alpha_\beta . \frac{\partial x'^i}{\partial x^\alpha}\right) \frac{\partial x^\beta}{\partial x'^j} = \left(\frac{\partial x'^i}{\partial x^\beta}\right)\left(\frac{\partial x^\beta}{\partial x'^j}\right)$$

$$= \frac{\partial x'^i}{\partial x'^j} = \delta^i_j$$

$\therefore \delta'^i_j = \delta^i_j$, showing thereby δ^i_j is an invariant tensor.

EXAMPLE3: If $A^i \; and \; B^j$ are contravariant vectors and $C_{ij}A^iB^j$ is an invariant. Prove that C_{ij} is a tensor of the second order.

SOLUTION: Suppose $A^i \; and \; B^j$ are contravariant vectors. Also suppose that $C_{ij}A^iB^j$ is an invariant so that

$$C_{ij}A^iB^j = C'_{ij}A'^iB'^j \qquad \text{... (1)}$$

To prove that C_{ij} is a tensor.

Equation (1) ⇒

$$C_{\alpha\beta}A^\alpha B^\beta = C'_{ij}A'^iB'^j = C'_{ij}A^\alpha \frac{\partial x'^i}{\partial x^\alpha} B^\beta \frac{\partial x'^j}{\partial x^\beta}$$

$$\Rightarrow \qquad A^\alpha B^\beta \left(C_{\alpha\beta} - C'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta}\right) = 0. Also A^\alpha, B^\beta \neq 0$$

$$\Rightarrow \qquad C_{\alpha\beta} = C'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta}$$

⇒ C_{ij} is a second rank covariant tensor.

EXAMPLE4: Transform $ds^2 = dx^2 + dy^2 + dz^2$ in polar and cylindrical co-ordinates.

SOLUTION: Let $ds^2 = dx^2 + dy^2 + dz^2$. Comparing this with

$$ds^2 = g_{ij}dx^idx^j \;\; (i, j = 1,2,3)$$

We obtain $g_{11} = g_{22} = g_{33} = 1 \; and \; g_{ij} = 0 for \; i \neq j$

$$g'_{ij} = g_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j}$$

$$= \sum_{a=1}^{3} g_{aa} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^a}{\partial x'^j} \quad for \; g_{ab} = 0 \; s.t. \, a \neq b$$

$$= \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^a}{\partial x'^j} \quad for \; g_{aa} = 1 \; \forall a$$

$$\therefore \qquad g'_{ij} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^a}{\partial x'^j}$$

1. To determine polar form of the given line element.
   Polar co-ordinates are

$$x = r sin\theta cos\phi$$
$$y = r sin\theta sin\phi$$
$$z = r cos\theta$$

$set \; x^1 = x, x^2 = y, x^3 = z, x'^1 = r, x'^2 = \theta, x'^3 = \phi$

From equation (1)

$$g'_{11} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^1} \frac{\partial x^a}{\partial x'^1}$$

$$= \frac{\partial x^1}{\partial x'^1} \frac{\partial x^1}{\partial x'^1} + \frac{\partial x^2}{\partial x'^1} \frac{\partial x^2}{\partial x'^1} + \frac{\partial x^3}{\partial x'^1} \frac{\partial x^3}{\partial x'^1}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= (sin\theta cos\phi)^2 + (sin\theta sin\phi)^2 + (cos\theta)^2$$

$$= 1$$

From equation (1)

$$g'_{22} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^2} \frac{\partial x^a}{\partial x'^2}$$

$$= \sum_{a=1}^{3} \left(\frac{\partial x^a}{\partial x'^2}\right)^2$$

$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= (r cos\theta cos\phi)^2 + (r cos\theta sin\phi)^2 + (-r sin\theta)^2$$

$$= r^2$$

From equation (1)

$$g'_{33} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^3} \frac{\partial x^a}{\partial x'^3}$$

$$= \sum_{a=1}^{3} \left(\frac{\partial x^a}{\partial x'^3}\right)^2$$

$$= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= (-rsin\theta sin\phi)^2 + (rsin\theta cos\phi)^2 + (0)^2$$

$$= r^2 sin^2\theta$$

From equation (1)

$$g'_{12} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^1} \frac{\partial x^a}{\partial x'^2}$$

$$= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}$$

$$= (sin\theta cos\phi)(rcos\theta cos\phi) + (sin\theta sin\phi)$$
$$(rcos\theta sin\phi) + (cos\theta)(-rsin\theta)$$
$$= rsin\theta cos\phi[cos^2\phi + sin^2\phi - 1] = 0$$

Similarly $g'_{23} = 0$, $g'_{31} = 0$.

Hence, $g'_{ab} = 0 \; for \; a \neq b$.

$$ds^2 = g'_{ab} dx'^a dx'^b = \sum_{a=1}^{3} g'_{aa} (dx'^a)^2$$

$$= g'_{11}(dx'^1)^2 + g'_{22}(dx'^2)^2 + g'_{33}(dx'^3)^2$$

$$ds^2 = \mathrm{dr}^2 + \mathrm{r}^2 \mathrm{d}\theta^2 + \mathrm{r}^2 \sin^2\theta \mathrm{d}\phi^2$$

2. To determine cylindrical form.

Cylindrical co-ordinates are

$$x = rcos\theta$$
$$y = rsin\theta$$
$$z = z$$

$set \; x^1 = x, x^2 = y, x^3 = z, x'^1 = r, x'^2 = \theta, x'^3 = z$

From equation (1)

$$g'_{11} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^1} \frac{\partial x^a}{\partial x'^1}$$

$$= \sum_{a=1}^{3} \left(\frac{\partial x^a}{\partial x'^3}\right)^2$$
$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$
$$= (cos\theta)^2 + (sin\theta)^2 + (0)^2$$
$$= 1$$

From equation (1)

$$g'_{22} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^2}\frac{\partial x^a}{\partial x'^2}$$
$$= \sum_{a=1}^{3} \left(\frac{\partial x^a}{\partial x'^2}\right)^2$$
$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$
$$= (-rsin\theta)^2 + (rcos\theta)^2 + (0)^2$$
$$= r^2$$

From equation (1)

$$g'_{33} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^3}\frac{\partial x^a}{\partial x'^3}$$
$$= \sum_{a=1}^{3} \left(\frac{\partial x^a}{\partial x'^3}\right)^2$$
$$= \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2$$
$$= (0)^2 + (0)^2 + (1)^2 = 1$$

From equation (1)

$$g'_{12} = \sum_{a=1}^{3} \frac{\partial x^a}{\partial x'^1}\frac{\partial x^a}{\partial x'^2}$$
$$= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}$$
$$= cos\theta(-rsin\theta) + sin\theta(rcos\theta) + 0 = 0$$

Similarly $g'_{23} = 0,\ g'_{31} = 0.$

Hence, $g'_{ab} = 0\ for\ a \neq b.$

$$ds^2 = g'_{ab} dx'^a dx'^b = \sum_{a=1}^{3} g'_{aa} (dx'^a)^2$$
$$= g'_{11}(dx'^1)^2 + g'_{22}(dx'^2)^2 + g'_{33}(dx'^3)^2$$
$$ds^2 = \mathrm{dr}^2 + \mathrm{r}^2\mathrm{d\theta}^2 + \mathrm{dz}^2$$

SELF CHECK QUESTIONS

1. Tensor equations are invariant under.
a) Energy equations
b) Velocity transformation
c) Momentum transformation
d) Co-ordinate transformation
2. What is the Riemann curvature tensor?
3. What is the covariant derivative?
4. How do the tools of differential geometry help describe spacetime in general relativity?

7.16 *SUMMARY*:-

In this unit, we have studied various fundamental concepts related to tensors and their operations. A tensor is a mathematical object that generalizes scalars, vectors, and matrices, and is used to represent physical quantities in multiple dimensions. We examined symmetric tensors, which remain unchanged when their indices are swapped and anti-symmetric tensors, which change sign when their indices are swapped. The addition of tensors involves combining tensors element-wise when they have the same rank, and the inner product of two vectors is a scalar product that measures the projection of one vector onto another. Multiplication of tensors includes various operations like contraction, where repeated indices are summed over, and the reciprocal symmetric tensor, which is a symmetric tensor whose inverse follows specific properties. We also studied relative tensors, which change with coordinate transformations, and the Riemannian metric, which defines distances in curved spacetime. Associate tensors are related through operations like contraction or multiplication, while the magnitude of a vector is computed using the metric, and the angle between two vectors is determined by the cosine of their inner product divided by their magnitudes. These concepts form the basis for analyzing geometrical and physical problems in curved spaces and spacetime.

7.17 *GLOSSARY*:-

- **Tensor:** A mathematical object that generalizes scalars, vectors, and matrices, capable of representing multi-dimensional data and transforming according to specific rules under coordinate changes.
- **Symmetric Tensor:** A tensor that remains unchanged when its indices are swapped, i.e., $T^{\mu v} = T^{v\mu}$.
- **Anti-symmetric Tensor:** A tensor that changes sign when its indices are swapped, i.e., $T_{\mu v} = T_{v\mu}$.
- **Addition of Tensors:** The operation where two tensors of the same rank and dimension are added element-wise, resulting in a new tensor where each component is the sum of the corresponding components of the two tensors.
- **Inner Product of Two Vectors:** The operation that combines two vectors to produce a scalar, defined as $A_\mu B^\mu$, where A_μ and B^μ are the components of the vectors.
- **Multiplication of Tensors:** The operation of combining tensors through different methods, such as the tensor product or contraction, to form a new tensor.
- **Contraction:** The operation of summing over repeated indices in a tensor, which reduces its rank by 2 and results in a scalar or lower-rank tensor.
- **Reciprocal Symmetric Tensor:** A symmetric tensor whose inverse also exhibits symmetry, meaning the inverse tensor maintains the property $T^{\mu v} = T^{v\mu}$.
- **Relative Tensor:** A tensor whose components transform according to specific rules when the reference frame or coordinate system is changed.
- **Riemannian Metric:** A mathematical tool in differential geometry that defines the geometry of a curved space by providing a way to measure distances between points, represented by the metric tensor $g_{\mu v}$.
- **Associate Tensors:** Tensors that are related through operations such as contraction or multiplication, leading to new tensors derived from the original ones.
- **Magnitude of a Vector:** The length or norm of a vector, calculated as
- $|V| = \sqrt{V_\mu V^\mu}$, where the components of the vector are contracted with the metric tensor.

- **Angle Between Two Vectors:** The angle between two vectors A and
- B, defined by $cos(\theta) = \frac{A_\mu B^\mu}{|A||B|}$, using the inner product and magnitudes of the vectors.
- **Geodesic:** The shortest path between two points in a curved space, representing the trajectory of a free-falling particle in general relativity.
- **Christoffel Symbols:** Connection coefficients that describe how vectors change during parallel transport in curved spaces, used in the calculation of covariant derivatives.
- **Covariant Derivative:** An extension of the partial derivative to curved spaces, which takes into account the curvature of the manifold and is used to differentiate tensors.
- **Riemann Curvature Tensor:** A tensor that describes how spacetime is curved due to mass and energy, and how vectors change as they are parallel transported around a closed loop.
- **Parallel Transport:** The process of moving a vector along a curve while keeping it parallel according to the connection in a curved space.
- **Metric Compatibility:** A property of a connection in which the covariant derivative of the metric tensor is zero, ensuring the preservation of distances and angles under parallel transport.
- **Conformal Transformation:** A transformation that preserves angles but not necessarily distances, often used in the study of scaling and geometry in curved spaces.

7.18 *REFERENCES*:-

- Tevian Dray(2023), Differential Forms and the Geometry of General Relativity , CRC Press.
- Iva Stavrov (2020),Curvature of Space and Time, with an Introduction to Geometric Analysis, American Mathematical Society.

7.19 *SUGGESTED READING*:-

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.

- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

7.20 *TERMINAL QUESTIONS:-*

(TQ-1) Show that the open product of two vectors is a tensor of rank 2.

(TQ-2) Show that Kronecker delta is a mixed tensor of rank two.

(TQ-3) Explain what is meant by covariant and mixed tensor.

(TQ-4) Show that the contraction of two suffixes in a tensor reduces its rank by two.

(TQ-5) Find the components of a vector in polar co-ordinates, whose components in Cartesian co-ordinates are $\dot{x}, \dot{y}$ *and* $\ddot{x}, \ddot{y}$.

(TQ-6) Show that $\sqrt{g dx^1 dx^2 \dots dx^n}$ is an invariant.

(TQ-7) If $B_{v\sigma}$ is any arbitrary covariant tensor, and $A(\mu, v) B_{v\sigma} = C_{\mu\sigma}$, where $C_{\mu\sigma}$ is a tensor, then show that $A(\mu, v)$ is a mixed tensor.

7.21 *ANSWERS:-*

SELF CHECK ANSWERS

1. d)
2. The Riemann curvature tensor measures the intrinsic curvature of a manifold, describing how vectors change when parallel transported around a closed loop.
3. The covariant derivative is a generalization of the partial derivative that accounts for curvature when differentiating tensors in curved spaces.
4. The tools of differential geometry, such as tensors and covariant derivatives, provide the mathematical framework for describing the curvature and geometry of spacetime in general relativity.

TERMINAL ANSWERS

(TQ-5) (i) Polar form of $\dot{x}, \dot{y}$ are $\dot{r}, \dot{\theta}$

(ii) Polar form of $\ddot{x}, \ddot{y}$ are $\ddot{r} + r\dot{\theta}^2$ *and* $\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r}$

UNIT 8:-Geodesic Equations and Their Applications

CONTENTS:

8.1 *INTRODUCTION:-*

Geodesic equations form a fundamental part of differential geometry and general relativity, describing the path that a particle or object follows when moving solely under the influence of spacetime curvature, without any external forces. In simple terms, geodesics represent the "straightest possible" lines in curved space or spacetime, generalizing the idea of a straight line in Euclidean geometry. These equations are derived from the principle of extremal action, typically minimizing the proper time or distance between two events. Mathematically, geodesics are expressed using second-order differential equations involving Christoffel symbols, which encode information about the curvature of the space. Applications of geodesic equations are vast and include predicting the motion of planets and light in gravitational fields, analyzing satellite orbits, understanding black hole dynamics, and modeling the structure of the universe in cosmology.

8.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To explain Christoffel symbols.
- To understand differential equation of geodesic.
- To solve the transformation law for Christoffel symbols.
- To discuss Principal normal, Geodesic co-ordinates and Natural co-ordinates.

8.3 *CHRISTOFFEL SYMBOLS:-*

We define

$$\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$$

$$\Gamma^{\sigma}_{\mu\nu} = g^{\sigma\beta}\Gamma_{\mu\nu,\beta}$$

The first one $\Gamma_{\mu\nu,\sigma}$ is known as the Christoffel symbol of the first kind, whiles the second one, $\Gamma^{\sigma}_{\mu\nu}$ is known as the Christoffel symbol or Christoffel's bracket of the second kind.

Note:

1. $\Gamma_{\mu\nu,\sigma} = \Gamma_{\nu\mu,\sigma}$
   This follows from the fact $g_{\mu\nu}$ is symmetric tensor.
2. $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$
   For $\Gamma^{\sigma}_{\mu\nu} = g^{\sigma\beta}\Gamma_{\mu\nu,\beta} = g^{\sigma\beta}\Gamma_{\nu\mu,\beta} = \Gamma^{\sigma}_{\nu\mu}$
3. The following notations are used by some authors:
   $$\Gamma_{\mu\nu,\sigma} = [\mu\nu,\sigma]$$
   $$\begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix} = \Gamma^{\sigma}_{\mu\nu}$$
4. $g_{kp}\Gamma^{k}_{ij} = \Gamma_{ij,p}$
   By definition
   $$\Gamma^{k}_{ij} = g^{kp}\Gamma_{ij,p}$$

$$\therefore \qquad g_{kr}\Gamma_{ij}^{k} = g_{kr}g^{kp}\Gamma_{ij,p} = \delta_r^p\Gamma_{ij,p}$$

$$or \qquad g_{kr}\Gamma_{ij}^{k} = \Gamma_{ij,r}$$

$$\Rightarrow \qquad g_{kp}\Gamma_{ij}^{k} = \Gamma_{ij,p}$$

Theorem:1. To prove that

$a)\ \Gamma_{ij,k} + \Gamma_{jk,i} = \dfrac{\partial g_{ik}}{\partial x^j}$

$or\ \ \dfrac{\partial g_{ij}}{\partial x^k} - \Gamma_{jk,i} - \Gamma_{ik,j} = 0$

$b)\ \Gamma_{ij}^{i} = \dfrac{\partial}{\partial x^j} log\sqrt{(-g)}$

$c)\ \Gamma_{ij}^{i} = \dfrac{\partial}{\partial x^j} log\sqrt{g}$

$d)\ \dfrac{\partial g_{ij}}{\partial x^k} = -g^{il}\Gamma_{lk}^{j} - g^{lj}\Gamma_{lk}^{i}$

Proof: a) By definition

$$\Gamma_{ij,k} + \Gamma_{jk,i} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) + \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

$$= \frac{\partial g_{ik}}{\partial x^j}$$

b) & c) We know that

$$\frac{\partial a}{\partial x} = A_j^i \frac{\partial a_i^j}{\partial x} \qquad \text{(in usual notation)}$$

In this case it becomes

$$\frac{\partial g}{\partial x^j} = (\text{cofector of } g_{ik})\frac{\partial g_{ik}}{\partial x^j}$$

$$= g g_{ik}\frac{\partial g_{ik}}{\partial x^j}\ ;\ g_{ik} = \frac{\text{cofector of } g_{ik}}{g}$$

$$or \qquad \frac{1}{g}.\frac{\partial g}{\partial x^j} = g_{ik}\frac{\partial g_{ik}}{\partial x^j} = g_{ik}\left[\Gamma_{ij,k} + \Gamma_{jk,i}\right]$$

$$= g_{ik}\Gamma_{ij,k} + g_{ik}\Gamma_{jk,i}$$

$$= \Gamma^i_{ij} + \Gamma^k_{jk} = \Gamma^i_{ij} + \Gamma^i_{ji} = 2\,\Gamma^i_{ij}$$

or $$\frac{1}{2g}\cdot\frac{\partial g}{\partial x^j} = \Gamma^i_{ij} \qquad \text{... (1)}$$

But $$\frac{1}{2g}\cdot\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j} log\sqrt{g} \qquad \text{... (2)}$$

Also $$\frac{1}{2g}\cdot\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j} log\sqrt{(-g)} \qquad \text{... (3)}$$

Equating equation (1) to (3), we get the result (b) i.e.

$$\Gamma^i_{ij} = \frac{\partial}{\partial x^j} log\sqrt{(-g)}$$

Equating equation (1) to (2), we get the result (c) i.e.

$$\Gamma^i_{ij} = \frac{\partial}{\partial x^j} log\sqrt{g}$$

d) We know that

$$g_{ij}g^{jk} = \delta^k_i = 1 \text{ } or \text{ } 0.$$

Differentiating it w.r.t. x^m,

$$g^{jk}\frac{\partial g_{ij}}{\partial x^m} + g_{ij}\frac{\partial g^{jk}}{\partial x^m} = 0$$

Multiplying it by g^{li} and noting that

$$g_{ij}g^{li} = \delta^l_j, \delta^l_j\frac{\partial g^{jk}}{\partial x^m} = \frac{\partial g^{lk}}{\partial x^m},$$

We obtain

$$g^{li}g^{jk}\frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{lk}}{\partial x^m} = 0$$

or $$\frac{\partial g^{lk}}{\partial x^m} + g^{li}g^{jk}\left[\Gamma_{im,j} + \Gamma_{jm,i}\right] = 0$$

or $$\frac{\partial g^{lk}}{\partial x^m} + g^{li}\Gamma^k_{im} + g^{jk}\Gamma^l_{jm} = 0$$

or $$\frac{\partial g^{lk}}{\partial x^m} + g^{lh}\Gamma^k_{hm} + g^{hk}\Gamma^l_{hm} = 0$$

In view of this, we have

$$-\frac{\partial g_{ij}}{\partial x^k} = g^{lj}\Gamma^i_{lk} + g^{il}\Gamma^j_{lk}$$

8.4 *GEODESIC:-*

It is a curve whose length stays constant for arbitrary displacements as long as the end points are held constant.
That is to say,

$$\int_A^B ds \ \ is \ stationary.$$

or $$\delta \int_A^B ds = 0$$

Theorem:2. Differential equation of a geodesic: Determine the differential equations of a geodesic, which is defined as a path of extremum distance between any two points on it.

or

To use a variational concept in a given space to find the differential equations of a geodesic.

Proof: A geodesic is an extremum-distance path that connects any two places on it. In other words, for a geodesic's differential equations, we obtain

$$\int_A^B ds \ \ is \ stationary \text{ i.e.} \ \delta \int_A^B ds = 0$$

We have $ds^2 = g_{ij}dx^i dx^j$

Taking differential of both sides,

$$2ds.\delta(ds) = \frac{\partial g_{ij}}{\partial x^k}\delta x^k.dx^i dx^j + g_{ij}\delta\left(dx^i\right)dx^j + g_{ij}dx^i\delta\left(dx^j\right)$$

Interchanging the dummy suffixes in the last term on R.H.S.

$$2ds.\delta(ds) = \frac{\partial g_{ij}}{\partial x^k}\delta x^k.dx^i dx^j + 2g_{ij}\delta\left(dx^i\right)dx^j$$

Dividing by $2ds$ and then integrating

$$\int_A^B \delta(ds) = \frac{1}{2}\int_A^B \frac{\partial g_{ij}}{\partial x^k}\delta x^k \frac{dx^i}{ds}\frac{dx^j}{ds}ds + \int_A^B g_{ij}\frac{\delta\left(dx^i\right)}{ds}\frac{dx^j}{ds}ds$$

But the geodesic $\delta\int_A^B ds = 0$

$$\therefore \qquad \frac{1}{2}\int_A^B \frac{\partial g_{ij}}{\partial x^k}\delta x^k \frac{dx^i}{ds}\frac{dx^j}{ds}ds + \int_A^B g_{ij}\frac{\delta\left(dx^i\right)}{ds}\frac{dx^j}{ds}ds = 0 \qquad \ldots(1)$$

But by integrating by parts

$$\int_A^B \frac{\partial g_{ij}}{\partial x^k}\delta x^k \frac{dx^i}{ds}\frac{dx^j}{ds}ds = \left[g_{ij}\delta x^i.\frac{dx^j}{ds}\right]_A^B - \int_A^B \frac{d}{ds}\left(g_{ij}\frac{dx^j}{ds}\right)\delta x^i.ds$$

$$= -\int_A^B \frac{d}{ds}\left(g_{ij}\frac{dx^j}{ds}\right)\delta x^i.ds$$

(since $\delta x^i = 0$ at both A and B)

$$= -\int_A^B \frac{d}{ds}\left(g_{kj}\frac{dx^j}{ds}\right)\delta x^k.ds$$

$$= -\int_A^B \left[\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} + g_{kj}\frac{d^2x^j}{ds^2}\right]\delta x^k.ds$$

Putting this in equation (1), we get

$$\int_A^B \left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} - g_{kj}\frac{d^2x^j}{ds^2}\right]\delta x^k.ds = 0$$

But δx^k is arbitrary and hence the integrand of the last integral vanishes.

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} - g_{kj}\frac{d^2x^j}{ds^2} = 0$$

$$or \quad g_{kj}\frac{d^2x^j}{ds^2} + \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

Interchanging the dummy suffixes $i\ and\ j$ in third term, we get

$$g_{kj}\frac{d^2x^j}{ds^2}+\frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds}+\frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds}-\frac{1}{2}\frac{\partial g_{ji}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}=0$$

or $$g_{kp}\frac{d^2x^p}{ds^2}+\Gamma_{ij,k}\frac{dx^i}{ds}\frac{dx^j}{ds}=0$$

or $$g_{kp}\frac{d^2x^p}{ds^2}+g_{kp}\Gamma^p_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0$$

or $$g_{kp}\left[\frac{d^2x^p}{ds^2}+\frac{dx^i}{ds}\frac{dx^j}{ds}\Gamma^p_{ij}\right]=0$$

But g_{kp} is arbitrary. Hence

$$\frac{d^2x^p}{ds^2}+\frac{dx^i}{ds}\frac{dx^j}{ds}\Gamma^p_{ij}=0$$

This is required differential equation of a geodesic. For $p=1,2,3,4$ this equation gives four equations to determine a geodesic.

Null geodesic: A geodesic is referred to as null if there is no distance between any two of its points i.e. the distance between any two points on geodesic is zero. The characteristics of the null geodesics are

$$g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}=0$$

and $$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0$$

Theorem: 3. Geodesics for Euclidean space are straight lines that are referred to as rectangular coordinates.

or

Show that the geodesics in S_n the Euclidean space of n dimensions are straight lines.

Proof: We know that differential equation of geodesic are

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^i}{ds}\frac{dx^j}{ds}=0 \qquad \dots(1)$$

In case of Euclidean space, $g_{ij}=constant\ \forall\ i\ and\ j$

$\Rightarrow \qquad \Gamma^{\alpha}_{\beta\gamma} = 0 \ \forall \ \alpha, \beta, \gamma$

From equation (1)

$$\frac{d^2x^{\alpha}}{ds^2} + 0 = 0$$

Integrating the above equation, we get

$$\frac{dx^{\alpha}}{ds} = a^{\alpha}$$

Again integrating, we get

$$x^{\alpha} = a^{\alpha}s + b^{\alpha} \qquad \text{... (2)}$$

Where a^{α} and b^{α} are integration constants.

Clearly, equation (2) of the type $y = mx + c$

Hence, equation (2) represents straight line. But equation (2) is the solution of equation (1). Hence geodesic are straight line in case of Euclidian space.

Theorem: 4. To determine the distance formula in S_n.

Proof: We know that geodesic is straight line in S_n. The Euclidian space of n dimension whose equation is

$$y^i = a^i s + b^i \qquad \text{... (1)}$$

Where b^i is constant of integration.

Equation (1) shows that a^i are components of unit tangent vector and so

$$1 = a^2 = a_i a_j = \sum_{i=1}^{n} (a_i)^2 \qquad \text{... (2)}$$

Let $P(y_1^i)\ and\ Q(y_2^i)$ be two points on the line (1) and l be the length of the line joining $P\ to\ Q$.

Then equation (1)

$\Rightarrow \qquad y_1^i = a^i s_1 + b^i, y_2^i = a^i s_2 + b^i$

$\Rightarrow \qquad y_2^i - y_1^i = a^i(s_2 - s_1) = a^i l$

$$\Rightarrow \qquad \sum_{i=1}^{n}(a_i l)^2 = \sum_{i=1}^{n}\left(y_2^i - y_1^i\right)^2$$

Using equation (2),

$$l^2 = \sum_{i=1}^{n}\left(y_2^i - y_1^i\right)^2$$

$$or \qquad l = \left\{\sum_{i=1}^{n}\left(y_2^i - y_1^i\right)^2\right\}^{1/2}$$

This is the required distance formula.

Theorem: 5. To obtain geodesic equations from Lagrangian equation.

Proof: We know that

$$ds^2 = g_{ij}dx^i dx^j$$

$$\left(\frac{ds}{dt}\right)^2 = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = g_{ij}\dot{x}^i\dot{x}^j$$

Where dot denote differentiation with respect to t.

$$ds = \sqrt{\left[g_{ij}\dot{x}^i\dot{x}^j\right]}dt$$

Let $I = \sqrt{\left[g_{ij}\dot{x}^i\dot{x}^j\right]}$, we get

$$ds = Idt \quad or \quad \frac{ds}{dt} = \dot{s} = I = \sqrt{\left[g_{ij}\dot{x}^i\dot{x}^j\right]}$$

$$\therefore \qquad \frac{\partial I}{\partial x^k} = \frac{1}{2\dot{s}}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j$$

$$and \qquad \frac{\partial I}{\partial \dot{x}^k} = 2\frac{1}{2\dot{s}}g_{ik}\dot{x}^i$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\left(\frac{\partial I}{\partial \dot{x}^k}\right) = -\frac{1}{\dot{s}^2}\ddot{s}g_{ik}\dot{x}^i + \frac{1}{\dot{s}}\frac{\partial g_{ik}}{\partial x^j}\dot{x}^j\dot{x}^i + \frac{1}{\dot{s}}g_{ik}\ddot{x}^i$$

For a geodesic, $\int_A^B ds$ is stationary. i. e. $\int_A^B Idt$ is stationary.

This is obtaining by putting,

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial \dot{x}^k}\right) - \frac{\partial I}{\partial x^k} = 0$$

Which are known as Euler- Lagrange equations.

$$-\frac{1}{\dot{s}^2}\ddot{s} g_{ik}\dot{x}^i + \frac{1}{\dot{s}}\frac{\partial g_{ik}}{\partial x^j}\dot{x}^j\dot{x}^i + \frac{1}{\dot{s}} g_{ik}\ddot{x}^i - \frac{1}{2\dot{s}}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = 0$$

or $$-\frac{1}{\dot{s}^2}\ddot{s} g_{ik}\dot{x}^i + \frac{1}{\dot{s}}\frac{\partial g_{ik}}{\partial x^j}\dot{x}^j\dot{x}^i + \frac{1}{\dot{s}} g_{pk}\ddot{x}^p - \frac{1}{2\dot{s}}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = 0$$

or $$g_{pk}\ddot{x}^p + \Gamma_{ij,k}\dot{x}^i\dot{x}^j = 0$$

or $$g_{pk}g^{kr}\ddot{x}^p + g^{kr}\Gamma_{ij,k}\dot{x}^i\dot{x}^j = 0$$

or $$\delta_p^r\ddot{x}^p + \Gamma_{ij}^r\dot{x}^i\dot{x}^j = 0$$

$$\ddot{x}^r + \Gamma_{ij}^r\dot{x}^i\dot{x}^j = 0$$

or $$\frac{d^2x^r}{ds^2} + \Gamma_{ij}^r\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

For $r = 1,2,3,4$ this gives four equations for determining a geodesic.

Theorem: 6. Transformation law for Christoffel symbols: Prove that Christoffel symbols are not tensors.

Proof: $\because$ g_{ij} is a second rank covariant tensor

$$g'_{ij} = g_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

Differentiate with respect to x'^k, we get

$$\frac{\partial g'_{ij}}{\partial x'^k} = \frac{\partial g_{ab}}{\partial x^c}\frac{\partial x^c}{\partial x'^k}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + g_{ab}\frac{\partial^2 x^a}{\partial x'^i\partial x'^k}\frac{\partial x^b}{\partial x'^j}$$

$$+ g_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial^2 x^b}{\partial x'^j\partial x'^k} \quad \ldots (1)$$

Similarly differentiation of g'_{jk}

$$= g_{bc}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}$$ with respect to x'^i, we get

$$\frac{\partial g'_{jk}}{\partial x'^i} = \frac{\partial g_{bc}}{\partial x^a}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k} + g_{bc}\frac{\partial^2 x^b}{\partial x'^j \partial x'^i}\frac{\partial x^c}{\partial x'^k} + g_{bc}\frac{\partial x^b}{\partial x'^j}\frac{\partial^2 x^c}{\partial x'^i \partial x'^k} \quad \ldots (2)$$

Similarly differentiation of g'_{ki}

$$= g_{ca}\frac{\partial x^c}{\partial x'^k}\frac{\partial x^a}{\partial x'^i} \text{ with respect to } x'^j\text{, we get}$$

$$\frac{\partial g'_{ki}}{\partial x'^j} = \frac{\partial g_{ca}}{\partial x^b}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\frac{\partial x^a}{\partial x'^i} + g_{ca}\frac{\partial^2 x^c}{\partial x'^k \partial x'^j}\frac{\partial x^a}{\partial x'^i} + g_{ca}\frac{\partial x^c}{\partial x'^k}\frac{\partial^2 x^a}{\partial x'^i \partial x'^j} \quad \ldots (3)$$

Equation(1) + equation (2) – equation(3), we get

$$\frac{\partial g'_{ij}}{\partial x'^k} + \frac{\partial g'_{jk}}{\partial x'^i} - \frac{\partial g'_{ki}}{\partial x'^j} = \frac{\partial g_{ab}}{\partial x^c}\frac{\partial x^c}{\partial x'^k}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + g_{ab}\frac{\partial^2 x^a}{\partial x'^i \partial x'^k}\frac{\partial x^b}{\partial x'^j}$$
$$+ g_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial^2 x^b}{\partial x'^j \partial x'^k} + \frac{\partial g_{bc}}{\partial x^a}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k} + g_{bc}\frac{\partial^2 x^b}{\partial x'^j \partial x'^i}\frac{\partial x^c}{\partial x'^k}$$
$$+ g_{bc}\frac{\partial x^b}{\partial x'^j}\frac{\partial^2 x^c}{\partial x'^i \partial x'^k} - \frac{\partial g_{ca}}{\partial x^b}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\frac{\partial x^a}{\partial x'^i} - g_{ca}\frac{\partial^2 x^c}{\partial x'^k \partial x'^j}\frac{\partial x^a}{\partial x'^i} - g_{ca}\frac{\partial x^c}{\partial x'^k}\frac{\partial^2 x^a}{\partial x'^i \partial x'^j}$$

$$2\Gamma'_{ij,k} = 2\Gamma_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k} + 2g_{ab}\frac{\partial^2 x^a}{\partial x'^i \partial x'^j}\frac{\partial x^b}{\partial x'^k}$$

$$or \qquad \Gamma'_{ij,k} = \Gamma_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k} + g_{ab}\frac{\partial^2 x^a}{\partial x'^i \partial x'^j}\frac{\partial x^b}{\partial x'^k} \quad \ldots (4)$$

Multiplying eq. (4) by g'^{kp}

$$g'^{kp} = g^{\alpha\beta}\frac{\partial x'^k}{\partial x^\alpha}\frac{\partial x'^p}{\partial x^\beta}$$

$$g'^{kp}\Gamma'_{ij,k} = g^{\alpha\beta}\Gamma_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\frac{\partial x'^k}{\partial x^\alpha}\frac{\partial x'^p}{\partial x^\beta}$$

$$+g_{ab}g^{\alpha\beta}\frac{\partial^2 x^a}{\partial x'^i \partial x'^j}\frac{\partial x^b}{\partial x'^k}\frac{\partial x'^k}{\partial x^\alpha}\frac{\partial x'^p}{\partial x^\beta}$$

$$or \qquad \Gamma'^p_{ij} = g^{\alpha\beta}\delta^c_\alpha \Gamma_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x'^p}{\partial x^\beta} + g_{ab}g^{\alpha\beta}\delta^b_\alpha \frac{\partial^2 x^a}{\partial x'^i \partial x'^j}\frac{\partial x'^p}{\partial x^\beta}$$

$$or \qquad \Gamma'^p_{ij} = \left[\Gamma^\beta_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + \frac{\partial^2 x^\beta}{\partial x'^i \partial x'^j}\right]\frac{\partial x'^p}{\partial x^\beta} \qquad \dots(5)$$

$for\ g^{\alpha\beta}\delta^c_\alpha \Gamma_{ab,c} = g^{\beta c}\Gamma_{ab,c} = \Gamma^\beta_{ab}\ and\ g_{ab}g^{\alpha\beta}\delta^b_\alpha = g_{ab}g^{b\beta} = \delta^\beta_a$

$$and \qquad \delta^\beta_a \frac{\partial^2 x^a}{\partial x'^i \partial x'^j} = \frac{\partial^2 x^\beta}{\partial x'^i \partial x'^j}$$

Equation (5) is also expressible as

$$\Gamma'^p_{ij} = \left[\Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}\right]\frac{\partial x'^p}{\partial x^c} \qquad \dots(6)$$

$$or \qquad \Gamma'^p_{ij} = \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x'^p}{\partial x^c} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}\frac{\partial x'^p}{\partial x^c} \qquad \dots(7)$$

Christoffel's bracket of the first kind is not tensor, according to equation (4). Christoffel's bracket of the second kind is not tensor, according to equation (7).

It is evident from the work done that Christoffel's brackets are not tensor components.

Remarks:

1. From equation (6), we have
$$\Gamma'^p_{ij}\frac{\partial x^c}{\partial x'^p} = \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}$$
This result is of important for further study.
2. If the linear transformation of the type
$$x^i = a^i_j x'^j + b^i$$
Is valid, then equation (4) and (7) becomes
$$\Gamma'_{ij,k} = \Gamma'_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}$$
$$\Gamma'^p_{ij} = \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x'^p}{\partial x^c}$$
Christoffel's brackets are tensors with respect to linear transformations, as demonstrated by these equations.
3. The transformation laws for Christoffel's brackets are found in equations (4) and (7).

Theorem: 7. Covariant derivative of a covariant vector: Define covariant derivative of a covariant vector and show that it is a tensor of rank 2.

Proof: Let A_i be a covariant vector, then by tensor law of transformation

$$A'_i = A_a \frac{\partial x^a}{\partial x'^i}$$

Differentiation of it with respect to x'^j, we get

$$\frac{\partial A'_i}{\partial x'^j} = \frac{\partial A_a}{\partial x^b}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + A_a \frac{\partial^2 x^a}{\partial x'^i \partial x'^j} \qquad \dots(1)$$

$$but \qquad \Gamma'^p_{ij}\frac{\partial\, x^c}{\partial\, x'^p} = \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}$$

Now the last equation becomes

$$\frac{\partial A'_i}{\partial x'^j} = \frac{\partial A_a}{\partial x^b}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + A_c\left[\Gamma'^p_{ij}\frac{\partial\, x^c}{\partial\, x'^p} - \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\right]$$

$$= \frac{\partial A_a}{\partial x^b}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + A'_p\Gamma'^p_{ij} - A_c\Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

$$or \qquad \frac{\partial A'_i}{\partial x'^j} - A'_p\Gamma'^p_{ij} = \left(\frac{\partial A_a}{\partial x^b} - A_c\Gamma^c_{ab}\right)\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

If we write

$$A_{a,b} = \frac{\partial A_a}{\partial x^b} - A_c\Gamma^c_{ab}$$

Then the last equation becomes

$$A'_{i,j} = A_{a,b}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

This says that is a second rank covariant tensor and this tensor is defined as $A_{a,b}$ covariant derivative of covariant vector A_a W.r.t. x^b.

Remark: If only linear transformation of the type

$$x^i = a^i_j x'^j + b^i \qquad \dots(2)$$

Is valid, then equation (1) becomes

$$\frac{\partial A'_i}{\partial x'^j} = \frac{\partial A_a}{\partial x^b}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

This demonstrates that, in relation to the linear transformation (2), the ordinary partial derivative of a covariant vector is a second rank covariant tensor.

Theorem: 8. Define covariant derivative of a contravariant vector and show that it is a tensor.

Proof: Let A^i be a contravariant vector, then by the tensor law of transformation

$$A^a = A'^i \frac{\partial x^a}{\partial x'^i}$$

Differentiate the above equation with respect to x^b

$$\frac{\partial A^a}{\partial x^b} = \frac{\partial A'^i}{\partial x'^j}\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b} + A'^i \frac{\partial^2 x^a}{\partial x'^i \partial x'^j}\frac{\partial x'^j}{\partial x^b} \dots (1)$$

$$but \qquad \Gamma'^p_{ij}\frac{\partial\, x^c}{\partial\, x'^p} = \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}$$

Now the last equation becomes

$$\frac{\partial A^a}{\partial x^b} = \frac{\partial A'^i}{\partial x'^j}\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b} + A'^i \frac{\partial x'^j}{\partial x^b}\left[\Gamma'^p_{ij}\frac{\partial\, x^a}{\partial\, x'^p} - \Gamma^a_{mc}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^c}{\partial x'^j}\right]$$

$$or \quad \frac{\partial A^a}{\partial x^b} + A'^i \frac{\partial x'^j}{\partial x^b}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^c}{\partial x'^j}\Gamma^a_{mc} = \frac{\partial A'^i}{\partial x'^j}\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b} + A'^p \frac{\partial x'^j}{\partial x^b}\frac{\partial x^a}{\partial x'^i}\Gamma'^i_{pj}$$

$$or \qquad \frac{\partial A^a}{\partial x^b} + A^m \delta^c_b \Gamma^a_{mc} = \left[\frac{\partial A'^i}{\partial x'^j} + A'^p \Gamma'^i_{pj}\right]\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b}$$

$$or \qquad \frac{\partial A^a}{\partial x^b} + A^m \Gamma^a_{mb} = \left[\frac{\partial A'^i}{\partial x'^j} + A'^p \Gamma'^i_{pj}\right]\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b}$$

If we write

$$A^a_{,b} = \frac{\partial A^a}{\partial x^b} + A^m \Gamma^a_{mb}$$

Then the last equation becomes

$$A^a_{,b} = A'^i_{,j}\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b}$$

This proves that $A^a_{,b}$ is a mixed tensor of rank two and this tensor is defined as covariant derivative of A^a w.r.t. x^b. Here covariant differentiation is denoted by subscript preceded by a comma.

Remark: If the linear transformation of the type

$$x^i = a^i_j x'^j + b^i \qquad \dots (2)$$

Is valid, then equation (1) becomes

$$\frac{\partial A^a}{\partial x^b} = \frac{\partial A'^i}{\partial x'^j}\frac{\partial x^a}{\partial x'^i}\frac{\partial x'^j}{\partial x^b}$$

In relation to the linear transformation (2), this demonstrates that the ordinary partial derivative of A^a w.r.t. x^b is a second rank tensor.

Theorem: 9. Define covariant derivative of covariant tensor of second order and show that it is a covariant tensor of rank three.

Proof: Let A_{ij} be a second rank covariant tensor, then by tensor law of transformation

$$A'_{ij} = A_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

Differentiation of it with respect to x'^k, we get

$$\frac{\partial A'_{ij}}{\partial x'^k} = \frac{\partial A_{ab}}{\partial x^c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k} + A_{ab}\frac{\partial^2 x^a}{\partial x'^i \partial x'^k}\frac{\partial x^b}{\partial x'^j}$$

$$+A_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial^2 x^b}{\partial x'^j \partial x'^k} \qquad \dots (1)$$

Now

$$A_{ab}\frac{\partial^2 x^a}{\partial x'^i \partial x'^k}\frac{\partial x^b}{\partial x'^j} = A_{pb}\frac{\partial^2 x^p}{\partial x'^i \partial x'^k}\frac{\partial x^b}{\partial x'^j}$$

$$= A_{pb}\frac{\partial x^b}{\partial x'^j}\left[\Gamma'^r_{ik}\frac{\partial x^p}{\partial x'^r} - \Gamma^p_{ac}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^c}{\partial x'^j}\right]$$

$$\left[for\ \Gamma'^p_{ij}\frac{\partial x^c}{\partial x'^j} = \Gamma^c_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}\right]$$

$$= A'_{rj}\Gamma'^r_{ik} - A_{pb}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Gamma^p_{ac} \qquad \dots (2)$$

$$and\ A_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial^2 x^b}{\partial x'^j \partial x'^k} = A_{ap}\frac{\partial x^a}{\partial x'^i}\frac{\partial^2 x^p}{\partial x'^i \partial x'^k}$$

$$= A_{pb}\frac{\partial x^a}{\partial x'^i}\left[\Gamma'^r_{jk}\frac{\partial x^p}{\partial x'^r} - \Gamma^p_{bc}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\right]$$

$$= A'_{ir}\Gamma'^{r}_{jk} - A_{ap}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Gamma^p_{bc} \qquad \ldots (3)$$

Writing equation (1) with the help of equation (2) and (3)

$$\frac{\partial A'_{ij}}{\partial x'^k} - A'_{ir}\Gamma'^{r}_{jk} - A'_{rj}\Gamma'^{r}_{ik} = \left[\frac{\partial A_{ab}}{\partial x^c} - A_{pb}\Gamma^p_{ac} - A_{ap}\Gamma^p_{bc}\right]\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}$$

If we write $A_{ab,c} = \frac{\partial A_{ab}}{\partial x^c} - A_{pb}\Gamma^p_{ac} - A_{ap}\Gamma^p_{bc}$ then the last equation

Becomes

$$A_{ij,k} = A_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}$$

This proves that $A_{ab,c}$ is a third rank covariant tensor and this tensor is defined as covariant derivative of A_{ab} w.r.t. x^c.

8.5 *COVARIANT DIFFERENTIATION OF TENSOR:-*

A subscript followed by comma or semicolon indicates the covariant differentiation of a tensor. We define

$$A_{a,b} = \frac{\partial A_a}{\partial x^b} - A_c\Gamma^c_{ab}$$

$$A^a, b = \frac{\partial A^a}{\partial x^b} + A^c\Gamma^a_{bc}$$

$$A_{ab,c} = \frac{\partial A_{ab}}{\partial x^c} - A_{pb}\Gamma^p_{ac} - A_{ap}\Gamma^b_{pc}$$

$$A^{ab}, c = \frac{\partial A_{ab}}{\partial x^c} + A^{pb}\Gamma^p_{ac} - A^{ap}\Gamma^b_{pc}$$

More generally,

$$A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m, b} = \frac{\partial A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m}}{\partial x^b} + A^{a i_2 \ldots i_l}_{j_1 j_2 \ldots j_m}\Gamma^{i_1}_{ab} + \cdots + A^{i_1 i_2 \ldots i_{l-1} a}_{j_1 j_2 \ldots j_m}\Gamma^{i_l}_{ab} - A^{i_1 i_2 \ldots i_l}_{a j_2 \ldots j_m}\Gamma^a_{j_i b} - \cdots - A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_{m-1}}\Gamma^a_{j_m b}$$

Theorem: 10. Show that covariant derivatives of the fundamental tensor and Kronecker delta vanish.

Proof: We have to prove that

$$g_{ij,k} = 0 \quad \text{... (1)}$$

$$g^{ij}{}_{,k} = 0 \quad \text{... (2)}$$

$$g^{i}_{j,k} = 0 \quad \text{... (3)}$$

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{aj}\Gamma^a_{ik} - g_{ia}\Gamma^a_{jk} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik,j} - \Gamma_{jk,i}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \left[\Gamma_{ik,j} + \Gamma_{jk,i}\right] = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} = 0$$

Refer theorem 1.

Hence $$g_{ij,k} = 0$$

Hence the equation (1) becomes

$$g_{ij}g^{jk} = \delta^k_i = 1 \; or \; 0$$

Differentiating it w.r.t. x^m, we get

$$g^{jk}\frac{\partial g_{ij}}{\partial x^m} + g_{ij}\frac{\partial g^{jk}}{\partial x^m} = 0$$

Multiplying the above equation by g^{li} and noting that

$$g_{ij}g^{li} = \delta^l_j, \delta^l_j\frac{\partial g^{jk}}{\partial x^m} = \frac{\partial g^{lk}}{\partial x^m}$$

We obtain

$$g^{li}g^{jk}\frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{lk}}{\partial x^m} = 0$$

or $$\frac{\partial g^{lk}}{\partial x^m} + g^{li}g^{jk}\left[\Gamma_{im,j} + \Gamma_{jm,i}\right] = 0$$

or $$\frac{\partial g^{lk}}{\partial x^m} + g^{li}\Gamma^k_{im} + g^{jk}\Gamma^l_{jm} = 0$$

$$or \qquad \frac{\partial g^{lk}}{\partial x^m} + g^{li}\Gamma^k_{im} + g^{ik}\Gamma^l_{im} = 0 \qquad \ldots(4)$$

$$or \qquad {g^{lk}}_{,m} = 0 \quad or \quad {g^{ij}}_{,k} = 0$$

$$\Rightarrow \qquad {g^{ij}}_{,k} = 0$$

Hence equation (2) proved.

$$g^i_{j,k} = \frac{\partial g^i_j}{\partial x^k} + g^m_j\Gamma^i_{mk} - g^i_a\Gamma^a_{jk}$$

$$= \frac{\partial g^i_j}{\partial x^k} + \Gamma^i_{jk} - \Gamma^i_{jk} = \frac{\partial g^i_j}{\partial x^k} = 0 \; as \; g^i_j = 0 \; or \; 1$$

Hence equation (3) proved.

Theorem: 11. To show that the covariant differentiation for products, sums, differences obeys the same rule as in the case of ordinary differentiation.

Proof: Let $A^i \; and \; B_{jk}$ be any two tensors, then their outer product A^iB_{jk} is also a tensor. Let

$$C^i_{jk} = A^iB_{jk}$$

$$\left(A^iB_{jk}\right)_{,l} = C^i_{jk,l}$$

$$= \frac{\partial C^i_{jk}}{\partial x^l} + C^a_{jk}\Gamma^i_{al} - C^i_{ak}\Gamma^a_{jl} - C^i_{ja}\Gamma^a_{kl}$$

$$= \frac{\partial A^iB_{jk}}{\partial x^l} + A^aB_{jk}\Gamma^i_{al} - A^iB_{ak}\Gamma^a_{jl} - A^iB_{ja}\Gamma^a_{kl}$$

$$= \frac{\partial A^i}{\partial x^l}.B_{jk} + A^i\frac{\partial B_{jk}}{\partial x^l} + A^aB_{jk}\Gamma^i_{al} - A^iB_{ak}\Gamma^a_{jl} - A^iB_{ja}\Gamma^a_{kl}$$

$$= B_{jk}\left(\frac{\partial A^i}{\partial x^l} + A^a\Gamma^i_{al}\right) + A^i\left(\frac{\partial B_{jk}}{\partial x^l} - B_{ak}\Gamma^a_{jl} - B_{ja}\Gamma^a_{kl}\right)$$

$$= B_{jk}A^i_{,l} + A^iB_{jk,l}$$

$$or \qquad \left(A^iB_{jk}\right)_{,l} = B_{jk}A^i_{,l} + A^iB_{jk,l} \qquad \ldots(1)$$

We can show that this approach works for all situations involving outer products by generalizing this finding.
The inner product of two tensors that are created via contraction and outer multiplication. Thus, it is a total of the products. Therefore, equation (1) also applies in this case. For example

$$\left(A_k^{ij}B_{lm}^k\right)_{,a} = A_{k,a}^{ij}B_{lm}^k + A_k^{ij}B_{lm,a}^k \qquad \dots(2)$$

Let A_{ij} and B_{ij} be any two tensors, each of the same rank and similar character. Then their sum is tensor of the same rank and similar character. Let

$$C_{ij} = A_{ij} + B_{ij}$$

$$\left(A_{ij} + B_{ij}\right)_{,l} = C_{ij,l}$$

$$= \frac{\partial C_{ij}}{\partial x^l} - C_{aj}\Gamma_{il}^a - C_{ia}\Gamma_{jl}^a$$

$$= \frac{\partial}{\partial x^l}\left(A_{ij} + B_{ij}\right) - \left(A_{aj} + B_{aj}\right)\Gamma_{il}^a - (A_{ia} + B_{ia})\Gamma_{jl}^a$$

$$= \frac{\partial A_{ij}}{\partial x^l} + \frac{\partial B_{ij}}{\partial x^l} - A_{aj}\Gamma_{il}^a - B_{aj}\Gamma_{il}^a - A_{ia}\Gamma_{jl}^a - B_{ia}\Gamma_{jl}^a$$

$$= \left(\frac{\partial A_{ij}}{\partial x^l} - A_{aj}\Gamma_{il}^a - A_{ia}\Gamma_{jl}^a\right) + \left(\frac{\partial B_{ij}}{\partial x^l} - B_{aj}\Gamma_{il}^a - -B_{ia}\Gamma_{jl}^a\right)$$

$$= A_{ij,l} + B_{ij,l}$$

$$\Rightarrow \left(A_{ij} + B_{ij}\right)_{,l} = A_{ij,l} + B_{ij,l} \qquad \dots(3)$$

Similarly we can show that

$$\left(A_{ij} - B_{ij}\right)_{,l} = A_{ij,l} - B_{ij,l} \qquad \dots(4)$$

From equation (1),(2),(3) and (4), it follows that covariant differentiation of products, sums, differences obeys the same rule as in the case of ordinary differentiation.

Theorem: 12. To show that covariant derivative of an invariant is the same as ordinary derivative.

Proof: Let I be an invariant and A_i be a covariant vector so that the product IA_i is a covariant vector.

To prove that

$$I_{,j} = \frac{\partial I}{\partial x^j}$$

By definition

$$(IA_i)_{,j} = \frac{\partial IA_i}{\partial x^j} - IA_a\Gamma_{ij}^a = \frac{\partial I}{\partial x^j}A_i + I\frac{\partial A_i}{\partial x^j} - IA_a\Gamma_{ij}^a$$

$$= I\left(\frac{\partial A_i}{\partial x^j} - A_a\Gamma_{ij}^a\right) + \frac{\partial I}{\partial x^j}A_i$$

$or \qquad (IA_i)_{,j} = IA_{i,j} + A_i\frac{\partial I}{\partial x^j} \qquad \ldots(1)$

$but \qquad (IA_i)_{,j} = I_{,j}A_i + IA_{i,j} \qquad \ldots(2)$

For covariant differentiation of products obeys the same rule as in the case of ordinary differentiation. Equating equation (1) to (2)

$$I_{,j}A_i + IA_{i,j} = IA_{i,j} + A_i\frac{\partial I}{\partial x^j}$$

$or \qquad \left(I_{,j} - \frac{\partial I}{\partial x^j}\right)A_i = 0$

$or \qquad I_{,j} - \frac{\partial I}{\partial x^j} = 0. For\ A_i\ is\ arbitrary$

$or \qquad I_{,j} = \frac{\partial I}{\partial x^j}$

8.6 *GRADIANT OF SCALAR*:-

The ordinary derivative of a scalar (or invariant) I is its gradient, and it is represented by $\nabla I \equiv \text{grad } I$.

Thus $\qquad \nabla I \equiv \text{grad } I = \frac{\partial I}{\partial x^i}$

But $\qquad \frac{\partial I}{\partial x^i} = I_{,i}$ (refer theorem 12)

$\therefore \qquad \nabla I \equiv \text{grad } I = \frac{\partial I}{\partial x^i} = I_{,i}$

8.7 *DERIVED VECTOR PROJECTION:-*

1. $a^i_{,j}b^i$ is the derived vector of vector a^i in the direction of b^i.
2. $a_{i,j}b^j$ is the derived vector of vector a_i in the direction of b^i.
3. $a^i b_i = a_i b^i$ is the projection of a^i in the direction of b^i.

Another name for the derived vector is the intrinsic derivative.

8.8 *TENDENCY OF VECTORS*:-

1. Tendency of a^i in the direction of b^i

$$= a^i_{,j}b_i b^j$$

2. Tendency of a_i in the direction of b^i

$$= a_{i,j}b^i b^j$$

3. $\frac{dx^j}{ds}$ is the intrinsic derivative of a^i in the direction of a curve $a^i_{,j}$.
4. $\frac{dx^j}{ds}$ is the intrinsic derivative of a_i in the direction of a curve$a_{i,j}$.

8.9 *CURL OF A VECTOR:-*

The curl of vector $\boldsymbol{A}$ is defined as

$$\boldsymbol{Curl\ A} = \boldsymbol{Curl}\ A_i = A_{i,j} - A_{j,i}$$

But $$A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - A_a\Gamma^a_{ij} - \left(\frac{\partial A_j}{\partial x^i} - A_a\Gamma^a_{ji}\right)$$

$$= \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

$\therefore$ $$\boldsymbol{Curl\ A} = A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

$$Curl_{ij}\{a_i\} = a_{i;k} - a_{k;i} = a_{i,k} - a_{k,i} = \frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i}$$

$$Curl_{ijk}\{F_{ij}\} = F_{ij;k} + F_{jk;i} + F_{ki;j} = \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j}$$

If F_{ij} is anti-symmetric tensor.

8.10 *DIVERGENCE OF A VECTOR:-*

Given a vector $\boldsymbol{A}$, its divergence can be defined as the contraction of its covariant derivative, or the divergence of its contravariant component A^i. Thus

$$div\ A = div\ A^i = A^i_{,i}$$

To prove that

$$A^i_{,i} = \frac{1}{\sqrt{g}} \frac{\partial\left(A^i\sqrt{g}\right)}{\partial x^i}$$

By definition

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + A^a \Gamma^i_{aj}$$

Putting $i = j$, we get

$$A^i_{,i} = \frac{\partial A^i}{\partial x^i} + A^a \Gamma^i_{ai}$$

or $$div\ A^i = \frac{\partial A^i}{\partial x^i} + A^a \frac{\partial\left(log\sqrt{g}\right)}{\partial x^a} = \frac{\partial A^i}{\partial x^i} + \frac{A^i}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^i}$$

or $$div\ A^i = \frac{1}{\sqrt{g}}\left(\sqrt{g}\frac{\partial A^i}{\partial x^i} + A^i \frac{\partial \sqrt{g}}{\partial x^i}\right) = \frac{1}{\sqrt{g}} \frac{\partial\left(A^i\sqrt{g}\right)}{\partial x^i}$$

or $$div\ A^i = A^i_{,i} = \frac{1}{\sqrt{g}} \frac{\partial\left(A^i\sqrt{g}\right)}{\partial x^i}$$

Theorem: 13. A necessary and sufficient condition that the first covariant derivative of a covariant vector be symmetric is that the vector be gradient.

Proof: Let A_i be a covariant vector such that the first covariant derivative of A_i is symmetric so that

$$A_{i,j} = A_{j,i}$$

We have to prove that $A_i = grad\phi$ where ϕ is scalar.

Equation (1) ⇒

$$\frac{\partial A_i}{\partial x^j} + A_a \Gamma^a_{ij} = \frac{\partial A_j}{\partial x^i} + A_a \Gamma^a_{ji}$$

⇒ $$\frac{\partial A_i}{\partial x^j} = \frac{\partial A_j}{\partial x^i}$$

⇒ $$\frac{\partial A_i}{\partial x^j} dx^j = \frac{\partial A_j}{\partial x^i} dx^j$$

⇒ $$\int \frac{\partial A_i}{\partial x^j} dx^j = \int \frac{\partial A_j}{\partial x^i} dx^j$$

⇒ $$\int dA_i = \frac{\partial}{\partial x^j} \int A_j dx^j$$

⇒ $$A_i = \frac{\partial \phi}{\partial x^i} = grad\phi$$

where $\int A_j dx^j = \phi = $ a scalar

$\Rightarrow$ $$A_i = grad\phi$$

Conversely let A_i be a covariant vector such that

$$A_j = grad\phi = \frac{\partial\phi}{\partial x^j}$$

Where ϕ is scalar.

We have to prove that

$$A_{i,j} = A_{j,i}$$

$$A_{i,j} - A_{j,i} = \left(\frac{\partial\phi}{\partial x^i}\right), j - \left(\frac{\partial\phi}{\partial x^j}\right), i$$

$$= \left(\phi_{,i}\right)_{,j} - \left(\phi_{,j}\right)_{,i} = \phi_{,ij} - \phi_{,ji} = 0$$

or $$A_{i,j} - A_{j,i} = 0$$

or $$A_{i,j} = A_{j,i}$$

8.11 *PARALLEL DISPLACEMENT OF VECTORS*:-

In a Riemannian V_n , let A^i be a vector with a constant magnitude that is defined along curve C. Along curve C, the vector A^i is said to suffer a parallel displacement if

$$A^i_{,j}\frac{dx^j}{ds} = 0 \qquad \text{... (1)}$$

At each point of C.

It is also expressed by saying that the vector A^i is parallel along C.

Equation (1) $\Rightarrow$

$$A_{i,j}\frac{dx^j}{ds} = 0 \qquad \text{... (2)}$$

Equation (1) multiplying by g_{ik}, we get

$$g_{ik}A^i_{,j}\frac{dx^j}{ds} = 0$$

or $$\left(g_{ik}A^i\right)_{,j}\frac{dx^j}{ds} = 0. for\ g_{ik,j} = 0$$

or $$A_{k,j}\frac{dx^j}{ds} = 0 \quad or \quad A_{i,j}\frac{dx^j}{ds} = 0$$

Writing equation (1) in full

$$\frac{\partial A^i}{\partial x^j}\frac{dx^j}{ds} + A^a\Gamma^i_{aj}\frac{dx^j}{ds} = 0$$

or $$\frac{dA^i}{ds} = -A^a\Gamma^i_{aj}\frac{dx^j}{ds}$$

or $$dA^i = -A^a\Gamma^i_{aj}dx^j \qquad \text{... (3)}$$

Similarly equation (2) gives

$$dA_i = A^a\Gamma^a_{ij}dx^j \qquad \text{... (4)}$$

Thus the increments in components of $A^i \; and \; A_i$ due to the displacement dx^j along C are given by equation (3) and (4) respectively. This concept is due to Levi and Cita.

Theorem: 14. To prove that the magnitude of all vectors of a field of parallel vectors is constant.

Proof: If a^i forms a field of parallel vectors along the curve $x^i = x^i(t)$, then we have

$$a^i_{,j}\frac{dx^j}{ds} = 0 \qquad \dots(1)$$

We have to prove that $a = 0$,where a is the magnitude of a^i

Equation (1) ⇒

$$a_{i,j}\frac{dx^j}{ds} = 0$$

$$\because \quad a^2 = a^i a_i$$

$$\therefore \quad \frac{da^2}{ds} = \frac{d}{ds}\left(a^i a_i\right) = \left(a^i a_i\right)_{,j}\frac{dx^j}{ds}$$

$$= \left(a^i_{,j}\frac{dx^j}{ds}\right)a_i + a^i\left(a_{i,j}\frac{dx^j}{ds}\right)$$

$$= (0)a_i + a^i(0) = 0$$

$$\Rightarrow \quad \frac{da^2}{ds} = 0$$

Integrating above equation, we get

$$a^2 = constant \; or \; a = constant$$

Theorem: 15. To prove that a vector of constant magnitude is orthogonal to its intrinsic derivative in any direction.

Proof: Let A be a vector of constant magnitude so that

$$A^2 = constant \qquad \dots(1)$$

Let a^i be any unit vector. Then the intrinsic derivative of A in the direction of a^i is $A^i_{,j}a^j$.

To prove that $A^i_{,j}a^j$ is orthogonal to A, we have to show that

$$\left(A^i_{,j}a^j\right)A_i = 0 \qquad \dots(2)$$

Equation (1) ⇒

$$A^i A_i = constant = \left(A^i A_i\right)_{,j} = 0$$

$$\Rightarrow \quad A^i_{,j}A_i + A^i A_{i,j} = 0$$

Since the dummy suffix has freedom to movement and therefore the last equation becomes

$$A^i_{,j}A_i + A_i A^i_{,j} = 0 \Rightarrow 2A^i_{,j}A_i = 0$$

$$\Rightarrow A^i_{,j}A_i = 0$$

Forming scalar product of this with a^j, we get

$$\left(A^i_{,j}A_i\right)a^j = 0$$

$$or \quad \left(A^i_{,j}a^j\right)A_i = 0$$

8.12 *PRINCIPAL NORMAL*:-

The derived vector of t^i in its own direction is known as the first curvature vector of C relative to V_n is represented by p^i. Let t^i be the unit tangent vector at any point $P(x^i)$ lie on a curve C in a V_n.

$$p^i = t^i_{,j}\frac{dx^j}{ds} \; ; \; t^i = \frac{dx^i}{ds} \qquad \ldots (1)$$

The magnitude of p^i is denoted by k, is defined as first curvature (or curvature simply) of the curve C relative to V_n. Then

$$k^2 = g_{ij}p^i p^j \qquad \ldots (2)$$

If n be unit vector along p^i, then

$$p^i = kn^i \qquad \ldots (3)$$

n^i, contravariant component of n is called unit principal normal.

From equation (1)

$$p^i = \left(\frac{\partial t^i}{\partial x^j} + t^a \begin{Bmatrix} i \\ aj \end{Bmatrix}\right)\frac{dx^j}{ds}$$

$$= \frac{dt^i}{ds} + \begin{Bmatrix} i \\ aj \end{Bmatrix}\frac{dx^a}{ds}\frac{dx^j}{ds}$$

or
$$p^i = \frac{d}{ds}\left(\frac{dx^i}{ds}\right) + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

or
$$p^i = \frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

or
$$n^i = \frac{p^i}{k} \;\; or \;\; n^i = \frac{1}{k}\left[\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}\right] \qquad \ldots (4)$$

This is required expression for principle normal.

Theorem: 16. Any vector of constant magnitude which undergoes a parallel displacement along a geodesic is inclined at a constant angle to a curve.

Proof: Let a vector α^i of constant magnitude undergoes a parallel displacement along a geodesic C, so that

$$\alpha^i_{,j}\frac{dx^i}{ds}=0 \qquad \ldots(1)$$

At each point of C.

Let t^i be the unit tangent vector to the curve C, so that

$$t^i_{,j}\frac{dx^j}{ds}=0 \qquad \ldots(2)$$

At each point of C. since geodesics are auto- parallel curves.

Let θ be the angle between the vector α^i and t^i. Hence

$$\alpha^i\, t^i=\alpha.1.cos\theta$$

$$\frac{d}{ds}(\alpha cos\theta\,)=\frac{d}{ds}\left(\alpha^i\, t^i\right)=\left(\alpha^i\, t^i\right)_{,j}\frac{dx^j}{ds}$$

$$or-\alpha sin\theta\frac{d\theta}{ds}=\ t^i\alpha^i{}_{,j}\frac{dx^i}{ds}+\alpha^i\, t^i{}_{,j}\frac{dx^j}{ds}$$

From equation (1) and (2), we get

$$\alpha sin\theta\frac{d\theta}{ds}=0.t^i+0.\alpha^i=0$$

$$or \qquad sin\theta\frac{d\theta}{ds}=0\ for\ \alpha\neq0$$

$$\Rightarrow \qquad sin\theta=0 \qquad or \qquad \frac{d\theta}{ds}=0$$

$$\Rightarrow \qquad \theta=0 \qquad or \qquad \theta=constant$$

$$\Rightarrow \qquad \theta=constant$$

8.13 *GEODESIC CO-ORDINATE:-*

With the pole at P_0 , the coordinate system x^i is referred to as a geodesic coordinate system if g_{ij} are locally constant in the neighbourhood of the point P_0. g_{ij} are said to be locally constant in the neighbourhood of P_0 if

$$and \qquad \frac{\partial g_{ij}}{\partial x^k}\neq0\ elsewhere$$

Or equivalently

$$\Gamma_{ij,k} = 0 = \Gamma_{ij}^{k} \ at \ P_0$$

To determine the necessary and sufficient condition for a given coordinate system to be a geodesic coordinate system with the pole at P_0.

$$\Gamma'^{a}_{ij} \frac{\partial x^l}{\partial x'^a} = \Gamma^l_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j} + \frac{\partial^2 x^l}{\partial x'^i \partial x'^j}$$

Interchanging co-ordinate system x^i and x'^i,

$$\Gamma^{a}_{ij} \frac{\partial x'^l}{\partial x^a} = \Gamma'^l_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} + \frac{\partial^2 x'^l}{\partial x^i \partial x^j}$$

$$or \qquad -\Gamma'^l_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} = -\Gamma^{a}_{ij} \frac{\partial x'^l}{\partial x^a} + \frac{\partial^2 x'^l}{\partial x^i \partial x^j} \qquad \ldots (1)$$

For a given value of l, x'^l is a scalar function of x^i and hence $\frac{\partial x'^l}{\partial x^i}$ is a covariant vector. Write

$$A_i = \frac{\partial x'^l}{\partial x^i} = x'^l, i$$

Then equation (1) becomes

$$-\Gamma'^l_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} = \frac{\partial A_i}{\partial x^j} - A_a \Gamma^a_{ij} = A_{i,j} = \left(x'^l_{,i}\right)_{,j}$$

$$or, \qquad x'^l_{,ij} = -\Gamma'^l_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \qquad \ldots (2)$$

Case ($\boldsymbol{i}$): Let x'^l be a geodesic co-ordinate system with the pole at P_0, then $\Gamma'^l_{\alpha\beta} = 0$ at P_0. In this even (2) shows that $x'^l_{,ij} = 0$ at P_0.

Case ($\boldsymbol{ii}$): Conversely suppose that $x'^l_{,ij} = 0$ at P_0.

Then equation (2) becomes

$$\Gamma'^l_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} = 0$$

$\because \qquad \left|\frac{\partial x'^{\alpha}}{\partial x^i}\right| \neq 0$

hence $\qquad \Gamma'^{l}_{\alpha\beta} = 0 \; at \;\; P_0$

This $\Rightarrow x'^l$ is a geodesic co-ordinate system with the pole at P_0.

This proves that a necessary and sufficient condition that a given co-ordinate system be geodesic co-ordinate system with the pole at P_0 are that all their second order covariant derivatives w.r.t. space co-ordinate vanish at P_0.

8.14*NATURAL CO-ORDINATE*:-

At the geodesic coordinate pole $\frac{\partial g_{ij}}{\partial x^k} = 0$. To obtain Galilean values for all g_{ij}, a transformation of coordinates can be introduced. A particle in motion at rest can also be created by using a Lorentz transformation. Such a coordinate system is referred to as a proper or natural coordinate system.

SOLVED EXAMPLE

EXAMPLE1: Show that all Christoffel symbols vanish at a point where g_{ij} are constants.

SOLUTION: $\because \;\; g_{ij} = constant \;\forall\; i \; and \; j$

so $\qquad \frac{\partial g_{ij}}{\partial x^k} = 0 \;\forall\; i, j, k$

$\Rightarrow \qquad \Gamma_{ij,k} = 0, \Gamma^k_{ij} = 0 \;\forall\; i, j, k$

$\Rightarrow$ All Christoffel symbols vanish.

EXAMPLE2: Show that if t^i is unit tangent to a geodesic, then $t^i_{,k} t^k = 0$, comma denoting covariant difference.

SOLUTION: The differential equation of a geodesic C is

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad \ldots (1)$$

Suppose t^i is unit tangent vector to the curve C is that

$$t^i = \frac{dx^i}{ds}$$

We have to prove that equation (1) is equivalent to the equation $t^i_{,k} t^k = 0$

Now equation (1) is expressible as

$$\frac{d}{ds}\left(\frac{dx^a}{ds}\right) + \Gamma^a_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$or \qquad \frac{dt^a}{ds} + \Gamma^a_{jk} t^j t^k = 0$$

$$or \qquad \frac{\partial t^a}{\partial x^k}\frac{dx^k}{ds} + \Gamma^a_{jk} t^j t^k = 0$$

$$or \qquad \left(\frac{\partial t^a}{\partial x^k} + \Gamma^a_{jk} t^j\right) t^k = 0$$

$$or \qquad t^a_{,k} t^k = 0$$

$$or \qquad t^i_{,k} t^k = 0$$

EXAMPLE3: If A_i is a vector show that, in general,

$\frac{\partial A_i}{\partial x^k}$ is not a tensor but that $\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$ is a tensor.

SOLUTION: If A_i is a vector so that, by tensor law of transformation,

$$A'_i = A_a \frac{\partial x^a}{\partial x'^i} \qquad \dots (1)$$

(i) We have to prove that $\frac{\partial A_i}{\partial x^k}$ is not a tensor, in general

Partially differentiating equation (1) with respect to x'^k, we get

$$\frac{\partial A'_i}{\partial x'^k} = \frac{\partial A_a}{\partial x^b}\frac{\partial x^b}{\partial x'^k}\frac{\partial x^a}{\partial x'^i} + A_a \frac{\partial^2 x^a}{\partial x'^i \partial x'^k} \quad \dots (2)$$

This says that if the term $\frac{\partial^2 x^a}{\partial x'^i \partial x'^k}$ were absent, then $\frac{\partial A_i}{\partial x^k}$ is a component of a tensor. But in general, (2)says that $\frac{\partial A_i}{\partial x^k}$ is not a tensor.

$$(ii)\frac{\partial A_i}{\partial x^k}-\frac{\partial A_k}{\partial x^i}=\frac{\partial A_i}{\partial x^k}-A_a\Gamma^a_{ik}+A_a\Gamma^a_{ik}-\frac{\partial A_k}{\partial x^i}$$

$$=\left(\frac{\partial A_i}{\partial x^k}-A_a\Gamma^a_{ik}\right)-\left(\frac{\partial A_k}{\partial x^i}-A_a\Gamma^a_{ik}\right)$$

$$\frac{\partial A_i}{\partial x^k}-\frac{\partial A_k}{\partial x^i}=A_{i,k}-A_{k,i} \qquad ...(3)$$

R.H.S. of equation (3) is a difference of two tensor each is second rank covariant tensor. Hence R.H.S. of equation (3) is a second rank covariant tensor. Therefore L.H.S. of equation (3) is also second rank covariant tensor.

$$i.e.\frac{\partial A_i}{\partial x^k}-\frac{\partial A_k}{\partial x^i} \text{ is second rank covariant tensor.}$$

EXAMPLE4: If A_{ik} is an anti-symmetric tensor of the second order, show that

$$\frac{\partial A_{ik}}{\partial x^m}+\frac{\partial A_{km}}{\partial x^i}+\frac{\partial A_{mi}}{\partial x^k} \text{ is a tensor.}$$

SOLUTION: Suppose A_{ik} is an anti-symmetric tensor so that

$$A_{ik}=-A_{ki}$$

$$Hence \qquad A_{ik}+A_{ki}=0 \qquad ...(1)$$

We claim

$$A_{ik,m}+A_{km,i}+A_{mi,k}=\frac{\partial A_{ik}}{\partial x^m}+\frac{\partial A_{km}}{\partial x^i}+\frac{\partial A_{mi}}{\partial x^k} \qquad ...(2)$$

L.H.S. of equation (2)

$$=\left(\frac{\partial A_{ik}}{\partial x^m}-A_{ak}\Gamma^a_{im}-A_{ia}\Gamma^a_{km}\right)-\left(\frac{\partial A_{km}}{\partial x^i}-A_{am}\Gamma^a_{ki}-A_{ka}\Gamma^a_{mi}\right)$$

$$+\left(\frac{\partial A_{mi}}{\partial x^k}-A_{ai}\Gamma^a_{mk}-A_{ma}\Gamma^a_{ikF}\right)$$

$$=\left(\frac{\partial A_{ik}}{\partial x^m}+\frac{\partial A_{km}}{\partial x^i}+\frac{\partial A_{mi}}{\partial x^k}\right)-(A_{ak}+A_{ka})\Gamma^a_{mi}$$

$$-(A_{ia}+A_{ai})\Gamma^a_{mk}-(A_{am}+A_{ma})\Gamma^a_{ik}$$

$$= \left(\frac{\partial A_{ik}}{\partial x^m} + \frac{\partial A_{km}}{\partial x^i} + \frac{\partial A_{mi}}{\partial x^k}\right)$$

From equation (1)

$$= R.H.S\ of\ equation\ (2)$$

Since L.H.S. of equation (2) is sum of tensors of rank 3 therefore L.H.S. and R.H.S. of equation (2) is also tensor of rank 3.

SELF CHECK QUESTIONS

A. The rank of the covariant derivative of a covariant tensor of second rank is:
a) One
b) Two
c) Three
d) Four

B. The geodesics in three dimensional Euclidean space are:
a) Straight lines
b) Spheres
c) Paraboloids
d) None of these

C. The differential equation of the geodesic is:

$$a)\ a^2\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\} = k^2 r^4$$

$$b)\ a^2\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\} = r^4$$

$$c)\ a^2\left\{\left(\frac{dr}{d\theta}\right)^2 - r^2\right\} = k^2 r^4$$

d) None of these

D. With usual symbols, the differential equation:

$$\frac{d^2x^p}{ds^2} + \frac{dx^i}{ds}\frac{dx^j}{ds}\Gamma^p_{ij} = 0$$

a) Riemannian equation
b) Newtonian equation
c) Geodesic equation
d) Metric equation

8.15 *SUMMARY*:-

In this unit, we have studied essential concepts in differential geometry and tensor calculus relevant to curved spaces and general relativity. **Christoffel symbols** were introduced as mathematical tools that help define how vectors change in curved spaces. We explored the concept of a **geodesic**, which is the shortest path between two points in a curved space, governed by geodesic equations. The unit also covered **covariant differentiation of tensors**, which extends the concept of differentiation to curved spaces while preserving tensorial properties. The **gradient of a scalar** gives the direction of the greatest rate of increase of the scalar field, while **derived vector projection** involves projecting vectors along specific directions. We studied the **tendency of a vector**, a notion capturing how a vector changes along a curve. Furthermore, we learned about the **curl** and **divergence** of a vector field, measuring the field's rotation and outward flux, respectively. The **parallel displacement of vectors** explained how vectors can be transported while maintaining their direction relative to the space. The concept of **principal normal** relates to curvature in a curve, helping to define the plane of curvature. Finally, we explored **geodesic coordinates**, where Christoffel symbols vanish at a point simplifying calculations, and **natural coordinates**, which are adapted to the geometry of a specific problem or surface. These concepts are crucial for understanding motion, forces, and geometry in curved spaces.

8.16 *GLOSSARY*:-

- **Geodesic:** The shortest path between two points in a curved space or spacetime, representing the natural trajectory of a free particle under no external forces.
- **Christoffel Symbols:** Mathematical expressions derived from the metric tensor, used to describe how coordinate bases change from point to point in a curved space.
- **Covariant Derivative:** A generalization of the derivative that accounts for curvature, allowing for the proper differentiation of tensors in curved spaces.
- **Gradient of a Scalar:** A vector field that points in the direction of the greatest rate of increase of a scalar function, defined as the covariant derivative of the scalar.

- **Derived Vector Projection:** The component of a vector projected in a specified direction, often used in describing motion along curves.
- **Tendency of a Vector:** Describes the change in direction and magnitude of a vector along a curve or field in a manifold.
- **Curl of a Vector:** A measure of the rotational tendency of a vector field; in curved space, defined using covariant derivatives.
- **Divergence of a Vector:** A scalar measure of how much a vector field spreads out or converges at a point; calculated using the covariant derivative.
- **Parallel Displacement:** The process of moving a vector along a curve while keeping it parallel according to the rules of curved geometry.
- **Principal Normal:** A unit vector perpendicular to the tangent of a curve, pointing in the direction of the curve's immediate turning.
- **Geodesic Coordinates:** A coordinate system in which the Christoffel symbols vanish at a point, simplifying the form of geodesic equations locally.
- **Natural Coordinates:** Coordinates chosen to simplify a problem based on the geometry or symmetry of the space, often aligned with curves or surfaces.
- **Affine Parameter (λ):** A parameter along the geodesic that preserves the form of the geodesic equation and is often proportional to proper time or arc length.
- **Manifold:** A mathematical space that locally resembles Euclidean space and allows the definition of tensors and geodesics.

8.17 *REFERENCES*:-

- **Sean M. Carroll(2019)**, *Spacetime and Geometry: An Introduction to General Relativity* (2nd Edition), Cambridge University Press.
- **José Natário(2021)**, *General Relativity Without Calculus*, Springer.

8.18 *SUGGESTED READING:-*

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.

- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

8.19 *TERMINAL QUESTIONS:-*

(TQ-1) Show that

$$\left\{ {i \atop ij} \right\} = \frac{\partial}{\partial x^j} log\sqrt{g}$$

(TQ-2) Define geodesic and obtain their equations with the help of variational principle.

(TQ-3) Show that fundamental tensor is covariant constant.

(TQ-4) Prove that intrinsic derivative of fundamental tensors g_{ij}, g^{ij}, g^i_j vanish.

(TQ-5) Find the condition of the tensor $A_{i,j}$ to be symmetric.

(TQ-6) Show that unit tangent to a geodesic suffers a parallel displacement along the geodesic.

(TQ-7) Show that

$$\Gamma_{ij,k} + \Gamma_{jk,i} + \Gamma_{ki,j} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j}\right)$$

8.20 *ANSWERS:-*

SELF CHECK ANSWERS

A. c)
B. a)
C. a)
D. c)

UNIT 9:-Tensor of Curvature

CONTENTS:

9.1 *INTRODUCTION:-*

The Tensor of Curvature, commonly known as the Riemann Curvature Tensor**,** is a fundamental object in differential geometry and general relativity that measures the intrinsic curvature of a differentiable manifold. It provides a precise mathematical description of how vectors change when parallel transported around infinitesimal loops, revealing the manifold's deviation from flatness. Denoted as $R^{\rho}_{\sigma\mu\nu}$, the tensor depends on the metric and its derivatives, and encapsulates the effects of gravitational fields in Einstein's theory. It is essential for defining other important curvature-related tensors, such as the Ricci Tensor and the Scalar Curvature**,** and plays a crucial role in the Einstein Field Equations**,** governing the dynamics of space-time under the influence of mass and energy.

9.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- T explain Riemannian Christoffel tensor.
- To solve xplain properties of covariant curvature tensor.
- To prove Bianchi identity.
- To discuss flat space time.

9.3 *RIEMANNIAN CHRISTOFFEL TENSOR:-*

Let A_i be a covariant vector.

Write $A_{i,j} = A_{ij}\ , A_{ij,k} = A_{ijk}$

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_a \Gamma^a_{ij} = A_{ij}$$

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - A_{aj}\Gamma^a_{ik} - A_{ia}\Gamma^a_{jk}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - A_a\Gamma^a_{ij}\right) - \Gamma^a_{ik}\left(\frac{\partial A_a}{\partial x^j} - A_b\Gamma^b_{aj}\right)$$

$$-\Gamma^a_{jk}\left(\frac{\partial A_i}{\partial x^a} - A_b\Gamma^b_{ia}\right)$$

$$= \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^a_{ij}\frac{\partial A_a}{\partial x^k} - A_a\frac{\partial \Gamma^a_{ij}}{\partial x^k} - \Gamma^a_{ik}\frac{\partial A_a}{\partial x^j} + A_b\Gamma^b_{aj}\Gamma^a_{ik} - \Gamma^a_{jk}\frac{\partial A_i}{\partial x^a}$$

$$+A_b\Gamma^b_{ia}\Gamma^a_{jk}$$

Rearranging the terms

$$A_{ij,k} = \left(\frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^a_{jk}\frac{\partial A_i}{\partial x^a} + A_b\Gamma^b_{ia}\Gamma^a_{jk}\right) - \Gamma^a_{ij}\frac{\partial A_a}{\partial x^k} - \Gamma^a_{ik}\frac{\partial A_a}{\partial x^j}$$

$$-A_a\frac{\partial \Gamma^a_{ij}}{\partial x^k} + A_b\Gamma^b_{aj}\Gamma^a_{ik} \quad \ldots (1)$$

Interchanging $j \ and \ k$ in the equation (1), we get

$$A_{ik,j} = \left(\frac{\partial^2 A_i}{\partial x^k \partial x^j} - \Gamma^a_{kj}\frac{\partial A_i}{\partial x^a} + A_b\Gamma^b_{ia}\Gamma^a_{kj}\right) - \Gamma^a_{ik}\frac{\partial A_a}{\partial x^j} - \Gamma^a_{ij}\frac{\partial A_a}{\partial x^k}$$

$$-A_a\frac{\partial \Gamma^a_{ik}}{\partial x^j} + A_b\Gamma^b_{ak}\Gamma^a_{ij} \quad \ldots (2)$$

Subtracting equation (2) from equation (1), we get

$$A_{ij,k} - A_{ik,j} = -A_a\frac{\partial \Gamma^a_{ij}}{\partial x^k} + A_b\Gamma^b_{aj}\Gamma^a_{ik} + A_a\frac{\partial \Gamma^a_{ik}}{\partial x^j} - A_b\Gamma^b_{ak}\Gamma^a_{ij}$$

$$A_{ij,k} - A_{ik,j} = A_a\left(-\frac{\partial \Gamma^a_{ij}}{\partial x^k} + \Gamma^a_{bj}\Gamma^b_{ik} + \frac{\partial \Gamma^a_{ik}}{\partial x^j} - \Gamma^a_{bk}\Gamma^b_{ij}\right)$$

Taking $$R^a_{ijk} = -\frac{\partial \Gamma^a_{ij}}{\partial x^k} + \Gamma^a_{bj}\Gamma^b_{ik} + \frac{\partial \Gamma^a_{ik}}{\partial x^j} - \Gamma^a_{bk}\Gamma^b_{ij}$$

We get $$A_{ij,k} - A_{ik,j} = A_a R^a_{ijk} \qquad \ldots(3)$$

A difference of two tensors of rank three each makes up the first element of (3). As a result, both the first and second members of (3) are covariant tensors of rank three. The inner product of $A_a \; and \; R^a_{ijk}$ is a tensor of rank three since the covariant vector A_a is outside the bracket, and the quantity inside the bracket is a mixed tensor of type R^a_{ijk} of rank four, according to the quotient law. The symbols R^a_{ijk} are known as Riemann's symbols of the second kind, and the tensor R^a_{ijk} is known as the Curvature tensor.
The following have the same meaning:
Riemann Christoffel's tensor, Riemann Christoffel curvature tensor, Curvature tensor.

9.4 *COVARIANT CURVATURE TENSOR:-*

We define

$$R_{hijk} = g_{ha} R^a_{ijk}$$

R_{hijk} is thus referred to as the covariant curvature tensor.
The symbols R_{hijk} are referred to as Riemann's symbol of the first kind.
Now,

$$\begin{aligned}
R_{hijk} &= g_{ha} R^a_{ijk} \\
&= g_{ha}\left(-\frac{\partial \Gamma^a_{ij}}{\partial x^k} + \frac{\partial \Gamma^a_{ik}}{\partial x^j} + \Gamma^b_{ik}\Gamma^a_{bj} - \Gamma^b_{ij}\Gamma^a_{bk}\right) \\
&= -\frac{\partial}{\partial x^k} g_{ha}\Gamma^a_{ij} + \Gamma^a_{ij}\frac{\partial g_{ha}}{\partial x^k} + \frac{\partial}{\partial x^j} g_{ha}\Gamma^a_{ik} - \Gamma^a_{ik}\frac{\partial g_{ha}}{\partial x^j} + \Gamma^b_{ik}\Gamma_{bj,h} \\
&\qquad - \Gamma^b_{ij}\Gamma_{bk,h} \\
&= -\frac{\partial}{\partial x^k}\Gamma_{ij,h} + \Gamma^a_{ij}\left(\Gamma_{hk,a} + \Gamma_{ak,h}\right) + \frac{\partial}{\partial x^j}\Gamma_{ik,h} \\
&\qquad - \Gamma^a_{ik}\left(\Gamma_{hj,a} + \Gamma_{aj,h}\right) + \Gamma^b_{ik}\Gamma_{bj,h} - \Gamma^b_{ij}\Gamma_{bk,h}
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{jh}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h}\right) - \Gamma^a_{ik}\Gamma_{hj,a} + \Gamma^a_{ij}\Gamma_{hk,a}$$

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k}\right) + g_{ab}\Gamma^a_{ij}\Gamma^b_{hk} - g_{ab}\Gamma^a_{ik}\Gamma^b_{hj}$$

This is required relation for R_{hijk}.

Theorem: 1. Properties of covariant curvature tensor. To show that covariant curvature tensor R_{hijk} is

a) Skew-symmetric in the first two indices.
b) Skew-symmetric in the last two indices.
c) Symmetric in two pairs of indices.

Proof: We know that

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k}\right) + g_{ab}\Gamma^a_{ij}\Gamma^b_{hk} - g_{ab}\Gamma^a_{ik}\Gamma^b_{hj} \quad ...(1)$$

We have to show that

a) $R_{hijk} = -R_{ihjk}$
b) $R_{hijk} = -R_{hikj}$
c) $R_{hijk} = R_{jkhi}$

Interchanging the suffixes $h \text{ and } i$ in equation (1), we get

$$R_{ihjk} = \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} + \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{hk}}{\partial x^j \partial x^i} - \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k}\right) + g_{ab}\Gamma^a_{hj}\Gamma^b_{ik} - g_{ab}\Gamma^a_{hk}\Gamma^b_{ij}$$

Comparing the above equation with equation (1), we get

$$R_{hijk} = -R_{ihjk}$$

Hence the result a)

Interchanging the suffixes $j\ and\ k$ in equation (1), we get

$$R_{hikj} = \frac{1}{2}\left(\frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{ij}}{\partial x^k \partial x^h} - \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j}\right) + g_{ab}\Gamma^a_{ik}\Gamma^b_{hj} - g_{ab}\Gamma^a_{ij}\Gamma^b_{hk}$$

Comparing the above equation with equation (1), we get

$$R_{hijk} = -R_{hikj}$$

Hence the result b)

Interchanging the suffixes $h\ and\ j$ in equation (1), we get

$$R_{jihk} = \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^h} + \frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{jh}}{\partial x^i \partial x^k}\right) + g_{ab}\Gamma^a_{ih}\Gamma^b_{jk} - g_{ab}\Gamma^a_{ik}\Gamma^b_{jh}$$

Again interchanging the suffixes $i\ and\ k$ in above equation, we get

$$R_{jkhi} = \frac{1}{2}\left(\frac{\partial^2 g_{kh}}{\partial x^j \partial x^i} + \frac{\partial^2 g_{ji}}{\partial x^k \partial x^h} - \frac{\partial^2 g_{ki}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i}\right) + g_{ab}\Gamma^a_{kh}\Gamma^b_{ji} - g_{ab}\Gamma^a_{ki}\Gamma^b_{jh}$$

Comparing the above equation with equation (1), we get

$$R_{hijk} = R_{jkhi}$$

Theorem: 2. Prove the cyclic property

$$R_{hijk} + R_{hjki} + R_{hkij} = 0$$

Proof: We know that

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k}\right) + g_{ab}\Gamma^a_{ij}\Gamma^b_{hk} - g_{ab}\Gamma^a_{ik}\Gamma^b_{hj}$$

From the above equation we have

$$R_{hjki} = \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^h \partial x^i} + \frac{\partial^2 g_{hi}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ji}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hk}}{\partial x^j \partial x^i}\right) + g_{ab}\Gamma^a_{jk}\Gamma^b_{hi}$$

$$-g_{ab}\Gamma^a_{ji}\Gamma^b_{hk}$$

$$and\ R_{hkij} = \frac{1}{2}\left(\frac{\partial^2 g_{ki}}{\partial x^j \partial x^h} + \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{kj}}{\partial x^h \partial x^i} - \frac{\partial^2 g_{hi}}{\partial x^k \partial x^j}\right) + g_{ab}\Gamma^a_{ki}\Gamma^b_{jh}$$

$$-g_{ab}\Gamma^a_{kj}\Gamma^b_{hi}$$

Adding the above three equation, we get

$$R_{hijk} + R_{hjki} + R_{hkij} = 0$$

Theorem: 3. Contraction of R^a_{ijk}. To show that the curvature tensor may be contracted in two ways. One of these leads to a zero tensor and the other method leads to Ricci tensor.

Proof: We have

$$R^a_{ijk} = -\frac{\partial \Gamma^a_{ij}}{\partial x^k} + \frac{\partial \Gamma^a_{ik}}{\partial x^j} - \Gamma^a_{bk}\Gamma^b_{ij} + \Gamma^b_{ik}\Gamma^a_{bj} \qquad \dots(1)$$

Three methods exist for contracting the curvature tensor. A zero tensor is the result of one of them.

i. Contraction of R^a_{ijk} with respect to $a\ and\ i$ in equation (1)

$$R^a_{ajk} = -\frac{\partial \Gamma^a_{aj}}{\partial x^k} + \frac{\partial \Gamma^a_{ak}}{\partial x^j} - \Gamma^a_{bk}\Gamma^b_{aj} + \Gamma^b_{ak}\Gamma^a_{bj}$$

$$= -\frac{\partial^2 log\sqrt{g}}{\partial x^k \partial x^j} + \frac{\partial^2 log\sqrt{g}}{\partial x^j \partial x^k} - \Gamma^a_{bk}\Gamma^b_{aj} + \Gamma^a_{bk}\Gamma^b_{aj}$$

(on interchanging dummy suffixes a and b in the last term)

$$\therefore \qquad R^a_{ajk} = 0$$

ii. Contraction of R^a_{ijk} with respect to $a\ and\ k$ in equation (1)

This approach produces the Ricci tensor, a significant tensor represented by R_{ij}, which is defined as

$$R_{ij} = R^a_{ija} = -\frac{\partial \Gamma^a_{ij}}{\partial x^a} + \frac{\partial \Gamma^a_{ia}}{\partial x^j} - \Gamma^a_{ba}\Gamma^b_{ij} + \Gamma^b_{ia}\Gamma^a_{bj}$$

$$R_{ij} = -\frac{\partial \Gamma^a_{ij}}{\partial x^a} + \frac{\partial^2 log\sqrt{g}}{\partial x^j \partial x^i} - \Gamma^b_{ij}\frac{\partial log\sqrt{g}}{\partial x^b} + \Gamma^b_{ia}\Gamma^a_{bj} \qquad \dots(2)$$

Interchanging $i\ and\ j$ in equation (2), we get

$$R_{ji} = -\frac{\partial \Gamma_{ji}^a}{\partial x^a} + \frac{\partial^2 log\sqrt{g}}{\partial x^i \partial x^j} - \Gamma_{ji}^b \frac{\partial log\sqrt{g}}{\partial x^b} + \Gamma_{ja}^b \Gamma_{bi}^a \quad \dots (3)$$

Comparing equation (2) and (3), we get

$$R_{ij} = R_{ji}$$

Hence it is symmetric tensor.

iii. Contraction of R_{ijk}^a with respect to a *and* k in equation (1)

Here we get the Ricci tensor with negative sign. For

$R_{iak}^a = -R_{ika}^a = -R_{ik} =$ negative of Ricci tensor

$for \quad R_{ijk}^a = -R_{ikj}^a$

Theorem: 4. Bianchi identity: To prove that

$$R_{ijk,l}^a + R_{ikl,j}^a + R_{ilj,k}^a = 0$$

or

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

Proof: We know that

$$R_{ijk}^a = -\frac{\partial \Gamma_{ij}^a}{\partial x^k} + \frac{\partial \Gamma_{ik}^a}{\partial x^j} - \Gamma_{bk}^a \Gamma_{ij}^b + \Gamma_{ik}^b \Gamma_{bj}^a$$

Introducing geodesic co-ordinates with the pole at P_0, then

$$\Gamma_{ij}^k = 0 = \Gamma_{ij,k} \ at \ P_0$$

At P_0, covariant derivative reduces to ordinary partial derivative.

Differentiating covariantly with respect to x^l and them imposing the condition of geodesic co-ordinates with the pole at P_0 , we get

$$R_{ijk,l}^a = -\frac{\partial^2 \Gamma_{ij}^a}{\partial x^l \partial x^k} + \frac{\partial^2 \Gamma_{ik}^a}{\partial x^l \partial x^j} \ at \ P_0$$

$$Similarly \ R_{ikl,j}^a = -\frac{\partial^2 \Gamma_{ik}^a}{\partial x^j \partial x^l} + \frac{\partial^2 \Gamma_{il}^a}{\partial x^j \partial x^k} \ at \ P_0$$

$$R_{ilj,k}^a = -\frac{\partial^2 \Gamma_{il}^a}{\partial x^k \partial x^j} + \frac{\partial^2 \Gamma_{ij}^a}{\partial x^k \partial x^l} \ at \ P_0$$

Adding the above three equations, we get

$$R_{ijk,l}^a + R_{ikl,j}^a + R_{ilj,k}^a = 0 \ at \ P_0 \quad \dots (1)$$

Multiplying equation (1) by g_{ha} where g_{ha} is constant under covariant differentiation, i.e.

$g_{ha}R^{a}_{ijk,l} = (g_{ha}R^{a}_{ijk}), l = R_{hijk,l}$, we get

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0 \quad \dots (2)$$

Given that each term in this equation is a tensor component, (2) is a tensorial equation. In other words, it is true in all coordinate systems. P_0 is also an arbitrary point of V_n. As a result, equation (2) holds for all coordinate systems in Riemannian space. As a tribute to its discoverer, Bianchi, it is known as Bianchi identity.
Hence equation (1) and (2) gives required result.

Theorem: 5. To show that curvature tensor has 20 components in four dimensional space.

Proof: Let V_n be a Riemannian space with n dimensions. It has n^4 components since R_{hijk} is of rank four. None of them are unrelated to the properties listed below that belong to R_{hijk}:

$$R_{hijk} = -R_{hikj} \ (antisymmetric\ property)$$

$$R_{hijk} = R_{jkhi} \ (symmetric\ property)$$

$$R_{hijk} + R_{hjki} + R_{hkij} = 0 \ (cyclic\ property)$$

Case I: When R_{hijk} has one unlike suffix, i.e. of the type R_{hhhh}.

By anti-symmetric property,

$$R_{hhhh} = -R_{hhhh}$$

$$\Rightarrow R_{hhhh} = 0$$

The curvature tensor itself vanishes, indicating that R_{hhhh} has no component.
Case II: When it contains two unlike suffixes, i.e., of the type, R_{hihi} h can be had in n ways. Once a specific value is assigned to h, the remaining $n-1$ values can be assigned to i. There are thus $n(n-1)$ ways to have $h\ and\ i$.
By anti-symmetric property,

$$R_{hihi} = -R_{ihhi} = R_{ihih}$$

$$or\ R_{hihi} = R_{ihih}$$

i.e. $i \ and \ h$ be interchanged.

Due to this property the number $n(n-1)$ is reduced to $(n/2)(n-1)$.

By cyclic property,

$$R_{hihi} + R_{hhii} + R_{hiih} = 0$$

$$or - R_{hiih} + 0 + R_{hiih} = 0$$

$$or \ 0 = 0$$

Proving the satisfaction of their cyclic property. Because of this property, there is no reduction. As a result, R_{hihi} has $(n\,/\,2)(n\,-\,1)$ independent components.

Case III: When it has three unlike suffixes, ie., of the type R_{hihj} . It can be easily shown that $h, i \ and \ j$ can be had $n(n\,-\,1)(n\,-\,2)$ ways. Due to symmetric property, this number is reduced to $1/2\; n(n\,-\,1)(n\,-\,2)$.

Consider the cyclic property,

$$R_{hihj} + R_{hhji} + R_{hjih} = 0$$

$$or \ R_{hihj} + 0 + R_{ihhj} = 0$$

$$or - R_{ihhj} + R_{ihhj} = 0$$

$$or \ 0 = 0$$

Therefore satisfying the cyclic property itself. Because of this characteristic, there is no reduction. This means that R_{hihj} has $(n\,/\,2)(n\,-\,1)(n\,-\,2)$ independent components.

Case IV: When it has four unlike suffixes. i.e. of the type R_{hijk}.

All the suffixes $h, i, j \ and \ k$ are unequal.

It can be shown that $h, i, j \ and \ k$ can be had in $n(n-1)(n-2)(n-3)$ ways. Due to anti-symmetric this no. reduces to

9.5 *FLAT SPACE TIME*:-

If a Galiean frame of reference can be constructed in a given area of the world, that area is considered flat or homogeneous. We know that Galilean coordinates can be generated and that the line element ds^2 in four-dimensional space simplifies to the sum of four squares where guy are constants. Therefore, if such coordinates can be discovered in the space-time for which $g_{\mu\nu}$are constants, then the space-time is said to be flat. This is an equivalent definition of flat-space time.

Furthermore, all of the three-index symbols disappear when guy is a constant. However, since 3-index symbols do not form a tensor, they generally do not also disappear when other coordinates are replaced in the same flat region. The Riemann-Christofell tensor, which is made up of products and derivatives of Christofell's 3-index symbols, will disappear once more when 8μv are constants. Because it is a tensor, it will also disappear when other coordinates are substituted in the same flat region. Therefore, the vanishing of the Riemann-Christofell tensor is a prerequisite for flat-space time. This condition will also be sufficient if the converse is also true, i.e., if the Riemann-Chris-tofell tensor vanishes, the space-time must be flat.

Theorem: To prove that vanishing of Riemann Christoffel tensor is a necessary and sufficient condition for the flat space-time (or or Euclideam space).

Proof: In above section we have shown that the construction of a uniform tor field by parallel displacement of a vector all over the region is posssible if

$$R^{\lambda}_{\mu\nu\sigma} = 0 \qquad \ldots(1)$$

Given four uniform vector fields $A^{\mu}_{(\alpha)}$ with the tensor suffix $\alpha = 1, 2, 3, 4$, eqn. (1) implies

$$A^{\mu}_{(\alpha)};\sigma = \frac{\partial A^{\mu}_{(\alpha)}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\lambda\sigma} A^{\lambda}_{(\alpha)}$$

$$\frac{\partial A^{\mu}_{(\alpha)}}{\partial x^{\sigma}} = -\Gamma^{\mu}_{\lambda\sigma} A^{\lambda}_{(\alpha)} \qquad \ldots(2)$$

Let's now examine the coordinate transformation law

$$dx^{\mu} = A^{\mu}_{(\alpha)} \overline{dx}^{\alpha} (\alpha = 1, 2, 3, 4) \qquad \ldots(3)$$

Since ds^2 is an invariant, we obtain

$$ds^2 = \bar{g}_{\alpha\beta} \overline{dx}^{\alpha} \overline{dx}^{\beta} = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

$$= g_{\mu\nu} A^{\mu}_{(\alpha)} \overline{dx}^{\alpha} A^{\mu}_{(\beta)} \overline{dx}^{\beta}$$

$$\bar{g}_{\alpha\beta} = g_{\mu\nu} A^{\mu}_{(\alpha)} A^{\mu}_{(\beta)} \qquad \ldots(4)$$

Differentiating above equation w.r.t.x^{σ}, we have

$$\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^{\sigma}} = g_{\mu\nu} A^{\mu}_{(\alpha)} \frac{\partial A^{\mu}_{(\beta)}}{\partial x^{\sigma}} + g_{\mu\nu} A^{v}_{(\alpha)} \frac{\partial A^{\mu}_{(\alpha)}}{\partial x^{\sigma}} + A^{\mu}_{(\alpha)} A^{\mu}_{(\beta)} \frac{\partial g_{\mu v}}{\partial x^{\sigma}}$$

Using (2), we have

$$\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^{\sigma}} = -g_{\mu\nu} A^{\mu}_{(\alpha)} A^{\lambda}_{(\beta)} \Gamma^{v}_{\lambda\sigma} - g_{\mu\nu} A^{v}_{(\alpha)} A^{\lambda}_{(\beta)} \Gamma^{\mu}_{\lambda\sigma} + A^{\mu}_{(\alpha)} A^{\mu}_{(\beta)} \frac{\partial g_{\mu v}}{\partial x^{\sigma}}$$

Changing the dummy suffix, we obtain

$$\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^{\sigma}} = A^{\mu}_{(\alpha)} A^{v}_{(\beta)} \left[-g_{\mu\nu} \Gamma^{v}_{\lambda\sigma} - g_{\lambda\nu} \Gamma^{\lambda}_{\mu\sigma} + \frac{\partial g_{\mu v}}{\partial x^{\sigma}} \right]$$

$$= A^{\mu}_{(\alpha)} A^{v}_{(\beta)} \left[-\Gamma_{\mu;\nu\sigma} - \Gamma_{v;\mu\sigma} + \frac{\partial g_{\mu v}}{\partial x^{\sigma}} \right]$$

$$= A^{\mu}_{(\alpha)} A^{v}_{(\beta)} \left[-\frac{\partial g_{\mu v}}{\partial x^{\sigma}} + \frac{\partial g_{\mu v}}{\partial x^{\sigma}} \right] = 0$$

Integrating, we have

$$\bar{g}_{\alpha\beta} = constant\ throughout\ the\ region$$

It is clear from definition that space-time is flat. For flat space time, the vanish-or is therefore a necessary and sufficient condition.

SELF CHECK QUESTIONS

1. Contraction of Riemann- Christoffel tensor leads to
   a) Ricci tensor only
   b) Zero tensor only
   c) Ricci and zero tensors
   d) None of the above
2. What does the Riemann curvature tensor represent?
   It represents the intrinsic curvature of a manifold and measures how vectors change under parallel transport around a loop.

3. What is the condition for a spacetime to be flat?
   A spacetime is flat if the Riemann curvature tensor is zero everywhere.
4. Which geometric space is used to model flat spacetime?
   Minkowski spacetime is used to model flat spacetime in special relativity.
5. What role does the curvature tensor play in general relativity?
   It describes how mass and energy curve spacetime and appears in the formulation of Einstein's field equations.
6. Is curvature always due to gravity?
   Yes, in general relativity, spacetime curvature is interpreted as the manifestation of gravity.
7. Which tensors are derived from the Riemann curvature tensor?
   The Ricci tensor and scalar curvature are derived by contracting the Riemann curvature tensor.
8. What is parallel transport in curved spacetime?
   It is the process of moving a vector along a curve while keeping it "parallel" according to the manifold's geometry.
9. Does the curvature tensor depend on the coordinate system?
   No, although its components may change, the curvature tensor itself is a geometric object independent of coordinates.

9.7 *SUMMARY*:-

In this unit, we have studied the Riemannian Christoffel Tensor**,** which provides the connection coefficients necessary for defining covariant derivatives in curved spacetime; the Covariant Curvature Tensor**,** more formally known as the Riemann Curvature Tensor**,** which measures the intrinsic curvature of a manifold and describes how vectors are affected by parallel transport; and the concept of Flat Spacetime**,** an idealized model with zero curvature where the Riemann tensor vanishes, typically represented by Minkowski spacetime in special relativity.

9.8 *GLOSSARY*:-

- **Riemann Curvature Tensor:** A fourth-rank tensor that measures the curvature of a manifold by describing how vectors change when parallel transported around a closed loop.

- **Christoffel Symbols:** Mathematical objects representing connection coefficients used to define covariant derivatives and geodesics in curved spaces.
- **Covariant Derivative**: A derivative that accounts for the curvature of space, allowing for differentiation of tensors in a coordinate-independent way.
- **Ricci Tensor**: A second-rank tensor obtained by contracting the Riemann curvature tensor, used in Einstein's field equations to describe gravitational effects.
- **Scalar Curvature**: A single number derived from the Ricci tensor that summarizes the overall curvature of spacetime at a point.
- **Parallel Transport**: The process of moving a vector along a curve on a manifold such that it remains parallel according to the manifold's connection.
- **Geodesic**: The generalization of a straight line to curved spaces, representing the shortest path between two points on a curved surface.
- **Flat Spacetime**: A spacetime with zero curvature where the Riemann curvature tensor vanishes, typically modeled by Minkowski geometry.
- **Metric Tensor**: A symmetric tensor that defines the geometric properties of space or spacetime, including distances and angles.
- **Bianchi Identities**: Mathematical identities involving the Riemann tensor that are crucial in deriving Einstein's field equations in general relativity.

9.9 *REFERENCES:-*

- Abraham, Ralph; Marsden, Jerrold E.; Ratiu, Tudor (2017), *Manifolds, Tensor Analysis, and Applications*. Springer.
- Millman, Richard S., & Parker, George D. (2019), *Elements of Differential Geometry*. Dover Publications

9.10 *SUGGESTED READING:-*

- Satya Prakash and K.P. Gupta (Nineteenth Edition, 2019), Relativistic Mechanics.
- Dr. J.K. Goyal and Dr. K.P. Gupta (Twenty Eight Edition, 2018), Theory of Relativity.

9.11 *TERMINAL QUESTIONS:-*

(TQ-1) Show that $U_{i,jk} = U_{i,kj}$ for all covariant vector U_i iff curvature tensor is zero. Prove that

$$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$$

(TQ-2) Show that the vanishing of Riemann-Christoffel tensor is a necessary condition for flat space time. Is this condition sufficient also ?

(TQ-3) Prove that divergence of $R^{ij} - \frac{1}{2}Rg^{ij}$ is zero where R_{ij} is Einstein's tensor.

(TQ-4) Show that algebraically independent components of curvature tensor R_{hijk} in a V_4 cannot exceed 20.

(TQ-5) Prove that

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

(TQ-6) Define a flat space time. Show that the vanishing of curvature tensor is a necessary and sufficient condition for a space time to be flat.

(TQ-7) Show that in general $A_{\mu,v\sigma} \neq A_{\mu,\sigma v}$

(TQ-8) Show that the divergence of $G^v_\mu - \frac{1}{2} g^\mu_v$ is identically zero.

(TQ-9) Define Riemann Christoffel curvature tensor and obtain an expression for it.

(TQ-10) Show that vanishing of the Reimann curvature tensor is a necessary and sufficient condition that the space be flat.

(TQ-11) Prove that Bianchi Identity

$$R^a_{ijk,l} + R^a_{ikl,j} + R^a_{ilj,k} = 0$$

or

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

(TQ-12) show that covariant curvature tensor R_{hijk} is

a) Skew-symmetric in the first two indices.
b) Skew-symmetric in the last two indices.
c) Symmetric in two pairs of indices.

(TQ-13) show that

a) $R_{hijk} = -R_{ihjk}$
b) $R_{hijk} = -R_{hikj}$
c) $R_{hijk} = R_{jkhi}$

9.12 *ANSWERS:-*

SELF CHECK ANSWERS

1. c)
2. It represents the intrinsic curvature of a manifold and measures how vectors change under parallel transport around a loop.
3. A spacetime is flat if the Riemann curvature tensor is zero everywhere.
4. Minkowski spacetime is used to model flat spacetime in special relativity.
5. It describes how mass and energy curve spacetime and appears in the formulation of Einstein's field equations.
6. Yes, in general relativity, spacetime curvature is interpreted as the manifestation of gravity.
7. The Ricci tensor and scalar curvature are derived by contracting the Riemann curvature tensor.
8. It is the process of moving a vector along a curve while keeping it "parallel" according to the manifold's geometry.
9. No, although its components may change, the curvature tensor itself is a geometric object independent of coordinates.

BLOCK III
GENERAL RELATIVITY

UNIT 10:-Introduction of the General Theory of Relativity

CONTENTS:

10.1 *INTRODUCTION:-*

The General Theory of Relativity (GTR)**,** proposed by Albert Einstein in 1915**,** is a fundamental theory of gravitation that describes gravity not as a force but as the curvature of space-time caused by mass and energy. It extends the Special Theory of Relativity to include acceleration and gravity, introducing key principles such as the Principle of Covariance**,** which ensures that the laws of physics hold in all coordinate systems, and the Principle of Equivalence**,** which states that locally, the effects of gravity are indistinguishable from acceleration. The theory replaces Newton's concept of gravitational force with the idea that massive objects bend space-time, influencing the motion of other objects along geodesic paths. General Relativity successfully explains several gravitational phenomena, including gravitational time dilation, the bending of light near massive bodies, gravitational waves, and black holes**,** and has been confirmed by numerous experiments such as the Mercury perihelion shift, gravitational Lansing.

The Special Theory of Relativity originated from the development of electrodynamics and is based on the principle that the motion of a body can only be detected and measured relative to other bodies, with no absolute motion being meaningful. It specifically considers the relativity of uniform translational motion in regions of free space where gravitational effects can be neglected. This leads to the conclusion that physical laws remain unchanged in inertial reference frames, where the law of inertia holds. However, to address phenomena such as the "clock paradox" and the universal law of gravitation, the theory had to be extended to non-inertial systems, which involve acceleration. This extension led to the General Theory of Relativity, which incorporates gravity into the relativistic framework by describing it as the curvature of spacetime. Despite its success, early theoretical predictions struggled to fully explain certain observed gravitational phenomena, necessitating further refinements and experimental verification.

These deviations arose due to the following reasons:

1. The theory fails for fixed particles in a gravitational field, as observed in the redshift of spectral lines. In such cases, atoms remain fixed, and the spectral lines emitted by these atoms are affected by strong gravitational and magnetic fields.
2. The theory is fails for phenomena involving velocities comparable to the speed of light, such as the bending of light rays under the influence of a massive attracting body.
3. According to the Special Theory of Relativity, the predicted bending of light rays passing near the Sun should be 0.88 arc seconds, whereas actual observations show a bending of 1.75 arc seconds.
4. The theory also fails in scenarios where both velocity and gravitational fields are present, as seen in the precession of the perihelion of Mercury.

The predictions of Special Relativity suggest an advance of 7.2 seconds of arc per century, but the observed value is 43 seconds of arc per century, indicating a discrepancy that requires modification. Special Relativity applies only to inertial reference frames**,** where physical laws remain invariant under Lorentz transformations. However, this invariance is restricted to such frames, meaning it does not account for gravitational effects or accelerated motion. Since real-world phenomena often involve non-inertial frames, Special Relativity alone is insufficient to describe nature comprehensively. To address this limitation, Einstein extended the principles of Special Relativity to include non-inertial reference frames**,** leading to the General Theory of Relativity (GR)**.** GR describes gravity not as a force but as the curvature of space-time caused by mass and energy. When applied to gravitational phenomena, GR predicts small deviations from Special Relativity, such as the anomalous precession of Mercury's orbit**.** These deviations, confirmed through experimental observations, validate the accuracy of General Relativity and demonstrate its superiority in explaining gravitational interactions beyond the scope of Special Relativity.

10.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- Define and apply the **Equivalence Principle** in different physical situations.
- Understand the **Principle of General Covariance** and how it ensures the consistency of physical laws in all reference frames.

10.3 *PRINCIPAL OF COVARIANCE:-*

When referring to different sets of Cartesian axes that are in uniform relative translatory motion, the laws describing any phenomenon in free space must have the space form and contents and be independent of the velocity of the specific observer making the measurements, according to the special theory of relativity. We fully utilize the basic concept of relativity for all types of motion in general theory. *Here the laws must be expressible in a form which is independent of the particular space time co-ordinate choosen or in other words laws of nature remain invariant w.r.t. any space time co-ordinate system. This statement is called the principle of genera covariance.*

Therefore, all of our laws must be expressed using covariant equations that do not require a specific coordinate system. Since the form of a tensor equation, which expresses a law, is precisely the same in all coordinate systems, we utilize tensor calculus to do this. As we can see, the equation's modified form

$$ds^2 = -(dx^2 + dy^2 + dz^2) + c^2dt^2$$

in tensor form is

$$ds^2 = g_{ij}\, dx^i\, dx^j \; (i, j = 1,2,3,4).$$

The fundamental tensor g_{ij} is co-variant tensor of rank two that transforms according to the law

$$g'_{ij} = g_{ab}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}$$

where g'_{ij} is the transformed metric in the new coordinate system x'^i.

Suppose the physical laws of the nature in x^i co-ordinate system are expressed by an equation involving tensors, such as

$$A^i_j = B^i_j$$

Then we can write the transformation law for this tensor as

$$A'^i_j - B'^i_j = A^\alpha_\beta\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j} - B^\alpha_\beta\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j}$$

$$\left(A^\alpha_\beta - B^\alpha_\beta\right)\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j} = 0.\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j} = 0$$

$$A'^i_j - B'^i_j = 0 \quad or \quad A'^i_j = B'^i_j$$

Thus, we see that tensorial quantities follow the general covariant laws, ensuring that physical equations retain their form under arbitrary coordinate transformations, which is a fundamental principle of General Relativity.

10.4 *PRINCIPAL OF EQUIVALANCE:-*

The principle of co-variance is the assumption that the physical laws may be stated in a way that is independent of the coordinate system, and the principle of equivalence is the actual hypothesis that introduces gravitational considerations into the development. It is now possible for us to examine the principle of equivalency in depth

The inertial mass is a coefficient which measures the resistance of intertia of the body opposing the action of force. If the acceleration given to the body by the force F is a, then inertial mass is and the gravitational mass is

$$\frac{F}{\alpha} = m_i$$

as well as the coefficient that determines the attractive force that a body experiences in the gravitational field. *According to Newtonian theory the* *gravitational mass and inertial mass are always equal. This is called the principle of equivalence.*

$$m_i = m_q$$

and

$$m_i \frac{d^2x}{dt^2} = m_q g$$

$$\frac{d^2x}{dt^2} = g$$

$$\frac{d^2x_k}{dt^2} = g_k$$

We may observe that the equation above is unaffected by the body's mass. Consequently, we assert that, regardless of mass, the rate at which all bodies fall under the influence of gravity is the same. Galleilo verified the same empirically. All bodies, regardless of mass, fall in the same manner in an evacuated laboratory, he claims (here the word evacuation is simply used to avoid via friction of air).

Relative deceleration, which occurs when a system of reference is subjected to accelerated motion, is similar to gravitational acceleration i.e., when an elevator is accelerated ward, a person in the elevator feels momentarily heavier and en the acceleration is in the downward direction, he feels Lighter. Thus Einstein noticed this fact and gave a very fundamental and important idea that the gravitational field produced by accelerating uniformly ah inertial frame of reference. According to Einstein the principle of equivalence can be stated as follows:

In the neighborhood of any given point, we can distinguish between the gravitational field produced by the attraction of masses and the field produced by accelerating uniformly ah inertial frame of reference.

Consequently, two fields are the same. The equivalency concept is also discovered to apply to electrical and optical phenomena. For example, a light beam that propagated rectilinearly with regard to the uniform motion's x" coordinate system was no longer rectilinear when compared to the accelerated motion's x" coordinate system. It follows from this that light beams propagate curvilinearly in gravitational fields normally.

10.5 *EQUALITY OF INERTIAL AND GRAVITATIONAL MASSES:-*

Inertial mass: The inertial mass of an object determines its resistance to acceleration a_i when a force F_i is applied, as described by Newton's second law:

$$F_i = ma_i$$

where:

- F_i is the applied force.
- m inertial is the inertial mass, and
- a_i is the resulting acceleration.

$$m_i = \frac{F_i}{a_i}$$

Thus the inertial mass of a body may be defined as the ratio of the inertial force acting on the body to the acceleration acquired.

Gravitational Mass: If g is a body's acceleration in a field of gravitational attraction F_g, then

$$F_g = mg$$

where m is the gravitational mass of the body and may be expressed as

$$m_g = \frac{F_g}{g}$$

Thus the gravitational mass of a body is defined as the ratio of the gravitational force to the gravitational acceleration of the body in the gravitational field.

Since the principle of equivalence states that the gravitational and inertial forces are of the same kind and subject to the same laws, and that a desired gravitational field can be created by selecting an appropriate accelerated frame of reference, hence

$$\frac{F_i}{a_i} = \frac{F_g}{g}$$

$$m_i = m_g$$

Consequently, the equality of the gravitational and inertial masses of the same body is implied by the principle of equivalence. This equality of gravitational and inertial masses has the effect of accelerating the fall of all bodies in the same gravitational field. The equality of inertial and gravitational masses has been confirmed experimentally with a high degree of accuracy by Dicke in 1962, Eotvos in 1896, and 1908.

Sometimes, the concept of equivalence refers to the idea that inertial and gravitational masses are equivalent.

SELF CHECK QUESTIONS

1. What is the Principle of Covariance?
2. Why is the Principle of Covariance important in General Relativity?
3. How does the Principle of Covariance differ from Galilean Invariance?
4. What mathematical tools are used to express physical laws covariantly?

10.6 *SUMMARY:-*

In this unit, we explored two fundamental principles of General Relativity: the Principle of Covariance and the Principle of Equivalence**.**

- The **Principle of Covariance** states that the laws of physics must be valid in all coordinate systems, meaning their form remains unchanged under smooth transformations. This ensures that Einstein's field equations are expressed in a covariant form using tensors, making them independent of the observer's frame of reference.
- The **Principle of Equivalence** establishes that locally, the effects of gravity are indistinguishable from those of acceleration. This implies that a uniform gravitational field is equivalent to a uniformly accelerated reference frame, leading to the conclusion that gravity is not a traditional force but rather a curvature of space-time caused by mass and energy. Together, these principles provide the conceptual framework for General Relativity, fundamentally redefining our understanding of gravity as the geometric deformation of space-time rather than a force acting at a distance.

10.7 *GLOSSARY:-*

- **Principle of Covariance** – A fundamental concept in General Relativity stating that the laws of physics must have the same form in all coordinate systems, ensuring their invariance under smooth transformations.
- **General Covariance** – The requirement that physical laws be expressed using tensors so that they remain valid in any reference frame or coordinate system.
- **Coordinate Independence** – The idea that the formulation of physical laws should not depend on a specific choice of coordinates, reinforcing the universal applicability of physical principles.
- **Einstein Field Equations (EFE)** – A set of covariant equations in General Relativity that describe the relationship between spacetime curvature and energy-momentum distribution.
- **Principle of Equivalence** – The assertion that locally, the effects of gravity are indistinguishable from acceleration, meaning that a uniform gravitational field is equivalent to a uniformly accelerated reference frame.
- **Weak Equivalence Principle (WEP)** – The principle stating that all objects, regardless of their mass or composition, fall at the same rate in a gravitational field.
- **Strong Equivalence Principle (SEP)** – An extension of the Weak Equivalence Principle that includes gravitational self-energy, stating that the laws of physics, including General Relativity, hold true in all freely falling reference frames.
- **Tensors** – Mathematical objects used in General Relativity to express physical laws in a covariant form, ensuring their validity in all coordinate systems.
- **Spacetime Curvature** – A concept in General Relativity describing how mass and energy distort spacetime, leading to what we perceive as gravitational attraction.
- **Geodesic Motion** – The trajectory of a freely falling object in curved spacetime, which follows the shortest path (geodesic) dictated by the curvature of spacetime rather than a direct force.

10.8 *REFERENCES:-*

- Robert J. A. Lambourne (2010). Relativity, Gravitation and Cosmology.
- Michel Janssen and Christoph Lehner (2014), The Cambridge Companion to Einstein.

- Asghar Qadir (2020), Einstein's General Theory of Relativity.

10.9 *SUGGESTED READING:-*

- S.P.Puri (2013), General Theory of relativity.
- Farook Rahman (2021), The General Theory of Relativity: A Mathematical Approach
- Goyal and Gupta (1975), Theory of Relativity.
- R.K.Pathria (2003), Theory of Relativity.

10.10 *TERMINAL QUESTIONS:-*

(TQ-1) Give an account of Einstein's principle of equivalence. What are the observable consequences of general theory of relativity?

(TQ-2) Explain Einstein's principle of equivalence. Give a detailed account of red shift of light. How has this been verified experimentally?

(TQ-3) Explain the principle of equivalence and give a concise account of the general theory of relativity. Discuss the experimental evidence in support of it.

(TQ-4) State and comment on the basic hypothesis and postulates of the general theory of relativity and discuss how the principle of equivalence and covariance follow from the guiding principle in the development of general relativity?

(TQ-5) Explain the principle of equivalence and the principle of general covariance

(TQ-6) Write notes on the following:

(a) Fundamental concepts of general theory of relativity.
(b) Principle of covariance.
(c) Postulates of general theory of relativity.

(TQ-7) State the principle of equivalence in general theory of relativity and discuss that it acts as a bridge to pass from special to general theory of relativity.

(TQ-8) What is the Principle of Equivalence? Discuss its role in the development of General Relativity.

(TQ-9) Compare and contrast the Principle of Covariance and the Principle of Equivalence.

(TQ-10) Write short note on 'principle of equivalence'.

(TQ-11) Write short note on principle of equivalence.

(TQ-12) State the basic postulates and principles of General Theory of relativity. Justify the statement that the principle of equivalence acts as a bridge to pass from, special theory of relativity to General Theory of Relativity

10.11 *ANSWERS:-*

SELF CHECK ANSWERS

1. The Principle of Covariance states that the laws of physics should take the same mathematical form in all coordinate systems.
2. It ensures that the laws of physics, particularly Einstein's field equations, are valid for all observers regardless of their state of motion or coordinate choice, reflecting the general nature of spacetime.
3. Galilean invariance applies only to Newtonian mechanics and inertial frames, while the Principle of Covariance applies to all frames, inertial or non-inertial, using the language of tensor calculus.
4. Tensors, covariant derivatives, Christoffel symbols, and metric tensors are key tools used to express physical laws in a covariant (coordinate-independent) form.

UNIT 11:-Relativistic Field Equations

CONTENTS:

11.1 *INTRODUCTION:-*

The energy-momentum tensor, also known as the stress-energy tensor, is a fundamental mathematical object in physics that describes the distribution and flow of energy and momentum in space-time. It is a second-rank tensor denoted by $T^{\mu\nu}$and plays a crucial role in General Relativity (GR) as the source of space-time curvature in Einstein's field equations. Each component of the tensor represents different physical quantities, such as energy density, momentum density, and stress (pressure and shear forces) in a given system. Depending on the type of matter or field, the energy-momentum tensor takes different forms, including those for perfect fluids, electromagnetic fields, and scalar fields. In relativistic hydrodynamics and astrophysical models, it is often extended to include viscosity and heat conduction, making it essential in studying radiating stars, neutron stars, and cosmology. Furthermore, the conservation $\nabla_\nu T^{\mu\nu} = 0$ ensures that energy and momentum are locally conserved, governing the motion of matter in curved spacetime. The energy-momentum tensor serves as the bridge between matter-energy content and the geometry of space-time, shaping our understanding of gravity and the evolution of astrophysical objects.

11.2 *OBJECTIVES:-*

After studying this unit, Lernear's will be able to

- To solve the Energy momentum tensor
- To derive the formula of Energy momentum tensor for perfect fluid.
- To solve the Einstein's Field Equation.
- To explain the derivation of Einstein's Field Equations
- To provide solutions to Poisson's Equations as an Approximation of Field Equation.
- To explain solutions to Newtonian Equation of Motion as an Approximation of Geodesic Equations.

11.3 *ENERGY MOMENTUM TENSOR:-*

The energy-momentum tensor or material energy-tensor or energy tensor T^{ij} is a mathematical object that describes the density and flow of energy and momentum in space-time, serving as the source of gravity in Einstein's field equations.

Let $\frac{dx^j}{ds}$ represent the speed of the matter in the gravitational system, that is, when the velocity of light $= c = 1$, and let ρ_0 represent the appropriate density of matter.

The energy momentum tensor, denoted by T^{ij}, is written as

$$T^{ij} = \rho_0 \frac{dx^i}{ds}\frac{dx^j}{ds} \qquad \ldots(1)$$

The Galilean coordinate system gives us

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2, when\; c = 1$$

$$\left(\frac{ds}{dt}\right)^2 = -\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 + 1$$

Taking

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

We obtain

$$\left(\frac{ds}{dt}\right)^2 = 1 - v^2 \qquad \ldots(2)$$

where $c = 1$.

Given the coordinate density of matter ρ and velocity v with respect to the Galilean coordinate system, we obtain

$$\rho = \frac{\rho_0}{1 - \frac{v^2}{c^2}} = \frac{\rho_0}{1 - v^2}, \quad (when\ c = 1)$$

$$\rho(1 - v^2) = \rho_0$$

$$\rho\left(\frac{ds}{dt}\right)^2 = \rho_0 \quad (from\ (2))$$

Applying this to (1), we get

$$T^{ij} = \rho_0 \frac{dx^i}{ds}\frac{dx^j}{ds} = \rho_0 \frac{dx^i}{dt}\frac{dt}{ds}\frac{dx^j}{dt}\frac{dt}{ds}$$

$$= \rho_0 \frac{dx^i}{dt}\frac{dx^j}{dt}\left(\frac{dt}{ds}\right)^2 = \frac{dx^i}{dt}\frac{dx^j}{dt}\left[\frac{\rho_0}{\left(\frac{ds}{dt}\right)^2}\right] = \rho \frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$T^{ij} = \rho \frac{dx^i}{dt}\frac{dx^j}{dt} \quad \dots (3)$$

This is the Galilean coordinate system expression for T^{ij}.

If we write $u = \frac{dx^1}{dt}, v = \frac{dx^2}{dt}, w = \frac{dx^2}{dt}$, then the equation(3) obtain

$$T^{ij} = \begin{bmatrix} \rho u^2 & \rho uv & \rho uw & \rho u \\ \rho uv & \rho v^2 & \rho vw & \rho v \\ \rho uw & \rho vw & \rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix}$$

THEOREM1: To derive the formula for energy momentum tensor for a perfect in the form

$$T_\mu^\nu = (\rho + p)v_\mu v^\nu - g_\mu^\nu p$$

SOLUTION: Let $T_0^{\mu\nu}$ represent the energy momentum tensor in the appropriate coordinate system, where the matter is assumed to be at rest at the origin, we have

$$T_0^{11} = T_0^{22} = T_0^{33} = p_0, T_0^{44} = \rho_0 \quad \dots (1)$$

The other elements all being zero.

In the appropriate coordinate system, p_0 and ρ_0 stand for pressure and density of a perfect, respectively. When used correctly, the Galilean coordinate system is applicable for which

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2, when\ c = 1$$

Let g_0^{ij} represent the Galilean coordinate system's basic tensor so that

$$g_0^{11} = g_0^{22} = g_0^{33} = -g_0^{44} = -1, g_0^{ij} = 0\ for\ i \neq j$$

In an arbitrary coordinate system, let T^{ij} and g^{ij} stand for the energy tensor and fundamental tensor, respectively. By the transformation's tensor law,

$$T^{ij} = T_0^{ab}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^b}$$

$$= \sum_{a=1}^{4} T_0^{ab}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^b} \qquad from\ (1)$$

$$T^{ij} = p_0\sum_{a=1}^{3}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^a} + \rho_0\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} \quad again\ from\ (1) \qquad \ldots(2)$$

$$g^{ij} = g_0^{ab}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^b} = \sum_{a=1}^{4} g_0^{aa}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^a}$$

$$= -\sum_{a=1}^{3}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^a} = -g^{ij} + \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4}$$

From (2), we get

$$T^{ij} = p_0\left(-g^{ij} + \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4}\right) + \rho_0\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4}$$

$$= (p_0 + \rho_0)\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} - p_0 g^{ij}$$

$$T^{ij} = (p_0 + \rho_0)\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} - p_0 g^{ij} \qquad \ldots(3)$$

Since the fluid is at rest in the proper-co-ordinate system and hence the velocity components can be taken as

$$\frac{dx_0^1}{ds} = \frac{dx_0^2}{ds} = \frac{dx_0^3}{ds} = 0, \frac{dx_0^4}{ds} = 1 \qquad \ldots(4)$$

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial x_0^j}\frac{dx_0^j}{ds} = \frac{\partial x^i}{\partial x_0^4}\frac{dx_0^4}{ds}$$

$$= \frac{\partial x^i}{\partial x_0^4}.1 = \frac{\partial x^i}{\partial x_0^4}$$

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial x_0^4}$$

Putting the above value in (3), we obtain

$$T^{ij} = (p_0 + \rho_0)\frac{dx^i}{ds}\frac{dx^j}{ds} - p_0 g^{ij} \qquad \ldots(5)$$

This is the required expression for T^{ij}

From (4) it follows that

$$T^{ij} = (p_0 + \rho_0)v^i v^j - p_0 g^{ij}$$

where $v^i = \frac{dx^i}{ds} = velocity\ component$ .Again the equation (5) can be written as

$$T_\mu^v = (p_0 + \rho_0)v^v v_\mu - p g_\mu^v \quad where\ p = p_0, \rho_0 = \rho$$

THEOREM2: Explain the construction of the energy momentum tensor T for matter composed of moving particles and show that the conditions of conservation of energy and momentum lead to the tensor equation $(T^{\mu v})_{,v} = 0$.

SOLUTION: Prove as in theorem 1 that

$$T^{ij} = (p_0 + \rho_0)\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} - p_0 g^{ij} \qquad \dots(3)$$

Then

$$T_0^{ij} = \begin{bmatrix} p_{0xx} & p_{0xy} & p_{0xz} & 0 \\ p_{0yx} & p_{0yy} & p_{0yz} & 0 \\ p_{0zx} & p_{0zy} & p_{0zz} & 0 \\ 0 & 0 & 0 & \rho_0 \end{bmatrix}$$

where p_{0xx}, p_{0xy} etc. Represent internal stresses.

Let the coordinate density and velocity of the matter consisting of a perfect fluid flowing with regard to the Galilean coordinate system be represented by ρ and q (u, v, w), respectively. Then

$$T^{ij} = \begin{bmatrix} p_{xx} + \rho u^2 & p_{xy} + \rho uv & p_{xz} + \rho uw & \rho u \\ p_{yx} + \rho vu & p_{yy} + \rho v^2 & p_{yz} + \rho vw & \rho v \\ p_{zx} + \rho wu & p_{zy} + \rho wv & p_{zz} + \rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix}$$

Let

$$\frac{\partial T^{\mu v}}{\partial x^v} = 0 \qquad \dots(4)$$

For $\mu = 4$, the equation (4) obtain

$$\frac{\partial T^{4v}}{\partial x^v} = 0$$

Now

$$\frac{\partial T^{41}}{\partial x^1} + \frac{\partial T^{42}}{\partial x^2}\frac{\partial T^{43}}{\partial x^3}\frac{\partial T^{44}}{\partial x^4} = 0$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \qquad \dots(5)$$

Putting $\mu = 1$, the equation (4) gives that

$$\frac{\partial T^{1v}}{\partial x^v} = 0$$

$$\frac{\partial T^{11}}{\partial x^1} + \frac{\partial T^{12}}{\partial x^2}\frac{\partial T^{13}}{\partial x^3}\frac{\partial T^{14}}{\partial x^4} = 0$$

or

$$\frac{\partial}{\partial x}(p_{xx} + \rho u^2) + \frac{\partial}{\partial x}(p_{xx} + \rho uv) + \frac{\partial}{\partial z}(p_{xz} + \rho uw) + \frac{\partial}{\partial x}(\rho u) = 0$$

or

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}=-\left[\frac{\partial}{\partial x}(\rho u^2)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial z}(\rho uw)+\frac{\partial}{\partial x}(\rho u)\right]$$

$$=-u\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial x}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial x}\right]$$

$$-\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right]$$

$$=-u.0-\frac{\rho du}{dt}=-\frac{\rho du}{dt}$$

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}=-\frac{\rho du}{dt} \qquad \dots(6)$$

where

$$\frac{d}{dt}=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$$

Similarly for $\mu=2{,}3$ the equation 4 obtains

$$\left.\begin{aligned}\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}=-\frac{\rho dv}{dt}\\ \frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}=-\frac{\rho dw}{dt}\end{aligned}\right\} \qquad \dots(7)$$

$\frac{du}{dt}\frac{dv}{dt}\frac{dw}{dt}$ represent components of acceleration of fluid particles.

The hydrodynamics equation of continuity is equation (5). The hydrodynamic motion equations in the absence of external forces are (6) and (7).

Thus, the conservation of mass and momentum is expressed by equations (5), (6), and (7). As a result, equation (4) in relation to Galilean coordinates expresses the concepts of mass and momentum conservation.

Also $\Gamma^{\sigma}_{\mu\nu}=0$ relative to Galilean coordinates.

Hence

$$T^{\mu\nu}_{,\nu}=\frac{\partial T^{\mu\nu}}{\partial x^{\nu}}$$

Therefore relative to Galilean coordinates, (4) is expressible as

$$(T^{\mu\nu})_{\nu}=T^{\mu\nu}_{,\nu}=0$$

In reality $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}}$ denoted the rate of creation of mass and momentum in unit volume.

11.4 *EINSTEIN'S FIELD EQUATIONS:-*

Einstein's law of gravitation, formally known as the **Einstein Field Equations (EFE)** . According to the Newton's theory of gravity states that the field equations in the presence of matter are as follows:

$$\nabla^2\phi = 4\pi G\rho \qquad \text{... (1)}$$

where G is the gravitational constant, ρ is the matter density, and is the gravitational potential ϕ . We must substitute the metric tensor guy for & in the relativistic theory of gravitation since g_{44} performs the function of gravitational potential in the non-relativistic limit. Consequently, it is necessary to describe the left-hand side in terms of the second-order derivatives of $g_{\mu\nu}$ based on equation (1). The right-hand side of the relativistic theory of gravitation must be stated in terms of the material energy tensor $T^{\mu\nu}$ in such a way that its divergence disappears since the density of matter, ρ is one of the components of the second rank energy momentum tensor. Therefore, if the classical equation (1) is to be generalized for the relativistic theory of gravity, it must be a tensor equation that satisfying the following conditions:

(i) The tensor equation should not contain the derivatives of guv higher than the second order.

(ii) It must be linear in the second differential coefficients.

(iii) Its covariant divergence must vanish identically.

We know that the covariant derivatives of $g_{\mu\nu}$ are known to be exactly zero and $R_{\mu\nu}$ and $R_{\mu\nu}(= R)$ are the tensors that are generated by contracting the curvature tensor $R_{\rho\mu\nu\sigma}$ once and twice. This is the sole tensor that involves $g_{\mu\nu}$ up to and second order. Therefore, the most appropriate tensor of the form required is the Einstein's tensor provided by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

The above eqn. (1) is generalized as

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -kT_{\mu\nu} \qquad \text{... (2)}$$

where k is a constant and is related to the gravitational constant. If Newton's theory and Einstein's relativistic theory are equivalent in the non-relativistic approximation. In relativistic units

$$k = 8\pi \qquad \text{... (3)}$$

The equation (2) is obtained as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \qquad \dots(4)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = -8\pi T_{\mu\nu} \qquad \dots(5)$$

where Λ is called cosmological constant.

Neglecting cosmological constant Λ, the equation (4) given as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \qquad \dots(6)$$

The other two forms of Einstein's field equations are obtained by raising the indices using the metric is

$$R_{\mu}^{\nu} - \frac{1}{2}g_{\mu}^{\nu}R + \Lambda g_{\mu}^{\nu} = -8\pi T_{\mu}^{\nu} \qquad \dots(7)$$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = -8\pi T^{\mu\nu} \qquad \dots(8)$$

Now multiplying (5) by$g^{\mu\nu}$,we obtain

$$R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}(R - 2\Lambda) = -8\pi g^{\mu\nu}T_{\mu\nu}$$

$$R - \frac{1}{2}4(R - 2\Lambda) = -8\pi T \ (since g^{\mu\nu}g_{\mu\nu} = \delta_{\mu}^{\nu} = 4)$$

$$R - 2\Lambda = 8\pi T$$

Since in the absence of matter $T_{\mu\nu} = 0$ so that $T = 0$. then

$$R = 4\,\Lambda$$

Putting these values in equation (5), we obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(4\,\Lambda - 2\Lambda) = -8\pi.0$$

$$R_{\mu\nu} = g_{\mu\nu}\Lambda$$

Therefore, the above equation defined the Einstein’s field equations in general theory of relativity in the absence of matter or Einstein;’s law of gravitation in empty space is

$$R_{\mu\nu} = 0$$

THEOREM2: To derive the field equation (in empty space) from Lagrangian density.

Or

To derive the field equation (in empty space) from variational principle.

PROOF: The relativistic field equation are obtained from

$$\delta \int_D \sqrt{-g} g^{\mu\nu}. R_{\mu\nu} dT = 0 \qquad ...(1)$$

We assume that variations in $g_{\mu\nu}$ or its first order derivatives stay arbitrary inside area D but disappear on the edge of the four-dimensional domain D. In other words,

$$\delta g_{ij} = 0 = \delta \Gamma_{ij}^{k} \text{ on the boundary D.} \qquad ...(2)$$

$$R_{\mu\nu} = R_{\mu\nu\sigma}^{a} = -\frac{\partial \Gamma_{\mu\nu}^{a}}{\partial x^{a}} + \frac{\partial \Gamma_{\mu a}^{a}}{\partial x^{\nu}} - \Gamma_{\mu\nu}^{b} \Gamma_{ab}^{a} + \Gamma_{\mu a}^{b} \Gamma_{b\nu}^{a}$$

$$\delta R_{\mu\nu} = -\frac{\partial \delta \Gamma_{\mu\nu}^{a}}{\partial x^{a}} + \frac{\partial \delta \Gamma_{\mu a}^{a}}{\partial x^{\nu}} - \delta \Gamma_{b\nu}^{b} \Gamma_{ba}^{a} - \Gamma_{\mu\nu}^{b} \delta \Gamma_{ba}^{a} + \delta \Gamma_{\mu a}^{b} \Gamma_{b\nu}^{a} + \Gamma_{\mu a}^{b} \delta \Gamma_{b\nu}^{a}$$

$$\delta R_{\mu\nu} = -\left[\frac{\partial \delta \Gamma_{\mu\nu}^{a}}{\partial x^{a}} + \delta \Gamma_{\mu\nu}^{b} \Gamma_{ba}^{a} - \delta \Gamma_{\mu b}^{b} \Gamma_{a\nu}^{b} - \Gamma_{\mu a}^{b} \delta \Gamma_{b\nu}^{a}\right]$$
$$+ \left[\frac{\partial}{\partial x^{\nu}} \left(\delta \Gamma_{\mu a}^{a}\right) - \delta \Gamma_{ba}^{a} \Gamma_{b\nu}^{b}\right]$$
$$= -\left(\delta \Gamma_{\mu\nu}^{a}\right)_{,a} + \left(\delta \Gamma_{\mu a}^{a}\right)_{,\nu}$$

$$\Rightarrow \qquad g^{\mu\nu} \delta R_{\mu\nu} = -g^{\mu\nu} \left(\delta \Gamma_{\mu\nu}^{a}\right)_{,a} + g^{\mu\nu} \left(\delta \Gamma_{\mu a}^{a}\right)_{,\nu}$$
$$= -\left(g^{\mu\nu} \delta \Gamma_{\mu\nu}^{a}\right)_{,a} + \left(g^{\mu\nu} \delta \Gamma_{\mu a}^{a}\right)_{,\nu} \qquad as\ g^{\mu\nu}, \sigma = 0$$
$$= -\left(g^{\mu\nu} \delta \Gamma_{\mu\nu}^{a}\right)_{,b} + \left(g^{\mu\nu} \delta \Gamma_{\mu a}^{a}\right)_{,b}$$

Or
$$g^{\mu\nu} \delta R_{\mu\nu} = \left[-g^{\mu\nu} \delta \Gamma_{\mu\nu}^{a} + g^{\mu b} \delta \Gamma_{\mu a}^{a}\right]_{,b}$$

But

$$\left(A_{,i}^{i}\right) \sqrt{-g} = \frac{\partial \left[A^{i} \sqrt{-g}\right]}{\partial x^{i}}$$

Hence

$$\sqrt{-g}g^{\mu v}\delta R_{\mu\nu} = \frac{\partial}{\partial x^b}\left[\sqrt{-g}\{-g^{\mu v}\delta\Gamma^a_{\mu v} + g^{\mu b}\delta\Gamma^a_{\mu a}\}\right]$$

Now integrating,

$$\int_D \sqrt{-g}g^{\mu v}\delta R_{\mu\nu}dT = \int_D \frac{\partial}{\partial x^b}\left[\sqrt{-g}\{-g^{\mu v}\delta\Gamma^a_{\mu v} + g^{\mu b}\delta\Gamma^a_{\mu a}\}\right]dT \quad \ldots (3)$$

According to the Gauss theorem, this volume integral on the R.H.S. of the above equations may be transformed into a surface integral, which disappears in line with (2). It indicates that

$$\int_D \sqrt{-g}g^{\mu v}\delta R_{\mu\nu}dT = 0 \quad \ldots (4)$$

The equation (1) can be written as

$$\int_D \sqrt{-g}g^{\mu v}\delta R_{\mu\nu}dT + \int_D R_{\mu\nu}\delta[\sqrt{-g}g^{\mu v}]dT = 0$$

$$\int_D R_{\mu\nu}\delta[\sqrt{-g}g^{\mu v}]dT = 0 \quad \ldots (5)$$

Now

$$\delta\left[\sqrt{-g}g^{\mu v}\right] = \delta g^{\mu v}.\sqrt{-g} + \frac{g^{\mu v}}{2\sqrt{-g}}\delta(-g)$$

$$-\sqrt{-g}\delta g^{\mu v} - \frac{g^{\mu v}}{2\sqrt{-g}}\; g g^{\alpha\beta}\delta g_{\alpha\beta}$$

$$\left[For\ \frac{\partial g}{\partial x^\sigma} = (g g^{\mu v})\frac{\partial g_{\mu v}}{\partial x^\sigma}\right]$$

$$\delta\left[\sqrt{-g}g^{\mu v}\right] = \sqrt{-g}\;\delta g^{\mu v} + \frac{g^{\mu v}}{2}\sqrt{-g}\left(-g^{\alpha\beta}\delta g_{\alpha\beta}\right)$$

$$R_{\mu\nu}\delta\left[\sqrt{-g}g^{\mu v}\right] = \sqrt{-g}.\delta g^{\mu v}R_{\mu\nu} + \frac{R}{2}\sqrt{-g}\left(-g^{\alpha\beta}\delta g_{\alpha\beta}\right)$$

$$= \int_D \surd(-g).\,\delta g^{\mu v}\left[R_{\mu\nu} - \frac{1}{2}\,R g_{\mu\nu}\right]dT = 0$$

$$\Rightarrow \qquad \int_D \sqrt{-g}.\delta g^{\mu v}\left[R_{\mu\nu} - \tfrac{1}{2}\,R g_{\mu\nu}\right]dT = 0$$

Since $\delta g^{\mu v}$ is arbitrary.

$$\Rightarrow \qquad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

11.5 *NEWTONIAN EQUATION OF MOTION AS AN APPROXIMATION OF GEODESIC EQUATIONS:-*

Geodesic equations reducible to Newtonian equations of motion in case of weak static field.

Or,

To discuss the motion of a free particle in case of weak static field.

PROOF: The motion of a test particle in a weak static gravitational field is governed by the geodesic equation,

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{dz} = 0 \qquad \dots (1)$$

The entire $g_{\mu\nu}$ is constant and independent of coordinate systems, and the line element in special relativity corresponds to the Euclidean (flat) space-time manifold. All of Christofell's symbols $\Gamma^\alpha_{\mu\nu}$ disappear as a result, and the geodesic equations of motion provided by (1) then reduce to the equations of straight lines, i.e.

$$\frac{d^2x^\lambda}{ds^2} = 0$$

The fact that the equations of motion and the line element are determined by the metric tensor $g_{\mu\nu}$ is relevant. In the first case, the geometry's structure is determined by the metric tensor $g_{\mu\nu}$ components, but in the second case, the test particle's trajectory is determined by the derivatives of these components as shown by Christofell's symbols. When comparing Newton's equations of motion with equations of motion (1), we conclude that $g_{\mu\nu}$ represents gravitational potential since the derivatives of potential occur in Newton's equations of motion.

In this case, the constant components of the metric tensor $g_{\mu\nu}$ in Euclidean space, represented by $\epsilon_{\mu\nu}$, are provided by

$$g_{\mu\nu} \to \epsilon_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Since

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= -dx^2 + dy^2 + dz^2 + c^2 dt^2$$

Now, let's assume that $g_{\mu\nu}$ are not constants but rather deviate by a negligible amount from the values provided by (3), i.e., in a weak static field

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$$

where $\eta_{\mu\nu}$ is a metric tensor for Galilean values and $\epsilon_{\mu\nu}$ is a function of x, y, z; but independent of time t.

$$\frac{d\eta_{\mu\nu}}{dx^4} = \frac{\partial g_{\mu\nu}}{\partial x^4} = 0$$

Then we obtain

$$\Gamma^\lambda_{\mu\nu} = g^{\rho\lambda} \Gamma_{\rho,\mu\nu}$$

$$= g^{\rho\lambda} \frac{1}{2} \left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right)$$

$$= \frac{1}{2} \left(\in^{\rho\lambda} + \eta^{\rho\lambda} \right) \left(\frac{\partial \eta_{\rho\mu}}{\partial x^\nu} + \frac{\partial \eta_{\rho\nu}}{\partial x^\mu} - \frac{\partial \eta_{\mu\nu}}{\partial x^\rho} \right)$$

Now neglecting second order terms in η, we get

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \left(\frac{\partial \eta_{\lambda\mu}}{\partial x^\nu} + \frac{\partial \eta_{\lambda\nu}}{\partial x^\mu} - \frac{\partial \eta_{\mu\nu}}{\partial x^\lambda} \right) \qquad \dots (2)$$

Now the Galilean Coordinates are

$$x^1 = x, x^2 = y, x^3 = z, x^4 = ct$$

$$ds^2 = -dx^2 + dy^2 + dz^2 + c^2 dt^2$$

$$-v^2dt^2 + c^2dt^2 = c^2dt^2\left(1 - \frac{v^2}{c^2}\right)$$

For the velocities $v \ll c$, then

$$ds \approx cdt = dx^4 \qquad \text{... (3)}$$

Since the field is static, i.e., it does not change with time. Consequently, velocity components might be interpreted as

$$\frac{dx^1}{ds}, \frac{dx^2}{ds}, \frac{dx^3}{ds} = 0 \ and \ \frac{dx^4}{ds} = 1 \qquad \text{... (4)}$$

By equation (1) given as

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{44}\left(\frac{dx^4}{ds}\right)^2 = 0$$

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{44} = 0$$

Using (2)above equation, we get

$$\frac{d^2x^\alpha}{ds^2} = -\Gamma^\lambda_{44} \approx -\frac{1}{2}\left(\frac{\partial \eta_{44}}{\partial x^\alpha}\right) \ for \ \alpha = 1,2,3$$

Using (2) the equation may be given as

$$\frac{d^2x^\alpha}{ds^2} = -\frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}c^2g_{44}\right)$$

Now the Newton's equations of motion are

$$\frac{d^2x^\alpha}{ds^2} = -\frac{\partial \phi}{\partial x^\alpha}$$

where ϕ is potential function.

From the above equations, we obtain

$$-\frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}c^2g_{44}\right) = -\frac{\partial \phi}{\partial x^\alpha}$$

Integrating, we obtain

$$\int \frac{\partial g_{44}}{\partial x^\alpha}dx^\alpha = \frac{2}{c^2}\int \frac{\partial \phi}{\partial x^\alpha}dx^\alpha + constant$$

$$g_{44} = \frac{2\phi}{c^2} + k$$

In flat space

$$g_{44} = 1, \phi = 0, so\ that\ \ k = 1$$

Then

$$g_{44} = 1 + \frac{2\phi}{c^2}$$

Therefore, in the case of a weak static field, geodesic equations may be reduced to Newton's equations of motion if

$$g_{44} = 1 + \frac{2\phi}{c^2}$$

11.6 *POISSON'S EQUATION AS AN APPROXIMATION OF GEODESIC EQUATIONS:-*

To prove that (Einstein's) field equations reduce in linear approximation to Newtonian equations (Poisson's equations)

$$\nabla^2\psi = -4\pi\rho \qquad \ldots (1)$$

Proof: Let us consider the motion of a test particle in a weak static field. A weak static field is

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu} \quad such\ that$$

Where $\eta_{\mu\nu}$ metric tensor is for Galilean line element and $\epsilon_{\mu\nu}$ is the function of x, y, z.

The deviation of the metric from unity is represented through $\epsilon_{\mu\nu}$. The quantities $\epsilon_{\mu\nu}$ are taken to be so small that the powers of $\epsilon_{\mu\nu}$ higher than the first are neglected. Here we obtain

$$\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{44} = -1, \eta_{\mu\nu} = 0 = g_{\mu\nu} for\ \mu \neq v$$

Since the field is static, i.e., it does not change with time. Consequently, velocity components might be interpreted as

$$\frac{dx^1}{ds}, \frac{dx^2}{ds}, \frac{dx^3}{ds} = 0\ and\ \frac{dx^4}{ds} = 1 \qquad \ldots (1)$$

Now the Galilean Coordinates are

$$x^1 = x, x^2 = y\ , x^3 = z, x^4 = ct$$

Now the geodesic equations are reduced to Newtonian equations of motion if

$$g_{44} = 1 + \frac{2\phi}{c^2} = 1 + 2\phi \ \ when\, c = 1 \qquad \ldots(2)$$

Each element of the energy tensor will be approximately equal to zero on its own, with the exception of

$$T_{44} = \rho \ \text{ so that } T = g^{\mu v} T_{\mu v} = g^{44} T_{44}$$

$$= \left(1 + \epsilon_{\mu v}\right)^{-1} \rho = \left(1 - \epsilon_{\mu v} + \cdots\right)\rho = \rho$$

$$\therefore \qquad T_{44} = \rho\ \ , T = \rho$$

Now the field equation is given by

$$R_{\mu v} - \frac{1}{2} R g_{\mu v} = -8\pi T_{\mu v}$$

From which we obtain

$$R_{\mu v} = -8\pi \left(T_{\mu v} - \frac{1}{2} g_{\mu v}\right)$$

$$R_{44} = -8\pi \left(T_{44} - \frac{1}{2} g_{44}\right) = -8\pi\rho \left(1 - \frac{1}{2} g_{44}\right)$$

$$= -8\pi\rho \left(1 - \tfrac{1}{2} \times 1\right) \text{ approximate}$$

$$R_{44} = -4\pi\rho$$

$$R^a_{\mu v\sigma} = -\frac{\partial \Gamma^a_{\mu v}}{\partial x^\sigma} + \frac{\partial \Gamma^a_{\mu\sigma}}{\partial x^v} - \Gamma^b_{\mu v}\Gamma^a_{b\sigma} + \Gamma^b_{\mu\sigma}\Gamma^a_{bv}$$

From above equation we have

$$R_{44} = R^a_{44a} = -\frac{\partial \Gamma^a_{44}}{\partial x^a} + \frac{\partial \Gamma^a_{4a}}{\partial x^4} - \Gamma^b_{44}\Gamma^a_{ba} + \Gamma^b_{4a}\Gamma^a_{b4}$$

Now we obtaining the first order approximation,

$$R_{44} = -\frac{\partial \Gamma_{44}^{a}}{\partial x^{a}} + \frac{\partial \Gamma_{4a}^{a}}{\partial x^{4}} = -= -\frac{\partial \Gamma_{44}^{a}}{\partial x^{\sigma}}$$

$$R_{44} = -\frac{\partial \Gamma_{44}^{a}}{\partial x^{a}} = 4\pi\rho \qquad \ldots (3)$$

But $\frac{\partial}{\partial x^4}\Gamma_{44}^4 = 0, since\ \frac{\partial g_{\mu v}}{\partial x^a} = 0,$

Hence

$$\frac{\partial \Gamma_{44}^{a}}{\partial x^{a}} = 4\pi\rho(a = 1,2,3) \qquad \ldots (4)$$

If $a = 1,2,3$, then

$$\Gamma_{4a}^{a} = g^{ab}\Gamma_{44,b} = g^{ab}\Gamma_{44,4} = \frac{1}{-1+\epsilon_{aa}}\left(-\frac{1}{2}\frac{\partial g_{44}}{\partial x^{a}}\right) \text{ as } g_{4a} = 0$$

$$= (1-\epsilon_{aa})^{-1}\frac{1}{2}\frac{\partial g_{44}}{\partial x^{a}}$$

$$= (1+\epsilon_{aa})\frac{1}{2}\frac{\partial g_{44}}{\partial x^{\sigma}} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^{a}}$$

From (4), we have

$$\frac{\partial}{\partial x^{a}}\left(-\frac{1}{2}\frac{\partial g_{44}}{\partial x^{a}}\right) = 4\pi\rho$$

$$\sum_{a=1}^{3}\frac{\partial^2 g_{44}}{\partial^2 x^{a}} = 8\pi\rho \ \ or\ \ \nabla^2 g_{44} = 8\pi\rho$$

$$\nabla^2(1+2\phi\ ) = 8\pi\rho\ by\ (3)$$

$$\nabla^2(2\phi\ ) = 8\pi\rho$$

$$\nabla^2(\phi\ ) = 4\pi\rho$$

This is Poisson's equation.

SELF CHECK QUESTIONS

1. What is the general form of Einstein's field equations (EFE)?
2. What does the Einstein tensor $G_{\mu v}$represent?
3. What role does the energy-momentum tensor $T_{\mu v}$ play in the field equations?

4. Why are Einstein's equations nonlinear?
5. What is the cosmological constant Λ in Einstein's equations?
6. How does Poisson's equation compare to the relativistic field equations?

11.7 *SUMMARY:-*

In this unit we have studied the Relativistic Field Equation, or Einstein Field Equations (EFE), describes gravity as the curvature of space-time caused by mass and energy, given by $G_{\mu v} = \frac{8\pi G}{c^4} T_{\mu v}$ where $G_{\mu v}$represents space-time curvature and $T_{\mu v}$ represents energy-momentum distribution. In the Newtonian limit, where gravity is weak and velocities are much smaller than the speed of light, EFE reduces to Newton's law of gravitation which describes gravity as force acting at a distance. Newton's theory can be further expressed through Poisson's equation,$\nabla^2(\phi \ ) = 4\pi\rho$

which relates the gravitational potential ϕ to mass density ρ. While Poisson's and Newton's equations are sufficient for classical physics, they fail in strong gravitational fields or relativistic conditions, where Einstein's equations are necessary.

11.8 *GLOSSARY:-*

- **Einstein Field Equations (EFE)** – A set of ten interrelated differential equations in General Relativity that describe how matter and energy influence space-time curvature.
- **Metric Tensor** $g_{\mu v}$– A mathematical function that defines the geometry of spacetime and determines distances and intervals in curved space-time.
- **Einstein Tensor** $g_{\mu v}$ – A tensor that represents the curvature of spacetime, given by $g_{\mu v} = R_{\mu v} - \frac{1}{2} R g_{\mu v}$
- **Ricci Tensor** $R_{\mu v}$ – A contraction of the Riemann curvature tensor that represents gravitational effects due to matter distribution.
- **Ricci Scalar (RRR)** – A scalar quantity obtained from the Ricci tensor, summarizing the curvature of spacetime.
- **Energy-Momentum Tensor** $T_{\mu v}$ – A tensor that represents the distribution of energy, momentum, and stress in spacetime, acting as the source of gravity in EFE.

- **Newtonian Limit** – The weak-field, slow-motion approximation of EFE, where they reduce to Newton's law of gravitation.
- **Cosmological Constant Λ**– A term introduced by Einstein to account for the expansion of the universe, modifying the field equations as $G_{\mu v} + \Lambda_{\mu v} = \frac{8\pi G}{c^4} T_{\mu v}$.
- **Weak Field Approximation** – The limit in which spacetime curvature is small, allowing EFE to be approximated by Poisson's equation $\nabla^2(\phi\ ) = 4\pi\rho$.
- **Geodesic Equation** – The equation describing the motion of a free-falling test particle in curved space-time, derived from the principle of least action in General Relativity.
- **Gravitational Waves** – Ripples in space-time predicted by EFE, generated by accelerating masses, such as merging black holes or neutron stars.
- **Schwarzschild Solution** – An exact solution of EFE that describes the space-time around a spherically symmetric, non-rotating massive object, leading to the concept of black holes.

- **Kerr Solution** – A solution to EFE describing the space-time around a rotating massive object, important for understanding astrophysical black holes.
- **Stress-Energy Conservation** – Expressed as $\nabla^2 T_{\mu\nu} = 0$, indicating the local conservation of energy and momentum in General Relativity.
- **Bianchi Identities** – Mathematical identities $\nabla^2 G_{\mu\nu} = 0$, that ensure the consistency of EFE with energy-momentum conservation

11.7 *REFERENCES:-*

- Ashok Das (2011), Lectures on Gravitation, University of Rochester, USA ,Saha Institute of Nuclear Physics, India.
- Richard Feynman (2018), Feynman Lectures On Gravitation.
- Neil Ashby, Stanley C. Miller (2019), principles of modern physics

11.8 *SUGGESTED READING:-*

- Farook Rahman (2021), The General Theory of Relativity: A Mathematical Approach

- Satya Prakash,Revised by K.P.Gupta, Ninteenth Edition (2019), Relativistic Mechanics.
- Dr. J.K.Goyal & Dr.K.P.Gupta (2018), Theory of Relaivity.

11.9 *TERMINAL QUESTIONS:*

(TQ-1) Define energy momentum tensor. Hence derive the formula for this tensor for a perfect fluid in the form

$$T_\mu^\nu = (\rho + p)v_\mu - g_\mu^\nu p$$

(TQ-2) Discuss the reason which led Einstein to choose field equations in the form

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi T_{ij}$$

(TQ-3) Show further that these field equations reduce under approximation to Poisson's equations

$$\nabla^2(\phi\ ) = 4\pi\rho$$

(TQ-4) In general relativity derive the expression for the energy momentum tensor T^{ij} for a perfect fluid distribution in the

$$T^{ij} = (\rho + p)v^i v^j - g^{ij} p$$

(TQ-5) Discuss the formulation of energy-momentum vector in special relativity.

(TQ-6) Define Material energy tensor. Show that in Galilean coordinates $T^{\mu\nu} = \rho \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$.

(TQ-7) Show that the divergence of the energy tensor vanishes and in the usual notation prove that $G\ =\ 8\pi T$.

(TQ-8) Derive the energy momentum tensor for a perfect fluid in the form.

$$T_\mu^\nu = (\rho + p)v^\mu v^\nu - g_\mu^\nu p$$

(TQ-9) Obtain Einstein's law of gravitation of the material world and deduce some of its consequences.

(TQ-10) Verify that the equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi T_{ij} \text{ and } T^{ij} = \rho_0 \frac{dx^i}{ds}\frac{dx^j}{ds}.$$

(TQ-11) To show that (Einstein's) field equations reduce in linear approximation to Newtonian equations (Poisson's equations)

(TQ-12) To prove that Geodesic equations reducible to Newtonian equations of motion in case of weak static field.

(TQ-13) To derive the motion of a free particle in case of weak static field.

(TQ-14) : Explain the construction of the energy momentum tensor T for matter composed of moving particles and show that the conditions of conservation of energy and momentum lead to the tensor equation $(T^{\mu v})_{,v} = 0$.

11.10 *ANSWERS:*

SELF CHECK ANSWERS

1. $G_{\mu v} = \frac{8\pi G}{c^4}$
2. $G_{\mu v} = R_{\mu v} - \frac{1}{2}Rg_{\mu v}$
3. The energy-momentum tensor $T_{\mu v}$ represents the distribution of energy, momentum, and stress in space-time. It acts as the source of the gravitational field.
4. Einstein's equations are nonlinear because the curvature of space-time (represented by the metric tensor $G_{\mu v}$ itself affects the distribution of energy and momentum, leading to a feedback loop.
5. The modified Einstein field equations with a cosmological constant are:

$$G_{\mu v} + \Lambda_{\mu v} = \frac{8\pi G}{c^4}T_{\mu v}$$

   where Λ the energy density of the vacuum, responsible for the accelerated expansion of the universe.

6. Poisson's equation is a weak-field, non-relativistic limit of Einstein's field equations. It describes gravity in the Newtonian framework, whereas Einstein's equations describe it in a fully relativistic context.

UNIT 12:-Schwarzschild Solution

CONTENTS:

12.1 *INTRODUCTION:-*

The **Schwarzschild solution** is one of the most important exact solutions to Einstein's field equations, describing the spacetime around a spherically symmetric, non-rotating, and uncharged massive object. It plays a crucial role in understanding gravitational phenomena, including planetary orbits, gravitational time dilation, and light bending due to gravity. The solution also predicts the existence of **black holes**, introducing the concept of the **Schwarzschild radius**, which defines the event horizon beyond which nothing can escape. In the weak-field limit, it reduces to **Newtonian gravity**, making it a bridge between classical and relativistic gravity. The Schwarzschild metric has been instrumental in verifying **General Relativity** through experiments such as the precession of Mercury's orbit and gravitational lensing, making it a cornerstone of modern gravitational physics.

12.2 *OBJECTIVES:-*

After studying this unit, Lernear's will be able to

- To solve the Einstein's law of gravitation in empty space.
- To solve the Schwarzschild exterior solution.
- To explain the Isotropic Coordinates
- To explain the Crucial tests of General Relativity.

12.3 *SCHWARZSCHILD'S EXTERIOR SOLUTION:-*

The law of gravitation in empty space is represented by Einstein's original field equations, which are

$$R_{\mu\nu} = 0 \qquad \dots(1)$$

However, Einstein's law of gravity in empty space is altered as follows if the cosmological constant Λ is included

$$R_{\mu\nu} = \alpha g_{\mu\nu} \qquad \dots(2)$$

Finding the line element for the interval in empty space around a gravitating point particle, which eventually corresponds to the field of an isolated particle continuously at rest at the origin, is all that is required to solve the aforementioned equations. Schwarzschild was the first to obtain this solution.

In the absence of mass point, space time would be flat, so that the line element in spherical polar coordinates to be written as

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + dt^2 \qquad \dots(3)$$

The line element would change if the mass point were present. However, the line element would be spherically symmetric about the point mass and is static since mass is isolated and static. One way to describe such a line element in its most generic form is as

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + e^{\nu} dt^2 \qquad \dots(4)$$

Here λ and ν are functions of r only; since for spherically symmetric isolated particle the field will depend on r alone and not on θ and ϕ.

At an infinite distance from the particle, the line element (4) must limit to the Galilean line element (3) because the gravitational field (i.e., the disruption from flat-space time) caused by the particle diminishes indefinitely. Hence λ and ν must tend to zero as tends to infinity.

The line element in general relativity is obtained by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \dots (5)$$

Here the coordinates are

$$x^1 = r, x^2 = \theta\,, x^3 = \phi, x^4 = t \qquad \dots (6)$$

Comparing (4) and (5) with the help of (6), we obtain

$$g_{\mu\nu} = \begin{bmatrix} -e^\lambda & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 sin^2\theta & 0 \\ 0 & 0 & 0 & e^\nu \end{bmatrix} \qquad \dots (7)$$

Then the $g_{\mu\nu}$ is

$$g = \left| g_{\mu\nu} \right| = e^\lambda(-r^2).(-r^2)(-r^2 sin^2\theta).e^\nu = -e^{\lambda+\nu} r^4 sin^2\theta$$

Using

$g_{\mu\nu} = \frac{cofactor\ of\ \ g_{\mu\nu}\ in\ g}{g}$, we obtain

$$g_{\mu\nu} = \begin{bmatrix} -e^\lambda & 0 & 0 & 0 \\ 0 & \left(\frac{1}{r^2}\right) & 0 & 0 \\ 0 & 0 & \left(\frac{1}{r^2 sin^2\theta}\right) & 0 \\ 0 & 0 & 0 & e^{-\nu} \end{bmatrix} \qquad \dots (8)$$

If μ, ν, σ ae different suffixes, then we have

$$\left.\begin{aligned} \Gamma^\mu_{\mu\mu} &= \frac{1}{2} g^{\mu\mu} \frac{\partial\, g_{\mu\mu}}{\partial x^\mu} = \frac{1}{2} \frac{\partial(log\ g_{\mu\mu})}{\partial x^\mu} \\ \Gamma^\nu_{\mu\mu} &= \frac{1}{2} g^{\nu\nu} \frac{\partial\, g_{\mu\mu}}{\partial x^\nu} \\ \Gamma^\nu_{\mu\nu} &= \frac{1}{2} g^{\nu\nu} \frac{\partial\, g_{\nu\nu}}{\partial x^\mu} = \frac{1}{2} \frac{\partial(log\ g_{\nu\nu})}{\partial x^\mu} \\ \Gamma^\sigma_{\mu\alpha} &= 0 \end{aligned}\right\} \qquad \dots (9)$$

Now we obtain the following nine independent non-vanishing 3-index symbols, all others being zero.

$$\left.\begin{array}{c}\Gamma_{11}^{1}=\frac{1}{2}\frac{\partial\lambda}{\partial r};\Gamma_{12}^{2}=\frac{1}{r};\Gamma_{13}^{3}=\frac{1}{r};\\ \Gamma_{14}^{4}=\frac{1}{2}\frac{\partial\nu}{\partial r};\Gamma_{23}^{3}=cot\theta;\Gamma_{22}^{1}=-re^{-\lambda};\\ \Gamma_{33}^{1}=-rsin^{2}\theta e^{-\lambda};\Gamma_{14}^{4}=\frac{1}{2}e^{\nu-\lambda}\frac{\partial\nu}{\partial r};\Gamma_{33}^{2}=-sin\theta cos\theta\\ \Gamma_{\mu\alpha}^{\sigma}=0\end{array}\right\} \quad \ldots(10)$$

We obtain

$$R_{\mu\nu}=\frac{\partial}{\partial x^{\nu}}\Gamma_{\mu\beta}^{\beta}-\frac{\partial}{\partial x^{\beta}}\Gamma_{\mu\nu}^{\beta}+\Gamma_{\mu\beta}^{\alpha}\Gamma_{\alpha\nu}^{\beta}-\Gamma_{\mu\nu}^{\alpha}\Gamma_{\alpha\beta}^{\beta}$$

$$=\frac{\partial^{2}log\sqrt{(|g|)}}{\partial x^{\mu}\partial x^{\nu}}\Gamma_{\mu\beta}^{\beta}-\frac{\partial}{\partial x^{\beta}}\Gamma_{\mu\nu}^{\beta}+\Gamma_{\mu\beta}^{\alpha}\Gamma_{\alpha\nu}^{\beta}-\Gamma_{\mu\nu}^{\alpha}\frac{\partial log\sqrt{(|g|)}}{\partial x^{\alpha}} \quad \ldots(11)$$

$$R_{11}=\frac{\partial^{2}log\sqrt{(|g|)}}{\partial x^{1}\partial x^{1}}\Gamma_{\mu\beta}^{\beta}-\frac{\partial}{\partial x^{\beta}}\Gamma_{11}^{\beta}+\Gamma_{1\beta}^{\alpha}\Gamma_{\alpha1}^{\beta}-\Gamma_{11}^{\alpha}\frac{\partial log\sqrt{(|g|)}}{\partial x^{\alpha}}$$

$$=\frac{\partial^{2}log\sqrt{(|g|)}}{\partial r^{2}}\Gamma_{\mu\beta}^{\beta}-\frac{\partial}{\partial r}\Gamma_{11}^{1}+\Gamma_{11}^{1}\Gamma_{11}^{1}++\Gamma_{12}^{2}\Gamma_{21}^{2}+\Gamma_{13}^{3}\Gamma_{31}^{3}+\Gamma_{14}^{4}\Gamma_{41}^{4}-\Gamma_{11}^{1}\frac{\partial log\sqrt{(|g|)}}{\partial r}$$

As$|g|=e^{\lambda+\nu}r^{4}sin^{2}\theta,\qquad i.e.,\sqrt{|g|}=e^{\frac{\lambda+\nu}{2}}r^{2}sin\theta,\ \ therefore$

$$R_{11}=\frac{\partial^{2}}{\partial r^{2}}\left(\frac{\lambda+\nu}{2}+2logr+logsin\theta\right)-\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial\lambda}{\partial r}\right)+\left(\frac{1}{2}\frac{\partial\lambda}{\partial r}\right)^{2}+\frac{1}{r^{2}}+\left(\frac{1}{r}\right)^{2}+\left(\frac{1}{2}\frac{\partial\nu}{\partial r}\right)^{2}-\frac{1}{2}\frac{\partial\lambda}{\partial r}\frac{\partial}{\partial r}\left(\frac{\lambda+\nu}{2}+2logr+logsin\theta\right)$$

$$R_{11}=\frac{\partial^{2}\nu}{\partial r^{2}}+\frac{1}{4}\left(\frac{\partial\nu}{\partial r}\right)^{2}-\frac{1}{4}\frac{\partial\lambda}{\partial r}\frac{\partial\nu}{\partial r}-\frac{1}{r}\frac{\partial\lambda}{\partial r}$$

$$R_{11}=\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^{2}}{4}-\frac{\lambda'}{r}$$

Similarly

$$R_{22}=e^{-\lambda}\left(1+\frac{1}{2}r\frac{\partial\nu}{\partial r}-\frac{1}{2}r\frac{\partial\lambda}{\partial r}\right)-1=e^{-\lambda}\left[1-r\left(\frac{\lambda'-\nu'}{2}\right)\right]-1$$

$$R_{33}=\left\{e^{-\lambda}\left(1+\frac{1}{2}r\frac{\partial v}{\partial r}-\frac{1}{2}r\frac{\partial \lambda}{\partial r}\right)-1\right\}sin^2\theta=R_{22}sin^2\theta$$

$$R_{44}=-\frac{1}{2}e^{\nu-\lambda}\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{2}\left(\frac{\partial v}{\partial r}\right)^2-\frac{1}{2}\frac{\partial \lambda}{\partial r}\frac{\partial v}{\partial r}+\frac{2}{r}\frac{\partial v}{\partial r}\right\}$$
$$=e^{\nu-\lambda}\left[-\frac{v''}{2}+\frac{\lambda' v'}{4}-\frac{v'^2}{4}-\frac{v'}{r}\right]$$

Additionally, for the line element mentioned above, all of $R_{\mu\nu}$off diagonal components are zero.

Hence

$$R_{\mu\nu}=0$$

$$R_{11}=\frac{\partial^2 v}{\partial r^2}+\frac{1}{4}\left(\frac{\partial v}{\partial r}\right)^2-\frac{1}{4}\frac{\partial \lambda}{\partial r}\frac{\partial v}{\partial r}-\frac{1}{r}\frac{\partial \lambda}{\partial r}=0 \quad \ldots(12)$$

$$e^{-\lambda}\left(1+\frac{1}{2}r\frac{\partial v}{\partial r}-\frac{1}{2}r\frac{\partial \lambda}{\partial r}\right)-1=0 \quad \ldots(13)$$

$$R_{33}=R_{22}sin^2\theta=0 \quad \ldots(14)$$

$$R_{44}=-\frac{1}{2}e^{\nu-\lambda}\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{2}\left(\frac{\partial v}{\partial r}\right)^2-\frac{1}{2}\frac{\partial \lambda}{\partial r}\frac{\partial v}{\partial r}+\frac{2}{r}\frac{\partial v}{\partial r}\right\}=0 \quad \ldots(14)$$

Therefore, the only Einstein's field equations that λ and v may satisfy for empty space are

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{4}\left(\frac{\partial v}{\partial r}\right)^2-\frac{1}{4}\frac{\partial \lambda}{\partial r}\frac{\partial v}{\partial r}-\frac{1}{r}\frac{\partial \lambda}{\partial r}=0 \quad \ldots(15)$$

$$e^{-\lambda}\left(1+\frac{1}{2}r\frac{\partial v}{\partial r}-\frac{1}{2}r\frac{\partial \lambda}{\partial r}\right)-1=0 \quad \ldots(16)$$

$$\frac{1}{2}e^{\nu-\lambda}\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{2}\left(\frac{\partial v}{\partial r}\right)^2-\frac{1}{2}\frac{\partial \lambda}{\partial r}\frac{\partial v}{\partial r}+\frac{2}{r}\frac{\partial v}{\partial r}\right\}=0 \ldots(17)$$

Now dividing (17) by $e^{\nu-\lambda}$ and then subtracting (16), we obtain

$$\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial \lambda}{\partial r}=0$$

$$\frac{\partial v}{\partial r}+\frac{\partial \lambda}{\partial r}=0$$

Integrating, we have

$$v + \lambda = A$$

where A is the integrating constant, and since $r = \infty, \lambda = 0$ and $v = 0$, it is possible to adjust it to 0 without losing generality. Therefore

$$\lambda = -v \qquad \text{... (18)}$$

$$e^{v}\left(1 + r\,\frac{\partial v}{\partial \lambda}\right) = 1$$

$$\frac{\partial}{\partial r}(re^{v}) = 1$$

Integrating, we get

$$re^{v} = r + B$$

B being constant of integration

$$e^{v} = -e^{-\lambda} = 1 - \frac{2m}{r} \qquad \text{... (19)}$$

where we have substitute $B = 2m$. Hence, from equation (4), the line element resulting from a static, isolated gravitating mass point is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2 \qquad \text{... (20)}$$

This solution is called to as the Schwarzschild line element because it was first identified by Schwarzschild. In the limit $r \to \infty$,, the Schwarzschild line element clearly reduces to the line element of special relativity's flat space time.

Schwarzschild singularity: The singularities of the Schwarzschild solution are observed to be as follows:

1) At $r = 0$, the Schwarzschild solution becomes singular, but Newton's (classical) theory also occurs this singularity.

2) When the distance r is provided by $1 - 2m = 0$, that is, $r = 2m$, the Schwarzschild solution once more becomes singular.This value of r is called Schwarzschild radius. For points $0 \le r \le 2m, ds^2 < 0$, i.e., the interval is purely space-like. Hence there is a finite singular region for $0 \le r \le 2m$. Thus $r = 2m$ represents the boundary of the isolated particle and the solution holds in empty space outside the spherical distribution of matter (or isolated particle) whose radius must be greater than $2m$. Hence egn. (20) is known as the Schwarzschild exterior solution for the gravitational field of an isolated particle.

Schwarzschild solution (21) corresponds to Einstein's original field equations for empty space

$$R_{\mu v} = 0$$

But when the cosmological constant Λ is taken into account, the Schwarzschild's solution for empty space that corresponds to field equations

$$R_{\mu v} = \Lambda g_{\mu v}$$

Leads to the line element or metric is

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 \qquad \dots (21)$$

This is required Birkhoff's Solution.

By comparing the line elements provided by equations (20) and (21) we can observe that the larger the region under consideration, the greater the effect of the Λ term on the field surrounding an attractive point particle. However, the cosmological constant Λ is so minuscule that, even if it deviates from zero, it has no discernible impact inside an area the size of the solar system.

For empty world, we set $m = 0$, we get

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)^{-\lambda} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1 - \frac{\Lambda r^2}{3}\right) dt^2 \qquad \dots (22)$$

This is known as **Schwarzschild exterior solution** for entirely empty world.This solution has a singularity at $r = \sqrt{\frac{3}{\Lambda}}$ because r is very large and the cosmological constant Λ is very small. It represents the It expresses the horizon of the world.

12.4 *RELATION BETWEEN M AND m:-*

The Schwarzschild exterior solution for gravitational field of an isolated particle is given as below

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2 \qquad \dots(1)$$

Suppose $r \gg 2m$. The field of distance r due to an attracting Mass M is below

$$g_{44} = 1 + \frac{2\phi}{c^2}$$

Where ϕ is Newtonian potential, i.e.,

$$\frac{2\phi}{c^2} = g_{44} - 1 = \left(1 - \frac{2m}{r}\right) - 1 = -\frac{2m}{r}$$

$$\phi = -\frac{2mc^2}{r^2} \qquad \dots(2)$$

If M is the mass of the particle and G the gravitational constant, then

$$\frac{\partial\phi}{\partial r} = \frac{GM}{r^2}$$

Putting the value of (2) in above equation, we get

$$\frac{mc^2}{r} = \frac{GM}{r^2}$$

$$m = \frac{GM}{r^2}$$

This is the relationship between Schwarzchild’s solution's constant m and the attracting mass M.

12.5 *ISOTROPIC COORDINATES:-*

Schwarzchild's exterior solution is obtained by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2 \qquad \dots (1)$$

Let the transformation

$$r = \left(1 + \frac{m}{2r_1}\right)^2 r_1 \qquad \dots (2)$$

So that

$$dr = \left(1 - \frac{m}{2r_1}\right)^2 dr_1$$

$$dr = \left(1 - \frac{m^2}{4{r_1}^2}\right)^2 dr_1$$

$$dr^2 = \left(1 - \frac{m}{2r_1}\right)^2 \left(1 + \frac{m}{2r_1}\right)^2 d{r_1}^2$$

$$\left(1 - \frac{m}{2r_1}\right) = 1 - \frac{2m}{\left(1 + \frac{m}{2r_1}\right)^2 r_1}$$

$$= \frac{\left(1 - \frac{m}{2r_1}\right)^2}{\left(1 + \frac{m}{2r_1}\right)^2}$$

Putting the above values in equation (1), we obtain

$$ds^2 = \frac{\left(1+\frac{m}{2r_1}\right)^2}{\left(1-\frac{m}{2r_1}\right)^2}\left(1 - \frac{m}{2r_1}\right)^2 \left(1 + \frac{m}{2r_1}\right)^2 d{r_1}^2 - \left(1 + \frac{m}{2r_1}\right)^4 {r_1}^2 d\theta^2 - \left(1 + \frac{m}{2r_1}\right)^4 {r_1}^2 sin^2\theta d\phi^2 + \frac{\left(1+\frac{m}{2r_1}\right)^2}{\left(1-\frac{m}{2r_1}\right)^2} dt^2$$

$$ds^2 = \left(1 + \frac{m}{2r_1}\right)^4 \left(d{r_1}^2 + {r_1}^2 d\theta^2 + {r_1}^2 sin^2\theta d\phi^2\right) + \frac{\left(1+\frac{m}{2r_1}\right)^2}{\left(1-\frac{m}{2r_1}\right)^2} dr^2 \qquad \dots (3)$$

This is known as isotropic line element. The coordinates r_1, θ, ϕ are called isotropic polar coordinates.

On applying the transformation

$$x = r_1 sin\theta cos\phi, \quad y = r_1 sin\theta sin\phi, \quad z = r_1 cos\theta$$

The line element (3) becomes

$$ds^2 = \left(1 + \frac{m}{2r_1}\right)^4 (dx^2 + dy^2 + dz^2) + \frac{\left(1 + \frac{m}{2r_1}\right)^2}{\left(1 - \frac{m}{2r_1}\right)^2} dr^2 \quad \ldots (4)$$

This is known as isotropic line element in Cartesian Coordinates.

12.6 *PLANETARY ORBITS:-*

We shall now consider the motion of the planets in the gravitational field of the sun. The planets' space-time trajectories, when considered as free particles, are determined by geodesic equations.

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \quad \ldots (1)$$

Since the sun is an attractive point particle, its gravitational field may be considered as the field of a single particle that is always at rest at the origin. As a result, the Schwarzschild's line element for empty space provides the space-time, i.e.

$$ds^2 = -e^\lambda dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + e^\nu dt^2 \quad \ldots (2)$$

Where

$$\lambda = -\nu, e^\nu = -e^{-\lambda} = 1 - \frac{2m}{r}$$

Now the Christoffel's Symbols are

$$\Gamma^1_{11} = \frac{\lambda'}{2}, \Gamma^1_{22} = -re^{-\lambda}, \Gamma^1_{44} = \frac{\nu'}{2} e^{\nu-\lambda}$$

$$\Gamma^2_{12} = \frac{1}{r}, \Gamma^3_{23} = cot\theta$$

$$\Gamma_{13}^{3} = \frac{1}{r}, \Gamma_{33}^{1} = -rsin^2\theta e^{-\lambda}$$

$$\Gamma_{14}^{4} = \frac{\nu'}{2}, \Gamma_{33}^{2} = -sin\theta cos\theta$$

Our coordinate are $x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t$. Taking $\alpha = 1$, we get

$$\frac{d^2x^1}{ds^2} + \Gamma_{\mu\nu}^{1}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

$$\frac{d^2x^1}{ds^2} + \Gamma_{12}^{1}\left(\frac{d^2x^1}{ds^2}\right)^2 + \Gamma_{22}^{1}\left(\frac{d^2x^2}{ds^2}\right)^2 + \Gamma_{33}^{1}\left(\frac{d^2x^3}{ds^2}\right)^2 + \Gamma_{44}^{1}\left(\frac{d^2x^4}{ds^2}\right)^2 = 0$$

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{\partial\lambda}{\partial r}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2 - rsin^2\theta e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 - \frac{1}{2}e^{\nu-\lambda}\frac{\partial\nu}{\partial r}\left(\frac{dt}{ds}\right)^2 = 0 \qquad \ldots(3)$$

For $\alpha = 2$, we obtain

$$\frac{d^2x^2}{ds^2} + \Gamma_{\mu\nu}^{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

$$\frac{d^2x^1}{ds^2} + \Gamma_{12}^{2}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma_{21}^{2}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma_{33}^{2}\left(\frac{dx^3}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{1}{r}\frac{dr}{ds}\frac{d\theta}{ds} + \frac{1}{r}\frac{d\theta}{ds}\frac{dr}{ds} + (-sin\theta cos\theta)\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - sin\theta cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \qquad \ldots(4)$$

Similarly for $= 3$ & 4 , we get

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \qquad \ldots(5)$$

$$\frac{d^2t}{ds^2} + \frac{\partial\nu}{\partial r}\frac{dr}{dx}.\frac{dt}{ds} = 0 \qquad \ldots(6)$$

Hence the equations (3),(4), (5) and (6) are the motion of planet.

The planet moves initially on a plane $\theta = \frac{\pi}{2}$, thus let's use the coordinate system so that

$$\cos\theta = 0, \sin\theta = 1, \frac{d\theta}{ds} = 0$$

Then from (4) given

$$\frac{d^2\theta}{ds^2} = 0 \qquad \dots (7)$$

According to this equation, the planet continues to move in the plane $\theta = \frac{\pi}{2}$. Consequently, we always have

$$\cos\theta = 0, \sin\theta = 1 \; and \; \frac{d\theta}{ds} = 0 \qquad \dots (8)$$

So that the equations (3), (5) and (6) become

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{\partial\lambda}{\partial r}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 + \frac{1}{2}e^{\nu-\lambda}\frac{\partial\nu}{\partial r}\left(\frac{dt}{ds}\right)^2 = 0 \qquad \dots (9)$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 \qquad \dots (10)$$

$$\frac{d^2t}{ds^2} + \frac{\partial\nu}{\partial r}\frac{dr}{dx}.\frac{dt}{ds} = 0 \qquad \dots (11)$$

From the equation (10) and (11) may be obtained as

$$\frac{1}{r^2}\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0, \; i.e., \qquad \frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0$$

$$\frac{1}{e^\nu}\frac{d}{ds}\left(e^\nu\frac{dt}{ds}\right) = 0, \; i.e., \qquad \frac{d}{ds}\left(e^\nu\frac{dt}{ds}\right) = 0$$

The above equations' integration instantly produces

$$\left.\begin{aligned} r^2\frac{d\phi}{ds} &= h \\ e^\nu\frac{dt}{ds} &= k \end{aligned}\right\} \qquad \dots (12)$$

where h and k are integration constants. The motion's angular momentum is measured by the constant h. Additionally, because integrating equation (9) is difficult, we utilize the line element (2) instead, which, when combined with equation (8), yields

$$-e^{\lambda}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 + e^{\nu}\left(\frac{dt}{ds}\right)^2 = 1$$

$$e^{\lambda}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - e^{\nu}\left(\frac{dt}{ds}\right)^2 + 1 = 0 \quad \ldots(13)$$

Now using (12), we get

$$-e^{-\nu}\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2} - \frac{k^2}{e^{\nu}} + 1 = 0$$

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}e^{\nu} - k^2 + e^{\nu} = 0 \quad \ldots(14)$$

We obtain

$$\frac{dr}{ds} = \frac{dr}{d\phi}\frac{d\phi}{ds} = \frac{h^2}{r^2}\frac{dr}{d\phi} \quad using\ (12)$$

and $\quad e^{\nu} = 1 - \frac{2m}{r} \quad from\ (3)$

Consequently equation (14) becomes

$$\left(\frac{h^2}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{2m}{r}\right) - k^2 + \left(1 - \frac{2m}{r}\right) = 0$$

Now substituting $u = \frac{1}{r}$ and rearranging, we obtain

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{k^2-1}{h^2} + \frac{2mu}{h^2} + 2mu^3 \quad \ldots(15)$$

Differentiating (15) w.r.t. ϕ, we obtain

$$2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} = \frac{2m}{h^2}\frac{du}{d\phi} + 6mu^2\frac{du}{d\phi}$$

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 \quad \ldots(16)$$

$$r^2\frac{d\phi}{ds} = h$$

The relativistic differential equation for the planet's trajectory is represented by equation (16). Here, ds is a component of the proper time as determined by a clock that moves with the planet, and r and ϕ are the special coordinates.

One can compare the relativistic equation (16) of the planet's orbit with the corresponding Newtonian equation, which is

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2}$$

with

$$r^2\frac{d\phi}{dt} = h$$

It is clear that the Newtonian equation of the planet's orbit has an additional component ($3mu^2$) due to the relativistic effects of gravity, and the proper time element ds takes the place of the time element dr. The additional term's ratio $3mu^2$ to $\frac{m}{h^2}$ is

$$\frac{3mu^2}{\left(\frac{m}{h^2}\right)} = 3h^2u^2 = 3\left(r\frac{d\phi}{ds}\right)^2$$

which is practically three times the square of the transverse velocity of the planet in relativistic units.

In the terms of speed of light

$$3h^2u^2 = 3.\left(\frac{r.\frac{d\phi}{ds}}{c}\right)^2 = 3\left(\frac{transverse\ velocity\ of\ planat}{velocity\ of\ light}\right)^2$$

The calculated value of this ratio for the earth is 3×10^{-8} at normal speed, it is insignificant. In real-world applications, this ratio indicates a negligible adjustment to Newtonian orbit.

EXAMPLE: Explain the statement that the mass of the sun which is 1.99×10^{33} gms. becomes in gravitational units 1.47 kilometers.

SOLUTION: It is given that

M Mass of the sun = 1.99 × 10^{33} gms.

To prove that mass of the sun gravitational unit. 1.47 kilometres in

We know that

$$mc^2 = \gamma M$$

$$m = \frac{\gamma M}{c^2}$$

where c = velocity of light = $3 \times \frac{10^{10} cm}{sec}$.

y = Gravitational constant 6.66×10^{-8}C.G.S. unit.

$$m = \frac{\gamma M}{c^2} = \frac{6.66 \times 10^{-8} \times 1.99 \times 10^{33}}{(3 \times 10^{10})^2}$$

$$= \frac{13.2539 \times 10^5}{9} = 1.4694 \times 10^5 cms = 1.4694\ kilometers$$

$$= 1.47\ kilometers(app)$$

12.7*CRUCIAL TEST IN RELATIVITY:-*

In relativity, the following are referred to as important tests.

(i) Advance of perihelion.
(ii) Gravitational deflection of light.
(iii) Shift in spectral lines.

(i) Advance of perihelion: To discuss the advance vanced sun, comparing the the perihelion of a planet's orbit around the sun, relativistic equations with those of classical mechanics.

Proof: The differential equation of the path of a planet is

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 \qquad ...(1)$$

with

$$r^2 \frac{d\phi}{ds} = h$$

Neglecting the small term $3mu^2$ as a first approximation, then we obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2}$$

So the solution is

$$u = \frac{m}{h^2}[1 + ecos(\phi - \omega)] \qquad ...(2)$$

where e and ω are integration constants that provide eccentricity and longitude of the perihelion. Applying this initial estimate to (1)'s R.H.S., we get

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + \frac{m^3}{h^4}[1 + ecos(\phi - \omega)]^2$$

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + \frac{3m^3}{h^4} + \frac{6m^3e}{h^4}cos(\phi - \omega) + cos^2(\phi - \omega)\frac{3m^3e^2}{h^4}$$

One of the other terms is the only one that can have an impact inside the term's observational range

$$\frac{6m^3e}{h^4}cos(\phi - \omega).$$

The P.I of the terms is

$$\frac{1}{1 + D^2}\frac{6m^3e}{h^4}cos(\phi - \omega) = \frac{6m^3e}{h^4}.\frac{1}{1 + D^2}.cos(\phi - \omega)$$

$$= \frac{6m^3e}{h^4}.\frac{\phi}{2}.sin(\phi - \omega) = \frac{3m^3e}{h^4}\phi sin(\phi - \omega)$$

[Here $\frac{1}{1+D^2}\ cosx = \frac{x}{2}sinx$]

Hence the solution of (1) to the second order of approximation is

$$u = \frac{m}{h^2}[1 + ecos(\phi - \omega)] + \frac{3m^3e}{h^4}\phi sin(\phi - \omega)\ \ by\ (2)$$

$$= \frac{m}{h^2} + \frac{me}{h^2}\left[cos(\phi - \omega) + \frac{3m^2}{h^2}\phi sin(\phi - \omega)\right]$$

Now taking $\frac{3m^2}{h^2}\phi = \delta\omega$ and observing $sin\delta\omega, cos\delta\omega = 1$. Since $\delta\omega$ is very small so

$$u = \frac{m}{h^2} + \frac{me}{h^2}[cos\delta\omega\ cos(\phi - \omega) + sin\delta\omega\ sin(\phi - \omega)]$$

$$u = \frac{m}{h^2} + \frac{me}{h^2}[\ cos(\phi - \omega - \delta\omega)]$$

With $\frac{3m^2}{h^2}\phi = \delta\omega$

This is the required solution of (1). A planet's perihelion advances a fraction of a revolution when it completes one orbit around the sun, which is equivalent to

$$\frac{\delta\omega}{\phi} = \frac{3m^2}{h^2} = \frac{3m^2}{ml} = \frac{3m^2}{ma(1-e^2)} = \frac{3m}{a(1-e^2)} \quad for\ l = \frac{b^2}{a}$$

$i.e.,$

$$\frac{\delta\omega}{\phi} = \frac{3m}{a(1-e^2)} \qquad \dots (3)$$

$[on\ using\ the\ well\ known\ formulah^2 = ml]$

From (3), we get

$$\delta\omega = \frac{3m\phi}{a(1-e^2)}$$

By Kepler's third law,

$$T = \frac{2\pi}{\sqrt{m}}a^{3/2}$$

From which $\quad m = \frac{4\pi^2 a^3}{T^2}$

Now

$$\delta\omega = \frac{3m\phi}{a(1-e^2)} = \frac{3\phi}{a(1-e^2)}\frac{4\pi^2 a^3}{T^2}$$

$$\delta\omega = \frac{12\pi^2 a^2\phi}{T^2(1-e^2)}$$

The velocity of the light into the consideration,

$$\delta\omega = \frac{12\pi^2 a^2\phi}{c^2T^2(1-e^2)}$$

Taking

$$\phi = 2\pi, \delta\omega = \frac{24\pi^3 a^2}{c^2 T^2 (1 - e^2)}$$

T being a time period.

Thus the relativistic theory leads to an advance of perihelion of a planetary orbit. In other words, this theory leads to planetary orbit with a slow rotation of perihelion instead of to be perfectly closed elliptical orbits of the Newtonian theory.

When analyzed mathematically, the perihelion advance of all planets is negligibly minor, with the exception of Mercury, whose $e = 0.2056, a = 0.6 \times 10^8\, km, c = 3 \times 10^8\, m/sec,$ and $T = 88$ days. This means that the perihelion advance of Mercury is 43 seconds of arc each century. This is the precise amount that has been scientifically measured for the orbit of Mercury; neither the Newtonian theory nor the special relativistic theory of gravity could account for this precession. The development of the perihelions of Venus and Earth has also been isolated from the influence of other perturbing agents in recent years with the use of electronic computers, and it has been observed that the theoretical formula (10) also coincides with the experimental data in these conditions. The general theory of relativity can thus be experimentally tested by the advancement of planet perihelions.

(ii) **Gravitational deflection of light(Binding of light rays)**: To show that the deflection in the path of light due to the relativistic field of a heavy mass is wice that predicted by the Newtonian theory.

OR

Assuming Schwarzchild’s solution for a particle, show that the relativistic deflection of light in the gravitational field of the sun, as observed by a terrestrial observer, is twice the corresponding Newtonian effect.

Proof: Suppose the binding of light rays in the gravitational field of gravitating mass m is written by

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 \qquad \dots(1)$$

with

$$r^2 \frac{d\phi}{ds} = h \qquad \dots(2)$$

If $ds = 0$. In (2), then we obtain $h = \infty$. Substituting $h = \infty$. in (1), we given

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \qquad \dots (3)$$

Now neglecting the small term $3mu^2$ as a first approximation, we get

$$\frac{d^2u}{d\phi^2} + u = 0$$

The solution of this is

$$u = Acos\phi + Bsin\phi \qquad \dots (4)$$

Since $\phi = 0, \frac{du}{d\phi} = 0$ and $\phi = 0, u = \frac{1}{R}$, putting these condition in above equation

$$\frac{1}{R} = A + B.0 = A$$

$$0 = \frac{du}{d\phi} = -Asin\phi + B\,cos\phi = -A.0 + B.1 = B$$

$$A = \frac{1}{R}, B = 0$$

Putting these values in (4), we obtain

$$u = \frac{1}{R}cos\phi$$

From (3), we get

$$\frac{d^2u}{d\phi^2} + u = \frac{3m}{R^2}cos^2\phi$$

The particular integral of $\frac{3m}{R^2}cos^2\phi$ is

$$\frac{1}{1+D^2}\,\frac{3m}{R^2}cos^2\phi = \frac{3m}{R^2}.\frac{1}{1+D^2}.\left(\frac{1+cos2\phi}{2}\right) = \frac{m}{2R^2}(3 - cos2\phi)$$

$$= \frac{m}{2R^2}(3cos^2\phi + 3sin^2\phi - cos^2\phi + sin^2\phi) = \frac{m}{R^2}(cos^2\phi + 2sin^2\phi)$$

$$= \frac{m}{r^2 R^2}(r^2 cos^2\phi + 2r^2 sin^2\phi)$$

Hence the complete solution of (3) to the second approximation is

$$\frac{1}{r} = u = \frac{1}{R}cos\phi + \frac{m}{r^2 R^2}(r^2 cos^2\phi + 2r^2 sin^2\phi)$$

Multiplying by rR

$$R = rcos\phi + \frac{m}{rR}(r^2 cos^2\phi + 2r^2 sin^2\phi)$$

Introducing the Cartesian coordinates are $x = rcos\phi, y = rsin\phi$

We obtain

$$R = x + \frac{m(x^2 + 2y^2)}{R\sqrt{x^2 + y^2}}$$

$$x = R - \frac{m(x^2 + 2y^2)}{R\sqrt{x^2 + y^2}} \qquad \ldots(5)$$

Now the first approximation is

$$\frac{1}{r} = u = \frac{1}{R}cos\phi \; or \; R = r\, cos\phi \;\; or \; x = R \qquad \ldots(6)$$

From (5) and (6), we obtain the second term = $\frac{m(x^2+2y^2)}{R\sqrt{x^2+y^2}}$ in (5).

Asymptotes to (5) are given by taking y very large compared to so that asymptotes to (5) are

$$x = R - \frac{m}{R}(\pm 2y)$$

$$x = R + \frac{2my}{R} \text{ and } x = R - \frac{2my}{R}$$

$$y = \frac{Rx}{2m} - \frac{R^2}{2m} \text{ and } y = -\frac{Rx}{2m} + \frac{R^2}{2m}$$

Let α be the angle between these asymptotes then we obtain

$$tan\alpha = \frac{\frac{R}{2m} - \left(\frac{R}{2m}\right)}{1 + \frac{R}{2m} \times \left(-\frac{R}{2m}\right)} = \frac{4mR}{4m^2 - R^2}$$

$$tan\alpha = \frac{4mR}{4m^2 - R^2}$$

Then

$$sin\alpha = \frac{4mR}{4m^2 + R^2}$$

Since $4m^2 << R^2$ and hence neglected

$$sin\alpha = \frac{4mR}{R^2} = \frac{4m}{R}$$

$$\alpha = \frac{4m}{R} = \frac{4 \times 1.47}{697000} = 1.75\ seconds$$

$$[\because sin\alpha = \alpha]$$

Deflection =1.75 seconds.

Treatment of Newtonian theory: Assume that a star's light ray is moving parallel to the y-axis and passing through mass m at a distance x = R. In the x-direction, the acceleration is provided by

$$\frac{d^2x}{dt^2} = -\frac{m}{r^2}.\frac{x}{r} = -\frac{mx}{(x^2 + y^2)^{3/2}} \qquad \text{... (7)}$$

For a light ray moving parallel to y-axis, we get

$$\frac{dy}{dt} = 1, \frac{d^2y}{dt^2} = 0$$

$$\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt},$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dy}.\frac{dy}{dt}\right) = \frac{d^2x}{dy^2}\left(\frac{dy}{dt}\right)^2 + \frac{dx}{dy}.\frac{d^2y}{dt^2}$$

$$= \frac{d^2x}{dy^2}1^2 + \frac{dx}{dy}.0 = \frac{d^2x}{dy^2}$$

$$\frac{d^2x}{dt^2} = \frac{d^2x}{dy^2}$$

Using (7), we obtain

$$\frac{d^2x}{dy^2} = -\frac{mx}{(x^2+y^2)^{3/2}} = -\frac{mR}{(R^2+y^2)^{3/2}}. \quad for\ x = R$$

$$\frac{d^2x}{dy^2} = -\frac{mR}{(R^2+y^2)^{3/2}}$$

Integrating w.r.t.y, we get

$$\frac{dx}{dy} = -\int \frac{mR}{(R^2+y^2)^{\frac{3}{2}}}\ dy = -mR\int \frac{R\ sec^2\theta d\theta}{R^3 sec^3\theta}, y = R.$$

$$= -\frac{m}{R}\int cos\theta d\theta = -\frac{m}{R} sin\theta + C$$

$$\frac{dx}{dy} = -\frac{m}{R} sin\theta + C = -\frac{my}{R\sqrt{x^2+y^2}} + C \qquad \ldots(8)$$

Hence

$$x = -\frac{m}{R}\sqrt{x^2+y^2} + Cy + C_1 \qquad \ldots(9)$$

From (8) and (9)

$$\frac{dx}{dy} = 0, x = R, y = 0$$

We obtain $C = 0$ and $R = -m + C_1$ $i.e.$, $C = 0$ and $C_1 = m + R$

Now the equation (9) becomes

$$x = R + \left(m - \frac{m}{R}\sqrt{x^2+y^2} + Cy\right) \qquad \ldots(10)$$

Newtonian theory states that this is the equation for a light beam's path. Derivation from the path $x = R$ is demonstrated by the second term, m divergence from the path $m - \frac{m}{R}\sqrt{x^2+y^2}$.

Now from (10) are written by taking y very large compared with x so

$$x = R + m - \frac{m}{R}(\pm y)$$

$$y = \frac{Rx}{m}(R+m)\ \ and\ \ y = -\frac{Rx}{m} + \frac{R}{m}(R+m)$$

Let β be the angle between these asymptotes then we obtain

$$tan\alpha = \frac{\frac{R}{m} - \left(-\frac{R}{m}\right)}{1 + \frac{R}{m} \times \left(-\frac{R}{m}\right)} = \frac{2mR}{m^2 - R^2}$$

$$tan\alpha = \frac{2mR}{m^2 - R^2}$$

Then

$$sin\beta = \frac{2mR}{m^2 + R^2}$$

Since $m^2 << R^2$ and hence m^2 is neglected

$$\beta = \frac{4mR}{R^2} = \frac{2m}{R}$$

But

$$\alpha = \frac{4m}{R} = 2.\left(\frac{2m}{R}\right) = 2\beta$$

$$\alpha = 2\beta$$

This proves that the deflection on the path of a light ray due relativistic field is twice that predicted by Newtonian theory.

(iii) **Gravitational Shift in spectral lines:** Obtain the formula for gravitational shift in spectral lines.

OR

Give the theatrical account of the red shift of spectral lines in gravitational fields.

Proof: We examine how the spectral lines of light generated by an atom in a gravitational field change when the light is seen on Earth's surface. Sodium atoms vibrate at a consistent frequency. Let dt be the equivalent periodic time and ds be the time interval between the start and finish of a single vibration. Imagine a spectator moving beside sodium atoms. For a brief moment, let the atom be in the coordinate system (r, θ, ϕ, t). So that by Schwarzschild the line element

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1-\frac{2m}{r}\right) dt^2 \quad \ldots (1)$$

For $dr = d\theta = d\phi = 0,$ then

$$ds^2 = \left(1-\frac{2m}{r}\right) dt^2$$

$$\frac{dt}{ds} = \left(1-\frac{2m}{r}\right)^{-\frac{1}{2}} = 1+\frac{m}{r}$$

Up to first approximation

We compare the periodic time of sodium atom at two places

i. On the surface of the sun.
ii. On the surface of the carth.

On the surface of the sun and earth, let dt and dt' represent the periodic periods of a sodium atom, respectively. Then

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{dt}{ds} = 1+\frac{m}{r} \quad \ldots (2)$$

$$\frac{dt'}{ds'} = 1$$

Using the fact that ds remains invariant under arbitrary co-ordinate transformation, we can be written as

$$\frac{dt'}{ds} = 1$$

From (2) and above equation, we get

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{dt}{dt'} = 1+\frac{m}{r}$$

$$\frac{\delta\lambda}{\lambda} = \frac{m}{r}$$

This expression is required for the spectral line shift.

12.8 *SCHWARZSCHILD'S INTERIOR SOLUTION:-*

To derive Schwarz child's interior solution for a sphere of incompressible perfect fluid of constant proper density ρ such that at the boundary $r = r_1$ of the sphere, the pressure is equal to zero and the solution agrees with the exterior solution.

Proof: We must determine an expression for the line cleme that holds inside a large body that is at rest at its origin. Additionally, we assume that the body is spherically symmetric since it contains an incompressible perfect fluid with the right density. A suitable pressure Po, we use the line element in the manner described below.

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + e^{\nu}dt^2 \qquad \dots (1)$$

Where λ and ν are function of r only, we get

$$T^{ij} = (\rho_0 + p_0)\frac{dx^i}{ds}\frac{dx^j}{ds} - g^{ij}p_0$$

From which we get

$$T_j^i = (\rho_0 + p_0)\frac{dx^i}{ds}\frac{dx^\mu}{ds}g_{\mu j} - \delta_j^i p_0 \qquad \dots (2)$$

So the velocity components are

$$\frac{dx^1}{ds}, \frac{dx^2}{ds}, \frac{dx^3}{ds} = 0 \text{ \& } \frac{dx^4}{ds} = e^{-\nu/2} \qquad \dots (3)$$

From(2), we have

$$T_1^1 = (\rho_0 + p_0)\frac{dx^1}{ds}\frac{dx^\mu}{ds}g_{\mu 1} - p_0$$

$$= (\rho_0 + p_0).0 - p_0$$

$$= 0 - p_0 = -p_0$$

Similarly

$$T_2^2 = -p_0, T_3^3 = -p_0$$

From (2),

$$T_4^4 = (\rho_0 + p_0)\frac{dx^4}{ds}\frac{dx^\mu}{ds}g_{\mu 4} - p_0$$

$$= (\rho_0 + p_0)\frac{dx^4}{ds}\frac{dx^4}{ds}g_{44} - p_0$$

$$= (\rho_0 + p_0)\left(e^{-\nu/2}\right)^2 e^{\nu} - p_0$$

$$= \rho_0 + p_0 - p_0 = \rho_0$$

Thus

$$T_1^1 = T_2^2 = T_3^3 = -p_0, T_4^4 = \rho_0 \qquad \dots(4)$$

The field equations in the interior are obtained by

$$R_j^i - \frac{1}{2}\delta_j^i R + \Lambda\delta_j^i = -8\pi T_j^i$$

From which we have

$$-8\pi T_j^i = g^{i\alpha}T_{j\alpha} - \tfrac{1}{2}\delta_j^i R + \Lambda\delta_j^i \qquad \dots(5)$$

The non vanishing components are

$$R_{11} = \frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r}$$

$$R_{22} = e^{-\lambda}\left[1 - r\left(\frac{\lambda' - \nu'}{2}\right)\right] - 1$$

$$R_{33} = R_{22} sin^2\theta$$

$$R_{44} = e^{\nu-\lambda}\left[-\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4} - \frac{\nu'}{r}\right]$$

Where denote differentiation w.r.t. r, we get

From (5), we obtain

$$8\pi p_0 - \Lambda = g^{11}R_{11} - \tfrac{1}{2}R \qquad \dots(6)$$

$$8\pi p_0 - \Lambda = g^{22}R_{22} - \tfrac{1}{2}R \qquad \dots(7)$$

$$8\pi p_0 - \Lambda = g^{33}R_{33} - \tfrac{1}{2}R \qquad \dots(7')$$

$$-8\pi p_0 - \Lambda = g^{44}R_{44} - \tfrac{1}{2}R \quad \dots (8)$$

The equation $(7')$ can be written as

$$8\pi p_0 - \Lambda = -\frac{1}{r^2 sin^2\theta} R_{22} sin^2\theta - \frac{1}{2}R$$

$$= -\frac{1}{r^2}R_{22} - \frac{1}{2}R = g^{22}R_{22} - \frac{1}{2}R$$

Since (7) and $(7')$ are identical.

So

$$R = g^{ij}R_{ij} = \sum_{i=1}^{4} g^{ii}R_{ii} = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44}$$

$$= g^{11}R_{11} + g^{22}R_{22} + \frac{g^{22}}{sin^2\theta}R_{22}sin^2\theta + g^{44}R_{44}$$

$$= g^{11}R_{11} + 2g^{22}R_{22} + g^{44}R_{44}$$

$$= -e^{-\lambda}\left(-\frac{v''}{2} - \frac{\lambda' v'}{4} + \frac{v'^2}{4} - \frac{\lambda'}{r}\right) - \frac{2}{r^2}\left[e^{-\lambda}\left[1 - r\left(\frac{\lambda' - v'}{2}\right)\right] - 1\right]$$
$$+ e^{v-\lambda}e^{v}\left[-\frac{v''}{2} + \frac{\lambda' v'}{4} - \frac{v'^2}{4} - \frac{v'}{r}\right]$$

$$-\frac{R}{2} = e^{-\lambda}\left(v'' + \frac{\lambda' v'}{2} - \frac{v'^2}{2} + \frac{(\lambda' - v')}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2} \quad \dots (9)$$

From (6) and (9), we obtain

$$8\pi p_0 - \Lambda = g^{11}R_{11} - \frac{1}{2}R$$

$$= -e^{-\lambda}\left(-\frac{v''}{2} - \frac{\lambda' v'}{4} + \frac{v'^2}{4} - \frac{\lambda'}{r}\right)$$
$$+ e^{-\lambda}\left(v'' + \frac{\lambda' v'}{2} - \frac{v'^2}{2} + \frac{(\lambda' - v')}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2}$$

$$8\pi p_0 = e^{-\lambda}\left[\frac{v'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} + \Lambda \quad \dots (10)$$

Similarly

From (7) and (9), we get

$$8\pi p_0 = e^{-\lambda}\left(-\frac{v''}{2}-\frac{\lambda' v'}{4}+\frac{v'^2}{4}-\frac{\lambda'-v\prime}{2r}\right)+\Lambda \qquad \dots(11)$$

From (8) and (9), we have

$$8\pi p_0 = e^{-\lambda}\left(\frac{\lambda'}{r}+\frac{v'^2}{4}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda \qquad \dots(12)$$

Adding (10) and (12), we get

$$8\pi(p_0+\rho_0) = e^{-\lambda}\frac{(\lambda'+v')}{r} \qquad \dots(13)$$

$$8\pi(p_0+\rho_0)\frac{v'}{2} = e^{-\lambda}\frac{(\lambda' v\prime + v'^2)}{2r} \qquad \dots(13')$$

$$8\pi\frac{dp_0}{dr} = e^{-\lambda}\left(\frac{v''}{r}-\frac{v'}{r^2}-\frac{2}{r^3}-\frac{\lambda' v'}{r}-\frac{\lambda'}{r^2}\right)+\frac{2}{r^3} \quad \dots(14)$$

Adding (13') and (14), we obtain

$$8\pi\left[\frac{dp_0}{dr}+(p_0+\rho_0)\frac{v'}{2}\right] = e^{-\lambda}\left(\frac{v''}{r}-\frac{\lambda' v'}{2r}-\frac{\lambda'+v'}{r^2}+\frac{v'^2}{2r}-\frac{2}{r^3}\right)+\frac{2}{r^3}$$

$$=\frac{2}{r}\left[e^{-\lambda}\left(\frac{v''}{r}-\frac{\lambda' v'}{2r}-\frac{\lambda'+v\prime}{2r}+\frac{v'^2}{4}-\frac{1}{r^2}\right)+\frac{1}{r^2}\right] \qquad \dots(15)$$

Now equating (10) to (11), we have

$$e^{-\lambda}\left(-\frac{v''}{2}-\frac{\lambda' v'}{4}+\frac{v'^2}{4}-\frac{\lambda'-v'}{2r}\right)+\Lambda = e^{-\lambda}\left[\frac{v'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}+\Lambda$$

$$e^{-\lambda}\left(\frac{v''}{2}-\frac{\lambda' v'}{4}+\frac{v'^2}{4}-\frac{\lambda'+v'}{2r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0$$

From (15) we obtain

$$8\pi\left[\frac{dp_0}{dr}+(p_0+\rho_0)\frac{v'}{2}\right]=\frac{2}{r}(0)$$

$$\frac{dp_0}{dr}+(p_0+\rho_0)\frac{v'}{2}=0 \qquad \dots(15')$$

Since we have to integrate (10), (12) and (15)

From(12),

$$8\pi p_0 + \Lambda = e^{-\lambda}\left(\frac{\lambda'}{r} + \frac{v'^2}{4} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$\frac{d}{dr}\left(re^{-\lambda}\right) = 1 - r^2(8\pi p_0 + \Lambda)$$

Integrating,

$$\left(re^{-\lambda}\right) = r - \frac{r^3}{3}(8\pi p_0 + \Lambda) + C_1$$

$$\left(re^{-\lambda}\right) = 1 - \frac{r^2}{3}(8\pi p_0 + \Lambda) + \frac{C_1}{r}$$

Taking $\frac{1}{R^2} = \frac{(8\pi p_0+\Lambda)}{3}$, we obtain

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} + \frac{C_1}{r}$$

We take $C_1 = 0$

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} \qquad \dots(16)$$

From$(15')$, we get

$$\frac{dp_0}{dr} + (p_0 + \rho_0)\frac{v'}{2} = 0$$

$$\frac{dp_0}{p_0 + \rho_0} = -\frac{dv}{2}$$

Integrating,

$$\log(p_0 + \rho_0) = -\frac{v}{2} + constant = logC_2e^{-v/2}$$

$$(p_0 + \rho_0) = C_2e^{-v/2}$$

$$8\pi(p_0 + \rho_0) = 8\pi C_2e^{-v/2} = C_3e^{-v/2}$$

$$e^{\frac{v}{2}}\left(\frac{\lambda' + v'}{r}\right)e^{-\lambda} = C_3\ by\ (13)$$

But $\quad e^{-\lambda}\left(\frac{\lambda'+v'}{r}\right) = e^{-\lambda}\frac{\lambda'}{r} + e^{-\lambda}\frac{v'}{r}$

$$= \frac{2}{R^2} + \frac{v'}{2}\left(1 - \frac{r^2}{R^2}\right), for \ -e^{-\lambda}\lambda' = -\frac{2r}{R^2}$$

$$e^{\frac{v}{2}}\left[\frac{2}{R^2} + \frac{v'}{2}\left(1 - \frac{r^2}{R^2}\right)\right] = C_3$$

Put $\quad e^{\frac{v}{2}} = u \ \ so \frac{du}{dr} = \frac{v'}{2} e^{\frac{v}{2}}$

$$\frac{2u}{R^2} + \frac{2}{r}\frac{du}{dr}\left(1 - \frac{r^2}{R^2}\right) = C_3$$

$$\frac{du}{dr} + \frac{r}{R^2 - r^2} u = \frac{rC_4}{R^2 - r^2} \qquad \dots (17) \qquad C_4 = \frac{C_3}{2} R^2$$

Now we know that the solution of

$$\frac{du}{dr} + uP(r) = Q(r)$$

is $\quad ue^{\int Pdr} = \int Qe^{\int Pdr} dr + const.$

On applying this method of (17), we obtain the final answer

$$\frac{1}{R^2} = \frac{8\pi\rho_0 + \Lambda}{3}$$

From(10), we have

$$8\pi p_0 = \left[\frac{v' e^{-\lambda}}{r} + e^{-\lambda}\frac{1}{r^2}\right] - \frac{1}{r^2} + \Lambda$$

$$= \frac{v' e^{-\lambda}}{r} + \frac{1}{r^2} - \frac{1}{R^2} - \frac{1}{r^2} + \Lambda$$

$$8\pi p_0 = \frac{v' e^{-\lambda}}{r} - \frac{1}{R^2} + \Lambda \qquad \dots (18)$$

And $\quad e^{\frac{v}{2}}\frac{v'}{2} = \frac{r}{R^2} \cdot \frac{B}{\sqrt{\left(1 - \frac{r^2}{R^2}\right)}}$

$$\frac{v'}{r} = \frac{2B/R^2}{\left[A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right] \times \left\{\left(1 - \frac{r^2}{R^2}\right)\right\}^{1/2}}$$

$$e^{-\lambda}\frac{v'}{r} = \frac{2B}{R^2}\frac{\left(1-\frac{r^2}{R^2}\right)}{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}\cdot\frac{1}{\left[A - B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right]}$$

$$e^{-\lambda}\frac{v'}{r} - \frac{1}{R^2} = \frac{3B\sqrt{\left(1-\frac{r^2}{R^2}\right)} - A}{R^2\left[A - B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right]}$$

Using (18), we have

$$8\pi p_0 = \frac{3B\sqrt{\left(1-\frac{r^2}{R^2}\right)} - A}{R^2\left[A - B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right]} + \Lambda \qquad \text{... (19)}$$

When the distance from the origin is significantly more than r_1, where r_1 is the radius of the massive body, factor Λ becomes significant.

Hence we take $\Lambda = 0$ for $r \leq r_1$

Also $p_0 = 0$ for $r = r_1$

Thus $p_0 = 0 = \Lambda$ for $r = r_1$

From (19), we get

$$A = 3B\left\{\left(1-\frac{{r_1}^2}{R^2}\right)\right\}^{\frac{1}{2}} \qquad \text{... (20)}$$

The line element for an interval in the interior of the massive body is

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left[A - B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right]^2 dt^2 \qquad \text{... (21)}$$

This is called Schwarzschild’s interior solution.

The exterior solution is

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left(1-\frac{2m}{r}\right) dt^2 \qquad \ldots(22)$$

$$1-\frac{{r_1}^2}{R^2} = 1-\frac{2m}{r} = \left[A - B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right]^2$$

$$1-\frac{{r_1}^2}{R^2} = 1-\frac{2m}{r_1} = 4B^2\left(1-\frac{r^2}{R^2}\right) \quad by\ (20)$$

$$1-\frac{{r_1}^2}{R^2} = 1-\frac{2m}{r_1}, \qquad 1-\frac{{r_1}^2}{R^2} = 4B^2\left(1-\frac{r^2}{R^2}\right)$$

$\Longrightarrow \qquad m = \frac{{r_1}^3}{2R^2}, \ \ 4B^2 = 1$

$\Longrightarrow \qquad \frac{4\pi}{3}{r_1}^3\rho_0 = \frac{{r_1}^3}{2R^2}, \ \ 2B = 1$

$\Longrightarrow \qquad \frac{8\pi\rho_0}{3} = \frac{1}{R^2}, B = \frac{1}{2}$

Now

$$\frac{1}{R^2} = \frac{8\pi\rho_0 + \Lambda}{3} \quad where\ \Lambda = 0 for\ r = r_1$$

$$A = \frac{3}{2}\sqrt{\left(1-\frac{r^2}{R^2}\right)}, \qquad B = \frac{1}{2}, \qquad \frac{1}{R^2} = \frac{8\pi\rho_0}{3}$$

So the interior solution will be real only if

$$\frac{2m}{r_1} < 1 \text{ or } \frac{r^2}{R^2} < 1,$$

$${r_1}^2 < \frac{3}{8\pi\rho_0}$$

This is complete.

SELF CHECK QUESTIONS

1. What is the Schwarzschild solution?
   It is the exact solution to Einstein's field equations in general relativity that describes the spacetime geometry outside a spherically symmetric, non-rotating, uncharged mass.
2. What is the Schwarzschild radius?
   The radius at which the escape velocity equals the speed of light, given by:
   $$r_s = \frac{2GM}{c^2}$$
   This is the radius of the event horizon of a non-rotating black hole.
3. Write the Schwarzschild metric.
   $$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2$$
4. Is there a true singularity in the Schwarzschild solution? Where?
   Yes, at $r = 0$. It is a true physical singularity, where spacetime curvature becomes infinite.
5. What kind of spacetime does the Schwarzschild solution describe?
   - Static
   - Spherically symmetric
   - Vacuum (i.e., outside any mass distribution)
6. Is the Schwarzschild solution valid inside a black hole?
   No, it breaks down at $r = r_s$ due to coordinate singularity. For $r < r_s$, a different coordinate system like Kruskal–Szekeres is used.
7. What is the significance of the time dilation in the Schwarzschild metric?
   Clocks closer to a massive object tick slower compared to clocks farther away. Near the event horizon, time appears to stop for a distant observer.
8. How does the Schwarzschild metric reduce at large distances (i.e., r→∞)?
   It reduces to the flat Minkowski metric, as gravitational effects vanish at infinity.
9. Does the Schwarzschild solution include charge or rotation?

No. For charge, you need the Reissner–Nordström solution; for rotation, the Kerr solution.

12.9 *SUMMARY:-*

In this unit, we explored the Schwarzschild exterior solution, which describes the spacetime geometry outside a static, spherically symmetric, and uncharged mass, leading to important concepts such as the Schwarzschild radius and event horizon. We examined the relation between the gravitational mass M that appears in the metric and the inertial mass m, reinforcing the equivalence principle in general relativity. The Schwarzschild metric was also expressed in isotropic coordinates to simplify the form of the spatial components for certain applications. A detailed analysis of planetary orbits revealed key relativistic corrections, including the famous perihelion precession of Mercury. We reviewed the classical tests of general relativity—light bending, gravitational redshift, and time delay—which provided crucial experimental confirmations of Einstein's theory. Finally, the Schwarzschild interior solution was studied to understand the spacetime geometry inside a spherically symmetric, static mass distribution, such as a star, giving insight into the pressure and density profiles necessary for hydrostatic equilibrium in relativistic stars.

12.10 *GLOSSARY:-*

- **Schwarzschild Solution:** An exact solution to Einstein's field equations representing the spacetime outside a static, spherically symmetric, and uncharged mass.
- **Schwarzschild Metric:** The line element that defines the geometry of spacetime in the Schwarzschild solution. It is given by:
  - $ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1-\frac{2m}{r}\right) dt^2$
- **Schwarzschild Radius ($\boldsymbol{r_s = r}$):** The radius at which the escape velocity equals the speed of light:
- $r_s = \frac{2GM}{c^2}$
- It marks the **event horizon** of a black hole.
- **Event Horizon:** A boundary in spacetime beyond which events cannot affect an outside observer. In the Schwarzschild case, it lies at $r_s = r$

- **Coordinate Singularity:** A point where the coordinates used in the metric break down (e.g., at $r_s = r$), but the spacetime itself is not singular.
- **Physical Singularity:** A point in spacetime where curvature becomes infinite and physical laws break down. In the Schwarzschild solution, this occurs at $r = 0$.
- **Isotropic Coordinates:** A coordinate transformation of the Schwarzschild metric where spatial components appear more symmetric, often used to simplify calculations or match boundary conditions.
- **Gravitational Mass (M):** The source of the gravitational field in the Schwarzschild metric. It can be interpreted as the total mass-energy of the system.
- **Inertial Mass (m):** The mass that resists acceleration when a force is applied; in general relativity, it is equivalent to gravitational mass.
- **Perihelion Precession:** The relativistic effect that causes the closest point in a planet's orbit around the Sun (the perihelion) to shift over time. Explained accurately by the Schwarzschild solution.
- **Gravitational Time Dilation:** The effect where time runs slower in stronger gravitational fields. In the Schwarzschild spacetime, clocks closer to the mass tick more slowly.
- **Light Bending:** The deflection of light as it passes near a massive object. One of the classic tests of general relativity derived from the Schwarzschild geometry.
- **Interior Schwarzschild Solution:** A solution to Einstein's field equations that describes the spacetime inside a static, spherically symmetric body of constant density.

12.11 *REFERENCES:-*

- Ashok Das (2011), Lectures on Gravitation, University of Rochester, USA ,Saha Institute of Nuclear Physics, India.
- Richard Feynman (2018), Feynman Lectures On Gravitation.
- Neil Ashby, Stanley C. Miller (2019), principles of modern physics

12.12 *SUGGESTED READING:-*

- Farook Rahman (2021), The General Theory of Relativity: A Mathematical Approach

- Satya Prakash,Revised by K.P.Gupta, Ninteenth Edition (2019), Relativistic Mechanics.
- Dr. J.K.Goyal & Dr.K.P.Gupta (2018), Theory of Relaivity.

12.13 *TERMINAL QUESTIONS:*

(TQ-1) State Einstein's law of gravitation (for empty space) sketch the method for obtaining the gravitational field isolated particle as given by Schwarzschild metric.

(TQ-2) Assuming Schwarzschild solution, show how the relativistic term $3mu^2$ arises in modifying the Newtonian equation of a planetary orbit,

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2}$$

(TQ-3) Deduce from it the differential equation of a planetary orbit and compare it with Newtonian orbit for the same.

(TQ-4) Obtain the formula for gravitational shift in spectral lines.

(TQ-5) Derive Schwarzschild's interior solution

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left[A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right]^2 dt^2$$

(TQ-5) Derive the expression for the motion of the perihelion of a planetary orbit round the sun.

(TQ-7) Obtain Schwarzschild's exterior solution of an isolated gravitating body.

(TQ-8) What are the crucial tests of General Relativity? Discuss one of them.

(TQ-9) Derive Schwarzschild's solution for an isolated particle continually at rest at the origin.

(TQ-10) Discuss the three crucial tests of general relativity.

(TQ-11) Derive Schwarzschild's interior solution of a spherically symmetric distribution of matter with constant density.

(TQ-12) Show how general relativity modifies the equation of planetary orbit and explain the advance of perihelion.

(TQ-13) Discuss the phenomenon of red shift in general relativity.

(TQ-14) Derive the Schwarzschild exterior solution for the gravitational field of an isolated mass particle at rest.

(TQ-15) Derive the expression for the motion of the perihelion of Mercury round the sun.

12.14 *ANSWERS:*

SELF CHECK ANSWERS

1. It is the exact solution to Einstein's field equations in general relativity that describes the spacetime geometry outside a spherically symmetric, non-rotating, uncharged mass.
2. The radius at which the escape velocity equals the speed of light, given by:

$$r_s = \frac{2GM}{c^2}$$

   This is the radius of the event horizon of a non-rotating black hole.

3. $ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \left(1-\frac{2m}{r}\right) dt^2$
4. Yes, at $r = 0$. It is a true physical singularity, where spacetime curvature becomes infinite.
5. The kind of
   - Static
   - Spherically symmetric
   - Vacuum (i.e., outside any mass distribution)
6. No, it breaks down at $r = r_s$ due to coordinate singularity. For $r < r_s$, a different coordinate system like Kruskal–Szekeres is used.
7. Clocks closer to a massive object tick slower compared to clocks farther away. Near the event horizon, time appears to stop for a distant observer.
8. It reduces to the flat Minkowski metric, as gravitational effects vanish at infinity.
9. No. For charge, you need the Reissner–Nordström solution; for rotation, the Kerr solution.

UNIT 13:-Cosmology

CONTENTS:

13.1 *INTRODUCTION:-*

Cosmological models refer to solutions of Einstein's field equations of general relativity that describe the geometric and dynamic properties of the universe on a large scale. These models are typically built on the assumption that the universe is homogeneous and isotropic, which leads to the use of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric. This metric simplifies Einstein's equations into a set of Friedmann equations, which relate the scale factor a(t) to physical quantities like energy density ρ, pressure p, and the cosmological constant Λ. Different cosmological models arise by choosing various values of curvature k, matter content, and the cosmological constant. For example, the Einstein-de Sitter model assumes k=0 and Λ=0, while the de Sitter universe considers a vacuum-dominated model with Λ>0. These mathematical models are essential for predicting cosmic expansion, the age of the universe, and the fate of cosmic evolution.

13.2 *OBJECTIVES:-*

After studying this unit, Lernear's will be able to

- To Solve Einstein's and De-Sitter line elements.
- To explain properties of Einstein Universe.
- To explain properties of De-Sitter Universe.

13.3 *COSMOLOGICAL MODELS:-*

Cosmology is the branch of science that deals with the study of the universe as a whole, including the distribution and motion of matter on a large scale. It aims to understand the origin, structure, evolution, and eventual fate of the universe. In cosmology, we construct mathematical models known as **cosmological models** or **world models** that describe the large-scale behavior of matter and the geometry of space-time. These models are then compared with observational data to evaluate how accurately they represent the actual universe.

Theories concerning the nature of the cosmos have existed for as long as humanity. It has long been known that applying Newton's gravitational theory to the entire cosmos presents significant challenges. At least as far as the dimensions of the solar system are concerned, the three crucial tests of the general theory of relativity show that it has significantly modified the Newtonian theory and provided a workable solution to the problem of a star's field in the empty space surrounding it. It then seems to be of great interest to extend the application of the general theory of relativity to the universe as a whole. Einstein originally addressed this question shortly after the general theory of relativity was developed. It has been the focus of numerous investigations ever since. Because several large-scale features of the cosmos may be compared to such a model of the universe and are known experimentally, this program is highly intriguing. The following are the most important of these properties.

(a)**Homogeneity of Matter Distribution:** On average, matter is distributed in a fairly uniform manner throughout the universe. The estimated average density of matter is approximately$\rho \approx 10^{-27} g/cm^3$.
(b) **Isotropy of the Universe:**
From the viewpoint of the solar system, the universe appears to be fairly isotropic that is, it looks the same in all directions on a large scale.
(c) **Redshift of Light from Distant Nebulae:** Light reaching us from distant nebulae is redshifted, and the amount of redshift is proportional to the distance the light has traveled. This relationship follows the law:

$$\frac{\delta\lambda}{\lambda} = kr, k = 6 \times 10^{-28} cm^{-1}$$

Assuming this redshift is due to the Doppler effect, we infer that distant nebulae (or galaxies) are receding from us. The speed of recession is proportional to their distance from us.

(d) According to measurements of the radioactive remains, some rocks in the crust of earth are at least 3.5 to 4 billion years old. Hence the universe is older than 4 billion years.

The modified field equations proposed by Einstein are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \qquad \dots (1)$$

The constant Λ has such a small effect on solar system or even our own galaxy phenomena, but it becomes significant when the entire universe is taken into account.

It is possible to construct alternative models of the cosmos by mixing different values of A with different possibilities of "Static cosmological models" are represented by the static solutions of equation (1). Here, we will first examine the static, isotropic, and homogeneous models of the cosmos that were first put forth by Einstein and de Sitter. Finally, we will discuss Robrtson's non-static, isotropic, and homogeneous model. The following presumptions form the basis of Einstein's de-Sitter's cosmological models.

1. The universe is static, i.e., in a proper co-ordinate system matter is at rest and the proper pressure Po and proper density Po are the same everywhere.
2. The universe is isotropic, i.e., all spatial directions are equivalent.
3. The universe is homogeneous, i.e., no part of the universe can be distinguished from any other.
4. For small values of r the line element should reduce to special relativity form for flat-space time since local gravitational fields can be neglected in small space-time regions.

The line element satisfying the condition of spherical symmetry is given by

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + e^{\nu} dt^2 \qquad \dots (2)$$

where λ and ν are functions of r only.

For the universe containing perfect fluids, then we have the following relations.

$$8\pi p_0 = e^{-\lambda}\left(\frac{\nu\prime}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}+\Lambda \qquad \ldots(3)$$

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda \qquad \ldots(4)$$

$$\frac{dp_0}{dr}+(p_0+\rho_0)\frac{\nu'}{2}=0 \qquad \ldots(5)$$

According to the assumption $\frac{dp_0}{dr}=0$, then we get

$$\frac{dp_0}{dr}+(p_0+\rho_0)\frac{\nu'}{2}=0$$

$$(p_0+\rho_0)\nu'=0$$

$$\left.\begin{matrix}\nu'=0 \text{ or } p_0+\rho_0=0 \\ p_0+\rho_0=0=\nu'\end{matrix}\right\} \qquad \ldots(a)$$

These solution (a) lead respectively to Einstein, De-Sitter and special relativity line element.

13.4 *EINSTEIN LINE ELEMENT:-*

This Einstein line element arises from the possibility

$$\nu'=0 \qquad \ldots(1)$$

Integrating

$$\nu=C_1=constant$$

Applying the condition $\lambda=\nu=0$ at $r=0$, we have

$$C_1=0$$

Thus

$$\nu'=0=\nu$$

From (4), we obtain

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda$$

$$(8\pi\rho_0+\Lambda)r^2 = e^{-\lambda}(\lambda' r-1)+1$$

$$1-(8\pi\rho_0+\Lambda)r^2=\frac{d}{dr}\left(re^{-\lambda}\right)$$

Integrating

$$re^{-\lambda}=r-\frac{r^3}{3}(8\pi\rho_0+\Lambda)+C$$

Applying the condition $\lambda=\nu=0$ at $r=0$, we obtain

$$0=0-0+C \quad or\ C=0$$

Hence

$$re^{-\lambda}=r-\frac{r^3}{3}(8\pi\rho_0+\Lambda)$$

$$e^{-\lambda} = 1 - \frac{r^2}{3}(8\pi\rho_0 + \Lambda)$$

Taking $\frac{l}{R^2} = \frac{(8\pi\rho_0+\Lambda)}{3}$, we get $e^{-\lambda} = 1 - \frac{r^2}{R^2}$

Now from (2), we have

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + dt^2$$

This line element is called Einstein line element for static, isotropic and homogeneous universe.

13.5 *PROERTIES OF EINSTEIN UNIVERSE:-*

i. Geometry of Einstein Universe: By the transformation coordinates,
The Einstein line element is

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + dt^2 \dots (1)$$

Consider the transformation

$$r = \frac{\rho}{\left(1 + \frac{\rho^2}{4R^2}\right)}$$

Then

$$r\left(1 + \frac{\rho^2}{4R^2}\right) = \rho$$

This obtain

$$dr = \frac{\left(1 - \frac{\rho r}{2R^2}\right)}{\left(1 + \frac{\rho^2}{4R^2}\right)} d\rho$$

and

$$\frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} = \frac{1}{\left(1 - \frac{r^2}{R^2}\right)} \left[\frac{\left(1 - \frac{\rho r}{2R^2}\right)}{\left(1 + \frac{\rho^2}{4R^2}\right)} d\rho\right]^2$$

Simplifying this we have

$$\frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} = \frac{d\rho^2}{\left(1 + \frac{\rho^2}{4R^2}\right)^2}$$

Now from (1) becomes

$$ds^2 = -\frac{1}{\left(1+\frac{\rho^2}{4R^2}\right)^2}[d\rho^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)] + dt^2$$

This can also be transformed into

$$ds^2 = -\frac{1}{\left(1+\frac{\rho^2}{4R^2}\right)^2}[dx^2 + dy^2 + dz^2] + dt^2$$

Let the second transformation

$$z_1 = R - \sqrt{\left(1-\frac{r^2}{R^2}\right)}\ , z_2 = rsin\theta cos\phi, z_3 = rsin\theta sin\phi, z_4 = rcos\theta$$

Putting these values in (1), we obtain

$$ds^2 = -({dz_1}^2 + {dz_2}^2 + {dz_3}^2 + {dz_4}^2) + dt^2$$

with

$${z_1}^2 + {z_2}^2 + {z_3}^2 + {z_4}^2 = R^2$$

This prove that the physical space of Einstein universe may be embedded in a Euclidean space of higher dimensions.

a. Spherical Space: By the transformation take (1), we have

$$ds^2 = -R^2(d\beta^2 + sin^2\beta(d\theta^2 + sin^2\theta d\phi^2) + dt^2$$

We already get $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$

With the remaining variables θ & and ϕ being arbitrary, we discover that this line element (2) stays the same for $\beta = 0$ and $\beta = \pi$. this indicates that there is a comparable occurrence at $\beta = \pi$ to one at $\beta = 0$. In other words, there is a mirror image at $\beta = \pi$ that corresponds to an event at $\beta = 0$. According to this interpretation, the Einstein cosmos is spherical. The proper volume V_0 of the spherical universe is

$$V_0 = \int_{\beta=0}^{\beta=\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(Rd\beta)(Rsin\beta d\theta)(R\ sin\theta sin\beta d\phi)$$

$$= 4\pi R^3\int_0^{\pi}\frac{1}{2}(1-cos2\beta)d\beta = 2\pi R^3\left[\beta - \frac{1}{2}sin2\beta\right]_0^{\pi}$$

$$= 2\pi R^3$$

and the total distance around the spherical universe is

$$l_0 = 2\int_0^{\pi}Rd\beta = 2\pi R$$

Hence the proper volume of the co called spherical universe is $2\pi R^3$.

b. Elliptical Space: The Elliptical space (4) provides the Einstein line element. Only when the $r < R_0$ is defined is the element provided by (4) real. The statement defines the spatial expansion of the physical space in Einstein's universe is

$$d\sigma^2 = \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

The proper volume of Einstein universe is then

$$V_0 = \int_{r=0}^{R_0}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{dr}{\sqrt{\left(1 - \frac{r^2}{R^2}\right)}}, rd\theta . rsin\theta d\phi = \pi R^3$$

and the total distance around the elliptical universe is

$$l_0 = 2\int_0^R \frac{dr}{\sqrt{\left(1 - \frac{r^2}{R^2}\right)}} = 2\int_0^{\pi/2} \frac{Rcos\eta d\eta}{cos\eta}, where \frac{r}{R} = sin\eta = \pi R$$

ii. Density and pressure of the matter in Einstein universe:

For the line element is

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + e^{\nu} dt^2$$

Then we have

$$8\pi p_0 = e^{-\lambda}\left(\frac{\nu\prime}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \qquad \dots(3)$$

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \qquad \dots(4)$$

Where $\nu' = 0 = \nu, e^{-\lambda} = 1 - \frac{r^2}{R^2}$ so that $e^{-\lambda}\frac{\lambda'}{r} = \frac{2}{R^2}$

From (3) becomes

$$8\pi p_0 = 0 + \left(1 - \frac{r^2}{R^2}\right)\frac{1}{r^2} - \frac{1}{r^2} + \Lambda = \frac{1}{R^2} + \Lambda$$

$$8\pi p_0 = \Lambda - \frac{1}{R^2} \qquad \dots(5)$$

Now from (4),we get

$$8\pi\rho_0 = \frac{2}{R^2} - \left(\frac{1}{r^2} - \frac{1}{R^2}\right) + \frac{1}{r^2} - \Lambda$$

$$8\pi\rho_0 = \frac{3}{R^2} - \Lambda \qquad \dots(6)$$

Adding (5) and (6), we have

$$8\pi(\rho_0 + p_0) = \frac{2}{R^2}$$

$$\rho_0 + p_0 = \frac{1}{4\pi R^2} \qquad \dots(7)$$

The equation (5) and (6) represent required expressions for density and pressure.

CaseI: Let us consider the universe is filled with fluid consisting of incoherent matter exerting no pressure. For example free particles (stars).

Then $p_0 = 0$.
Now the equation (7), we have

$$\rho_0 = \frac{1}{4\pi R^2}$$

Mass of the spherical universe $= V_0\rho_0 = 2\pi^2 R^2 . \frac{1}{4\pi R^2} = \frac{\pi R}{2}$
Mass of the elliptical universe $= V_0\rho_0 = \pi^2 R^3 . \frac{1}{4\pi R^2} = \frac{\pi R}{4}$.

CaseII: When the universe is filled with radiation,$\rho_0 = 3p_0$.
From(7), we get

$$p_0 = \frac{1}{16\pi R^2} = \frac{\rho_0}{3}$$

Mass of the spherical universe $= V_0\rho_0 = 2\pi^2 R^3 \pi . \frac{3}{16\pi R^2} = \frac{3}{8}\pi R$
Mass of the elliptical universe $= \frac{3}{16}\pi R$.
CaseIII: When the universe is completely empty

$$\rho_0 = 0 = p_0$$

From (5) and (6), we have

$$\Lambda = \frac{1}{R^2}, \Lambda = \frac{3}{R^2}$$

$$\Lambda = \frac{1}{R^2} = 0$$

Therefore

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} = 1$$

This shows that for flat space time, the Einstein element would degenerate into a line element of special relativity type.

iii. Motion of attest particle in the Einstein Universe: The Einstein line element is

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + dt^2 \quad \ldots (1)$$

The motion of the particle is described by the geodesic equations:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

With

$$x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t$$

For the sake of simplicity, we assume the particle was initially at rest so that velocity components can be calculated as

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0 \quad \ldots (2)$$

From (2), we get

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{44}\left(\frac{dt}{ds}\right)^2 = 0$$

But

$$\Gamma^\alpha_{44} = \frac{1}{2}\frac{g_{44}}{\partial x^\alpha} = 0 \quad since\ g_{44} = 1$$

Hence

$$\frac{d^2r}{ds^2} = \frac{d^2\theta}{ds^2} = \frac{d^2\phi}{ds^2} = 0.$$

i.e., the particle has zero acceleration. Hence in Einstein universe a particle at rest remains at rest.

iv. Shift in Spectral lines (Doppler's effect in Einstein Universe): According to zero acceleration for stationary particles, consider an observer at $r = 0$ and a light source, such as a star, at $r = r$. Both are always at rest with regard to spatial coordinates.

$$ds = d\theta = d\phi = 0$$

So that from (1), we get

$$\frac{dr}{dt} = \pm\left(1 - \frac{r^2}{R^2}\right)^{1/2}$$

At time t_1, let a light pulse exit the star. The observer would get it at time t_2 , which is provided by

$$\int_{t_1}^{t_2} dt = -\int_{r_1}^{0} \frac{dr}{\left(1 - \frac{r^2}{R^2}\right)^{1/2}} = \int_{0}^{r_1} \frac{dr}{\left(1 - \frac{r^2}{R^2}\right)^{1/2}}$$

$$t_2 - t_1 = R sin^{-1}\frac{r_1}{R}$$

$$t_2 = t_1 + R sin^{-1}\frac{r_1}{R}$$

Since in Einstein universe particle at rest remain at rest.i.e., r_1 is constant so we have

$$\delta t_2 - \delta t_1 = 0$$

$$\frac{\delta t_2}{\delta t_1} = 1$$

13.6 *DE-SITTER'S LINE ELEMENT:-*

The de-Sitter's line element arises from the possibility

$$\rho_0 + p_0 = 0$$

Adding (3) and (4), we obtain

$$8\pi 0 = e^{-\lambda}\left(\frac{\lambda' + v'}{r}\right) \quad or \quad e^{-\lambda}\left(\frac{\lambda' + v'}{r}\right) = 0$$

$$\lambda' + v' = 0$$

On integration $\lambda + \nu = C$

Now the subject to condition $\lambda = \nu = 0$ at $r = 0$, we obtain

$$C = 0$$

Hence

$$\lambda + \nu = C = 0 \;\; or\, \lambda + \nu = 0 \;\; or \;\; \lambda = -\nu$$

From (40, we have

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda$$

$$(8\pi\rho_0 + \Lambda)r^2 - 1 = e^{-\lambda}(\lambda' r - 1)$$

$$1 - (8\pi\rho_0 + \Lambda)r^2 = \frac{d}{dr}\,(re^{-\lambda})$$

Integrating

$$re^{-\lambda} = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda) + C_1$$

Now the subject to condition $\lambda = \nu = 0$ at $r = 0$, we obtain

$$C_1 = 0$$

Consequently

$$re^{-\lambda} = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda)$$

$$e^{-\lambda} = 1 - \frac{r^2}{3}(8\pi\rho_0 + \Lambda)$$

Taking $\frac{1}{R^2} = \frac{(8\pi\rho_0 + \Lambda)}{3}$, we obtain $e^{-\lambda} = 1 - \frac{r^2}{R^2}$

$$e^{\nu} = e^{-\lambda} = 1 - \frac{r^2}{R^2}$$

Now from (2), we have

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left(1 - \frac{r^2}{R^2}\right)dt^2$$

This line element is called De-Sitter line element for static, isotropic and homogeneous universe.

13.7 *PROPERTIES OF DE-SITTER'S UNIVERSE:-*

i. Geometry of de-Sitter's universe:

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1-\frac{r^2}{R^2}\right) dt^2 \dots (1)$$

can be written into several forms. We make the transformation

$$\frac{r}{R} = \sin\beta.$$

As a result of which (1) becomes

$$ds^2 = -R^2[d\beta^2 + \sin^2\beta(d\theta^2 + \sin^2\theta d\phi^2)] + \cos^2\beta dt^2. \qquad (2)$$

On applying the transformation

$$\alpha = r\sin\theta\cos\phi, \delta - \varepsilon = \mathrm{Re}^{-t/R}/\sqrt{\left(1-\frac{r^2}{R^2}\right)}$$

$$\beta = r\sin\theta\sin\phi,$$

$$\gamma = r\cos\theta, \delta + \varepsilon = \mathrm{Re}^{t/R}/\left(1-\frac{r^2}{R^2}\right)^{1/2}$$

we find that (1) is reduced to

$$ds^2 = -[d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2] + d\varepsilon^2. \qquad (3)$$

Further taking $\alpha = iz_1, \beta = iz_2, \gamma = iz_3, \delta = iz_4, \varepsilon = iz_5$.
We obtain $ds^2 = dz_1{}^2 + dz_2{}^2 + dz_3{}^2 + dz_4{}^2 + dz_5$

with $z_1^2 + z_2^2 + z_3{}^2 + z_4^2 + z_5^2 = (iR)^2$.

The de-Stitter universe's physical space may be embedded in a higher-dimensional Euclidean space, according to equation (3). It also demonstrates that this universe's geometry is based on a sphere's surface embedded in a five-dimensional Euclidean space. Lemaitre Robertson transformation

$$r' = -\frac{re^{-t/R}}{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}, t' = t + R\log\left\{\left(1-\frac{r^2}{R^2}\right)\right\}^{1/2}$$

This transformation (1) helps to take the shape

$$ds^2 = -e^{2r'R}[dr^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2)] + dt'^2$$

Dropping dashes, we obtain

$$ds^2 = -e^{2t/R}[dr^2 + r^2(d\theta^2 + \sin^2\theta + d\phi^2)] + dt^2$$

Taking $k = 1/R$, we have

$$ds^2 = -e^{2kt}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + dt^2$$

Its Cartesian equivalent is

$$ds^2 = -e^{2kt}[dx^2 + dy^2 + dz^2] + dt^2$$

Therefore, we can see that a static line element may be changed into a non-static one using this transformation.

ii. Pressure and density of matter in de-Sitter universe: The de-Sitter line element is based on the assumption

$$\rho_0 + p_0 = 0 \qquad \ldots(4)$$

Since $\rho_0 \geq 0$ and therefore we obtain

$$\rho_0 = 0 = p_0 \qquad \ldots(5)$$

This is the only way to solve (4). The de-Sitter universe is implied to be entirely empty by equation (5). It is devoid of radiation and substance.

iii. Motion of a test particle in de-Sitter universe. Geodesic equations describe the motion of a test particle.

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{ij}\frac{dx^j}{ds}\frac{dx^j}{ds} = 0 \qquad (6)$$

The line element is taken into consideration in the general form

$$ds^2 = -e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^\nu dt^2 \qquad (7)$$

With$e^{-\lambda} = e^\nu = 1 - \frac{r^2}{R^2}$

The elements of Christoffel's brackets of the second type that are currently disappearing are

$$\Gamma^1_{11} = \lambda'/2, \Gamma^3_{23} = \cot\theta$$
$$\Gamma^2_{12} = 1/r = \Gamma^3_{13}, \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}$$
$$\Gamma^4_{14} = v'/2, \Gamma^2_{33} = -\sin\theta\cos\theta$$
$$\Gamma^1_{22} = -re^{-\lambda}, \Gamma^1_{44} = v'e^{v-\lambda}/2$$

where the dashes indicate the difference with respect to r.

For$\alpha = 2$ $\quad , \frac{d^2x^2}{ds^2} + \Gamma^2_{ij}\frac{dx^j}{ds}\frac{dx^j}{ds} = 0$

or,

$$\frac{d^2x^2}{ds^2} + \Gamma^2_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^2_{33}\frac{dx^3}{ds}\frac{dx^3}{ds} = 0$$

or,

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \qquad (8)$$

For $\alpha = 3,$ $\quad \frac{d^2x^3}{ds^2} + \Gamma^3_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$

or,

$$\frac{d^2x^3}{ds^2} + 2\Gamma^3_{13}\frac{dx^1}{ds}\frac{dx^2}{ds} + 2\Gamma^3_{23}\frac{dx^2}{ds}\frac{dx^3}{ds} = 0$$

or,

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \qquad (9)$$

For $\alpha = 4,$ $\quad \frac{d^2x^4}{ds^2} + \Gamma^4_{ij}\frac{dx}{ds}\frac{dx^j}{ds} = 0$

or,

$$\frac{d^2x^4}{ds^2} + 2\Gamma^4_{14}\frac{dx^1}{ds}\frac{dx^4}{ds} = 0 \qquad (10)$$

or, $\frac{d^2t}{ds^2} + v'\frac{dr}{ds}\cdot\frac{dt}{ds} = 0$

Let $\theta = \frac{\pi}{2}$ initially, then $\sin\theta = 1, \cos\theta = 0 = \frac{d\theta}{ds}$

Substituting these values in (8), (9) and (10), we obtain

$$\frac{d^2\theta}{ds^2} = 0$$
$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0$$

or,

$$\frac{d^2t}{ds^2} + v'\frac{dr}{ds}\frac{dt}{ds} = 0$$

A particle that begins moving in the plane $\theta = \frac{\pi}{2}$ will continue to move in the same plane, as demonstrated by the equation (8').

From (9'),

$$r^2\frac{d^2\phi}{ds^2} + 2r\frac{dr}{ds}\frac{d\phi}{ds} = 0$$
$$\text{or, } \frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0$$

This provides a solution

$$r^2\frac{d\phi}{ds} = h$$

h being a constant of integration.
From $(10')$

$$e^{\nu}\frac{d^2t}{ds^2} + v'e^{\nu}\frac{dr}{ds}\frac{dt}{ds} = 0$$

or,

$$\frac{d}{ds}\left(e^{\nu}\frac{dt}{ds}\right) = 0$$

Upon integration, $e^{\nu}\frac{dt}{ds} = k, k$ being a constant of integration Instead of taking $\alpha = 1$, we shall consider the line element (7). For $\theta = \frac{\pi}{2}$, (7) becomes, $ds^2 = -e^{\lambda}dr^2 - r^2d\phi^2 + e^{\nu}dt^2$
or,

$$e^{\lambda}\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\phi}{ds}\right)^2-e^{\nu}\left(\frac{dt}{ds}\right)^2+1=0$$

Substituting expressions for

$$\frac{d\phi}{ds} \text{ and } \frac{dt}{ds}$$

we get

$$e^{\lambda}\left(\frac{dr}{ds}\right)^2+r^2\cdot\frac{h^2}{r^4}-e^{\nu}\cdot k^2e^{-2\nu}+1=0$$

$$\text{or, } e^{\nu+\lambda}\left(\frac{dr}{ds}\right)^2+\frac{h^2e^{\nu}}{r^2}-k^2+e^{\nu}=0$$

$$\text{or. } \left(\frac{dr}{ds}\right)^2+\frac{h^2}{r^2}\left(1-\frac{r^2}{R^2}\right)-k^2+\left(1-\frac{r^2}{R^2}\right)=0. \text{ For } \lambda+\nu''$$

or

$$\left(\frac{dr}{ds}\right)^2=k^2-1+\frac{r^2}{R^2}-\frac{h^2}{r^2}+\frac{h^2}{R^2}$$

Taking positive square root.

$$\frac{dr}{ds}=\left(k^2-1+\frac{r^2}{R^2}-\frac{h^2}{r^2}+\frac{h^2}{R^2}\right)^{1/2}$$

$\frac{dr}{dt}=\frac{dr}{ds}\cdot\frac{ds}{dt}=\frac{e^{\nu}}{k}\frac{dr}{ds}$. For $e^{\nu}\frac{dt}{ds}=k$

$$=\frac{1}{k}\left(1-\frac{r^2}{R^2}\right)\frac{dr}{ds} \tag{11}$$

or, $\frac{dr}{dt}=\frac{1}{k}\left(1-\frac{r^2}{R^2}\right)\left(k^2-1+\frac{r^2}{R^2}-\frac{h^2}{r^2}+\frac{h^2}{R^2}\right)^{1/2}$

Differentiating with respect to t, we get

$$\frac{d^2r}{dt^2}=\frac{1}{k}\left(-\frac{2r}{R^2}\right)\frac{dr}{dt}\left(k^2-1+\frac{r^2}{R^2}-\frac{h^2}{r^2}+\frac{h^2}{R^2}\right)^{1/2}$$
$$+\frac{1}{k}\left(1-\frac{r^2}{R^2}\right)\frac{1}{2}\left(k^2-1+\frac{r^2}{R^2}-\frac{h^2}{r^2}+\frac{h^2}{R^2}\right)^{-1/2}\left(\frac{2r}{R^2}+\frac{2h^2}{r^3}\right)\frac{dr}{dt}$$
$$=\left(-\frac{2r}{R^2}\right)\cdot\left(\frac{dr}{dt}\right)^2\cdot\frac{1}{(1-r^2/R^2)}+\frac{1}{k^2}\left(1-\frac{r^2}{R^2}\right)^2\cdot\frac{1}{(dr/dt)}\cdot\left(\frac{r}{R^2}+\frac{h^2}{r^3}\right)\frac{dr}{dt}$$

or, $\frac{d\phi}{dt} = \frac{h}{k}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)$
Putting $\frac{dr}{dt} = 0$ in (12),

$$\frac{d^2r}{dt^2} = \left(\frac{1 - r^2/R^2}{k}\right)^2 \left(\frac{r}{R^2} + \frac{h^2}{r^3}\right)$$

This indicates that for $\frac{dr}{dt} = 0$, $\frac{d^2r}{dt^2} > 0$. It indicates that a particle will never return to the perihelion after it has reached it and begun to travel away from it at t_0.

Substituting $\frac{dr}{dt} = 0$ in (11), we obtain

$$k^2 - 1 + \frac{r^2}{R^2} - \frac{h^2}{r^2} + \frac{h^2}{R^2} = 0$$

This obtain the value of r at perihelion.Fo $r = R$, (11) and (13) are reduced to

$$\frac{dr}{dt} = 0 = \frac{d\phi}{dt}$$

This illustrates that all motion will cease inside a radius R. The perceived horizon of the cosmos is the name given to this radius.

For a particle at rest at origin with $h = 0$ we find that $\frac{d^2x^\alpha}{ds^2} = 0, (\alpha = 1,2,3)$. This demonstrates that the particle's acceleration is zero. It means that in de-Sitter world a particle at rest at origin with $h = 0$ remains at rest.

Step IV. Shift in spectral lines.

When a light beam from a far-off star travels in a radial path in the direction of the origin,
or,

$$ds, d\theta, d\phi = 0$$

Consequently $0 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + \left(1 - \frac{r^2}{R^2}\right) dt^2$

$$\frac{dr}{dt} = \pm\left(1 - \frac{r^2}{R^2}\right)$$

The required path is obtained by

$$dt = -\frac{dr}{\left(1 - \frac{r^2}{R^2}\right)}$$

If t is the amount of time it takes for a light beam to move from $r = 00^{10} r = R$, then

$$\begin{aligned} t &= \int_0^R \frac{dr}{\left(1 - \frac{r^2}{R^2}\right)} \\ &= \int_0^{\pi/2} \frac{R\cos\psi d\psi}{\cos^2\psi}, \frac{r}{R} = \sin\psi \\ &= R\int_0^{\pi/2} \sec\psi d\psi \\ &= R[\log(\sec\psi + \tan\psi)]_0^{\pi/2} \\ &= R[\infty - 0] = \infty \\ &= \infty \end{aligned}$$

It implies that a light beam would travel between the origin and the horizon in an indefinite amount of time as observed by an observer at the origin; in other words, the observer would never be aware of what was happening at the horizon.

Let δt_1 be the separation between two consecutive wavecrests that are emitted from a far-off star, and δt_2 be the corresponding time that an observer at rest at the origin receives them, so that

$$\delta t_2^0 = \delta t_2$$

$$\int_{t_1}^{t_2} dt = \int_r^0 -\frac{dr}{(1 - r^2/R^2)}$$

From which, $t_2 - t_1 = \int_0^r \frac{dr}{(1-r^2/R^2)}$.

Differentiating w.r.t. t_1, $\frac{\delta t_2}{\delta t_1} - 1 = \frac{dr/dt}{1-r^2/R^2}$

where $\frac{dr}{dt}$ denotes the radial velocity at $t = t_1$.

Thus $\frac{\delta t_2}{\delta t_1} = 1 + \frac{dr/dt}{1-r^2/R^2}$

where$\frac{dr}{dt} = \left(\frac{dr}{dt}\right)_{t=t_1}$
We obtain $e^{\nu}\frac{dt}{ds} = k$.

$$\therefore \left(1 - \frac{r^2}{R^2}\right) dt = kds$$

From which, $\left(1 - \frac{r^2}{R^2}\right)\delta t_1 = k\delta t_1^0$
or

$$\delta t_1^0 = \frac{1}{k}\left(1 - \frac{r^2}{R^2}\right) = \delta t_1$$

Also we have seen $\delta {t_2}^0 = \delta t_2$.Dividing, $\frac{\delta {t_1}^0}{\delta {t_2}^0} = \frac{1}{k}(1 - r^2/R^2) \cdot \frac{\delta t_1}{\delta t_2}$
or

$$\frac{\delta t_2^0}{\delta t_1^0} = \frac{\delta t_2}{\delta t_1} \cdot \frac{k}{(1 - r^2/R^2)}$$
$$\frac{\delta t_2^0}{\delta t_1^0} = \frac{k}{1 - r^2/R^2} + \frac{k(dr/dt)}{(1 - r^2/R^2)^2}$$

or
Since $k > 0$ and $1 - \frac{r^2}{R^2} > 0$.
It means that the sign of $\frac{\delta {t_1}^0}{\delta {t_2}^0}$ depends upon the sign of dr/dt, which is radial velocity at time $t = t_1$.

When $\frac{dr}{dt} > 0$, then $\frac{\delta t_2^0}{\delta t_1^0} > 0$, meaning thereby there exists red shift.

If $\frac{dr}{dt} < 0$, then $\frac{\delta t_2^0}{\delta t_1^0} < 0$, showing thereby there exists violet shift. For $\frac{dr}{dt}$ is so large that it makes the R.H.S. of (15) to be negative if $\frac{dr}{dt} < 0$.

Thus we see that there is a possibility of both red and void shifts. But the possibility of red shift is more prominent.
$065^{55^{10}} \log 9$
Let λ_0 and $\lambda_0 + \delta\lambda_0$ be the wave lengths of waves noresponding to the time $\delta {t_1}^0$ and $\delta {t_2}^0$.

Then $\frac{\delta t_2^0}{\delta {t_1}^0} = \frac{c\delta {t_2}^0}{c\delta {t_1}^0} = \frac{\lambda_0+\delta\lambda_0}{\lambda_0} = 1 + \frac{\delta\lambda_0}{\lambda_0}$

II, $\frac{\delta {t_2}^0}{\delta {t_1}^0} = 1 + \frac{\delta\lambda_0}{\lambda_0}$

Consider an alternate form of de-Sitter line element given by

$$ds^2 = -e^{2t/R}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + dt^2$$

Here we find that

$\frac{\delta {t_2}^0}{\delta {t_1}^0} = e^{(t_2-t_1)/R} = e^{r/R} = 1 + \frac{r}{R}$ upto first approximation.

For distance travelled in time $t_2 - t_1$ is r.

or,

$$1 + \frac{r}{R} = \frac{\delta {t_2}^0}{\delta {t_1}^0} = 1 + \frac{\delta\lambda_0}{\lambda_0}$$

$$\frac{\delta\lambda_0}{\lambda_0} = \frac{r}{R}$$

If we assume $c = 1,$ this demonstrates that red shift is proportionate to the distance measured from the origin. It also confirms Weyl's theory, which states that nebulae are moving away from us at a speed proportional to their distance. As a result, we may observe that de-Sitter forecasts nebula recession despite being entirely empty.

Problem 1. To show that Einstein universe is not an Einstein space where as de-Sitter's universe is

Solution. The characteristic is what defines an Einstein space

$$R_{ij} = \frac{1}{n} R g_{ij}, \qquad \ldots (1)$$

where n stands for dimension of the space.(i) To examine Einstein universe.

Einstein line element is obtain by

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + dt^2$$

Here we have

$$R_{\mu\mu} = -\frac{2}{R^2} g_{\mu\mu}$$

$R_{44} = 0, R_{\mu\nu} = 0$ for $\mu \neq \nu$. where $\mu = 1,2,3$.

$$R = g^{\mu\nu} R_{\mu\nu} = \sum_{\mu=1}^{4} g^{\mu\mu} R_{\mu\mu} = \sum_{\mu=1}^{4} \frac{R_{\mu\mu}}{g_{\mu\mu}}$$

$$= \frac{R_{11}}{g_{11}} + \frac{R_{22}}{g_{22}} + \frac{R_{33}}{g_{33}} + \frac{R_{44}}{g_{44}} = -\frac{2}{R^2}(1+1+1+0). \text{ by } (2).$$

$$\frac{R}{3} = -\frac{2}{R^2}.$$

This means that $R_{\mu\mu} = \frac{R}{3} g_{\mu\mu}$ in (2). Additionally, for $\mu \neq \nu$, $R_{44} = 0, g_{44} \neq 0$, and $R_{\mu\nu} = 0$. According to these facts, $R_{\mu\nu} \neq \frac{1}{4} R g_{\mu\nu}$. Einstein µ niverse is not an Einstein space, according to this.

(ii) We now examine de-Sitter's universe, de-Sitter's line element is obtaind by

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\theta^2) + \left(1 - \frac{r^2}{R^2}\right) dt^2.$$

Here we have $R_{\mu\mu} = \frac{3}{R^2} g_{\mu\nu}$

where $\mu = 1,2,3,4$.

$$R_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

Then $R = g^{\mu''} R_{\mu\nu} = g^{11} R_{11} + g^{22} R_{22} + g^{23} R_{33} + g^{44} R_{44}$

$$= \frac{R_{11}}{g_{11}} + \frac{R_{22}}{g_{22}} + \frac{R_{33}}{g_{33}} + \frac{R_{44}}{g_{44}} = \frac{3}{R^2}(1+1+1+1), \text{ by} \qquad (3)$$

$$\therefore \frac{R}{4} = \frac{3}{R^2} \qquad (3)$$

Now (3) is reduced to

$$R_{\mu\mu} = \frac{R}{4} g_{\mu\mu}$$

Also

$R_{\mu\nu} = 0$ for $\mu \neq v, g_{\mu\nu} = 0$ for $\mu \neq v$

Hence we can write

$$R_{\mu\mu} = \frac{R}{4} g_{\mu\mu}$$

13.8 *COMPARISON OF EINSTEIN MODEL WITH ACTUAL UNIVERSE:-*

To conclude our brief discussion of the properties of the Einstein universe we must now make some comparison with the properties of the actual universe.

The Einstein model's agreement with a cosmos that might in reality contain a finite concentration of uniformly distributed matter is its most satisfying aspect. It provides us with a cosmology that is better than the de-Sitter model in this regard. This benefit is only obtained by adding the extra cosmological term Aguy to Einstein's original field equations. This is a mechanism that is comparable to the change made to Poisson's equation to allow for a uniform static distribution of matter in flat space according to Newtonian theory.

The fact that there is no basis for expecting any consistent shift in the wave length of light from distant objects is the most unsatisfying aspect of the Einstein model as a foundation for the cosmology of the real universe. However, Hubble and Humason's research in the real cosmos reveals a clear red shift in the nebulae's light that gets stronger with distance. Naturally, this is the primary factor in favoring non-static universe theories as the foundation for real cosmology.

13.9 *COMPARISON OF DE-SITTER MODEL WITH ACTUAL UNIVERSE:-*

The linear relationship between red shift and distance that Hubble and Humason found for the light from nebulae in the real universe is provided by the de-Sitter model, which includes the distribution of moving particles. In this situation, the cosmological constant A is significantly higher in the de-Sitter universe than in the Einstein universe.

The line element, when strictly interpreted, corresponds to a completely empty university that contains neither matter nor radiations, which is the most disappointing aspect of the de-Sitter model as a foundation for the cosmology of the actual universe. The successful and poor aspects of the two initial static models can be concluded in the final section. The Einstein model accounts for any red shift in the light from far-off particles, but it does not account for the universe's limited matter concentration. The observed finite concentration of the real cosmos is not supported by the model, which only allows for a red shift in the light from distant particles.

SELF CHECK QUESTIONS

1. What is meant by a "static universe"?
2. What is the main reason the Einstein model is considered outdated today?
3. What does a redshift in the light from a galaxy tell us?
4. What role does general relativity play in cosmological models?

13.10 *SUMMARY:-*

In this unit, we explored the fundamental cosmological models used to describe the structure and evolution of the universe. We examined the Einstein line element and the associated Einstein universe**,** which assumes a static, closed cosmos with uniformly distributed matter and a cosmological constant to counterbalance gravity. We also studied the de Sitter line element**,** which describes an empty, expanding universe driven purely by the cosmological constant. The properties of the Einstein universe highlighted its attempt to maintain a static cosmos, though it fails to explain the observed redshift of distant galaxies. In contrast, the properties of the de Sitter universe allow for expansion and redshift but lack realistic matter content. We compared both models with the actual universe**,** finding that while the Einstein model is outdated due to its static nature, the de Sitter model offers better agreement with observational evidence such as Hubble's redshift–distance relation. These comparisons emphasize the need for non-static, dynamic models**,** paving the way toward more accurate representations like the FLRW and ΛCDM models used in modern cosmology.

13.11*GLOSSARY:-*

- **Einstein Universe:** A static, closed model of the universe proposed by Albert Einstein in 1917, where the universe is filled with uniformly distributed matter and maintained in a stable, unchanging state by a cosmological constant.
- **Cosmological Constant (Λ):** A term introduced by Einstein in his field equations of General Relativity to counteract gravity,

enabling a static universe. Later, it became associated with dark energy, which is responsible for the accelerated expansion of the universe.

- **de Sitter Universe:** A model of the universe proposed by Willem de Sitter in 1917, describing an empty, expanding universe with a large cosmological constant but no matter. It explains the redshift of distant galaxies but lacks the presence of matter.
- **Friedmann-Lemaître-Robertson-Walker (FLRW) Model:** A family of solutions to Einstein's field equations that assumes a homogeneous, isotropic universe. These models describe an expanding or contracting universe filled with matter and energy, and they serve as the basis for modern cosmological models like the ΛCDM model.
- **Redshift:** The phenomenon where light from distant objects in the universe appears shifted toward longer wavelengths due to the expansion of the universe. The redshift increases with the distance of the object, as observed by Hubble.
- **Hubble's Law:** A key observation that the velocity of galaxies moving away from us is proportional to their distance, indicating the expansion of the universe.
- **Expansion of the Universe:** The concept that space itself is stretching, causing galaxies to move farther apart over time. This phenomenon is supported by observations of the redshift in light from distant galaxies.
- **Static Universe:** A universe that remains unchanged over time, neither expanding nor contracting. The Einstein static universe was an early attempt to model this idea, though it was later discarded due to the discovery of the expanding universe.
- **Dark Energy:** A form of energy associated with the cosmological constant (Λ), which is responsible for the accelerated expansion of the universe. It makes up a significant portion of the universe's total energy content.
- **Cold Dark Matter (CDM):** A form of matter that does not emit, absorb, or reflect light, making it undetectable by electromagnetic radiation. It interacts with regular matter through gravity and is believed to be responsible for the formation of large-scale structures in the universe.
- **ΛCDM Model:** The standard model of cosmology that includes dark energy (Λ), cold dark matter (CDM), and normal matter. It

explains the large-scale structure of the universe and its accelerated expansion.

- **Line Element:** A mathematical expression that describes the geometry of space-time. In cosmology, the line element is used to express the distances between points in a universe model, such as the Einstein line element or the de Sitter line element.
- **Isotropy:** The property of the universe where it looks the same in every direction. This assumption is fundamental in the FLRW model.
- **Homogeneity:** The property of the universe where the same physical properties are present everywhere on a large scale. It is another assumption in the FLRW model.
- **Curvature of Space:** Refers to the bending of space-time caused by the presence of mass and energy, described by general relativity. A universe can have positive curvature (closed), zero curvature (flat), or negative curvature (open).

13.12 *REFERENCES:-*

- Edward J. Corbelli, et al.(2019), "Observational Tests of the ΛCDM Model: Status and Prospects" *Astrophysical Journal.*
- Antonio De Felice, et al. (2017), "Cosmological Models with Modified Gravity: Theories, Applications, and Observations" Springer Briefs in Physics".

13.13 *SUGGESTED READING:-*

- Farook Rahman (2021), The General Theory of Relativity: A Mathematical Approach
- Satya Prakash,Revised by K.P.Gupta, Ninteenth Edition (2019), Relativistic Mechanics.
- Dr. J.K.Goyal & Dr.K.P.Gupta (2018), Theory of Relaivity.

13.14 *TERMINAL QUESTIONS:*

(TQ-1) Describe Einstein's model of universe. Show that Einstein's universe is neither an Einstein space nor a constant curvatur.

(TQ-2) Discuss Einstein's model of universe and compare it w actual universe.

(TQ-3) Write an essay on static cosmological models.

(TQ-4) Describe the three possibilities of a static model of universe and bring out the similarity and difference be them.

(TQ-5) Obtain the equations of the geodesics from a varia principle.

(TQ-6) Discuss the three crucial tests of general relativity.

(TQ-7) Show how the general relativity modifies the equation d planetary orbit and explain the advance of the perihelion.

(TQ-8) Derive de-Sitter's model of the universe and discuss physical properties.

(TQ-9) Derive Einstein's model of the universe and discuss properties.

(TQ-10) Describe the salient features of Einstein's the de-Sitter cosmological models, and discuss the inadequancy of static models.

(TQ-11) Write short notes on 'cosmological models'.

(TQ-12) Show that in de-Sitter's universe there may be both red violet shift, but the tendency of red shift is more prominent.

(TQ-13) Obtain the line element for Einstein's universe and discuss its properties.

(TQ-14) Discuss the physical properties of de-Sitter universe and compare it with those of the actual universe.

(TQ-15) Obtain the line elements for Einstein and de-Sitter's cosmological models.

(TQ-16) Obtain the line element for de-Sitter's cosmological model and discuss fully the motion of a particle in this universe by investigating the shape of its orbit, and its velocity and acceleration in the orbit.

(TQ-17) Compare and contrast de-Sitter's world with Einstein's world.

(TQ-18) Deduce an expression for the Einstein line element for a static universe, stating the assumptions made, and find the total mass of universe.

(TQ-19) Obtain the line element for Einstein's and de Sitter's compological models.

(TQ-20) Indicate the unsatisfactory features of Einstein's model as compared with actual universe.

(TQ-21) Show that de-Sitter's model corresponds to a completely empty universe without matter or radiation.

13.15 *ANSWERS:*

SELF CHECK ANSWERS

1. A static universe is one in which the overall size and geometry do not change over time neither expanding nor contracting.
2. Because it fails to explain the redshift of distant galaxies and does not account for the expansion of the universe, which has been clearly observed.
3. It tells us that the galaxy is moving away from us, indicating that the universe is expanding.
4. General relativity provides the mathematical framework for modeling the universe's structure and evolution, forming the basis for solutions like the Einstein, de Sitter, and FLRW models.

UNIT 14:-Electrodynamics

CONTENTS:

14.1 *INTRODUCTION: -*

Electrodynamics, from a mathematical perspective, is the study of how electric and magnetic fields interact and evolve in space and time, governed by Maxwell's equations, which are a set of four coupled partial differential equations. These equations expressed using vector calculus describe how electric fields $(\vec{E})$ and magnetic fields $(\vec{B})$, arise from charge distributions (ρ) and currents $(\vec{J})$, and how they propagate as electromagnetic waves in free space or media. The mathematics of electrodynamics involves solving these equations using tools like gradient, divergence, curl, and Laplacian operators, often within the framework of boundary conditions and gauge choices. This rigorous formulation provides the foundation for understanding phenomena such as wave propagation, radiation, and the behavior of circuits and materials under electromagnetic fields.

14.2 *OBJECTIVES: -*

After studying this unit, the lernear's will be able to

- To understand the behavior of electric and magnetic fields in space and time, and how they influence each other.
- To develop and solve Maxwell's equations in various physical situations, using vector calculus and boundary conditions.

- To understanding the Electromagnetic Problems with the help mathematical techniques.

14.3 *THEOREMS: -*

Theorem 1. To show that a charge in motion is accompanied by a magnetic field.

Or

To show that an electromagnetic field is produced by an electric Field.

Proof: Orested observed that an ordinary electric current of density J might generate a magnetic field of strength H. Maxwell proposed that shifting electric displacement may create a magnetic field. For unoccupied space, the Morcover electric displacement is linearly proportional to the electric field strength E. It indicates that magnetic fields are produced by electric fields.

"A long straight stationary wire carrying a current sets a magnetic field."

Assume a system S contains electric charges. Assume that a system S is traveling along the X-axis with velocity u in relation to S. In such case, an observer in S will only see electric fields; in contrast, an observer in S will see both magnetic and electric fields. Thus, the electric field in system S generates an electromagnetic field in system S'. Lorentz transformation

$$E_x = E'_x, E_y = \beta\left(E'_y + uH'_z\right), E_z = \beta\left(E'_z - uH'_y\right)$$
$$H_x = H'_x, H_y = \beta\left(H'_y - uE'_z\right), H_z = \beta\left(H'_z + uE'_y\right)$$
$$\beta = \sqrt{\left\{\left(1 - \frac{u^2}{c^2}\right)\right\}}$$

As a result, the Lorentz-transformation makes it possible to understand that magnetic and electric fields cannot be changed independently. This amount to saying that a change in motion is accompanied by a magnetic field.

Theorem 2. To prove the existence of vector potential A and scalar potential ϕ With the help of Maxwell Lorentz equations.

Proof. Maxwell's electromagnetic field equations for unoccupied space are provided by

$$\text{div E} = 4\pi\rho \quad ..(1)$$

$$\text{div H} = 0 \quad ...(2)$$

$$curl\, E = \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \quad ...(3)$$

$$curl\, \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} j \quad ...(4)$$

where $E, H,$ and J stand for current density, magnetic field intensity, and electric field intensity, respectively. Additionally, if an electric charge with a density of ρ moves with velocity u, $J = \rho u$ (2) implies the existence of a vector A, also known as a vector potential, such that

$$\mathbf{H} = \text{curl } \mathbf{A} \qquad \ldots(5)$$

For div $curl \ \equiv 0$.

$$\left(\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$

Now (3) becomes curl $E = -\frac{1}{c}\frac{\partial}{\partial t}\text{curl}A$

$$\text{curl}\left(-\frac{1}{c}\cdot\frac{\partial \mathbf{A}}{\partial t} - \mathbf{E}\right) = 0.$$

This implies that there exists a scalar potential ϕ such that

$$grad\phi = -1/c \ \ \partial \boldsymbol{A}/\partial t - \boldsymbol{E}. \qquad \ldots(6)$$

To prove

$$\Box^2\mathbf{A} = -\frac{4\pi\rho\mathbf{u}}{c} \qquad \ldots(7)$$

$$\Box^2\phi = -4\pi\rho, \qquad \ldots(8)$$

$$\frac{4\pi}{c}\rho\mathbf{u} = \text{curl}\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \text{ by (4)}$$

$$= \text{curl curl}\mathbf{A} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$= curl\ curl\boldsymbol{A} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + grad\phi\right) \quad by\ (6)$$

$$= curl\ curl\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + grad\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right).$$

Utilizing the vector calculus formula,

$$grad\ div\boldsymbol{a} - \nabla^2\boldsymbol{a} = curl\ curl\ \boldsymbol{a} \qquad \ldots(*)$$

we obtain

$$\frac{4\pi}{c}\rho\boldsymbol{u}\ grad\ div\boldsymbol{A} \text{ -}\nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + grad\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

$$= grad\ \left(\text{div } \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) - \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}$$

$$= grad\ \left(div\ \boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)\Box^2 \text{ - } \Box^2\boldsymbol{A}.$$

Choosing ϕ such that

$$div\ \boldsymbol{A} + \left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = 0 \qquad \ldots(9)$$

we obtain $\frac{4\pi}{c}\rho u = \Box^2 A$

$$\Box^2\mathbf{A} = \frac{4\pi}{c}\rho\mathbf{u}$$

Thus, the equation (7) is proved.

From (6), $\nabla\nabla\emptyset \;=\; \nabla.\left(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) - E$

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\right)\nabla\mathrm{A} - \nabla\mathbf{E} = \left(\frac{1}{c}\frac{\partial}{\partial t}\right)\left(-\frac{1}{c}\frac{\partial\phi}{\partial t}\right) - 4\pi\rho, \text{ by (1) and (9),}$$

$$\nabla^2\emptyset - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 4\pi\rho$$

Or $\Box^2\emptyset = -4\pi\rho$

Consequently, equation (8) is proved.

14.3 *GAUGE TRANSFORMATION*:-

The vector potential A and scalar potential ϕ solutions for E and H are as follows:

$$H \;=\; curl\,A \qquad \ldots(1)$$

$$E = \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - grad\,\phi \qquad \ldots(2)$$

are not unique as any scalar function s gradient can be added to A and defined

$$A' \;=\; A \;+\; grad\,s \qquad \ldots(3)$$

Then the new vector $H' \;=\; curl\,A' \;=\; curl\,(A + grad\,s)$

$H' = \; curl\,A \;+\; curl\,grad\,s \;=\; curl\,A \;=\; H$ as $curl\,grad\,s \;= 0$

So $H' = H$

In the previous calculation, ϕ must be replaced by a new function if we further demand that the new electric field intensity E' stay constant when A is substituted by A'.

$$\phi' \text{ s.t. } \phi' = \phi - \frac{1}{c}\frac{\partial S}{\partial t} \qquad \ldots(4)$$

Hence $\cdot\ \mathbf{E}' = -\frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t} - \mathrm{grad}\phi'$ according to (2),

$$= -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{A} + \mathrm{grad}s) - \mathrm{grad}\left(\phi - \frac{1}{c}\frac{\partial s}{\partial t}\right),$$

By (3) and (4)

$$\mathbf{E}' = \left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\phi\right) - \text{grad}\left(-\frac{1}{c}\frac{\partial s}{\partial t} + \frac{1}{c}\frac{\partial s}{\partial t}\right)$$
$$= \mathbf{E} - \text{grad}(0) = \mathbf{E}.$$

As a result, we can observe that E and H do not change despite the changes.

$$\mathbf{A}' = \mathbf{A} + \text{grad}s$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial s}{\partial t}$$

These transformations are known as Gauge Transformations.

Question 1. **In case of free space, prove that**

$$\Box^2\mathbf{E} = \mathbf{0} = \Box^2\mathbf{H}.$$

Proof. In case of free space, $\rho = 0$.
Now Maxwell's equations for free space are

$$\text{div}\mathbf{E} = 0 \qquad \ldots(1)$$
$$\text{div}\mathbf{H} = 0 \qquad \ldots(2)$$
$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \ldots(3)$$
$$\text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \ldots(4)$$

For

$$J = \rho u = 0u = 0$$

Taking curl in (3),

$$curl\ curl\ \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}curl\ \boldsymbol{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right),\ by\ (4)$$

or, $$curl\ curl\ \boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

But $grad\ div\ \boldsymbol{E} = \nabla^2\boldsymbol{E} + curl\ curl\ \boldsymbol{E}$. Then the last gives or,

$$\therefore\ grad\ div\ \boldsymbol{E} - \nabla^2\boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0.$$

or, $grad(0) - \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E} = \mathbf{0}$, by (1).

$$0 - \Box^2\boldsymbol{E} = \mathbf{0}$$

Taking curl in (4), $curl\ curl\ \boldsymbol{H} = \frac{1}{c}\frac{\partial}{\partial t}curl\ \boldsymbol{E} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2}$, by (3) *or*,

$$curl\ curl\ \boldsymbol{H} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = \boldsymbol{0}.$$

But $$grad\ div\ \boldsymbol{H} = \nabla^2\ \boldsymbol{H} + curl\ curl H$$.

Henc

$$grad\ div H - \nabla^2 H + \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = 0$$

the last gives

or,

$$\text{grad}(0) - \Box^2 H = 0.\ \text{by (2)}$$

or,

$$0 - \Box^2\mathbf{H} = \mathbf{0}$$

or,

$$\Box^2\mathbf{H} = \mathbf{0} \qquad (6)$$

Combining (5) and (6),

$$\Box^2\mathbf{H} = 0 = \Box^2\mathbf{E}.$$

14.4 *TRANSFORMATION EQUATIONS FOR DIFFERENTIAL OPERATORS:*

Questions2. To prove invariance of D' Alembert operator $\Box^2$ with respect to Lorentz transformation.

Or. Prove the invariance of

$$\boldsymbol{\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}$$

Proof. Examine at two systems, S and S', where S', ' is moving relative to S with velocity v in the positive direction of the X-axis. Assume that the coordinates of an event in S and $S'$are Let (x, y, z, t) and $(\ x', y', z', t'\ )$ respectively.

D' Alembertain operator in $S = \Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

D' Alembertain operator in $S' = \Box'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$

Lorentz Transformation are

$$x' = \beta(x - vt), y' = y, z' = z, t' = \beta\left(t - \frac{vx}{c^2}\right), \beta = 1/\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

Lorentz inverse Transformation are

$$x = \beta(x' - vt'), \mathrm{y} = y', \mathrm{z} = z', \mathrm{t} = \beta\left(t - \frac{vx}{c^2}\right), \beta = \left(\mathrm{t}' + \frac{vx'}{c^2}\right)$$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial x'}\cdot\frac{\partial}{\partial t}$$

$$= \beta\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z} + \beta\cdot\frac{\mathrm{v}}{c^2}\frac{\partial}{\partial t} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\text{or}$$

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial y'} = \frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial y'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial y'}\cdot\frac{\partial}{\partial t}$$

$$= 0\frac{\partial}{\partial x} + 1\cdot\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z} + 0\frac{\partial}{\partial t} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}.$$

Similarly $\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial t'}\cdot\frac{\partial}{\partial y} + \frac{\partial z}{\partial t'}\cdot\frac{\partial}{\partial z} + \frac{\partial t}{\partial t'}\cdot\frac{\partial}{\partial t}$$

$$= v\beta\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial t} = \beta\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

Thus, $\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \frac{\partial}{\partial z'} = \frac{\partial}{\partial z},$

$$\frac{\partial}{\partial t'} = \beta\cdot\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial x'^2} = \frac{\partial}{\partial x'}\frac{\partial}{\partial x'} = \beta^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial^2}{\partial y'^2} = \frac{\partial}{\partial y'}\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} = \frac{\partial^2}{\partial y^2}.$$

Similarly $\frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z^2}$

$$\frac{\partial}{\partial t'^2} = \frac{\partial}{\partial t'}\frac{\partial}{\partial t'} = \beta^2\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

$$= \beta^2\left[v^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x\partial t}\right]$$

$$\therefore \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \beta^2\left[\frac{v^2}{c^2}\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\cdot\frac{\partial^2}{\partial x\partial t}\right]$$

$$\square'^2 = \frac{\partial^2}{x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^3}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

$$= \beta^2\left[\frac{\partial^2}{\partial x^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t}\right] + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$-\beta^2\left[\frac{v^2}{c^2}\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t}\right]$$

$$= \beta^2\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x^2} - \frac{\beta^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \text{ For } \beta^2\left(1 - \frac{v^2}{c^2}\right) = 1.$$

$$= \square^2$$

$$\therefore\ \square'^2 = \square^2$$

This proves that $\square^2$ is invariant under Lorentz transformation.

Question3. Prove that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is not invariant under Lorentz transformation.

Solution. Adding (1), (2) and (3) of Theorem 3, we get

$$\frac{\partial^2}{\partial' x^2} + \frac{\partial^2}{\partial' y^2} + \frac{\partial^2}{\partial' z^2}$$

$$= \beta^2\left[\frac{\partial^2}{\partial x^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t}\right] + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This proves that ∇^2 is not invariant under Lorentz transformation.

Theorem 1. To find Lorentz transformations of electric field *E* and magnetic Field component H.

Proof. Let S and S'be two systems, where S' is moving with velocity v in the X-axis's positive direction with respect to S. In the S system, let A_x, A_y, A_z be components of A, and in the S' system, let A_x', A_y', A_z' be components of A. Lorentz transformations state

$$A'_x = \beta\left(A_x - \frac{v}{c}\phi\right), A'_y = A_y, A'_z = A_z, \phi' = \beta\left(\phi - \frac{v}{c}A_x\right),$$

where $\beta = 1/\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$.

A being electromagnetic vector potential and ϕ scalar potential.

We obtain

$$\text{grad}\phi = -\mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{H} = \text{curl}\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{2}$$

Lorentz inverse transformations

$$x = \beta(x' + vt'), y = y', z = z', t = \beta\left(t' + \frac{v}{c^2}x'\right)$$

$$\beta = 1/\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

Then

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \frac{\iota}{\partial t'} = \beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

From (1), we have

$$\text{grad}'\phi' = -\mathbf{E}' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial \iota'}$$

or,

$$\mathbf{E}' = -\frac{1}{c}\frac{\partial \mathbf{A}'}{\partial \boldsymbol{t}'} - \text{grad}'\phi \qquad \ldots(1')$$

From which $E'_x = -\frac{1}{c}\frac{\partial A'_x}{\partial t'} - \frac{\partial \phi'}{\partial x'}$ or

$$E'_x = -\frac{\beta}{c}\left(\frac{\partial}{\partial t} + \frac{v\partial}{\partial x}\right)\beta\left(A_x - \frac{v\phi}{c}\right)$$
$$-\beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\beta\left(\phi - \frac{v}{c}A_x\right)$$

$$= -\beta^2 \left[\frac{1}{c}\left(\frac{\partial A_x}{\partial t} + \frac{v\partial A_x}{\partial x} - \frac{v}{c}\frac{\partial \phi}{\partial t} - \frac{v^2}{c}\frac{\partial \phi}{\partial x}\right) + \left(\frac{\partial \phi}{\partial x} - \frac{v}{c}\frac{\partial A_x}{\partial x} + \frac{v}{c^2}\frac{\partial \phi}{\partial t} - \frac{v^2}{c^3}\frac{\partial A_x}{\partial t}\right)\right]$$

$$= -\beta^2 \left[\frac{1}{c}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial A_x}{\partial t} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial \phi}{\partial x}\right]$$

$$= -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = E_x$$

From (1') ${E_y}' = -\frac{1}{c}\frac{\partial A'_y}{\partial t'} - \frac{\partial \phi'}{\partial y'}$ or

$$E'_y = -\frac{\beta}{c}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)A_y - \frac{\partial}{\partial y}\beta\left(\phi - \frac{vA_x}{c}\right)$$

$$= -\beta\left[\left(\frac{1}{c}\frac{\partial A_y}{\partial t} + \frac{\partial \phi}{\partial y}\right) + \frac{v}{c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \beta\left(E_y - \frac{v}{c}H_z\right), \text{ by (1) and (2).}$$

$$E'_z = -\frac{1}{c}\frac{\partial A'_z}{\partial t'} - \frac{\partial \phi'}{\partial z'}, \text{ by } (1')$$

$$= -\frac{\beta}{c}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)A_z - \frac{\partial}{\partial z}\beta\left(\phi - \frac{v}{c}A_x\right)$$

$$= \beta\left[\left(-\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z}\right) + \frac{v}{c}\left(-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}\right)\right]$$

$$= \beta\left[E_z + \frac{v}{c}H_{y'}\right]$$

From (2), ${H_x}' = \left(\frac{\partial {A_z}'}{\partial y'} - \frac{\partial {A_y}'}{\partial z'}\right) = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = H_\chi,$

$$H'_{y'} = -\frac{\partial A'_z}{\partial x'} + \frac{\partial A'_x}{\partial z'} = -\beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)A_z + \frac{\partial}{\partial z}\beta\left(A_x - \frac{v\phi}{c}\right)$$

$$= \beta\left[\left(-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}\right) + \frac{v}{c}\left(-\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z}\right)\right] = \beta\left[H_y + \frac{v}{c}E_z\right]$$

$$H'_z = \frac{\partial A'_y}{\partial x'} - \frac{\partial A'_x}{\partial y'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)A_y - \frac{\partial}{\partial y}\beta\left(A_x - \frac{v\phi}{c}\right)$$

$$= \beta\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \frac{v}{c}\left(\frac{1}{c}\frac{\partial A_y}{\partial t} + \frac{\partial \phi}{\partial y}\right) = \beta\left[H_z - \frac{v}{c}E_y\right]$$

Thus the transformation equations are

$$E'_x = E_x, E'_y = \beta\left(E_y - \frac{v}{c}H_z\right), E'_z = \beta\left[E_z + \frac{v}{c}H_y\right]$$
$$H'_x = H_x, H'_y = \beta\left[H_y + \frac{v}{c}E_z\right], H'_z = \beta\left[H_z - \frac{v}{c}E_y\right]$$

The inverse transformations are

$$E_x = E'_x, E_y = \beta\left({E_y}' + \frac{v}{c}H'_z\right), E_z = \beta\left(E'_z - \frac{v}{c}H'_y\right)$$
$$H_x = H'_x, H_y = \beta\left[H'_y - \frac{v}{c}E'_z\right), H_z = \beta\left[H'_z + \frac{v}{c}E'_y\right].$$

Remark. Take $v = (v, 0,0)$. Then the above transformations can be but in vector form as

$$\mathbf{E}' = \beta\mathbf{E} + (1-\beta)\frac{\mathrm{v}}{v^2}(\mathrm{v}\cdot\mathbf{E}) + \frac{\beta}{c}(\mathrm{v}\times\mathbf{H})$$
$$\mathbf{H}' = \beta\mathbf{H} + (1-\beta)\frac{\mathbf{v}}{v^2}(\mathbf{v}\cdot\mathbf{H}) - \frac{\beta}{c}(\mathrm{v}\times\mathbf{E})$$

and the inverse transformations are

$$\mathbf{E} = \beta\mathbf{E}' + (1-\beta)\frac{\mathbf{v}}{v^2}(\mathrm{v}\cdot\mathbf{E}') - \frac{\beta}{c}(\mathrm{v}\times\mathbf{H}')$$
$$\mathbf{H} = \beta\mathbf{H}' + (1-\beta)\frac{\mathrm{v}}{v^2}(\mathrm{v}\cdot\mathbf{H}) + \frac{\beta}{c}(\mathrm{v}\times\mathbb{E}')$$

Theorem 2 : To prove that $J^\mu\rho = \rho_0 v^\mu$

Proof. Suppose a charge of density ρ moving with velocity v produces a current of density J.Let a current of density J be produced by a charge of density ρ which is moving with velocity v.

Then $J = \rho v.\ Let\ ui + vj + wk = v.$

$$J_x = \rho u, J_y = \rho v, J_z = \rho w.$$

We define

$$J^\mu = \left(J_x, J_y, J_z, \rho\right) = (\rho u, \rho v, \rho w, \rho)$$
$$= \rho(u, v, w, 1) = \rho\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1\right)$$
$$= \rho\frac{ds}{dt}\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dt}{ds}\right)$$
$$= \rho\frac{ds}{dt}\left(\frac{dx^1}{ds}, \frac{dx^2}{ds}, \frac{dx^3}{ds}, \frac{dx^4}{ds}\right)$$

Taking $\rho_0 = \rho\frac{ds}{dt} =$ proper density of electric charge, we have or

$$J^\mu = \rho_0\left(\frac{dx^1}{ds}, \frac{dx^2}{ds}, \frac{dx^3}{ds}, \frac{dx^4}{ds}\right) = \rho_0 v^\mu$$

$J^{\mu} = \rho_0 v^{\mu}$ and v^{μ} stands for four-dimensional velocity.

Theorem 3. To prove Maxwell's equations are invaria, (covariant) under Lorentz transformations.

Proof. **Step I**. We will first identify the transformation equations for **E** and **H**.

$$\text{Thus } \frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{dt}\right), \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \frac{\partial}{\partial z'} = \frac{\partial}{\partial z},$$

$$E'_x = E_x, E'_y = \beta\left(E_y - \frac{\partial}{c}H_z\right), E'_z = \beta\left(\frac{\partial}{\partial t'} + \frac{v}{\partial t} + \frac{v}{c}H_y\right),$$

$$H'_x = H_x, H'_y = \beta\left(H_y + \frac{v}{c}E_z\right), H'_z = \beta\left(H_z - \frac{v}{c}E_y\right).$$

$$\frac{\partial}{\partial x} = \beta\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right), \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}, \frac{\partial}{\partial t} = \beta\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)$$

$$E_x = E'_x, E_y = \beta\left(E_y{}' + \frac{v}{c}H'_z\right), E_z = \beta\left(E'_z - \frac{v}{c}H'_y\right),$$

$$H_x = H'_x, H_y = \beta\left(H'_y - \frac{v}{c}E'_{z'}\right), H_z = \beta\left(H'_z + \frac{v}{c}\cdot E'_y\right).$$

Step II. To prove Maxwell's equations are invariant under Lorentz transformations.

$$divE = 4\pi\rho \quad \ldots(1)$$
$$div\, H = 0 \quad \ldots(2)$$
$$\text{curl E} = \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \quad \ldots(3)$$
$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}j \quad \ldots(4)$$

Cartesian equivalent of these equations are

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \quad ..(1')$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0. \quad \ldots(2')$$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{i}H_x + \boldsymbol{j}H_y + \boldsymbol{k}H_z\right) \quad \ldots(3')$$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{i}E_x + \boldsymbol{j}E_y + \boldsymbol{k}E_z\right) + \frac{4\pi}{c}\boldsymbol{J} \quad \ldots(4')$$

Substituting values in (2'),

$$\beta\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)H'_x + \frac{\partial}{\partial y'\left(H'_y - \frac{v}{c}E'_z\right)}\beta + \frac{\partial}{\partial z'}\left(H'_z + \frac{v}{c}E'_y\right)\beta = 0$$

Taking
$$P = \frac{\partial H'_x}{\partial x'} + \frac{\partial H_y{}'}{\partial y'} + \frac{\partial H_z{}'}{\partial z'}$$
$$Q = \frac{1}{c} \cdot \frac{\partial H'_x}{\partial t'} + \frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'}$$
We get $\beta \left[P - \frac{v}{c} Q\right] = 0$

Dividing by β, we obtain $P = \frac{v}{c} Q$ ... (5′)

From (3'), $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial H_x}{\partial t} = 0$ ... (3′a′)

Substituting values of E_z, E_y, H_x etc.,
$$\frac{\partial}{\partial y'} \beta (E'_z - \frac{v}{c} H'_y)$$
$$- \frac{\partial}{\partial z'} \beta \left(E''_y + \frac{v}{c} H'_z\right) + \frac{1}{c} \beta \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'}\right) H'_x = 0$$
or,
$$\beta \left[Q - \frac{v}{c} P\right] = 0$$
or,
$$Q = \frac{v}{c} P.$$
Using (5), we have
$$Q = \frac{v}{c}\frac{v}{c} Q$$

or,
$$\left(1 - \frac{v^2}{c^2}\right) Q = 0$$
or
$$Q = 0,\ For\ 1 - \frac{v^2}{c^2} \neq 0$$
Using this in (5), $P = 0$.

Thus
$$P = 0 = Q$$
i.e.,
$$\frac{\partial H'_x}{\partial x'} + \frac{\partial H'_y}{\partial y'} + \frac{\partial H'_z}{\partial z'} = 0 \qquad \ldots (2'')$$
$$\frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} = -\frac{1}{c} \cdot \frac{\partial H'_x}{\partial t'} \qquad \ldots (3'a'')$$
When combined, the equations (2′) and (2′′) suggest that equation (2) is invariant with respect to the Lorentz transformation. Equation (3) is invariant with respect to Lorentz transformation because the equations ($3'a'$) and ($3'a''$) taken together imply the equation ($3'a'$) and consequently (3′). As a result, we have demonstrated that the Lorentz transformation does not affect equations (2) and (3). Likewise, we can

demonstrate that equations (1) and (4), the other two are likewise Lorentz invariant.

Theorem 4. To prove that the Maxwell equations for empty space are represented by the two equations

$$\boldsymbol{J^{\mu} = F^{\mu\nu}_{,\mu}}$$

$$\boldsymbol{F_{\mu\nu,\sigma} + F_{\nu\sigma,\nu} + E_{\sigma\mu,\nu} = 0}$$

To derive "Maxwell's equation in tensor form.

Proof. A framework for explaining Maxwell's equations in a manner that complies with physics' principles is offered by the special theory of relativity. Special relativity describes the behavior of electromagnetic fields in spacetime, but it does not take into account interactions between spacetime and electrodynamics in the sense of altering the geometry of spacetime. The framework used to describe Maxwell's equations is

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2$$

Holds so that $\Gamma^{k}{}_{ij} = 0 = \Gamma_{ij,k}$

Maxwell's equations for empty space are

$$\text{div}\mathbf{E} = \rho \qquad \text{... (1)}$$

$$\text{div}\mathbf{H} = 0 \qquad \text{... (2)}$$

$$\text{curl}\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \qquad \text{... (3)}$$

$$\text{curl}\mathbf{H} = -\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \qquad \text{... (4)}$$

Assuming that the speed of light is one. Here, the component 4π is eliminated from the equations above by using the Heavy Lorentz unit of change.

$$\text{Set } J = (\sigma_x, \sigma_y, \sigma_z), J^{\mu} = (\sigma_x, \sigma_y, \sigma_z, \rho)$$

where ρ stands for charge density and $\sigma_x, \sigma_y, \sigma_z$ denote components of current density.

There are scalar potential ϕ and electromagnetic potential $\mathbf{A}$ such that

$$\mathbf{H} = \text{curl}\mathbf{A}, \qquad \text{grad}\phi = -\frac{\partial \mathbf{A}}{\partial t} - E$$

i.e.,

$$\mathbf{i}H_x + \mathbf{j}H_y + \mathbf{k}H_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad \text{... (5)}$$

$$\nabla\phi = -\frac{\partial(\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)}{\partial t} - (\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z) \qquad \text{... (6)}$$

Cartesian equivalent of (3) and (4) are

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial}{\partial t}(\mathbf{i}H_x + \mathbf{j}H_y + \mathbf{k}H_z)$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -\frac{\partial}{\partial t}\left(\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z\right) + \left(\mathbf{i}\sigma_x + \mathbf{j}\sigma_y + \mathbf{k}\sigma_z\right) \quad \ldots (4')$$

We define generalized potential k'' as

$$k'' = \left(A_x, A_y, A_z, \phi\right)$$

in terms of ordinary electromagnetic potential A and scalar potential ϕ.
The associate covariant vector k_μ of k^μ is defined as

$$k_\mu = g_{\mu l}k^l = g_{\mu\mu}k^\mu. \text{ For } g_{ij} = 0 \text{ for } i \neq j$$

$$\therefore\ k_\mu = g_{\mu\mu}k^\mu.$$

$$\text{This } \Rightarrow\ k_1 = g_{11}k^1 = -k^1, k_2 = -k^2, k_3 = -k^3$$

$$k_4 = g_{44}k^4 = k^4$$

Therefore, we must have

$$k_\mu = \left(-A_x, -A_y, -A_z, \phi\right) \quad \ldots (7)$$

We define an electromagnetic tensor F_{ij} as

$$F_{ij} = k_{i,j} - k_{j,i}$$

This is equivalent to

$$F_{ij} = \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x^i}\cdot \text{ For } \Gamma_{ij}^k = 0\ \forall i, j \text{ and } k.$$

$$\text{This } \Rightarrow F_{ij} = -F_{ji}, F_{ii} = 0 \text{ so that } F^{ii} = 0.$$

$$F_{14} = \frac{\partial k_1}{\partial x^4} - \frac{\partial k_4}{\partial x^1} = -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = E_x, \text{ by (6).}$$

$$F_{24} = \frac{\partial k_2}{\partial x^4} - \frac{\partial k_4}{\partial x^2} = -\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial y} = E_y, \text{ by (6).}$$

$$F_{34} = \frac{\partial k_3}{\partial x^4} - \frac{\partial k_4}{\partial x^3} = -\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z} = E_z, \text{ by (6).}$$

$$F_{23} = \frac{\partial k_2}{\partial x^3} - \frac{\partial k_3}{\partial x^2} = -\frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y} = H_x, \text{ by (5).}$$

$$F_{31} = \frac{\partial k_3}{i x^1} - \frac{\partial k_1}{1.3} = \frac{\partial(-A_2)}{\partial x} - \frac{\partial(-A_x)}{\partial z} = H_y, \text{ by (5)}$$

Similarly $\mathrm{F}_{12} = \mathrm{H}_2$.
Thus, we have proved that

$$F_{14} = E_x, F_{24} = E_y, F_{34} = E_z$$

$$F_{23} = H_x, F_{31} = H_y, F_{12} = H_z$$

Consider the tensor equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \quad \ldots (8)$$

$$F_{,j}^{ij} = J^i \quad \ldots (9)$$

In our frame work, these equations become

$$J^i = \frac{\partial F^{ij}}{\partial x^j} \quad \ldots (8')$$

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0 \quad \ldots (9')$$

$$F^{12} = g^{1\alpha}F_{\alpha\beta}g^{\beta 2} = g^{11}g^{22}F_{12} = (-1)(-1)F_{12} = F_{12}$$
$$F^{14} = g^{1\alpha}g^{4\beta}F_{\alpha\beta} = g^{11}g^{44}F_{14} = (-1)(1)F_{14} = -F_{14}.$$

The final result is

$$F^{12} = F_{12} = H_z = -F_{21}, F^{23} = F_{23} = H_x = -F_{32},$$
$$F^{31} = F_{31} = A_y = -F_{13}$$
$$F^{14} = -F_{14} = -E_x = F_{41}, F^{24} = -F_{24} = -E_y = F^2$$
$$F^{34} = --_{34} = -E_z = F_{43}$$

I. Putting i=1 in (8′)

$$J^1 = \frac{\partial F^{1j}}{\partial x^j} = \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\hat{c}t^j} + \frac{\partial F^{14}}{\partial x^4}$$
$$0 + \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{\partial E_x}{\partial t} = \sigma_x.$$
$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\partial E_x}{\partial t} + \sigma_x$$

For i =2, (8′)gives

$$\frac{\partial F^{2j}}{\partial x^j} = J^2$$
$$\frac{\partial F^{21}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{23}}{\partial x^3} + \frac{\partial F^{24}}{\partial x^4} = \sigma_y$$

or,

$$-\frac{\partial H_z}{\partial x} + 0 + \frac{\partial H_x}{\partial z} - \frac{\partial E_y}{\partial t} = \sigma_y$$

or,

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\partial E_y}{\partial t} + \sigma_y.$$

or,

For $i = 3$,.

$$\frac{\partial F^{3j}}{\partial x^j} = J^3$$
$$\frac{\partial F^{31}}{\partial x^1} + \frac{\partial F^{32}}{\partial x^2} + \frac{\partial F^{33}}{\partial x^3} + \frac{\partial F^{34}}{\partial x^4} = J^3$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} + 0 - \frac{\partial E_z}{\partial t} = \sigma_z$$

or,

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\partial E_z}{\partial t} + \sigma_z.$$

or,

For $i = 4$, (8′) gives

$$\frac{\partial F^{4j}}{\partial x^j} = J^4$$
$$\frac{\partial F^{41}}{\partial x^1} + \frac{\partial F^{42}}{\partial x^2} + \frac{\partial F^{43}}{\partial x^3} + \frac{\partial F^{44}}{\partial x^4} = \rho,$$

or,

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}+0=\rho. \qquad \dots(1')$$

Together, the equations ($4''$), ($4'''$) and ($4''''$) yield (4^'). For the equation (4). Equation (1) is reflected in equation (1').
As a result, equation (8) represents equations (1) and (4).
Taking $i=1, j=2, k=3\ in$ ($9'$), we get

$$\frac{\partial F_{12}}{\partial x^3}+\frac{\partial F_{23}}{\partial x^1}+\frac{\partial F_{31}}{\partial x^2}=0$$

$$\frac{\partial H_z}{\partial z}+\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}=0$$

$Div\ H\ =0$, which is the equation (2),
Taking $i=1, j=2, k=4\ in$ ($9'$), we obtained

$$\frac{\partial F^{12}}{\partial x^4}+\frac{\partial F_{24}}{\partial x^1}+\frac{\partial F_{41}}{\partial x^2}=0$$

or,

$$\frac{\partial H_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0$$

or,

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial H_z}{\partial t}$$

Taking $i=\ 2, j=3, k=4\ in$ (9’),we have

$$\frac{\partial F_{23}}{\partial x^4}+\frac{\partial F_{34}}{\partial x^2}+\frac{\partial F_{42}}{\partial x^3}=0$$

$$\frac{\partial H_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=0$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial H_z}{\partial t}.$$

Taking $i=\ 1, j=\ 3, k=4$, we have

$$\frac{\partial F_{13}}{\partial x^4}+\frac{\partial F_{34}}{\partial x^1}+\frac{\partial F_{41}}{\partial x^3}=0$$

$$-\frac{\partial H_y}{\partial t}+\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}=0$$

or,

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial H_y}{\partial t}$$

The equation ($3'$), or (3), is represented by the sum of the equations ($3''$), ($3'''$), and($3'''$) .
As a result, (9), represents equations (2) and (3).
Consequently, the Maxwell equations are represented by

$$J^\mu=F^{\mu\nu}_{,\nu}$$

$$F_{\mu\nu,\sigma}+F_{\nu\sigma,\mu}+F_{\sigma\mu,\nu}=0$$

14.5*LORENTZ FORCE ON A MOVING CHARGE:-*

Question 4. If E and **H** are the electric and magnetic field intensities respectively of an electromagnetic field, show that the electromagnetic force **f** experienced by a single charged particle carrying a electric charge e moving with instantaneous velocity V is given by

$$\mathbf{f} = e\left[E + \frac{1}{c}(\mathbf{V} \times \mathbf{H})\right]$$

Proof. Let an observer S' be moving with velocity v along X-axis w.r.t. an observer S. Let a particle carrying an electric charge e be moving with the same velocity v along X-axis relative tothe observer S so that

$$u_x = v, u_y = 0, u_z = 0$$

For the observer $S^{'}$, the same particle seems to be at rest.

$$u'_x = 0, u'_y = 0, u'_z = 0.$$

As a result, the field in S' will be entirely electrostatic. therefore

$$\dot{H}'_x = 0, H'_y = 0, H'_z = 0$$

The charge and electric field strength $E'\left(E'_x, E'_y, E'_z\right)$ will be multiplied to determine the force $\mathbf{F}'\left(F'_x, F'_y, F'_z\right)$ acting on the charge e in relation to the cbserver S'.So that

$$F'_x = e'E'_x, F'_y = e'E'_y, F'_z = e'E'_z$$

Because the charge e is not affected by the Lorentz translation,

$$e' = e.$$

Consequently

$F'_x = cE'_x, F'_y = cE'_y, F'_z = cE'_z.$
In system S, the same field seems to be electromagnetic in nature. In relation to S, the force components are provided by

$$F_x = F'_x, F_y = \sqrt{\left(1 - \frac{v^2}{c^2}\right)}F'_y, F_z = F'_z\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

[This follows from Lorentz transformation for a force].

$$F_x = cE'_x, F_y = cE'_y\sqrt{\left(1-\frac{v^2}{c^2}\right)}, F_z = cE'_z\sqrt{\left(1-\frac{v^2}{c^2}\right)}$$

or, $F_x = eE'_x, F_y = \frac{e}{\beta}E'_y, F_z = \frac{e}{\beta}E'_z$
or, $F_x = eE_x, F_y = \frac{e\beta}{\beta}\left(E_y - \frac{v}{c}H_z\right), F_z = \frac{e}{\beta}\beta\left(E_z + \frac{v}{c}H_y\right).$

or, $F_x = eE_x, F_y = e\left(E_y - \frac{u_xH_z}{c}\right), F_z = e\left(E_z + \frac{u_xH_y}{c}\right).$

In general, the force components are provided by

$$\mathbf{F} = e\left[E + \frac{1}{c}(\mathbf{u}\times\mathbf{H})\right]$$

where the velocity of the charge is denoted by $\mathbf{u}(u_x, u_y, u_z)$ rather than v

According to the specified issue, we obtain

$$\mathbf{F} = \mathbf{f}, \mathbf{u} = \mathbf{V}$$

so that the last obtains

$$\mathbf{f} = e\left[\mathbf{E} + \frac{1}{c}(\mathbf{V}\times\mathbf{H})\right]$$

This is the required result.
Note. The force $\mathbb{F}$ per unit volume is obtained by

$$\mathbf{F} = \rho\left[\mathbf{E} + \frac{1}{c}(\mathbf{u}\times\mathbf{H})\right] = \rho\mathbf{E} + \frac{\rho}{c}(\mathbf{u}\times\mathbf{H})$$

Taking the velocity of light to be unity,

$$\mathbf{F} = \rho\mathbf{E} + \rho(\mathbf{u}\times\mathbf{H})$$

$$\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z = \rho(\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z) + \rho\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_x & u_y & u_z\\ H_x & H_y & H_z\end{vmatrix}$$

Since $\sigma_x = \rho u_x, \sigma_y = \rho u_y$ etc.

The last one is therefore equal to the following series of equations:

$$F_x = \rho E_x + (\sigma_y H_z - \sigma_z H_y)$$
$$F_y = \rho E_y + (\sigma_z H_x - \sigma_x H_z)$$
$$F_z = \rho E_z + (\sigma_x H_y - \sigma_y H_x)$$

The rate at which work is being done, let's assume W, is provided by $i.e.$,

$$W = \frac{\text{Force } \times \text{ distance}}{\text{time}} = \text{ Force } \times \text{ velocity}$$
$$= \rho E_x u_x + \rho E_y u_y + \rho E_z u_z$$
$$W = \rho\left(E_x u_x + E_y u_y + E_z \cdot u_z\right)$$

Because it acts perpendicular to the direction of the current, the magnetic component of the force is ineffective.

Define

$$h_\mu = F_{\mu\nu} J^\nu.$$

Then $$h_1 = F_{1\nu} J''$$

$$= F_{11}J^1 + F_{12}J^2 + F_{13}J^3 + F_{14}J^4$$
$$= 0.\sigma_x + H_z\sigma_y + \left(-H_y\sigma_z\right) + E_x\rho$$
$$= \rho E_x + \left(H_z\sigma_y - H_y\sigma_z\right) = F_x$$

Similarly, $h_2 = F_y, h_3 = F_z$

$$h_4 = F_{4\nu}J^\nu = F_{41}J^1 + F_{42}J^2 + F_{43}J^3 + F_{44}J^4$$
$$= -E_x \cdot \sigma_x + \left(-E_y\right)\sigma_z + (-E_{zi})\sigma_z + 0.\rho$$
$$= -\left(E_x \cdot \sigma_x + E_y \cdot \sigma_y + \sigma_z \cdot E_z\right)$$
$$= -\rho\left(E_x \cdot u_x + E_y \cdot u_y + E_z \cdot u_z\right) = -W$$

It is obvious from what has been done that

$$h_\mu = F_{\mu\nu}J^\nu = \left(F_x, F_y, F_z, -W\right)$$

14.6*ELECTROMAGNETIC ENERGY MOMENTUM TENSOR:-*

Question 5.To prove $\boldsymbol{E_j^i = -F^{ik}F_{jk} + \frac{1}{4}f_j^i F_{kj}F^{ki}}$
or

$$\boldsymbol{T_i^j = -F^{jk}F_{ik} + \frac{1}{4}\delta_l^j F^{\alpha\beta}F_{\alpha\beta}}$$

Proof. The line element is taken into consideration here.

$$ds^2 = -\chi^2 - dy^2 - dz^2 + dt^2$$

In order for each covariant derivative to decrease to its matching partial derivative.
The following relations must be used in order to explicitly calculate the value of the electromagnetic energy momentum tensor T^{μ}_{ν}:

$$K^{\mu} = A_x, A_y, A_z, \phi \qquad \ldots (1)$$

:where **A** and ϕ are vector potential and scalar potential

$$F_{ij} = K_{i,j} - K_{j,i} \qquad \ldots (2)$$

where F_{ij} is field tensor and is antisymmetric

$$J^{i} = F^{ij}_{,j} = \frac{\partial F^{ij}}{\partial x^{i}} \qquad \ldots (3)$$

where J^{i} is current vector.

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = \frac{\partial F_{ij}}{\partial x^{k}} + \frac{\partial F_{jk}}{\partial x^{i}} + \frac{\partial F_{ki}}{\partial x^{j}} = 0 \quad \ldots (4)$$

$$h_{\mu} = F_{\mu\nu} v' \qquad \ldots (5)$$

By(3)and (5), $h_{\mu} = F_{\mu\nu} J^{\nu} = F_{\mu\nu} F^{\nu\sigma}_{,\sigma} = F_{\mu\nu} \frac{\partial F^{\nu\sigma}}{\partial x^{\sigma}}$.
The electromagnetic energy momentum tensor is defined.

$$T^{\mu}_{\nu} \text{ as } h_{\mu} = T^{\nu}_{\mu,\nu} \qquad \ldots (6)$$

$$T^{\nu}_{\mu,\nu} = F_{\mu\nu} \frac{\partial F^{\nu\sigma}}{\partial x^{\sigma}} \qquad \ldots (7)$$

This differential equation's solution is

$$T^{\nu}_{\mu} = -F^{\nu\sigma} F_{\mu\sigma} + \frac{1}{4} \delta^{\nu}_{\mu} F^{\alpha\beta} F_{\alpha\beta} \qquad \ldots (8)$$

To verify this, we take into account the divergence of both sides, taking into account that covariant differentiation obeys the normal distributive law and that δ^{ν}_{μ} is current.

$$T^{\nu}_{\mu,\nu} = -\left(F^{\nu\sigma}_{,\nu} F_{\mu\sigma} + F^{\nu\sigma} F_{\mu\sigma,\nu}\right) + \frac{1}{4} \delta^{\nu}_{\mu} \left(F^{\alpha\beta} F_{\alpha\beta,\nu} + F_{\alpha\beta} F^{\alpha\beta}_{,\nu}\right)$$

Using the fact $A_{\alpha\beta} B^{\alpha\beta} = A^{\alpha\beta} B_{\alpha\beta}$, we have

$$T^{\nu}_{\mu,\nu} = -\left(F^{\nu\sigma}_{,\nu} F_{\mu\sigma} + F^{\alpha\beta} F_{\mu\beta,\alpha}\right) + \frac{1}{2} \delta^{\nu}_{\mu} F^{\alpha\beta} F_{\alpha\beta,\nu}$$

$$= -F^{\nu\sigma}_{,\nu} F_{\mu\sigma} - \frac{1}{2} F^{\alpha\beta} F_{\mu\beta,\alpha} - \frac{1}{2} F^{\beta\alpha} F_{\mu\alpha,\beta} + \frac{1}{2} F_{\alpha\beta,\mu} F^{\alpha\beta}$$

$$= -F_{\mu\sigma}F^{\nu\sigma}_{,\nu} + \frac{1}{2}F^{\alpha\beta}\left(F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta}\right)$$

$$\left[\text{ as } F^{\alpha\beta} = -F^{\beta\alpha}, F_{\mu\alpha} = -F_{\alpha\mu}\right]$$

$$= F_{\mu\sigma}F^{\sigma\nu}_{,\nu} + 0, \text{ by virtue of (4)}$$

$$= F_{\mu\sigma}{}'^{\sigma} = h_\mu, \text{ by (3) and (5)}$$

or, $h_\mu = T^{\nu}_{\mu,\nu}$ which is true by virtue of (5).
Hence the solution (8) is a correct solution of (7).

14.7 *LAW OF GRAVITATIONAL IN ELECTROMEGNETIC FIELD:-*

Question 6.To derive field equations in electromagnetic field.
Proof. In terms of the field tensor F_{ij}, the electromagnetic energy momentum tensor is defined as
with

$$T^i_j = -F_{j\alpha}F^{i\alpha} + \frac{1}{4}\delta^i_j F_{\alpha\beta}F^{\alpha\beta}$$

$$T^{ij} = g^{\alpha j}T^i_\alpha \text{ and } T = T^i_i$$

Here, some writers substitute the sign E^{ij} for T^{ij}.

$$T = T^i_i = -F_{i\alpha}F^{i\alpha} + \frac{1}{4}\delta^i_i F_{\alpha\beta}F^{\alpha\beta}$$

$$= -F_{\beta\alpha}F^{\beta\alpha} + \frac{1}{4}4F_{\alpha\beta}F^{\alpha\beta} \qquad \text{... (1)}$$

$$T = 0. [i.e., \text{Trace of encrgy momentum tensor } = 0]$$

The field equations are obtain by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi T_{ij} \qquad \text{... (2)}$$

From which, $\quad g^{ij}R_{ij} - \frac{1}{2}Rg_{ij}g^{ij} = -8\pi T_{ij}g^{ij}$

or, $\quad R - \frac{1}{2}R4 = -8\pi T$

or, $\quad -R = -8\pi T = -8\pi(0) = 0$, by (2)

or, $\quad -R = 0$ or $R = 0$.

Using this in (2), $\quad R_{ij} = -8\pi T_{ij}$.

This is the required formulation for field equations in electrodynamics.

14.8 *ENERGY AND MOMENTUM OF THE ELECTRO-MAGNETIC FIELD:-*

The rate at which forces are performing work W is determined by

$$\frac{dW}{dt} = \frac{\text{Force} \times \text{distance}}{\text{time}} = \text{Force} \times \text{velocity}$$

$$= \rho E_x u_x + \rho E_y \cdot u_y + \rho E_z \cdot u_z$$

as work = Froce × distance

or,

$$\frac{dW}{dt} = \rho\mathbf{E}.\mathbf{u} \text{ or, } \frac{dW}{dt} = \rho\mathbf{u}\cdot\mathbf{E} = \mathbf{J}.\mathbf{E}.$$

Since the magnetic part of the force does not work as it acts in a direction perpendicular to the direction of current. Also, current density

$$J = \rho\mathbf{u}$$

Therefore, the rate of work in a space with a fixed volume is given by

$$\frac{dW}{dt} = \int \mathrm{J}.\mathrm{E}dv \qquad \ldots(1)$$

By Maxwell's equations,

$$\text{curl } E = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \ldots(2)$$

$$\text{curl } H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\mathrm{J} \qquad \ldots(3)$$

BY (3) $\quad \text{curl } H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\mathbf{J}$

With the Heavy Lorentz change unit, we obtain

$$\frac{J}{c} = \text{ curl } H - \frac{1}{c}\frac{\partial E}{\partial t} \cdot (\text{ Now the factor } 4\pi \text{ is disappeared })$$

or,

$$\mathbf{J}\cdot E = \left(c\text{curl } H - \frac{\partial E}{\partial t}\right)\cdot \mathbb{E} - \mathbb{H}\cdot\left(\text{curl } E + \frac{\partial \mathbf{H}}{\partial t}\right)$$

or,

$$\frac{J}{c} = \text{ curl } H - \frac{1}{c}\frac{\partial E}{\partial t} \cdot (\text{ Now the factor } 4\pi \text{ is disappeared })$$

or,

$$\mathbf{J}\cdot E = \left(c\text{curl}H - \frac{\partial E}{\partial t}\right)\cdot E - H\cdot\left(c\text{curl}\mathbb{E} + \frac{\partial \mathbf{H}}{\partial t}\right)$$

$$\frac{J}{c} = \text{ curl } H - \frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} \cdot (\text{ Now the factor } 4\pi \text{ is disappeared })$$

or,

$$\mathbf{J}\cdot E = \left(c\text{curl } H - \frac{\partial E}{\partial t}\right)\cdot E - H\cdot\left(c\text{curl}E + \frac{\partial \mathbf{H}}{\partial t}\right)$$

or,

$$\frac{dW}{dt} = -\int \frac{\partial}{\partial t}\left(\frac{E^2 + H^2}{2}\right)dv$$

$$(\mathbf{H}.\mathit{curl}\ \mathbf{E} - \mathbf{E}.\text{curl } \mathbf{H})dv$$

or,

$$\frac{dW}{dt} = -\frac{\partial}{\partial t}\int\left(\frac{E^2 + H^2}{2}\right)dv - c\int[\mathbf{E}\times\mathbf{H}]_n ds \quad \ldots(4)$$

Where n denotes outward normal component and $[\mathrm{E} \times \mathrm{H}]_n$ denotes normal component of the vector in $\mathbf{E} \times \mathbf{H}$.
And

$$\operatorname{div}(E \times H) = \mathbf{H}.\operatorname{curl} E - E.\operatorname{curl} \mathbf{H}$$

$$\int \operatorname{div}(E \times H) \mathrm{dV} = \int \quad \hat{n}.(\mathbf{E} \times \mathbf{H}) ds$$

The rate of energy change in the electromagnetic field inside the volume under consideration is the first term on R.H.S. of (4). The rate at which energy flows over this volume's surface is the second term in R.H.S. of (4). Consequently, we can take

$$\frac{1}{2}(E^2 + H^2) = \text{ density of electromagnetic energy,}$$

$$c(E \times H) = \text{ density of energy flow.}$$

or, We can use $g = c\left(\frac{\mathbf{E}\times\mathbf{H}}{c}\right) = \frac{\mathbf{E}\times\mathbf{H}}{c^2}$ as the density of momentum and $\rho = \frac{1}{c^2}\left(\frac{E^2+H^2}{2}\right)$ as the density of electromagnetic mass.

14.9 *ELECTROMAGNETIC STRESS:-*

We know that the Maxwell's equations are

$$\operatorname{div} \mathbf{E} = \rho \qquad \ldots (1)$$

$$\operatorname{div} \mathbf{H} = 0 \qquad \ldots (2)$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \ldots (3)$$

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{\rho \mathbf{u}}{c} \qquad \ldots (4)$$

In this case, the component 4π is eliminated by using the Heavy Lorentz unit of change. The momentum's density g is determined by

$$g = \frac{1}{c}(\mathbf{E} \times \mathbf{H}) \qquad \ldots (5)$$

According to the theorem, the Lorentz force F acting on an electric charge e moving at velocity u is $\mathbf{F} = \left\{\mathbf{E} + \frac{1}{c}(\mathbf{u} \times \mathbf{H})\right\} e$.
The force **F** per unit volume is written by

$$\mathbf{F} = \rho\left\{\mathbf{E} + \frac{1}{c}(\mathbf{u} \times \mathbf{H})\right\} \qquad (6)$$

If **G** is the charged particle's momentum, the rate at which the electromagnetic field charges it is determined by

$$\frac{d\mathbf{G}}{dt} = \int \mathbf{F} dv = \int \rho\left[\mathbf{E} + \frac{1}{c}(\mathbf{u} \times \mathbf{H})\right] dv$$

where the integral is taken over a fixed volume in the space

$$\frac{d\mathbf{G}}{dt} = \int \left[\rho\mathbf{E} + \frac{1}{c}\rho\mathbf{u} \times \mathbf{H}\right] dv$$
$$= \int \left[\mathbf{E}\text{div}\mathbf{E} + \left(\text{curl}\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial\mathbf{t}}\right) \times \mathbf{H}\right] dv$$

[This follows from (1) and (4)]

or, $\frac{d\mathbf{G}}{dt} = \int \left[\{\mathbf{E}\text{div}\mathbf{E} + (\text{curl}\mathbf{H}) \times \mathbf{H}\} - \frac{1}{c}\left\{\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{H}) - \mathbf{E} \times \frac{\partial\mathbf{H}}{\partial\mathbf{t}}\right\}\right] dv$

$$= \int \left[\{\mathbf{E}\text{div}\mathbf{E} + (\text{curl}\mathbf{H}) \times \mathbf{H}\} - \frac{\partial\mathbf{g}}{\partial t} + \frac{1}{c}\mathbf{E} \times \frac{\partial\mathbf{H}}{\partial\mathbf{t}}\right] dv, \text{ by (5)}$$
$$= \int \left[\{\mathbf{E}\text{div}\mathbf{E} + (\text{curl}\mathbf{H}) \times \mathbf{H}\} - \frac{\partial\mathbf{g}}{\partial t} - \mathbf{E} \times \text{curl}\mathbf{E}\right] dv, \text{ by (5)}$$

or, $\frac{d\mathbf{G}}{dt} = \int \left[\mathbf{E}\text{div}\mathbf{E} + \{(\text{curl}\mathbf{H}) \times \mathbf{H} + (\text{curl}\mathbf{E}) \times \mathbb{E}\} - \frac{\partial\mathbf{g}}{\partial t}\right] dv \ldots (6)$

Writing the x-component of this equation,

$$\frac{dG_x}{dt} = \int \left[\left[\frac{1}{2}\frac{\partial}{\partial x}\left(E_x^2 - E_y^2 - E_z^2 + H_x^2 - H_y^2 - H_z^2\right) + \frac{\partial}{\partial y}(E_x E_y + H_x H_y) + \frac{\partial}{\partial t}(E_x E_z + H_x H_z) - \frac{\partial g_x}{\partial t}\right] dv \quad \ldots (7)$$

If we now categorize the electromagnetic field's stress components as

$$p_{ii} = -\frac{1}{2}\left(E_i^2 - E_j^2 - E_k^2 + H_i^2 - H_j^2 - H_k^2\right)$$
$$p_{ij} = -\left(E_i E_j + H_i H_j\right)$$

then (7) can be expressed as

$$\frac{dG_x}{dt} = \iiint -\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) dxdydz.$$

or,

$$\frac{dG_x}{dt} + \frac{\partial g_x}{\partial t} = -\iiint \left(-\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) dxdydz.$$

Question 7: Calculate T_j^i in terms of $\overline{\mathbf{E}}$ and H.

Solution. We know that

$$T_j^i = -F_{j\alpha}F^{i\alpha} + \frac{1}{4}\delta_j^i F_{\alpha\beta}F^{\alpha\beta} \quad \ldots (1)$$
$$F_{\alpha\beta} = -F_{\beta\alpha}, F_{14} = E_x, F_{24} = E_y, F_{34} = E_z$$
$$F_{23} = H_x, F_{31} = H_y, F_{12} = H_z$$
$$F^{12} = F_{12}, F^{13} = F_{13}, F^{32} = F_{23}$$
$$F^{14} = -F_{14}, F^{24} = -F_{24}, F^{34} = -F_{34}$$

In accordance with (1),

$$T_j^i = -F_{j\alpha}F^{i\alpha} \text{ For } i \neq j \quad \ldots (2)$$
$$T_j^i = -F_{j\alpha}F^{i\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \text{ for } i \neq j \quad \ldots (3)$$
$$F_{\alpha\beta}F^{\alpha\beta} = 2(F_{12}F^{12} + F_{13}F^{13} + F_{14}F^{14} + F_{23}F^{23} + F_{24}F^{24} + F^{34}F_{34}) + (F_{11}F^{11} + F_{22}F^{22} + F_{33}F^{33} + F_{44}F^{44})$$

$$= 2(F_{12}F_{12} + F_{13}F_{13} - F_{14}F_{14} + F_{23}F_{23} - F_{24}F_{24}$$
$$= 2\left(H_z^2 + H_y^2 - E_x^2 + H_x^2 - E_y^2 - E_z^2\right) \quad \dots (4)$$
$$= 2\left\{\left(H_x^2 + H_y^2 + H_z^2\right) - \left(E_x^2 + E_y^2 + E_z^2\right)\right\}$$
$$\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2}\left(\left\{H_x^2 + H_y^2 + H_z^2\right) - \left(E_x^2 + E_y^2 + E_z^2\right)\right\}$$

From (3), $T_1^1 = -F_{1\alpha}F^{1\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$

$$\left.\begin{aligned} T_2^2 &= -F_{2\alpha}F^{2\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \\ T_3^3 &= -F_{3\alpha}F^{3\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \\ T_4^4 &= -F_{4\alpha}F^{4\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \end{aligned}\right\} \quad \dots (5)$$

$$F_{1\alpha}F^{1\alpha} = F_{11}F^{11} + F_{12}F^{2}$$
$$= 0 + F_{12}F_{12} + F_{13}F_{13} - F_{14}F_{14}$$
$$= H_z^2 + H_y^2 - E_x^2$$
$$F_{2\alpha}F^{2\alpha} = F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23} + F_{24}F^{24}$$
$$= F_{21}F_{21} + 0 + F_{23}F_{23} - F_{24}F_{24}$$
$$= H_z^2 + H_x^2 - E_y^2$$
$$F_{3\alpha}F^{2\alpha} = F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33} + F_{34}F^{24}$$
$$= F_{31}F_{31} + F_{32}F_{32} + 0 - F_{34}F_{34}$$
$$= H_y^2 + H_x^2 - E_z^2$$
$$F_{4\alpha}F^{4\alpha} = F_{41}F^{41} + F_{42}F^{42} + F_{43}F^{43} + F_{44}F^{44}$$
$$= -F_{41}F_{41} - F_{42}F_{42} - F_{43}E_{43} + 0$$
$$= -\left(E_x^2 + E_y^2 + E_z^2\right)$$

With these values (5), they become

$$T_1^1 = \left(-H_z^2 + H_y^2 - E_x^2\right) + \tfrac{1}{2}\left[\left(H_x^2 - H_y^2 + H_z^2\right) - \left(E_X^2 + E_Y^2 - E_z^2\right)\right]$$
$$= \tfrac{1}{2}\left(E_x^2 + E_y^2 - E_z^2\right) + \left(H_x^2 - H_y^2 - H_z^2\right)$$
$$T_2^2 = -F_{2\alpha}F^{2\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$
$$= -\left(H_z^2 + H_x^2 - E_y^2\right) + \tfrac{1}{2}\left\{\left(H_x^2 + H_y^2 + H_z^2\right) - \left(E_X^2 + E_Y^2 - E_z^2\right)\right\}$$
$$= \frac{1}{2}\left[\left(E_y^2 - E_x^2 - E_z^2\right) + \left(H_y^2 - H_x^2 - H_z^2\right)\right]$$
$$T_3^3 = -F_{3\alpha}F^{3\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$
$$= -\left(H_y^2 + H_x^2 - E_z^2\right) + \frac{1}{2}\left\{\left(H_x^2 + H_y^2 + H_z^2\right) - \left(E_x^2 + E_y^2 + E_z^2\right)\right\}$$
$$= \frac{1}{2}\left[\left(E_z^2 - E_x^2 - E_y^2\right) + \left(H_z^2 - H_x^2 - H_y^2\right)\right]$$

$$T_4^4 = -F_{4\alpha}F^{4\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$
$$= \left(E_x^2 + E_y^2 + E_z^2\right)$$
$$+ \frac{1}{2}\left\{\left(H_x^2 + H_y^2 + H_z^2\right) - \left(E_x^2 + E_y^2 + E_z^2\right)\right\}$$
$$= \frac{1}{2}\left\{\left(H_x^2 + H_y^2 + H_z^2\right) - \left(E_x^2 + E_y^2 + E_z^2\right)\right\}$$

From (2), $T_2^1 = -F_{2\alpha}F^{1\alpha}$

$$= -(F_{21}F^{11} + F_{22}F^{12} + F_{23}F^{13} + F_{24}F^{14})$$
$$= -(0 + 0F^{12} + F_{13}F_{23} - F_{24}F_{14})$$
$$= -\left(0 + 0 - H_xH_y - E_yE_x\right) = H_xH_y + E_xE_y$$

Similarly, $T_1^2 = H_xH_y + E_xE_y$

$T_1^4 = -F_{1\alpha}F^{4\alpha} = -(F_{11}F^{41} + F_{12}F^{42} + F_{13}F^{43} + F_{14}F^{44})$

$$= -(0 - F_{12}F_{42} - F_{13}F_{43} + 0)$$
$$= -\left[H_zE_y + H_y(-E_z)\right] = H_yE_z - E_yH_z$$

$T_4^1 = -F_{4\alpha}F^{1\alpha} = -[F_{41}F^{11} + F_{42}F^{12} + F_{43}F^{13} + F_{44}F^{14}]$

$$= -(0 + F_{42}F_{12} + F_{43}F_{13} + 0) = -\left(-E_yH_z + E_zH_y\right)$$
$$= E_yH_z - H_yE_z = -T_1^4$$

Thus $T_1^1 = \frac{1}{2}\left[\left(E_x^2 - E_y^2 - E_z^2\right) + \left(H_x^2 - H_y^2 - H_z^2\right)\right]$

$$T_2^2 = \frac{1}{2}\left[\left(E_y^2 - E_z^2 - E_x^2\right) + \left(H_y^2 - H_z^2 - H_x^2\right)\right]$$
$$T_3^3 = \frac{1}{2}\left[\left(E_z^2 - E_y^2 - E_x^2\right) + \left(H_z^2 - H_y^2 - H_x^2\right)\right]$$
$$T_4^3 = \frac{1}{2}\left\{\left(E_x^2 + E_y^2 + E_z^2\right) + \left(H_x^2 + H_y^2 + H_z^2\right)\right\}$$
$$T_2^1 = T_1^2 = H_xH_y + E_xE_y$$
$$T_4^1 = -T_1^4 = E_yH_z - H_yE_z$$

The electromagnetic field's energy is represented by T_4^A. Momentum is represented by T_4^1.

The field's stresses are represented by T_1^1 and T_2^1, among others. The formulas in each of these situations match those found in the classical theory.

14.10 *GRAVITATIONAL FIELD DUE TO AN ELECTRON OR CHARGED PARTICLE:-*

Theorum 4: To obtain the gravitational field of an electron (on charged particle) and show that the gravitational effect of the electronic energy is very slight.

Proof. Consider a charged particle at rest at its origin, such as an electron. It is expected that this electron produces a spherically symmetric field. The line element that satisfies the spherical symmetry criterion is provided by

$$s^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu} dt^2 \qquad \dots (1)$$

where λ and ν are functions of r only such that

$$\lambda = 0 = \nu \text{ at } r = \infty.$$

We assume that the field is unquestionably electrostatic. As a result of that

$$H_x, H_y, H_z = 0. \qquad \dots (2)$$

Since

$$K_\mu = (-A_x, -A_y, -A_z, \phi)$$

$$F_{\mu\nu} = K_{\mu,\nu} - K_{\nu,\mu} = \frac{\partial K_\mu}{\partial x^\nu} - \frac{\partial K_\nu}{\partial x^\mu}$$

$$\mathbf{H} = \text{curl}\mathbf{A}.$$

In view of this, (2)

$$\Rightarrow \qquad A_x, A_y, A_z = 0 \qquad .$$

For, vanishing electromagnetic vector potential implies vanishing magnetic field intensity.

The aforementioned arguments demonstrate that ϕ is just a function of r, that is,

$$\frac{\partial \phi}{\partial \theta} = 0 = \frac{\partial \phi}{\partial \phi}$$

$$F_{14} = \frac{\partial K_1}{\partial t} - \frac{\partial K_4}{\partial r} = -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial r} = 0 - \frac{\partial \phi}{\partial r}$$

$$F_{24} = \frac{\partial K_2}{\partial t} - \frac{\partial K_4}{\partial \theta} = -\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial \theta} = 0 - 0 = 0$$

$$F_{34} = \frac{\partial K_3}{\partial t} - \frac{\partial K_4}{\partial \phi} = -\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial \phi} = 0 - 0 = 0$$

(2) can be expressed as

$$F_{23}, F_{31}, F_{12} = 0$$

Thus we have proves that

$$F_{12}, F_{23}, F_{31}, F_{24}, F_{34} = 0 \text{ and } F_{14} = -\partial\phi/\partial r.$$

This indicatesF_{14} is the sole non-vanishing component of F_{ij}. Give this fact the designation (*).

$$F^{14} = F_{\alpha\beta} g^{1\alpha} g^{4\beta} = g^{11} g^{44} F_{14} = -e^{-\lambda} e^{-\nu} (-\partial\phi\ /\partial r)$$

$$= e^{-(\lambda+\nu)} \frac{\partial\phi}{\partial r}$$

$$g = -e^{\lambda+\nu} \cdot r^4 \sin^2\theta = |g_{ij}| = g_{11} g_{22} g_{33} g_{44}$$

$$\sqrt{(-g)} = e^{(\lambda+\nu)/2} r^2 \sin\theta.$$

$$J^{\mu} = F^{\mu\nu}, \nu = \frac{F^{\mu\nu}}{\partial x^{\nu}} + F^{a\nu}\Gamma^{\mu}_{a\nu} + F^{\mu a}\Gamma^{\nu}_{a\nu}$$

$$= \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} + 0 + F^{\mu a} \frac{\partial}{\partial x^{a}} \log\sqrt{(-g)} = \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} + F^{\mu\nu} \frac{\partial}{\partial x^{\nu}} \log\sqrt{(-g)}$$

$$= \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} + \frac{F^{\mu\nu}}{\sqrt{(-g)}} \frac{\partial\sqrt{(-g)}}{\partial x^{\nu}}$$

or,

$$\begin{bmatrix} F^{a\nu}\Gamma^{\mu}_{a\nu} & = -F^{\nu a}\Gamma^{\mu}_{a\nu.}. \text{ For } F^{\nu a} \text{ is antisymmetric.} \\ & = -F^{\nu a}\Gamma^{\mu}_{\nu a}. \text{ For } \Gamma^{\mu}_{\nu a} = \Gamma^{\mu}_{a\nu} \\ & = -F^{a\nu}\Gamma^{\mu}_{a\nu}, \text{ by interchanging } a \text{ and } \nu. \\ & \text{or, } 2F^{a\nu}\Gamma^{\mu}_{a\nu} = 0, \text{ or } F^{a\nu}\Gamma^{\mu}_{a\nu} = 0. \end{bmatrix} \quad \ldots (3)$$

From (3),

$$\sqrt{(-g)} J^4 = \frac{\partial}{\partial x^{\nu}} \left(\sqrt{(-g)} F^{4\nu}\right) = \frac{\partial}{\partial x^1} \left(\sqrt{(-g)} F^{41}\right)$$

or,

$$\sqrt{(-g)}\rho = \frac{\partial}{\partial r} \left(\sqrt{(-g)} F^{41}\right)$$

gives
to the state when there is no charge or current other than at the origin

Hence the last gives
or,

$$\frac{\partial}{\partial r}\sqrt{(-g)}F^{41}\Big) = 0$$
$$\frac{\partial}{\partial r}\left[-e^{(\lambda+\nu)/2}r^2 \sin\theta \cdot e^{-(\lambda+\nu)}\frac{\partial\phi}{\partial r}\right] = 0$$

Dividing by $-\sin\theta$,

$$\frac{\partial}{\partial r}\left[r^2 e^{-(\lambda+\nu)/2}\frac{\partial\phi}{\partial r}\right] = 0.$$

Integrating, we get $e^{-(\nu+\nu)/2} \cdot r^2 \cdot \frac{\partial\phi}{\partial r} =$ const. $= \varepsilon$ (say), ε being an absolute constant.

Then

$$\frac{\partial\phi}{\partial r} = \frac{\varepsilon}{r^2}e^{(\lambda+\nu)/2}$$

$$F_{14} = -\frac{\partial\phi}{\partial r} = -\frac{\varepsilon e^{(\lambda+\nu)/2}}{r^2}$$

This $\Rightarrow$ $F^{14} = e^{-(\lambda+\nu)\frac{\partial\phi}{\partial r} - \frac{\varepsilon}{r^2}e^{-(\lambda+\nu)/2}}$.

At a great distance from the attracting particle, $g_{44} = 1 + \frac{2\psi}{c^2}$, where ψ is the Newtonian potential, occurs when the field is weak and static.

This gives $1 - \frac{2m}{r} + \frac{4\pi\varepsilon^2}{r^2} = 1 + 2\psi$ if $c = 1$ or $\psi = -\frac{m}{r} + \frac{2\pi\varepsilon^2}{r^2}$

force $= \frac{\partial\psi}{\partial r} = \frac{m}{r^2} - \frac{4\pi\varepsilon^2}{r^2}$

If $m = 0$, then the last obtains

$$= -\frac{4\pi\varepsilon^2}{r^3}, \text{ i.e., force } \propto \frac{1}{r^3}$$

This is not possible.

Consequently, m cannot be zero. In this case, we designate $4\pi\varepsilon$ as the electron's charge and m as its associated mass.

For an electron of mass m,

$$m = -7 \times 10^{-56}\text{cm}.$$
$$a = \frac{2\pi\varepsilon^2}{m} = 1.5 \times 10^{-12}\text{cm}$$

It is assumed that this number, a, is on the order of the electron's radius magnitude. At all locations outside the electron, $\frac{m}{r}$ is at least 10^{-40}.

As a result, we may observe that the electrical energy's gravitational influence is minimal.

SELF CHECK QUESTIONS

1. Write down Maxwell's equations and explain their physical significance.
2. Derive the electromagnetic wave equation from Maxwell's equations.
3. Discuss the mathematical formulation of electrodynamics, including Maxwell's equations and the electromagnetic wave equation.
4. Obtain the gravitational field of a stationary electron in vacuum.

14.11*SUMMARY: -*

In this unit, we discussed several important concepts in electrodynamics, including Gauge Transformation, Transformation Equations for Differential Operators, Maxwell's Equations, Lorentz Force on a Moving Charge, and the Electromagnetic Energy-Momentum Tensor. A Gauge Transformation refers to a change in the scalar and vector potentials that leaves the physical electric and magnetic fields unchanged, reflecting a fundamental symmetry of electromagnetism. The Transformation Equations for Differential Operators explain how mathematical operations like gradient, divergence, and curl behave under changes of coordinates, which is crucial for expressing physical laws consistently across different reference frames. Maxwell's Equations are a set of four differential equations that govern the behavior of electric and magnetic fields, their sources, and how they propagate through space. The Lorentz Force describes the force experienced by a charged particle when moving in the presence of electric and magnetic fields, combining both fields into a single expression. Finally, the Electromagnetic Energy-Momentum Tensor provides a compact and powerful way to describe the density and flow of energy and momentum carried by electromagnetic fields, playing a central role in the interaction between fields and matter, especially in the context of special relativity.

14.12*GLOSSARY:-*

- **Electric fields $(\vec{E})$**: A vector field representing the force per unit charge exerted on a stationary test charge.
- **Magnetic fields $(\vec{B})$** : A vector field representing the force per unit charge exerted on a moving charge; it arises due to moving charges (currents).
- **Maxwell's Equations**: Four fundamental equations that describe how electric and magnetic fields are generated and altered by charges and currents.
- **Gauge Transformation**: A method of changing the potentials (ϕ, A) without affecting the physical electric and magnetic fields.
- **Lorentz Force**: The total force on a charged particle moving through electric and magnetic fields, given by $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.
- **Vector Potential $(\vec{A})$**A vector field whose curl gives the magnetic field, $\vec{B} = \nabla \times (\vec{A})$.
- **Scalar Potential (ϕ)**: A scalar field whose negative gradient gives the electric field in electrostatics, $\vec{E} = -\nabla\phi$.
- **Transformation Equations for Differential Operators**: Rules describing how operators like gradient, divergence, and curl change under coordinate transformations.
- **Electromagnetic Waves**: Oscillating electric and magnetic fields that propagate through space, predicted by Maxwell's equations.
- **Poynting Vector $(\vec{S})$**: A vector representing the directional energy flux (the rate of energy transfer per unit area) of an electromagnetic field, $\vec{S} = \vec{E} \times \frac{\vec{B}}{\mu_0}$.
- **Electromagnetic Energy-Momentum Tensor**: A tensor that describes the distribution of energy, momentum, and stress in electromagnetic fields.
- **Continuity Equation**: A mathematical expression of the conservation of electric charge.
- **Displacement Current**: A term added by Maxwell to Ampère's law, accounting for a changing electric field as a source of the magnetic field.
- **Boundary Conditions**: Conditions that electric and magnetic fields must satisfy at the interface between different materials.
- **Retarded Potentials**: Solutions for the potentials that take into account the finite speed of light, representing the fields at a point due to earlier positions of the sources.

14.13*REFERENCES: -*

- Robert J. A. Lambourne (2010). Relativity, Gravitation and Cosmology.

- Michel Janssen and Christoph Lehner (2014), The Cambridge Companion to Einstein.
- Asghar Qadir (2020), Einstein's General Theory of Relativity.

14.14*SUGGESTED READING: -*

- S.P.Puri (2013), General Theory of relativity.
- Farook Rahman (2021), The General Theory of Relativity: A Mathematical Approach
- Goyal and Gupta (1975), Theory of Relativity.
- R.K.Pathria (2003), Theory of Relativity.

14.15 *TERMINAL QUESTIONS: -*

(TQ-1). Derive the electromagnetic wave equation from Maxwell's equations and explain its physical significance.

(TQ-2). Discuss the mathematical formulation of electrodynamics, including the role of vector calculus and differential equations.

(TQ-3). Explain how Maxwell's equations describe the behavior of electric and magnetic fields.

(TQ-4). Discuss the application of electrodynamics in the design of electrical systems, such as power transmission lines or antennas.

(TQ-5). Explain how electrodynamics is used in medical imaging techniques, such as MRI.

(TQ-6). Establish the invariance of Maxwell's field equations inn different spaces moving with uniform relative velocity.

(TQ-7). Derive the gravitational field of an electron.

(TQ-8). To prove $E_j^i = -F^{ik}F_{jk} + \frac{1}{4}f_j^i F_{kj}F^{ki}$

or

$$T_i^j = -F^{jk}F_{ik} + \frac{1}{4}\delta_l^j F^{\alpha\beta}F_{\alpha\beta}$$

(TQ-9). To prove that the Maxwell equations for empty space are represented by the two equations

$$J^\mu = F^{\mu\nu}_{,\mu}$$

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,v} + E_{\sigma\mu,v} = 0$$

(TQ-10). Prove Maxwell's equations are invariant under Lorentz transformations.

(TQ-11). Prove that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is not invariant under Lorentz transformation.

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