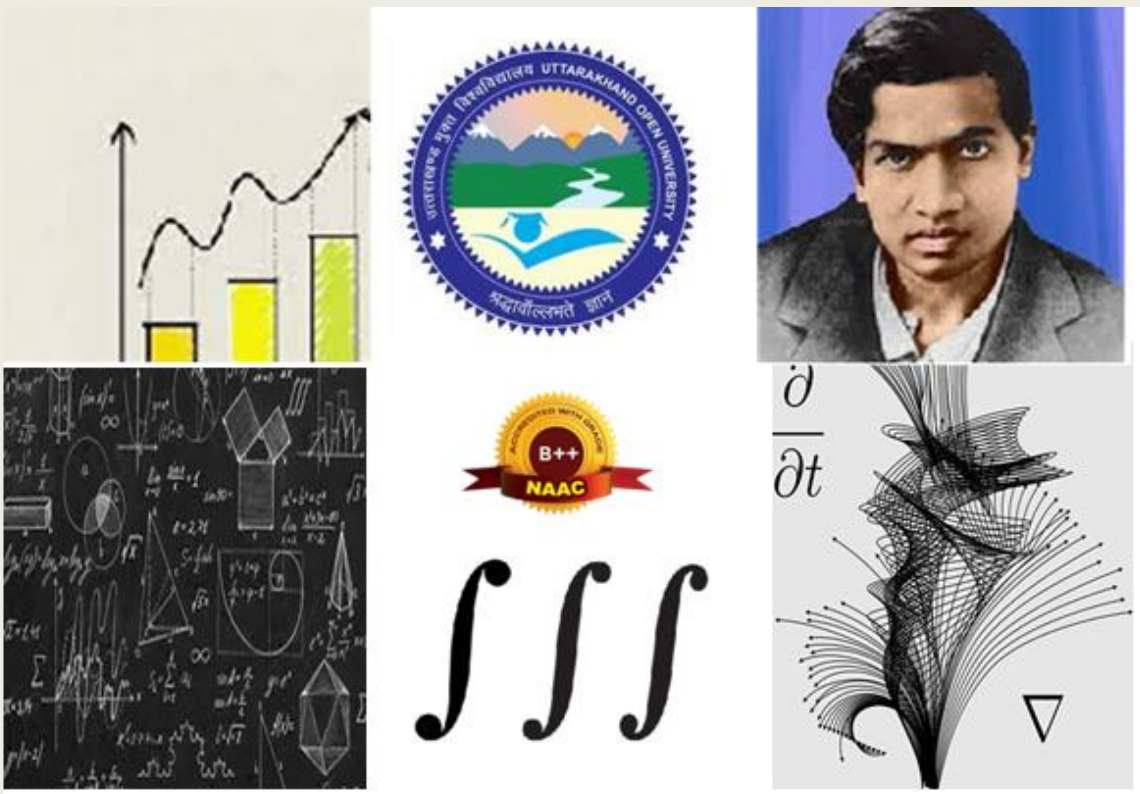
Master of Science

MATHEMATICS

Second Semester

MAT 509
MATHEMATICAL METHODS



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: MATHEMATICAL METHODS

COURSE CODE: MAT 509

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India,
263139

BOARD OF STUDIES - 2023

Editor

Dr. Arvind Bhatt

Associate Professor

Department of Mathematics

Uttarakhand Open University

Haldwani, Uttarakhand

Unit Writer	Blocks	Units
Dr. Deepak Kumar Sharma Assistant Professor Department of Mathematics Uttarakhand Open University	I, II, III and IV	1 to 14

CONTENTS
MAT-509

COURSE INFORMATION

The present self-learning material **“Mathematical Methods”** has been designed for M.Sc. (Second Semester) learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

First block is **Fourier series,** in this block Fourier series, Generalized Fourier series, Fourier Cosine series, Fourier Sine series, Fourier integral, Fourier transform and inverse Fourier Transform defined clearly.

Second block is **Integral transform,** in this block Laplace transform, convolution theorem and inverse Laplace transform and application in solving differential equation defined clearly.

Third block is **Integral equations,** in this block Volterra integral equations, Fredholm integral equations, Volterra and Fredholm equations of first and second kind, Volterra and Fredholm equations with regular kernels. Degenerate kernel, Fredholm Theorem, Method of Successive approximation. Concept and calculation of Green's function, Approximate Green's function, modified Green’s function, Green's function method for differential equations, Green's function in integral equations are defined.

Fourth block is **Calculus of Variation,** in this block concept of extrema of a functional, variation and its properties. Variational problems with fixed boundaries, The Euler equation, The fundamental lemma of calculus of variations. Variational problems with moving boundaries, Sufficient conditions for an extremum, Field of extremals, Jacobi conditions, Legendre Condition, Rayleigh- Ritz method, Galerkin’s methos are defined.

Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

Course Name: MATHEMATICAL METHODS Course Code: MAT 509

BLOCK-I

UNIT 1: FOURIER SERIES I

CONTENTS

1.1 *INTRODUCTION*

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines. Fourier Series makes use of the orthogonality relationships of the sine and cosine functions. Baron Jean Baptiste Joseph Fourier (1768−1830) first introduced the idea that any periodic function can be represented by a series of sines & cosines waves in 1828. A periodic signal is just a signal that repeats its pattern at some period. The primary reason that we use Fourier series is that we can better analyse a signal in another domain rather in the original domain.

Fig.1.1.
Ref:
https://en.wikipedia.org/wiki/File:Fourier2_-_restoration1.jpg

1.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) Periodic Function

(ii) Even and odd functions

(iii) Euler's Formulae

(iv) Fourier Series

(v) Dirichlet's conditions

1.3 *PERIODIC FUNCTION*

A function f(x) which satisfies the relation $f(x + T) = f(x)$ for all real x and some fixed T is called a periodic function. The smallest positive number T, for which this relation holds is called the period of f(x).

If T is the period of f(x),

then f(x) = f(x + T) = f(x + 2T) = … = f(x + nT) = ……

Also f(x) = f(x - T) = f(x - 2T) = ….. = f(x - nT) = ……

∴ f(x) = f(x $\pm$ nT), where n is a positive integer.

Thus, f(x) repeats itself after periods of T.

For example, sinx, cosx, secx and cosecx are periodic functions with periodic functions with period 2π.

Since $\tan(\theta + \pi) = \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$

And $\cot(\theta + \pi) = \frac{\cos(\theta+\pi)}{\sin(\theta+\pi)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta$.

Therefore $\tan\theta$ and $\cot\theta$ are periodic functions with period π.

The function $\sin nx\ and\ \cos nx$are periodic with period $\frac{2\pi}{n}$.

Note: 1. The sum of a number of periodic functions is also periodic.

2. if T_1 and T_2 are the periods of f(x) and g(x), then the period of a

f(x) + b g(x) is the least common multiple of T_1 and T_2.

For Example: cosx, cos2x and cos3x are periodic functions with periods 2π , π and $\frac{2\pi}{3}$ respectively.

∴ f(x) = cosx + $\frac{1}{2}$ cos2x + $\frac{2}{3}$ cos3x is also periodic with periodc with period 2π, the L.C.M. of 2π, π and $\frac{2\pi}{3}$.

1.4 *FOURIER SERIES*

Expansion of a function f(x) in a series of sines and cosines of multiples of x was developed by French Mathematician and physicist Jacques Fourier. We have seen how a function can be expanded in power of x by Maclaurin’s theorem but that expansion was possible only when the function and its derivatives are continuous. A need arises to expand functions which have discontinuities in their values or derivatives.

By Fourier series, we can expand both type of functions under certain conditions as an infinite series of sines and cosines of x and its integral multiples.

Fourier series for the function f(x) in the interval $c < x < c + 2\pi$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ……… (1)

where $a_0 = \frac{1}{\pi}\int_c^{c\,+\,2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_c^{c\,+\,2\pi} f(x)\ cosnx\ dx$$

$$b_n = \frac{1}{\pi}\int_c^{c\,+\,2\pi} f(x)\ sinnx\ dx$$

Above formulae are also called **Euler's formulae**. Constants a_0 , a_n and b_n are called Fourier coefficients of f(x).

Note: To determine a_0 , a_n and b_n , we shall use the following results

(m and n are integers).

I. $\int_c^{c\,+\,2\pi} sinnx\ dx = \left(\frac{-\ cosnx}{n}\right)_c^{c\,+\,2\pi} = 0,\ n \neq 0$ and

$\int_c^{c\,+\,2\pi} cosnx\ dx = \left(\frac{sinnx}{n}\right)_c^{c\,+\,2\pi} = 0,\ n \neq 0$

II. $\int_c^{c\,+\,2\pi} sinmx\ cosnx\ dx = 0,\ m \neq n$

III. $\int_c^{c\,+\,2\pi} cosmx\ cosnx\ dx = 0,\ m \neq n$

IV. $\int_c^{c\,+\,2\pi} sinmx\ sinnx\ dx = 0,\ m \neq n$

V. $\int_c^{c\,+\,2\pi} cos^2nx\ dx = \pi,\ n \neq 0$; $\int_c^{c\,+\,2\pi} sin^2nx\ dx = \pi,\ n \neq 0$

VI. $\int_c^{c\,+\,2\pi} sinnx\ cosnx\ dx = 0,\ n \neq 0$

VII. $\int e^{ax} \sin bx\ dx = \frac{e^{ax}}{a^2 + b^2}$ (a sinbx – b cosbx) + c

VIII. $\int e^{ax} \cos bx\ dx = \frac{e^{ax}}{a^2 + b^2}$ (a cosbx + b sinbx) + c

IX. Sin $n\pi = 0$ and cos $n\pi = (-1)^n$

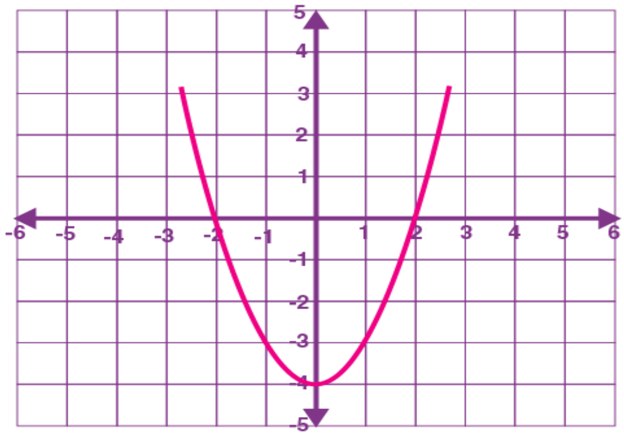
X. **Even and odd functions**

A function f(x) is said to be even if f(-x) = f(x). for example x^4 , cosx, $\sin^2 x$ are even functions.

The graph of an even function is symmetrical about the y-axis.

Here y-axis is a mirror for the reflection of the curve.

$$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$$

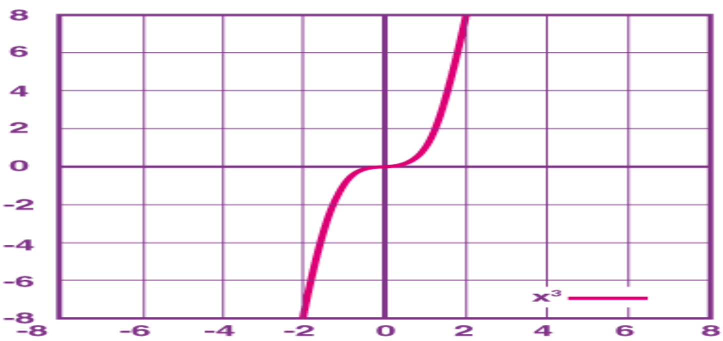


Graphs of even functions

A function f(x) is said to be odd if f(-x) = - f(x). for example x^3 , sinx , $\tan^3 x$ are odd functions.

The graph of an odd function is symmetrical about the origin.

$$\int_{-\pi}^{\pi} f(x)dx = 0$$



Graphs of odd functions

1.5 *EULER'S FORMULAE*

The Fourier series for the function f(x) in the interval c < x < c + 2π is given by

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ **........ (1)**

in finding the coefficients a_0 , a_n and b_n , we assume that the series on the right hand side of the equation (i) is uniformly convergent for c < x < c + 2π and it can be integrated term by term in the given interval.

To find a_0 , integrate both sides of (1) w.r.t. x between the limits c to c + 2π.

$$\int_c^{c\,+\,2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c\,+\,2\pi} dx + \int_c^{c\,+\,2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx$$

$$+ \int_c^{c\,+\,2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\,\pi - c) + 0 + 0 \qquad \text{(By note I)}$$

$$= a_0\,\pi$$

$$\therefore \qquad \mathbf{a_0 = \frac{1}{\pi}\int_c^{c\,+\,2\pi} f(x)\,dx}$$

***To* find a_n ,** multiply both sides of (1) by $\cos nx$ and integrate w.r.t. x between the limits c to c + 2π.

$$\int_c^{c\,+\,2\pi} f(x)\,cosnx\,dx = \frac{a_0}{2}\int_c^{c\,+\,2\pi} cosnx\,dx$$

$$+ \int_c^{c\,+\,2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right) cosnx\,dx$$

$$+ \int_c^{c\,+\,2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right) cosnx\,dx$$

$$= 0 + a_n\pi + 0$$

$$= a_n\pi$$

$$\therefore \qquad \mathbf{a_n = \frac{1}{\pi}\int_c^{c\,+\,2\pi} f(x)\,cosnx\,dx}$$

***To* find b_n ,** multiply both sides of (1) by $\sin nx$ and integrate w.r.t. x between the limits c to c + 2π.

$$\int_c^{c+2\pi} f(x)\, sinnx\, dx = \frac{a_0}{2}\int_c^{c+2\pi} sinnx\, dx$$
$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right) sinnx\, dx$$
$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right) sinnx\, dx$$
$$= 0 + 0 + b_n\pi$$
$$= b_n\pi$$

$$\therefore \qquad \mathbf{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, sinnx\, dx}$$

Note: These values of a_0 , a_n *and* b_n are called **Euler's formulae.**

Corollary 1. If c = 0, the interval becomes 0 < x < 2π, and the formulae reduce to

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cosnx\, dx$ and

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, sinnx\, dx$

Corollary 2. If c = - π, the interval becomes $-\pi$ < x < π, and the formulae reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cosnx\, dx \text{ and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sinnx\, dx$$

Corollary 3. When f(x) is odd function then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$

Since cosnx is an even function, therefore f(x) cosnx is an odd function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cosnx\, dx = 0$$

Since sinnx is an odd function, therefore f(x) sinnx is an even function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sinnx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\, sinnx\, dx$$

Hence, **if a periodic function f(x) is odd, its Fourier expansion contains only sine terms.**

i.e., $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, sinnx\, dx$

Corollary 4. When f(x) is an even function then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

Since cosnx is an even function, therefore f(x) cosnx is an even function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cosnx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\, cosnx\, dx$$

Since sinnx is an odd function, therefore f(x) sinnx is an odd function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sinnx\, dx = 0$$

Hence**, if a periodic function f(x) is even, its Fourier expansion contains only cosine terms**.

i.e., $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$ *and*

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\ dx\ .$$

1.6 *DIRICHLET'S CONDITIONS*

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions. All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function f(x) can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n, b_n are constants provided

(i) f(x) is periodic, single valued and finite.

(ii) f(x) has finite number of finite discontinuities in any one period.

(iii) F(x) has a finite number of maxima and minima.

(iv) When these conditions are satisfied, the fourier series converges to f(x) at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left

i.e. $\frac{1}{2}[f(x+0)+f(x-0)]$

where $f(x+0)$ and $f(x-0)$ denotes the limit on the right and the limit on the left respectively.

ILLUSTRATIVE EXAMPLES

Example 1. Obtain the fourier series to represent f(x) = $\frac{1}{4}(\pi - x)^2$ in the interval $0 \leq x \leq 2\pi$.

Hence obtain the following relations:

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

Sol. Let f(x) = $\frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

…. (1)

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2\,dx = \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{6} \text{ and}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cosnx\, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2\, cosnx\, dx$$

$$= \frac{1}{4\pi}\left[\left\{(\pi - x)^2 \frac{sinnx}{n}\right\}_0^{2\pi} + \int_0^{2\pi} 2(\pi - x)\frac{sinnx}{n}\,dx\right]$$

$$= \frac{1}{4\pi}\cdot\frac{2}{n}\left[\left\{(\pi - x)\left(\frac{-cosnx}{n}\right)\right\}_0^{2\pi} - \int_0^{2\pi}(-1)\left(\frac{-cosnx}{n}\right)dx\right]$$

$$= \frac{-1}{2\pi n^2}(-\pi - \pi) = \frac{1}{n^2} \text{ and}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, sinnx\, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2\, sinnx\, dx$$

$$= \frac{1}{4\pi}\left[\left\{(\pi - x)^2 \frac{cosnx}{n}\right\}_0^{2\pi} - \int_0^{2\pi} 2(\pi - x)\frac{cosnx}{n}\, dx\right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + \frac{\pi^2}{n}\right) - \frac{2}{n}\int_0^{2\pi}(\pi - x) cosnx\, dx\right]$$

$$= -\frac{1}{2\pi n}\left[\left\{(\pi - x)\frac{sinnx}{n}\right\}_0^{2\pi} - \int_0^{2\pi}(-1)\frac{sinnx}{n}\, dx\right]$$

$$= \frac{-1}{2\pi n^2}\left(\frac{-cosnx}{n}\right)_0^{2\pi} = 0$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{cosnx}{n^2} = \frac{\pi^2}{12} + \frac{cosx}{1^2} + \frac{cos2x}{2^2} + \frac{cos3x}{3^2} + \ldots \qquad \textbf{...... (2)}$$

(i) putting x = 0 in equation (2), we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right)$$

$$\Longrightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \textbf{...... (3)}$$

(ii) putting x = π in equation (2), we get

$$0 = \frac{\pi^2}{12} + \left[\left(\frac{-1}{1^2}\right) + \frac{1}{2^2} + \left(\frac{-1}{3^2}\right) + \frac{1}{4^2} + \ldots\right]$$

$$\Longrightarrow \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad \textbf{....... (4)}$$

(iii) Adding equation (3) and (4), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$ Hence the results.

Example 2. Obtain the Fourier series to represent f(x) = e^{-x} in the interval 0< $x < 2\pi$.

Sol. Let f(x) = $e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

…. (1)

Here, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cosnx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x} cosnx\,dx$

$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-cosnx + nsinnx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, sinnx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x} sinnx\,dx$

$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-sinn - ncosnx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi} \cdot \frac{n}{1+n^2}$

$\therefore\ f(x) = e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{cosnx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{nsinnx}{1+n^2}$.

Example 3. Expand f(x) = x sinx, $0 < x < 2\pi$ as a Fourier series.

Sol. Let f(x) = x sinx = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} xsinx\,dx$

$= \frac{1}{\pi}\left[x(-\cos x) - 1(-sinx)\right]_0^{2\pi} = \frac{1}{\pi}[-2\pi]$ = -2

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cosnx\,dx = \frac{1}{\pi}\int_0^{2\pi} xsinx\, cosnx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2cosnx\, sinx)\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$

$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos2(n+1)\pi}{n+1} + \frac{\cos2(n-1)\pi}{n-1}\right\}\right]$

$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}$, n ≠ 1

When n = 1, we have

$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x \, dx$

$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx \sin x) \, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x \, [\cos(n-1)\, x - \cos(n+1)\, x] dx$

$= \frac{1}{2\pi}\left[x\left\{-\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$

$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0$, n ≠ 1

Ehen n = 1, we have

$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x) \, dx$

$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$

$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$

$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$.

Example 4. Find the Fourier series for the function $f(x) = x + x^2$, $-\pi < x < \pi$.

Hence show that

(i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

Sol. Let the Fourier series be

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\, dx \; + \; \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx\right]$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^2$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\, cosnx\, dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\, dx \; + \; \int_{-\pi}^{\pi} x^2 cosnx\, dx\right] = \frac{2}{\pi}\int_{-\pi}^{\pi} x^2\, cosnx\, dx$

$= \frac{2}{\pi}\left[\left(x^2\frac{sinnx}{n}\right)_0^{\pi} - \int_0^{\pi} 2x.\frac{sinnx}{n}\, dx\right]$

$= -\frac{4}{\pi n}\int_0^{\pi} xsinnx\, dx$

$= -\frac{4}{\pi n}\left[\left\{x\left(\frac{-cosnx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1.\left(\frac{-cosnx}{n}\right) dx\right]$

$= \frac{4}{\pi n}\left(-\frac{\pi}{n} cosn\pi\right) = \frac{4}{n^2}\cos nx = \frac{4}{n^2}(-1)^n$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\; (x + x^2) sinnx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x\, sinnx\, dx [\because \int_{-\pi}^{\pi}\; x^2 sinnx\, dx = 0]$

$= \frac{2}{\pi}\left(-\frac{\pi}{n} cosn\pi\right) = -\frac{2}{n}(-1)^n$, as above

$\therefore$ from equation (1),

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\, cosx + \frac{1}{2^2}\, cos2x - \frac{1}{3^2}\, cos3x + \cdots\right]$$

$$-2\left[\frac{-1}{1}\, sinx + \frac{1}{2}\, sin2x - \frac{1}{3}\, sin3x + \cdots\right] \quad \ldots\ldots (2)$$

We observe that the series on the R.H.S. given by equation (2) always represents $x + x^2$ for all values of x except the end points - π or π.

At the point of discontinuity,

$$f(-\pi) = \frac{1}{2}\,(\text{L.H.L.} + \text{R.H.L.}) = \frac{1}{2}\,[f(-\pi - 0) + f(-\pi + 0)]$$

$$= \frac{1}{2}\,[f(\pi - 0) + f(\pi + 0)]\ \ [\because f(x) \text{ is periodic with period } 2\pi]$$

$$= \frac{1}{2}\,[\pi + \pi^2 + (-\pi) + (-\pi)^2] = \pi^2$$

Putting $x = -\pi$ in equation (2), we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \qquad \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Again, putting x = 0 in equation (2), we get

$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\Rightarrow \qquad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$ Hence the results.

Example 5. Express f(x) = |x|, - π < x < π, as a Fourier series. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

Sol. Since f(-x) = |−x| = |x| = f(x)

∴ f(x) is even function and hence $b_n = 0$

Let $\quad f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} |x| dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} x dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi \qquad \text{and}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cosnx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, cosnx\, dx = \frac{2}{\pi}\int_{0}^{\pi} |x| cosnx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\, cosnx\, dx = \frac{2}{\pi}\left[x\left(\frac{sinnx}{n}\right) - 1\left(-\frac{cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{cosn\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1] = f(x) = \begin{cases} 0\,, & if\ n\ is\ even \\ \frac{-4}{\pi n^2}, & if\ n\ is\ odd \end{cases}$$

$$\therefore f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(cosx + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + \cdots\right)$$

Putting x = 0 in the above result, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Trigonometric functions are periodic function.

Problem 2. In the Fourier series of the function f(x) in the interval 0 to 2π, The value of a_0 is defined by $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$.

Problem 3. If the periodic function is odd, its Fourier series contains only sine terms.

Problem 4. F(x) = |x| is odd function.

Problem 5. If the periodic function is even, its Fourier series contains only cosine terms.

1.7 *SUMMARY*

1. The Fourier series can be thought of as analyzing the periodic extension (bottom graph) of the original function. The Fourier series is always a periodic function, even if original function wasn't.

2. Any function f(x) can be expressed as a Fourier series

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0 , a_n , b_n are constants.

3. If a periodic function f(x) is even, its Fourier expansion contains only cosine terms.

i.e., $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$ *and*

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \ dx$$

1.8 GLOSSARY

Periodic Functions

Integration

Even, odd functions

Trigonometric functions

1.9 *REFERENCES*

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1.10 *SUGGESTED READING*

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1.11 *TERMINAL AND MODEL QUESTIONS*

Q 1. Expand f(x) = |cosx| as a Fourier series in the interval $-\pi < x < \pi$.

Q 2. Find the Fourier series of f(x) = x^3 in $(-\pi, \pi)$.

Q 3. Expand in a Fourier series the function f(x) = x in the interval $0 < x < 2\pi$.

Q 4. Obtain the Fourier series to represent e^x in the interval $0 < x < 2\pi$.

Q 5. Find the Fourier series expansion for f(x) = x + $\frac{x^2}{4}$, $-\pi \leq x \leq \pi$.

Q 6. Express f(x) = $\frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$. Also prove $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Q7. Prove that in the range $-\pi < x < \pi$,cosh(ax) = $\frac{2a}{\pi}$ sinh $a\pi$ $\left[\frac{1}{2a^2} + \sum_1^{\infty} \frac{(-1)^n}{n^2+a^2} cosnx\right]$.

Q 8. Prove that for all values of x between -π and π , $\frac{1}{2}$ x = sinx - $\frac{1}{2}$ sin2x + $\frac{1}{3}$ sin3x - $\frac{1}{4}$ sin4x + …

1.12 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. False

CYQ 3. True

CYQ 4. False

CYQ 5. True

TERMINAL QUESTIONS

TQ 1. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{cos2x}{3} - \frac{cos4x}{15} + \cdots\right)$

TQ 2. $f(x) = 2\sum_1^\infty \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)(-1)^n \sin nx$

TQ 3. $f(x) = \pi - 2\sum_1^\infty \frac{sinnx}{n}$

TQ 4. $e^x = \frac{\boldsymbol{e^{2\pi}-1}}{2\pi} + \frac{\boldsymbol{e^{2\pi}-1}}{\pi}\sum_1^\infty \left(\frac{cosnx}{1+n^2} - \frac{n}{1+n^2} sinnx\right)$

TQ 5. $f(x) = \frac{\pi^2}{\mathbf{12}} + \sum_1^\infty \frac{(-1)^n}{n^2} \cos nx - 2\sum_1^\infty \frac{(-1)^n}{n} \sin nx$

TQ 6. $f(x) = \sum_1^\infty \frac{sinnx}{n}$

UNIT 2: FOURIER SERIES II

CONTENTS

2.1 *INTRODUCTION*

Fourier series representation of such function has been studied, and it has been pointed out that, at the point of discontinuity, this series converges to the average value between the two limits of the function about the jump point. so for a step function, this convergence occurs at the exact value of one half. Fourier series is used to describe a periodic signal in terms of cosine and sine waves. In other words, it allows us to model any arbitrary periodic signal with a combination of sines and cosines. A Fourier series is an expansion of a periodic function f(x) in term of infinite sum of sines and cosines. Fourier Series makes use of the orthogonality relationships of the sine and cosine functions. In this unit learner are learn about the Fourier series for discontinuous function, Fourier series for change of variable, Fourier series for even and odd function.

2.2 OBJECTIVE

At the end of this topic leaner will be able to understand:

(i) Fourier Series for discontinuous functions
(ii) Fourier Series for change of variable
(iii) Fourier Series for even and odd functions

2.3 DISCONTINUOUS FUNCTION

A function in algebra is said to be a **discontinuous function** if it is not a continuous function. Just like a continuous function has a continuous curve, a discontinuous function has a discontinuous curve. In other words, we can say that the graph of a discontinuous function cannot be made with a single stroke of the pen, i.e., once we put the pen down to draw the graph of a discontinuous function, we must pick it up at least once before the graph is complete. A discontinuous function has breaks/gaps on its graph and hence, in its range on at least one point.

Some of the examples of a discontinuous function are:

- f(x) = 1/ (x - 2)
- f(x) = tan x.
- f(x) = x^2 - 1, for x < 1 and f(x) = x^3 - 5 for 1 < x < 2.

2.4 FOURIER SERIES FOR DISCONTINUOUS FUNCTION

Fourier series representation of such function has been studied, and it has been pointed out that, at the point of discontinuity, this series converges to the average value between the two limits of the function about the jump point. So for a step function, this convergence occurs at the exact value of one half.

In the last unit we derived Euler's formulae for $a_0,\ a_n,\ b_n$ on the assumption that f(x) is continuous in (c, c + 2π). However, if f(x) has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for $a_0,\ a_n,\ b_n$ are to be evaluated by breaking up the range of integration.

Let f(x) be defined by f(x) = $f(x) = \begin{cases} f_1(x),\ c < x < x_0 \\ f_2(x),\ x_0 < x < c + 2\pi \end{cases}$, where x_0 is the point of finite discontinuity in the interval (c, c + 2π).

The values of $a_0,\ a_n,\ b_n$ are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)dx + \int_{x_0}^{c+2\pi} f_2(x)dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)cosnx\ dx + \int_{x_0}^{c+2\pi} f_2(x)cosnx\ dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)sinnx\ dx + \int_{x_0}^{c+2\pi} f_2(x)sinnx\ dx\right]$$

At x_0, there is an infinite jump in the graph of the function. Both the limits f($x_0 - 0$) and

$f(x_0 + 0)$ exist but unequal. The sum of the Fourier series

$= \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)] = \frac{1}{2}[AB + AC] =$ AM, where M is the mid-point of BC.

ILLUSTRATIVE EXAMPLES

Example 1. Find the Fourier series to represent the function f(x) given by

$$f(x) = \begin{cases} x, & for\ 0 \le x \le \pi \\ 2\pi - x, & for\ \pi \le x \le 2\pi \end{cases},$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

Sol. Let f(x) = $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ …….. (1)

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi}(2\pi - x)\, dx\right]$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} + \left|2\pi x - \frac{x^2}{2}\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

And $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ cosnx\ dx$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\ cosnx\ dx + \int_{\pi}^{2\pi}(2\pi - x)\ cosnx\ dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\frac{sinnx}{n}\right|_0^{\pi} - \int_0^{\pi} 1.\frac{sinnx}{n}dx + \left|(2\pi - x)\frac{sinnx}{n}\right|_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\frac{sinnx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{cosnx}{n^2}\right)_0^{\pi} - \left(\frac{cosnx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1 - 1 + (-1)^n]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} -\frac{4}{\pi n^2}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

And $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ sinnx\ dx$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\ sinnx\ dx + \int_{\pi}^{2\pi}(2\pi - x)\ sinnx\ dx\ \right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-cosnx}{n}\right)\right\}_0^{\pi} + \int_0^{\pi}\frac{cosnx}{n}\ dx + \left\{(2\pi - x)\left(\frac{-cosnx}{n}\right)\right\}_{\pi}^{2\pi} - \int_{\pi}^{2\pi}(-1)\left(\frac{-cosnx}{n}\right)\ dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}cosnx + \frac{\pi}{n}cosnx\right] = 0$$

$$\therefore \text{f(x)} = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{cosx}{1^2} + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + \cdots\right)$$

Putting x = 0, we get $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

$$\Longrightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 2. If f(x) = $\begin{cases} 0, & for -\pi \le x \le 0 \\ sinx, & for\ 0 \le x \le \pi \end{cases}$

prove that f(x) = $\frac{1}{\pi} + \frac{1}{2}\ sinx - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{cos2nx}{4n^2-1}$.

Hence show that (i) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots = \frac{1}{2}$

(ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi-2}{4}$.

Sol. Let f(x) $= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ dx + \int_0^{\pi}\ sinxdx\ \right] = \frac{2}{\pi}$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ cosnx\ dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ dx + \int_0^{\pi}\ sinx\ cosnx\ dx\ \right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\ cosnx\ sinx\ dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)\,x - \sin(n-1)\,x]\ dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi},\ n \ne 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \frac{1}{2\pi}\left(-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right), & when\ n\ is\ odd \\ \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right), & when\ n\ is\ even \end{cases}$$

$$= \begin{cases} 0, & when\ n\ is\ odd, i.e., n = 3, 5, 7, \ldots \\ -\frac{2}{\pi(n^2-1)}, & when\ n\ is\ even \end{cases}$$

When n = 1, we have

$$a_1 = \frac{1}{\pi}\int_0^{\pi} sinx\ cosx\ dx = \frac{1}{2\pi}\int_0^{\pi} sin2x\ dx = \frac{1}{2\pi}\left[-\frac{cos2x}{2}\right]_0^{\pi} = 0$$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ sinnx\ dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ dx + \int_0^{\pi} sinx\ sinnx\ dx\right]$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\ sinnx\ sinx\ dx = \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)\,x - \cos(n+1)\,x]\ dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0,\ n \neq 1$$

When n = 1 , we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} sinx\ sinx\ dx = \frac{1}{2\pi}\int_0^{\pi}(1 - cos2x)\ dx$$

$$= \frac{1}{2\pi}\left[x - \frac{sin2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

$\therefore$ $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{cos2x}{2^2-1} + \frac{cos4x}{4^2-1} + \frac{cos6x}{6^2-1} + \cdots\right] + \frac{1}{2}\sin x$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{cos2nx}{4n^2-1} \qquad \ldots \textbf{(1)}$$

Putting x = 0 in (1), we have

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\Rightarrow \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$$

Putting $x = \frac{\pi}{2}$ in (1) , we have

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{cosn\pi}{4n^2-1}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{\pi} = - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \frac{\pi-2}{4} = - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = - \left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right)$$

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi-2}{4}.$$

Example 3. Find the Fourier series to represent the function f(x) given by

$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$, and hence show that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Sol. Let f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ……. (1)

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} x\, dx + \int_0^{\pi} -x\, dx\right]$

$$= \frac{1}{\pi} \left[\left(\frac{x^2}{2}\right)_{-\pi}^{0} - \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left(0 - \frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = -\pi$$

And $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, cosnx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \text{xcosnx}\, dx + \int_0^{\pi} -x\, cosnx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\frac{sinnx}{n}\right)_{-\pi}^{0} - \int_{-\pi}^{0} 1.\frac{sinnx}{n}\, dx + \left(-x\frac{sinnx}{n}\right)_0^{\pi} - \int_0^{\pi}(-1)\left(\frac{sinnx}{n}\right) dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(cosnx)_{-\pi}^{0} - \frac{1}{n^2}(cosnx)_0^{\pi}\right] = \frac{1}{\pi}\left[\left\{\frac{1-(-1)^n}{n^2}\right\} - \left\{\frac{(-1)^n-1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left\{2.\frac{1-(-1)^n}{n^2}\right\}\right] = \frac{2}{\pi n^2}\{1-(-1)^n\} = = \begin{cases} 0, & when\ n\ is\ even \\ -\frac{4}{\pi n^2}, & when\ n\ is\ odd \end{cases}$$

And $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, sinnx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\, sinnx\, dx + \int_0^{\pi}(-x)\, sinnx\, dx\right]$

$$= \frac{1}{\pi} \left[\left\{x.\left(-\frac{cosnx}{n}\right)\right\}_{-\pi}^{0} - \int_{-\pi}^{0} 1.\left(-\frac{cosnx}{n}\right) dx + \left\{(-x)\left(-\frac{cosnx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}(-1).\left(-\frac{cosnx}{n}\right) dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{1}{n}.\pi(-1)^n\right] = 0$$

$\therefore$ from (1), f(x) = $-\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{cosx}{1^2}+\frac{cos3x}{3^2}+\frac{cos5x}{5^2}+\cdots\right)$ …(2)

At point of discontinuity,

$$F(0)=\frac{1}{2}\left[f(0-0)+f(0+0)\right]=\frac{1}{2}(0-0)=0$$

Putting x = 0 is above expression, we get

$$0=-\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$\Longrightarrow \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

CHECK YOUR PROGRESS

MCQ Questions

Problem 1. What is the Fourier series expansion of the function f(x) in the interval (c, c+2π)?

a) $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

b) $\frac{a_0}{3}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

c) $\frac{a_0}{4}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

d) $a_0+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

Problem 2. f the function f(x) is even, then which of the following is zero?

a) a_n

b) b_n

c) a_0

d) None

Problem 3. Who discovered Fourier series?

a) Jean Baptiste de Fourier

b) Jean Baptiste Joseph Fourier

c) Fourier Joseph

d) Jean Fourier

Problem 4. What are the two types of Fourier series?

a) Trigonometric only

b) Trigonometric and logarithmic

c) Exponential and logarithmic

d) Trigonometric and exponential

2.5 *FOURIER SERIES FOR CHANGE OF INTERVAL*

In many questions, it is described to expand a function in a fourier series over an interval of length 2l and not 2 π. In order to apply forgoing theory, this interval must be transformed into an interval of length 2 π. This can be achieved by a transformation of the variable.

Consider a periodic function f(x) defined in the interval c < x < c + 2l.

to change the interval into one of length 2 π , we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \quad \text{so that}$$

When x = c, $z = \frac{\pi c}{l} = d$ (say)

And when x = c + 2l, $z = \frac{\pi(c+2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi$.

Thus the function f(x) of period 2l in (c, c + 2l) is transformed to the function $f\left(\frac{lz}{\pi}\right) = F(z), say, of\ period\ 2\pi$ in $(d, d + 2\pi)$ and the later function can be expressed as the Fourier series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Where $a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)dz$ and $a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)cosnzdz$ and

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)sinnzdz$$

Now making the inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{x}{l}dx$

When z = d, x = c and when $z = d + 2\pi$, x = c + 2l.

The expression (1) becomes

$$F(z) = F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

And the coefficient a_0, a_n, b_n from (2) reduce to

$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$; $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$;

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$

Hence the fourier series f(x) in the interval c < x <c + 2l is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

Corollary 1. If we put c = 0, in the interval becomes 0 < x < 2l , and the above result reduce to

$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$; $a_n = \frac{1}{l}\int_0^{2l} f(x)\ \cos\frac{n\pi x}{l}dx$; $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$

Corollary 2. if we put c = - l, the interval become - l < x < l and the above result reduce to

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$; $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\ \cos\frac{n\pi x}{l}dx$; $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

Corollary 3. If f(x) is even function, we have

$a_0 = \frac{2}{l}\int_0^{l} f(x)dx$; $a_n = \frac{2}{l}\int_0^{l} f(x)\ \cos\frac{n\pi x}{l}dx$; $b_n = 0$

Corollary 4. If f(x) is odd function,

we have $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}dx$

ILLUSTRATIVE EXAMPLES

Example 1. Obtain the fourier series expansion of

$f(x) = \left(\frac{\pi - x}{2}\right)$ for 0 < x < 2.

Sol. Let $f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here $l = 1$

$\therefore \frac{\pi-x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ ...(1)

Here, $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx = \int_0^2 \left(\frac{\pi-x}{2}\right) dx$

$= \frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)_0^2 = \frac{1}{2}(2\pi - 2) = \pi - 1$

And $a_n = \frac{1}{l}\int_0^{2l} f(x) \cos n\pi x\, dx = \int_0^2 \left(\frac{\pi-x}{2}\right) \cos n\pi x\, dx$

$= \frac{1}{2}\left[\left\{(\pi - x)\frac{sin n\pi x}{n\pi}\right\}_0^2 - \int_0^2 (-1)\frac{sin n\pi x}{n\pi} dx\right]$

$= \frac{1}{2n\pi}\left(\frac{-Cos n\pi x}{n\pi}\right)_0^2 = 0$

and $b_n = \frac{1}{l}\int_0^{2l} f(x) \sin n\pi x\, dx = \int_0^2 \left(\frac{\pi-x}{2}\right) \sin n\pi x\, dx$

$= \frac{1}{2}\left[\left\{(\pi - x)\frac{-Cos n\pi x}{n\pi}\right\}_0^2 - \int_0^2 (-1)\left(\frac{-Cos n\pi x}{n\pi}\right) dx\right]$

$= -\frac{1}{2n\pi}[(\pi - 2) - \pi] = \frac{1}{n\pi}$

Hence, from (1)

$\frac{\pi-x}{2} = \frac{\pi-1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} \sin n\pi x$.

Example 2. Find the Fourier series for the function $f(x) = x - x^2$,

$-1 < x < 1$.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_{-1}^{1}(x - x^2)dx = \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2 dx = 0 - 2\int_0^1 x^2 dx$

$= -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$

And $a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x\, dx$

$= \int_{-1}^{1} x \cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$

$= 0 - 2\int_0^1 x^2 \cos n\pi x\, dx = -2\left[\left\{x^2\frac{sin n\pi x}{n\pi}\right\}_0^1 - \int_0^2 2x\left(\frac{sin n\pi x}{n\pi}\right) dx\right]$

$= \frac{4}{n\pi}\int_0^1 x\, sin\, n\pi x\, dx$

$= \frac{4}{n\pi}\left[x.\left\{\frac{-cos n\pi x}{n\pi}\right\}_0^1 - \int_0^1\left(\frac{-cos n\pi x}{n\pi}\right) dx\right]$

$= -\frac{4}{n^2\pi^2}\, cos n\pi = -\frac{4(-1)^n}{n^2\pi^2}$

and $b_n = \int_{-1}^{1}(x - x^2)\sin n\pi x\, dx = \int_{-1}^{1} x \sin n\pi x\, dx - \int_{-1}^{1} x^2 \sin n\pi x\, dx$

$= 2\int_0^1 x\ sin n\pi x dx - 0 = 2\left[x.\left(\frac{-cosn\pi x}{n\pi}\right) - 1.\left(\frac{-\ sinn\pi x}{n^2\pi^2}\right)\right]_0^1$

$= 2\left(\frac{-cosn\pi x}{n\pi}\right) = -\frac{2(-1)^n}{n\pi}$

$\therefore$ x − $x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right)$

$+\frac{4}{n\pi}\left(\frac{\text{Sin}\ \pi x}{1} - \frac{\text{Sin}\ 2\pi x}{2} + \frac{\text{Sin}\ 3\pi x}{3} - \cdots\right)$

Example 3. Find the Fourier series for the function f(x) = x^2 - 2,

-2≤ x ≤ 2.

Sol. Since f(x) is even function, $b_n = 0$.

Let f(x) = x^2 - 2 = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

Then $a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)$dx = $\left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$

And $a_n = \frac{2}{2}\int_0^2 (x^2 - 2)\cos\frac{n\pi x}{2}$ dx

$= \left\{(x^2 - 2)\frac{\text{Sin}\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right\}_0^2 - \int_0^2 (2x)\frac{\text{Sin}\frac{n\pi x}{2}}{\frac{n\pi}{2}}dx$

$= -\frac{4}{n\pi}\int_0^2 x\ \text{Sin}\frac{n\pi x}{2}dx$

$= -\frac{4}{n\pi}\left[\left\{x.\frac{-\text{Cos}\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right\}_0^2 - \int_0^2\left(-\frac{\text{Cos}\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)dx\right]$

$= \frac{8}{n^2\pi^2}(2\cos n\pi) = \frac{16\cos n\pi}{n^2\pi^2} = -\frac{16(-1)^n}{n^2\pi^2}$

$\therefore (x^2 - 2) = -\frac{2}{3} - \frac{16}{\pi^2}\left(\text{Cos}\frac{\pi x}{2} - \frac{1}{4}cos\pi x + \frac{1}{9}\text{Cos}\frac{3\pi x}{2} - \cdots\right).$

Example 4. Obtain the Fourier series for the function

$$f(x) = \begin{cases} \pi x, & for\ 0 \le x \le 1 \\ \pi(2-x), & for\ 1 \le x \le 2 \end{cases}$$

Sol. Let f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)$dx = $\int_0^1 \pi x$dx + $\int_1^2 \pi(2-x)$dx

$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$

$= \pi\frac{1}{2} + \pi\left[(4-2) - (2 - \frac{1}{2})\right] = \pi$

And $a_n = \int_0^2 f(x)\cos n\pi x\, dx$

$= \int_0^1 \pi x \cos \mathrm{n\pi x}\, dx + \int_1^2 \pi(2x) \cos \mathrm{n\pi x}\, dx$

$= \left[\pi x \frac{sin n\pi x}{n\pi} - \pi\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\frac{sin n\pi x}{n\pi} - (-)\pi\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$

$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}$ (cosnπ – 1)

$= \frac{2}{n^2\pi}[(-1)^n - 1]$

= 0 or $\frac{4}{n^2\pi}$ according as n is even or odd.

$and\ b_n = \int_0^2 f(x) \sin n\pi x\, dx = \int_0^1 n\pi \sin n\pi x\, dx + \int_1^2 \pi(2-x) \sin n\pi x\, dx$

$= \left[\pi x \frac{-cos n\pi x}{n\pi} - \pi\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)(\frac{-cos n\pi x}{n\pi}) - (-)\pi\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$

$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$

∴ f(x) = $\frac{\pi}{2}$ - $\frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$

CHECK YOUR PROGRESS

True and False questions

Problem 5. The Fourier series expansion of f(x) = x^3 in the interval -1 < x < 1 with periodic continuation has only sine terms.

Problem 6. The value of b_n in the Fourier series expansion of f(x) in the interval c < x < c + 2 l in given by $\frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{\mathrm{n\pi x}}{l} dx$.

Problem 7. The value of a_n in the Fourier series expansion of f(x) in the interval 0 < x < 2 l in given by $\frac{1}{l}\int_0^{2l} f(x) \cos \frac{\mathrm{n\pi x}}{2l} dx$.

Problem 8. The value of a_0 in the Fourier series expansion of f(x) in the interval - l < x < l in given by $\frac{1}{l}\int_{-1}^{1} f(x) dx$.

Problem 9. The function f(x) = 1/x is continuous at x = 0.

Problem 10. $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Problem 11. f(x) = x^3 + 2 is even function.

Problem 12. f(x) = sinx in odd function.

2.6 SUMMARY

1. Fourier series for Discontinuous function:

Let f(x) be defined by f(x) = $f(x) = \begin{cases} f_1(x), \ c < x < x_0 \\ f_2(x), \ x_0 < x < c + 2\pi \end{cases}$, where x_0 is the point of finite discontinuity in the interval (c, c + 2π).

The values of $a_0, \ a_n, \ b_n$ are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)dx + \int_{x_0}^{c+2\pi} f_2(x)dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)cosnx\, dx + \int_{x_0}^{c+2\pi} f_2(x)cosnx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)sinnx\, dx + \int_{x_0}^{c+2\pi} f_2(x)sinnx\, dx\right]$$

2. Fourier series for change of interval:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

3. let $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$; $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$ then

Corollary 1. If we put c = 0, in the interval becomes 0 < x < 2l , and the above result reduce

to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx \ ; \ a_n = \frac{1}{l}\int_0^{2l} f(x)\ \cos\frac{n\pi x}{l}dx \ ; \ b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$$

Corollary 2. if we put c = - l, the interval become - l < x < l and the above result reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \ ; \ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\ \cos\frac{n\pi x}{l}dx \ ; \ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Corollary 3. If f(x) is even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)dx \ ; \ a_n = \frac{2}{l}\int_0^{l} f(x)\ \cos\frac{n\pi x}{l}dx \ ; \ b_n = 0$$

Corollary 4. If f(x) is odd function, we have $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

4. Discontinuous function.

5. Fourier series for the function f(x) = x^2 - 2, -2≤ x ≤ 2 is given by

$F(x) = -\frac{2}{3} - \frac{16}{\pi^2}\left(\text{Cos}\frac{\pi x}{2} - \frac{1}{4}cos\pi x + \frac{1}{9}\text{Cos}\frac{3\pi x}{2} - \cdots\right).$

6. Fourier series for the function f(x) = x – x^2 , -1< x < 1.

$$f(x) = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right) + \frac{4}{n\pi}\left(\frac{\text{Sin}\,\pi x}{1} - \frac{\text{Sin}\,2\pi x}{2} + \frac{\text{Sin}\,3\pi x}{3} - \cdots\right)$$

7. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

8. The Fourier series expansion for f(x) = πx from x = -c to x = c is

$f(x) = 2c\left[\sin\left(\frac{\pi x}{c}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{c}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{c}\right) - \cdots\right]$

2.7 *GLOSSARY*

Discontinuous functions

Periodic Functions

Integration

Even, odd functions

Trigonometric functions

Integrations

Series

2.8 *REFERENCES*

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2.10 *TERMINAL AND MODEL QUESTIONS*

Q 1. Find the Fourier series for f(x) = 1 - t^2 when -1$\leq t \leq 1$.

Q 2. Find the Fourier series in the interval (0, 2) if

$$f(x) = \begin{cases} x, & for\ 0 < x < 1 \\ 0, & for\ 1 < x < 2 \end{cases}.$$

Q 3. Find the Fourier series expansion for the function f(x) = x – x^3 in the interval -1 < x < 1.

Q 4. Find the Fourier series for the function given by

$$f(x) = \begin{cases} t, & for\ 0 < x < 1 \\ 1 - t, & for\ 1 < x < 2 \end{cases}.$$

Q 5. Find the Fourier series expansion for f(x) = πx from x = -c to x = c.

Q 6. Obtain the Fourier series of the function

$$f(x) = \begin{cases} 1 + \frac{2x}{l}, & for -l < x < 0 \\ 1 - \frac{2x}{l}, & for\ 0 < x < l \end{cases}.$$

Q 7. Find the Fourier series expansion of the periodic function whose definition in one period is f(x) = 4 – x^2 , -2 $\leq x \leq 2$. Also prove that

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$.

Q 8. Find the Fourier series to represent the function

$f(x) = \begin{cases} -k, & for -\pi < x < 0 \\ k, & for\ 0 < x < \pi \end{cases}$, Also deduce that

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Q 9. Find the Fourier series for f(x) in the interval $(-\pi, \pi)$ when

$$f(x) = \begin{cases} \pi + x, & for -\pi < x < 0 \\ \pi - x, & for\ 0 < x < \pi \end{cases}$$

Q 10. Find the Fourier series to represent the periodic function

$$f(x) = \begin{cases} x, & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

2.11 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. (a)

CYQ 2. (b)

CYQ 3. (b)

CYQ 4. (d)

CYQ 5. True

CYQ 6. True

CYQ 7. False

CYQ 8. True

CYQ 9. False

CYQ 10. True

CYQ 11. False

CYQ 12. True

TERMINAL QUESTIONS

TQ 1. f(x) $= \frac{2}{3} + \frac{4}{\pi^2}\left(\cos\pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \cdots\right)$

TQ 2. f(x) = $\frac{1}{4} - \frac{2}{\pi^2}\left(\cos\pi + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$

$$+ \frac{\mathbf{1}}{\boldsymbol{\pi}}\left(\sin\pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right)$$

TQ 3. f(x) = $\frac{12}{\pi^3}\left(\sin\pi x - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} - \cdots\right)$

TQ 4. f(t) $= -\frac{4}{\pi^2}\left(\cos\pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} - \cdots\right)$

$$+ \frac{\mathbf{2}}{\boldsymbol{\pi}}\left(\sin\pi t + \frac{\sin 3\pi t}{3} + \cdots\right)$$

TQ 5. f(x) = 2c$\left[\sin\left(\frac{\pi x}{c}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{c}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{c}\right) - \cdots\right]$

TQ 6. F(x) = $\frac{4}{\pi^2}\sum_{n=1}^{\infty}\{1-(-1)^n\}\frac{\cos\frac{n\pi x}{l}}{n^2}$

TQ 7. f(x) = $\frac{8}{3}$ - $\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$

TQ 8. f(x) = $\frac{4k}{\pi}\left(sinx + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \ldots\right)$

TQ 9. f(x) = $\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$

TQ 10. f(x) = $\frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right)$

BLOCK-II

UNIT 3: Laplace Transform I

Contents

3.1 *INTRODUCTION*

A transformation is a mathematical device which convert one function into another. It transforms one variable at a time. The Laplace transform of a function f(t) is designated as L[f(t)], with the variable t covers a spectrum of (0, ∞). where s is the parameter of the Laplace transform, and F(s) is the expression of the Laplace transform of function f(t) with $0 \le t < \infty$. Laplace transformation is directly gives the solution of differential equations with given initial conditions without the necessary of first finding the general solution and then evaluating the arbitrary constants.

French Mathematician Pierre De Laplace (1749 – 1827) used this transform much earlier in 1799 while developing the theory of probability.

3.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) Laplace transform

(ii) Transforms of Discontinuous functions

(iii) Initial-value theorem

(iv) Final value problem

(v) Existence theorem

3.3 *DEFINITION OF LAPLACE TRANSFORM*

Let F(t) be a function of t defined all $t \geq 0$. Then the Laplace transform of F(t) , denoted by L{F(t)}, is defined by

$$\boxed{L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt}$$

Provided that the integral exists, 'p' is a parameter which may be real or complex.

L{F(t)} is said to exist if the above integral converges for some value of p otherwise not. The function f(p) is called the Laplace transform or the image of the objective function F(t).

Note: ■ Some authors use the letter 's' for the parameter instead of p. therefore we may also write

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)dt \text{ .}$$

Note: ■ In general, we will denote the object function by a capital letter and its transform by the same letter in lower case. But other notations that distinguish between functions and their transforms are sometimes preferable

i.e. $L\{F(t)\} = \varphi(p)$ or $L\{y(t)\} = \bar{y}(p)$ or $L\{f(t)\} = \bar{f}(p)$

3.4 *LINEARITY PROPERTY*

If c_1, c_2 are constants and f, g are functions of t, then

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{F(t)\} + c_2 L\{g(t)\}$$

By definition,

$$L\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-pt}\{c_1 f(t) + c_2 g(t)\}dt$$

$$= c_1 \int_0^\infty e^{-pt} f(t)dt + c_2 \int_0^\infty e^{-pt} g(t)dt$$

$$= c_1 L\{F(t)\} + c_2 L\{g(t)\}$$

The result can easily be generalized.

3.5 LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

(1) $L\{1\} = \frac{1}{p}$, p > 0

Proof. $L\{1\} = \int_0^\infty e^{-pt}.1dt = \left[-\frac{e^{-pt}}{p}\right]_0^\infty = \frac{1}{p}$, if $p > 0$.

(2) $L\{t^n\} = \frac{n!}{p^{n+1}}$, where n is positive integer.

Proof. $L\{t^n\} = \int_0^\infty e^{-pt}.t^n dt = \int_0^\infty e^{-x}\left(\frac{x}{p}\right)^n \frac{dx}{p}$, on putting pt = x

$$= \frac{1}{p^{n+1}}\int_0^\infty x^n e^{-pt} dx = \frac{\Gamma(n+1)}{p^{n+1}}$$

provided that p > 0 and n + 1 > 0

If n is a positive integer, $\Gamma(n+1) = n!$

Therefore $L\{t^n\} = \frac{n!}{p^{n+1}}$

Note: ■ For n = 1, $L\{t\} = \frac{1}{p^2}$

(3) $L\{e^{at}\} = \frac{1}{p-a}$, p > a

Proof. $L\{e^{at}\} = \int_0^\infty e^{-pt}.e^{at}dt = \int_0^\infty e^{-(p-a)t}dt = \left[-\frac{e^{-(p-a)t}}{p-a}\right]_0^\infty = \frac{1}{p-a}$, p > a

(4) $L\{sinat\} = \frac{a}{p^2+a^2}$, p > 0

Proof. $L\{sinat\} = \int_0^\infty e^{-pt} sinat\, dt$

$$= \left[\frac{e^{-pt}}{p^2+a^2}(-psinat - acosat)\right]_0^\infty = \frac{a}{p^2+a^2}$$

(5) L$\{cosat\}$ = $\frac{p}{p^2+a^2}$, **p > 0**

Proof. L$\{cosat\} = \int_0^{\infty} e^{-pt} cosat\, dt$

$$= \left[\frac{e^{-pt}}{p^2+a^2}(-pcosat - asinat)\right]_0^{\infty} = \frac{p}{p^2+a^2}$$

(6) L$\{sinh\ at\}$ = $\frac{a}{p^2-a^2}$, **p >** $|a|$

Proof. L$\{sinh\ at\} = \int_0^{\infty} e^{-pt} sinh\ at\ dt = \int_0^{\infty} e^{-pt}\left[\frac{e^{at}-e^{-at}}{2}\right] dt$

$$= \left[\int_0^{\infty} e^{-(p-a)t}\ dt - e^{-(p+a)t}\ dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{p-a} - \frac{1}{p+\_a}\right] = \frac{a}{p^2-a^2} \text{ ,for p > } |a|$$

Note: ■We can also prove it by using linear property.   Thus L$\{sinh\ at\} = L\left\{\frac{1}{2}e^{at} - e^{-at}\right\} = \frac{1}{2}L(e^{at})$ - $\frac{1}{2}L(e^{-at})$ $= \frac{1}{2}\left(\frac{1}{p-a}\right)$ - $\frac{1}{2}\left(\frac{1}{p+a}\right) = \frac{a}{p^2-a^2}$

(7) L$\{sinh\ at\}$ = $\frac{\mathrm{P}}{\mathrm{p}^2-\mathrm{a}^2}$, **p >** $|a|$

Proof. L$\{cosh\ at\} = L\left\{\frac{1}{2}e^{at} + e^{-at}\right\} = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$

$$= \frac{1}{2}\left(\frac{1}{p-a}\right) + \frac{1}{2}\left(\frac{1}{p+a}\right) = \frac{p}{p^2-a^2} \text{ for p > } |a|.$$

Note: The Laplace transforms of various elementary functions have been listed in the following table:

F(t)	L{F(t)} = f(p)
1	$\frac{1}{p}$, $p > 0$

t	$\frac{1}{p^2}$, $p > 0$
t^n, n *is a positive integers*	$\frac{n!}{p^{n+1}}$, $p > 0$
$t^n, n > -1$	$\frac{\Gamma(n+1)}{p^{n+1}}$, $p > 0$
e^{at}	$\frac{1}{p-a}$, $p > a$
e^{-at}	$\frac{1}{p+a}$
$\sin at$	$\frac{a}{p^2+a^2}$ p > 0
$\cos at$	$\frac{p}{p^2+a^2}$, p > 0
$\sinh at$	$\frac{a}{p^2-a^2}$, p > $\lvert a \rvert$
$\cosh at$	$\frac{p}{p^2-a^2}$, p > $\lvert a \rvert$

3.6 *TRANSFORM OF DISCONTINUOUS FUNCTIONS*

The Laplace transform of F(t) will exist even if the object function F(t) is discontinuous, provided the integral in the definition of L{F(t)} exists.

3.7 *FIRST TRANSLATION PROPERTY OR FIRST SHIFTING PROPERTY*

If L{F(t)} = f(p) then L{e^{at}F(t)} = f(p - a)

L{e^{at}F(t)} = $\int_0^\infty e^{-pt} e^{at} F(t) dt$ (By definition)

= $\int_0^\infty e^{-(p-a)t} F(t) dt$ = f(p – a).

Note: ■ L{e^{at}F(t)} = f(p + a)

■ L{e^{at}F(bt)} = $\frac{1}{b} f\left(\frac{p-a}{b}\right)$.

Applying this property to the elementary functions of Art. 3.5, we get the following useful results:

(1) $L\{e^{at}t^n\} = \frac{n!}{(p-a)^{n+1}}$ **;** n is a positive integer**.**

(2) $L\{e^{at}\sin bt\} = \frac{b}{(p-a)^2 + b^2}$

(3) $L\{e^{at}\cos bt\} = \frac{p-a}{(p-a)^2 + b^2}$

(4) $L\{e^{at}\sinh bt\} = \frac{b}{(p-a)^2 - b^2}$

(5) $L\{e^{at}\cosh bt\} = \frac{p-a}{(p-a)^2 - b^2}$

3.8 SECOND TRANSLATION PROPERTY OR HEAVISIDE'S SHIFTING THEOREM

■ If L{F(t)} = f(p) and G(t) = $f(x) = \begin{cases} F(t-a), & t > a \\ 0 & , t < a \end{cases}$

Then, L{G(t)} = $e^{-ap}f(p)$.

Proof. L{G(t)} = $\int_0^\infty e^{-pt}.G(t)dt = \int_0^a e^{-pt}.G(t)dt + \int_a^\infty e^{-pt}.G(t)dt$

$= 0 + \int_a^\infty e^{-pt}.F(t-a)dt = \int_a^\infty e^{-pt}.F(t-a)dt$

Put t – a = u ⟹ dt = du

$= \int_0^\infty e^{-p(u+a)}.F(u)du = e^{-pa}\int_0^\infty e^{-pu}.F(u)du$

$= e^{-pa}\int_0^\infty e^{-pt}.F(t)dt = e^{-pa}f(p).$

3.9 *CHANGE OF SCALES PROPERTY*

■ If L{F(t)} = f(p) then L{F(at)} = $\frac{1}{a}f\left(\frac{p}{a}\right)$.

Proof: L{F(at)} = $\int_0^\infty e^{-pt}.F(at)dt$

Put at = u ⇒ dt = $\frac{du}{a}$

$= \int_0^\infty e^{-p\frac{u}{a}}.F(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)u}.F(u)du$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)t}.F(t)dt = \frac{1}{a}f\left(\frac{p}{a}\right).$$

ILLUSTRATIVE EXAMPLES

Example 1. Find the Laplace transform of

$$7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2.$$

Sol. $L(7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2)$

$$= 7L(e^{2t})+9L(e^{-2t})+5L(\cos t)+7L(t^3)+5L(\sin 3t)+2L(1)$$

$$= 7.\frac{1}{p-2}+9.\frac{1}{p+2}+5.\frac{p}{p^2+1}+7.\frac{3!}{p^4}+5.\frac{3}{p^2+9}+2.\frac{1}{p}$$

$$= \frac{7}{p-2}+\frac{9}{p+2}+\frac{5p}{p^2+1}+\frac{42}{p^4}+\frac{15}{p^2+9}+\frac{2}{p}.$$

Example 2. Find the Laplace transforms of

(i) Sin 2t cos 3t (ii) $\sin^3 2t$

(ii) $\cosh^3 2t$ (iv) $(1+te^{-t})^3$

Sol. (i) Since $\sin 2t\cos 3t = \frac{1}{2}(2\cos 3t\sin 2t) = \frac{1}{2}(\sin 5t - \sin t)$

$$\therefore L(\sin 2t\cos 3t) = L\{\tfrac{1}{2}(\sin 5t-\sin t)\} = \tfrac{1}{2}[L(\sin 5t) - L(\sin t)]$$

$$= \frac{1}{2}\left[\frac{5}{p^2+5^2}-\frac{1}{p^2+1^2}\right] = \frac{2(p^2-5)}{(p^2+25)(p^2+1)}$$

(ii) $\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\therefore \sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$$\therefore L(\sin^3 2t) = L\{\tfrac{3}{4}(\sin 2t - \tfrac{1}{4}\sin 6t)\}$$

$$= \frac{3}{4}L(\sin 2t) - \frac{1}{4}L(\sin 6t)$$

$$= \frac{3}{4}\frac{2}{p^2+2^2} - \frac{1}{4}\frac{6}{p^2+6^2} = \frac{48}{(p^2+4)(p^2+36)}$$

(iii) Since cosh 6t = 4 cos^3 2t – 3 cosh 2t

∴ cosh^{3}2t = $\frac{3}{4}$ cosh 2t + $\frac{1}{4}$ cosh 6t

∴ L(cosh^{3}2t) = L{$\frac{3}{4}$ cosh 2t + $\frac{1}{4}$ cosh 6t}

$= \frac{3}{4}$ L(cosh 2t) + $\frac{1}{4}$ (cosh 6t)

$$= \frac{3}{4}\frac{P}{p^2-2^2} + \frac{1}{4}\frac{P}{p^2-6^2} = \frac{p(p^2-28)}{(p^2-4)(p^2-36)}.$$

(iv) $(1 + te^{-t})^3 = 1 + t^3e^{-3t} + 3te^{-t}(1 + te^{-t})$

$= 1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}$

∴ $L\{(1 + te^{-t})^3\} = L(1) + L(t^3e^{-3t}) + 3L(te^{-t}) + 3L(t^2e^{-2t})$ …(1)

Now first we find the following

Determination of $L(t^3e^{-3t})$:

$L(t^3) = \frac{3!}{p^4}$ then $L(t^3e^{-3t}) = \frac{3!}{(p+3)^4} = \frac{6}{(p+3)^4}$ (Using first shifting property)

Determination of $L(t\,e^{-t})$:

$L(t) = \frac{1}{p^2}$ then $L(t\,e^{-t}) = \frac{1}{(p+1)^2}$ (Using first shifting property)

Determination of $L(t^2e^{-2t})$:

$L(t^3) = \frac{2!}{p^3}$ then $L(t^2e^{-2t}) = \frac{2!}{(p+2)^3} = \frac{2}{(p+2)^3}$ (Using first shifting property)

Also, $L(1) = \frac{1}{p}$ Now, from (1)

$$L\{(1 + te^{-t})^3\} = \frac{1}{p} + \frac{6}{(p+3)^4} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3}.$$

Example 3. Find the Laplace transforms of e^{-3t} (cos 4t + 3 sin 4t).

Sol. L(cos 4t + 3 sin 4t) = L(cos 4t) + 3 L(sin 4t)

$$= \frac{P}{p^2+16} + \frac{12}{p^2+16} = \frac{P+12}{p^2+16}$$

∴ L{e^{-3t} (cos 4t + 3 sin 4t)} = $\frac{(P+3)+12}{(p+3)^2+16}$ (Using first shifting property)

$$= \frac{P+15}{p^2+6p+25}.$$

Example 4. Find the Laplace transforms of

(i) F(t) = $\begin{cases} \cos t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ (ii) F(t) = $\begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & 2 \le t < \infty \end{cases}$

(iii) F(t) = $\begin{cases} t^2, & 0 < t < 2 \\ t-1, & 1 < t < 3 \\ 7, & t > 3 \end{cases}$

Sol. (i) L{F(t)} = $\int_0^\infty e^{-pt}.F(t)dt$ = $\int_0^\pi e^{-pt}.cost dt$ + $\int_\pi^\infty e^{-pt}.0\, dt$

$$= \left[\frac{e^{-pt}}{p^2+1}(-pcost + sint)\right]_0^\pi = \left[\frac{e^{-p\pi}}{p^2+1}p - \frac{1}{p^2+1}(-p)\right]$$

$$= \frac{p(1+e^{-p\pi})}{p^2+1}.$$

(ii) L{F(t)} = $\int_0^\infty e^{-pt}.F(t)dt$

$$= \int_0^1 e^{-pt}dt + \int_1^2 te^{-pt}dt + \int_2^\infty t^2 e^{-pt}dt$$

$$= \left(\frac{e^{-pt}}{-p}\right)_0^1 + \left(t\frac{e^{-pt}}{-p} - \frac{e^{-pt}}{p^2}\right)_1^2 + \left(t^2\frac{e^{-pt}}{-p}\right)_2^\infty - \int_2^\infty 2t\frac{e^{-pt}}{-p}dt$$

$$= \left(\frac{1-e^{-p}}{p}\right) + \left(\frac{-2}{p}e^{-2p} - \frac{e^{-2p}}{p^2}\right) - \left(\frac{e^{-p}}{-p} - \frac{e^{-p}}{p^2}\right) + \frac{4}{p}e^{-2p} + \frac{2}{p}\int_2^\infty te^{-pt}dt$$

$$= \frac{1}{p} + \frac{2}{p}e^{-2p} + \frac{e^{-p}}{p^2} - \frac{e^{-2p}}{p^2} + \frac{2}{p}\left[\left(t\frac{e^{-pt}}{-p}\right)_2^\infty - \int_2^\infty 1.\frac{e^{-pt}}{-p}dt\right]$$

$$= \frac{1}{p} + \frac{2}{p}e^{-2p} + \frac{e^{-p}}{p^2} - \frac{e^{-2p}}{p^2} + \frac{2}{p}\left[\frac{2}{p}e^{-2p} + \frac{1}{p}\left(\frac{e^{-pt}}{-p}\right)_2^\infty\right]$$

$$= \frac{1}{p} + \frac{2}{p}e^{-2p} + \frac{e^{-p}}{p^2} + \frac{3}{p^2}e^{-2p} + \frac{2}{p^3}e^{-2p}.$$

(iii) $L\{F(t)\} = \int_0^\infty e^{-pt}.F(t)dt$

$$= \int_0^2 t^2e^{-pt}dt + \int_2^3 (t-1)e^{-pt}dt + \int_3^\infty 7e^{-pt}dt$$

$$= \left(t^2\frac{e^{-pt}}{-p}\right)_0^2 - \int_0^2 2t.\frac{e^{-pt}}{-p}dt + \left\{(t-1)\frac{e^{-pt}}{-p}\right\}_2^3 - \int_2^3 \frac{e^{-pt}}{-p}dt + 7\left(\frac{e^{-pt}}{-p}\right)_3^\infty$$

$$= -\frac{4}{p}e^{-2p} + \frac{2}{p}\left[\left(t\frac{e^{-pt}}{-p}\right)_0^2 - \int_0^2 1.\frac{e^{-pt}}{-p}dt\right] + \frac{1}{p}e^{-2p} - \frac{2}{p}e^{-3p} + \frac{1}{p}\left(\frac{e^{-pt}}{-p}\right)_2^3 + \frac{7}{p}e^{-3p}$$

$$= -\frac{4}{p}e^{-2p} - \frac{2}{p^2}(2e^{-2p}) + \frac{2}{p^2}\left(\frac{e^{-pt}}{-p}\right)_0^2 + \frac{1}{p}e^{-2p} - \frac{2}{p}e^{-3p} - \frac{1}{p^2}e^{-3p} + \frac{1}{p^2}e^{-2p} + \frac{7}{p}e^{-3p}$$

$$= -\frac{4}{p}e^{-2p} - \frac{4}{p^2}e^{-2p} + \frac{2}{p^3} - \frac{2}{p^3}e^{-3p} + \frac{1}{p}e^{-2p} - \frac{2}{p}e^{-3p} - \frac{1}{p^2}e^{-3p} + \frac{1}{p^2}e^{-2p} + \frac{7}{p}e^{-3p}$$

$$= \frac{2}{p^3} - \frac{e^{-2p}}{p^3}(2 + 3p + 3p^2) + \frac{e^{-2p}}{p^2}(5p-1).$$

Example 5. Find L{F(t)} if

(i) $F(t) = \begin{cases} \sin\left(t-\frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$ (ii) $F(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases}$

Sol. (i) $L\{F(t)\} = e^{-p\frac{\pi}{3}}$ L(sin t) $\left(\because a = \frac{\pi}{3}\right)$

$$= e^{-p\frac{\pi}{3}}.\frac{1}{p^2+1}$$ (Using Second Shifting property)

(ii) $L\{F(t)\} = e^{-p}\, L(t^2)$ $(\because a = 1)$

$$= e^{-p} \cdot \frac{2}{p^3}. \qquad \text{(Using Second Shifting property)}$$

Example 6. Given that $L\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{p}$, find $L\left(\frac{\sin at}{t}\right)$.

Sol. By change of scale property ,

$$L\left(\frac{\sin at}{t}\right) = \frac{1}{a}\tan^{-1}\left(\frac{1}{p/a}\right)$$

$$\frac{1}{a}\, L\left(\frac{\sin at}{t}\right) = \frac{1}{a}\tan^{-1}\left(\frac{a}{p}\right)$$

$$\Rightarrow \quad L\left(\frac{\sin at}{t}\right) = \tan^{-1}\left(\frac{a}{p}\right).$$

3.10 *FUNCTIONS OF EXPONENTIAL ORDER*

A function F(t) is said to be of exponential order as t → ∞, if there exist constants M and b and a fixed value t_0 of t such that

$$|F(t)| = Me^{bt} \text{ for } t \geq t_0$$

We also write F(t) = O (e^{bt}) , t → ∞ to mean that F(t) is of exponential order.

From the definition, it is clear that if a constant b exists, such that $\lim_{t\to\infty} e^{-bt}F(t)$ exists or the value of limit is finite then function F(t) is of exponential order.

3.11 *A FUNCTIONS OF CLASS 'A'*

A function which is piecewise continuous over every finite interval in the range t ≥ 0 and is of exponential order as t → ∞ is termed as a function of class A. A function F(t) is said to be piecewise continuous in any interval [a , b] if it is defined on that interval and is such that the interval can be broken up into a finite number of subinterval in each of which F(t) is continuous.

3.12 *EXISTANCE THEOREM*

■ If F(t) is piecewise continuous for t ≥ 0 and is of exponential order b, then

L{F(t)} = f(p) exist for p > b. in the other words, if F(t) is a function of class A , L{F(t)} exists.

Proof: $\int_0^\infty e^{-pt}.F(t)dt = \int_0^{t_0} e^{-pt}.F(t)dt + \int_{t_0}^\infty e^{-pt}.F(t)dt = I_1 + I_2$ (say)

I_1 exist since F(t) is piecewise continuous in every finite interval

$0 \leq t \leq t_0$.

$$|I_2| \leq \int_{t_0}^\infty |e^{-pt}.F(t)|dt \leq \int_0^\infty |e^{-pt}.F(t)|dt$$

$$\leq \int_0^\infty e^{-pt}.Me^{pt}dt \text{(as F(t) is exponential function of order b)}$$

$$\leq \int_0^\infty e^{-(p-b)t}.Mdt = \frac{M}{p-b}.$$

Thus the Laplace transform exist for p > b.

Note: ■ The condition of the theorem are sufficient but not necessary for the existence of Laplace transform.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that t^n is of exponential order as t → ∞.

Sol. $\lim_{t\to\infty}(e^{-bt}.t^n) = \lim_{t\to\infty}\frac{t^n}{e^{bt}}$

$$= \lim_{t\to\infty}\frac{nt^{n-1}}{be^{bt}} = \lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{b^2e^{bt}} = 0$$

Since t^n = O (e^{bt}) , t → ∞ for any fixed positive value of b.

Therefore t^n is of exponential order.

Example 2. Prove that e^{t^2} is not of exponential order as t → ∞.

Sol. $\lim_{t\to\infty}(e^{-bt}.e^{t^2}) = \lim_{t\to\infty}\frac{e^{t^2}}{e^{bt}} = \lim_{t\to\infty}(e^{t^2-bt})$

If b ≤ 0, this limit is infinite.

If b > 0, $\lim_{t\to\infty} e^{(t-b)} = \infty$

Thus, whatever be the value of b, this limit is not finite hence we can not find a number M such that $e^{t^2} < Me^{bt}$.

∴ e^{t^2} is not of exponential order as t → ∞.

3.13 *LAPLACE TRANSFORM OF DERIVATIVE*

■ **Theorem 1.** If F(t) is continuous for all t ≥ 0 and of exponential order b as t → ∞, and if F'(t) is of class A, then Laplace transform of the derivative F'(t) exists when p > b and

L{F'(t)} = pL{F(t)} – F(0) = pf(p) – F(o) , if L{F(t)} = f(p).

Proof: L{F'(t)} = $\int_0^\infty e^{-pt}F'(t)dt$ **…..(1)**

$$= [e^{-pt}F(t)]_0^\infty + \int_0^\infty e^{-pt}F(t)dt \text{ (integrating by parts)}$$

$$= \lim_{t\to\infty} e^{-pt}F(t) - F(0) + pL\{F(t)\} \quad \textbf{…..(2)}$$

Since F(t) is of exponential order b as t → ∞ then for p > b, $e^{-pt}F(\mathrm{t}) \to 0$ as → ∞

∴ From (2) L{F'(t)} = pL{F(t)} – F(0) = pf(p) – F(o) ,   if L{F(t)}= f(p)

Note: ■ if F(t) fails to be continuous at t = 0 but $\lim_{t\to\infty}$F(t) = F(0+0) exists, then L{F'(t)} = pL{F(t)} – F(0 + 0)

■ **Theorem 2.** If F(t) is continuous, except for an ordinary discontinuity at t = a (a > 0) as given in figure, then L{F'(t)} = pL{F(t)} – F(0) - e^{-ap}[F(a + 0) – F(a – 0)], where F(a + 0) and F(a – 0) are the limits of F at t = a as t approaches a from right and left respectively. The quantity

F(a + 0) – F(a – 0) is called jump discontinuity at t = a, and

$e^{-pt}F(\mathrm{t}) \to 0$ as t → ∞.

Proof: L{F'(t)} = $\int_0^\infty e^{-pt}.F'(\mathrm{t})dt$

$$= \int_0^a e^{-pt}.F'(\mathrm{t})dt + \int_a^\infty e^{-pt}.F'(\mathrm{t})dt$$

$$= [e^{-pt}F(\mathrm{t})]_0^a + p\int_0^a e^{-pt}F(\mathrm{t})dt + \{e^{-pt}.F(\mathrm{t})\}_0^\infty + p\int_0^\infty e^{-pt}F(\mathrm{t})dt$$

$$= e^{-at}\,\mathrm{F(a-0) - F(0)} + p\int_0^\infty e^{-pt}F(\mathrm{t})dt + \lim_{t\to\infty} e^{-pt}F(\mathrm{t}) - e^{-at}F(\mathrm{a}+0)$$

$$= \mathrm{L}\{F'(\mathrm{t})\} = \mathrm{pL}\{\mathrm{F(t)}\} - \mathrm{F(0)} - e^{-ap}[\mathrm{F(a+0) - F(a-0)}] \quad (\lim_{t\to\infty} e^{-pt}F(\mathrm{t}) = 0)$$

Note:

■**Generalization** if F(t) and its first (n – 1) derivatives are continuous functions for all t ≥ 0 and are of exponential order b as t → ∞ and if $F^{(n)}(t)$ is of class A then Laplace transformation of $F^{(n)}(t)$ exists when p > b given by

L{$F^{(n)}(t)$} = p^nL{f(t)} - $p^{n-1}F(0)$ - $p^{n-2}F'(0)$ - . . . $F^{(n-1)}(0)$.

3.14 *INITIAL – VALUE THEOREM*

If F(t) is continuous for all t ≥ 0 and is of exponential order as t → ∞ and if $F'(t)$ is of class A then $\lim_{t\to 0} F(t) = \lim_{p\to\infty} pL\{F(t)\}$.

3.15 *FINAL – VALUE THEOREM*

If F(t) is continuous for all t ≥ 0 and is of exponential order as t → ∞ and if $F'(t)$ is of class A then $\lim_{t\to\infty} F(t) = \lim_{p\to 0} pL\{F(t)\}$.

Example. If $L\{F(t)\} = \frac{1}{p(p+\beta)}$ then, find (i) $\lim_{t\to\infty} F(t)$ (ii) $\lim_{t\to 0} F(t)$

Sol. (i) Using final value theorem,

$$\lim_{t\to\infty} F(t) = \lim_{p\to 0} pL\{F(t)\} = \lim_{p\to 0} \frac{1}{p+\beta} = \frac{1}{\beta}$$

(i) Using initial-value theorem,

$$\lim_{t\to 0} F(t) = \lim_{p\to\infty} pL\{F(t)\} = \lim_{p\to\infty} \frac{p}{p(p+\beta)} = \lim_{p\to\infty} \frac{1}{p+\beta} = 0.$$

3.16 *LEIBNITZ RULE*

To develop the theory of Laplace transforms further, we state the following results for differentiation under the integral sign.

Let $\emptyset(\alpha) = \int_{u_1}^{u_2} f(x,\alpha)dx$, $a \le \alpha \le b$, where u_1 and u_2 may depend on the parameter α then,

$$\frac{d\emptyset}{d\alpha} = \int_{u_1}^{u_2} \frac{\partial f}{\partial \alpha} dx + f(u_2,\alpha)\,\frac{du_2}{d\alpha} - f(u_1,\alpha)\,\frac{du_1}{d\alpha}$$

For a $\leq \alpha \leq$ b if $f(x, \alpha)$ and $\frac{\partial f}{\partial \alpha}$ are continuous in both x and α in some region of x α plane including, $u_1 \leq \alpha \leq u_2$, a $\leq \alpha \leq$ b and u_1 and u_2 are continuous and have continuous derivatives in interval (a, b).

Note: if u_1 and u_2 are constants, the last two terms in (1) are zero and so $\frac{d\emptyset}{d\alpha} = \int_{u_1}^{u_2} \frac{\partial f}{\partial \alpha} dx$

3.17 *LAPLACE TRANSFORM OF INTEGRALS*

■ If L{F(t)} = f(p), then L{$\int_0^t F(t)dt$} = $\frac{1}{P}$ f(p)

Proof: let G(t) = $\int_0^t F(t)dt$, then $G'(t)$ = F(t) and G(0) = 0

Taking Laplace transform, we get

L{$G'(t)$ } = p L{G(t)} – G(0) = p L{G(t)}

∴ L{G(t)} = $\frac{1}{P}$ L{$G'(t)$ } = $\frac{1}{P}$ L{G(t)} = $\frac{1}{P}$ f(p)

i.e. L{$\int_0^t F(t)dt$} = $\frac{1}{P}$ f(p)

3.18 *MULTIPLICATION BY t^n*

■ **If L{F(t)} = f(p), then L{t^n F(t)} = $(-1)^n \frac{d^n}{dp^n}$[f(p)],**

where n = 1, 2, 3, ……..

Proof: we prove the theorem by Mathematical induction

If L{F(t)} = f(p) $\Rightarrow$ $\int_0^\infty e^{-pt}.F(\mathrm{t})dt$ = f(p)

Differentiating both sides w.r.t. p (using Leibnitz's rule) , we have

$$\frac{d}{dp}\int_0^\infty e^{-pt}.F(\mathrm{t})dt = \frac{d}{dp}\ [\mathrm{f(p)}]$$

Or $$\int_0^\infty \frac{\partial}{\partial p}e^{-pt}.F(\mathrm{t})dt = \frac{d}{dp}\ [\mathrm{f(p)}]$$

Or $$\int_0^\infty -t\ e^{-pt}.F(\mathrm{t})dt = \frac{d}{dp}\ [\mathrm{f(p)}]$$

Or $$\int_0^\infty e^{-pt}[\ tF(\mathrm{t})]dt = -\frac{d}{dp}\ [\mathrm{f(p)}]$$

Or $$\mathrm{L\{F(t)\}} = -\frac{d}{dp}\ [\mathrm{f(p)}]$$

Therefore theorem is true for n = 1.

Now assume the theorem to be true for n = m, so that

$$\mathrm{L}\{t^m\ \mathrm{F(t)}\} = (-1)^m\ \frac{d^m}{dp^m}[\mathrm{f(p)}]$$

Or $$\int_0^\infty e^{-pt}t^m\ F(\mathrm{t})dt = (-1)^m\ \frac{d^m}{dp^m}[\mathrm{f(p)}]$$

Differentiating both sides w.r.t. p, we have

$$\frac{d}{dp}\int_0^\infty e^{-pt}t^m\ F(\mathrm{t})dt = (-1)^m\ \frac{d^{m+1}}{dp^{m+1}}[\mathrm{f(p)}]$$

Or $$\int_0^\infty \frac{\partial}{\partial p}e^{-pt}t^m\ F(\mathrm{t})dt = (-1)^m\ \frac{d^{m+1}}{dp^{m+1}}[\mathrm{f(p)}]$$

Or $$\int_0^\infty -te^{-pt}t^m\ F(\mathrm{t})dt = (-1)^m\ \frac{d^{m+1}}{dp^{m+1}}[\mathrm{f(p)}]$$

Or $$\int_0^\infty e^{-pt}t^{m+1}\ F(\mathrm{t})dt = (-1)^{m+1}\ \frac{d^{m+1}}{dp^{m+1}}[\mathrm{f(p)}]$$

Or $$\mathrm{L}\{t^{m+1}\ \mathrm{F(t)}\} = (-1)^{m+1}\ \frac{d^{m+1}}{dp^{m+1}}[\mathrm{f(p)}]$$

Which shows that the theorem is true for n = m + 1.

Hence by Mathematical induction, the theorem is true for all positive integer n.

3.19 *DIVISION BY t*

■ **If L{F(t)} = f(p), then L$\left\{\frac{1}{t} F(t)\right\} = \int_p^\infty f(p)dp$ provided the integral exists.**

Proof: we have f(p) = $\int_0^\infty e^{-pt}.F(t)dt$

Integrating both sides w.r.t. p from p to ∞ , we have

$$\int_p^\infty f(p)dp = \int_p^\infty \left[\int_0^\infty e^{-pt}.F(t)dt\right]dp$$

Since p and t are independent, changing the order of integration on the right-hand side, we have

$$\int_p^\infty f(p)dp = \int_0^\infty \left[\int_p^\infty e^{-pt}.F(t)dt\right]dp$$

$$= \int_0^\infty \left[\frac{e^{-pt}}{-t}\right]_p^\infty F(t)\,dt = \int_0^\infty e^{-pt}\frac{F(t)}{t}\,dt = L\left\{\frac{1}{t} F(t)\right\}.$$

ILLUSTRATIVE EXAMPLES

Example 1. If L{t sin ωt } = $\frac{2\omega t}{(p^2+\omega^2)^2}$, evaluate

(i) L$\{\omega t\cos\omega t + \sin\omega t\}$ (ii) L$\{2\cos\omega t - \omega t\sin\omega t\}$

Sol. Let F(t) = t sin ωt then

$$F'(t) = \omega t\cos\omega t + \sin\omega t \text{ and } F''(t) = 2\omega\cos\omega t - \omega^2 t\sin\omega t$$

Also F(0) = 0, $F'(0) = 0$, $F''(0) = 2\omega$

Given f(p) = $\frac{2\omega p}{(p^2+\omega^2)^2}$

(i) L{$F'(t)$} = p f(p) – F(0)

$\Rightarrow$ L{$\omega t \cos \omega t + \sin \omega t$ } = p. $\frac{2\omega p}{(p^2+\omega^2)^2}$ - 0 = $\frac{2\omega p^2}{(p^2+\omega^2)^2}$.

(ii) L{$F''(t)$} = p^2 f(p) – p F(0) - $F'(0)$

$\Rightarrow$ L{$2\omega \cos \omega t - \omega^2 t \sin \omega t$ } = $p^2 . \frac{2\omega p}{(p^2+\omega^2)^2}$ - p.0 - 0 = $\frac{2\omega p^3}{(p^2+\omega^2)^2}$

$\therefore$ L{$2 \cos \omega t - \omega t \sin \omega t$ } = $\frac{p^3}{(p^2+\omega^2)^2}$.

Example 2. If F(t) = $\frac{e^{at} - \cos \omega t}{t}$, find the Laplace transform of F(t).

Sol. L(e^{at}) = $\frac{1}{p-a}$

L($\cos bt$) = $\frac{p}{p^2+b^2}$

$\therefore$ L($e^{at} - \cos bt$) = $\frac{1}{p-a} - \frac{p}{p^2+b^2}$

Now, L($\frac{e^{at} - \cos \omega t}{t}$) = $\int_p^\infty \left(\frac{1}{p-a} - \frac{p}{p^2+b^2}\right)$dp

$$= \left[\log(p-a) - \frac{1}{2}\log(p^2+b^2)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log(p-a)^2 - \log(p^2+b^2)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{(p-a)^2}{p^2+b^2}\right]_p^\infty = \frac{1}{2}\left[\log\frac{1-\frac{a}{p}}{1+\frac{b^2}{p^2}}\right]_p^\infty$$

$$= -\frac{1}{2}\log\{\frac{(p-a)^2}{p^2+b^2}\} = \frac{1}{2}\log\{\frac{p^2+b^2}{(p-a)^2}\}.$$

Example 3. Find the Laplace transform of $\frac{\sin at}{t}$, Does the Laplace transform of $\frac{\text{Cos}\, at}{t}$ exist ?

Sol. Since $\lim_{t\to 0}\left(\frac{\sin at}{t}\right)$ = a, the Laplace transform of $\frac{\sin at}{t}$ exists.

Now, L($\sin at$) = $\frac{a}{p^2+a^2}$

$\therefore$ L($\frac{\sin at}{t}$) = $\int_p^\infty \frac{a}{p^2+a^2}\, dp = \left[\tan^{-1}\left(\frac{p}{a}\right)\right]_p^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{a}\right)$

$= \cot^{-1}\left(\frac{p}{a}\right) = \tan^{-1}\left(\frac{a}{p}\right)$.

Example 4. Find the Laplace transform of

(i) t^3e^{-3t} (ii) t Sin^2 3t (iii) $\frac{1-Cost}{t^2}$

Sol. (i) L$\{e^{-3t}\} = \frac{1}{p\,+\,a}$

$\therefore$ L$\{t^3e^{-3t}\} = (-1)^3\frac{d^3}{dp^3}\left(\frac{1}{p\,+\,3}\right) = -\frac{(-1)^3\, 3!}{(p+3)^4} = \frac{6}{(p+3)^4}$.

(ii) $\sin^2$ 3t = $\frac{1-Cos6t}{2}$

$\therefore$ L$\{sin^2 3t\} = \frac{1}{2}$ [L(1) – L(cos 6t)] = $\frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+36}\right) = \frac{18}{p(p^2+36)}$

$\therefore$ L$\{tsin^2 3t\} = -\frac{d}{dp}\left[\frac{18}{p(p^2+36)}\right]$ = (-18) (-1) $(p^3+36p)^{-2}(3p^2+36)$

$= \frac{54(p^2+12)}{p^2(p^2+36)^2}$.

(iii) L(1 – cos t) = $\frac{1}{p} - \frac{p}{p^2+1}$

L$\left(\frac{1-\cos t}{t}\right) = \int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+1}\right) dp$

$$= \left[\log p - \tfrac{1}{2}\log(p^2+1)\right]_p^\infty = \tfrac{1}{2}\left[\log(\tfrac{p^2}{p^2+1})\right]_p^\infty$$

$$= \tfrac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{p^2}}\right)\right]_p^\infty = -\tfrac{1}{2}\log(\tfrac{p^2}{p^2+1}) = \tfrac{1}{2}\log(\tfrac{p^2+1}{p^2})$$

Now, $L\left(\frac{1-\cos t}{t^2}\right) = \int_p^\infty \tfrac{1}{2}\log(\tfrac{p^2+1}{p^2})dp$

$$= \tfrac{1}{2}\int_p^\infty [\log(p^2+1) - 2\log p]dp$$

$$= \tfrac{1}{2}\left[\{\log(p^2+1) - 2\log p\}.p - \int\left(\frac{2p}{p^2+1} - \frac{2}{p}\right).pdp\right]_p^\infty$$

$$= \left[\tfrac{p}{2}\log(\tfrac{p^2+1}{p^2})\right]_p^\infty + \int_p^\infty \frac{1}{p^2+1}dp = -\tfrac{p}{2}\log(1+\tfrac{1}{p^2}) + \tfrac{\pi}{2} - \tan^{-1} p$$

$\Longrightarrow L\left(\frac{1-\cos t}{t^2}\right) = \cot^{-1} p - \frac{p}{2}\log(1+\frac{1}{p^2})$.

Example 5. Find the Laplace transform of the following functions:

(i) $\frac{e^{-t}\sin t}{t}$ (ii) $\frac{1-\cos 2t}{t}$

Sol. (i) $L(e^{-t}\sin t) = \frac{1}{(p+1)^2+1}$

$\Longrightarrow L(\frac{e^{-t}\sin t}{t}) = \int_p^\infty \frac{1}{(p+1)^2+1}dp = [\tan^{-1}(p+1)]_p^\infty$

$$= \tfrac{\pi}{2} - \tan^{-1}(p+1) = \cot^{-1}(p+1).$$

(ii) $L(1-\cos 2t) = \frac{1}{P} - \frac{p}{p^2+2^2}$

$\Longrightarrow L\left(\frac{1-\cos 2t}{t}\right) = \int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+4}\right)dp$

$$= \left[\log p - \tfrac{1}{2}\log(p^2+4)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{p^2}{p^2+4}\right]_p^\infty = \frac{1}{2}\left[\lim_{p\to\infty}\log\left(\frac{1}{1+\frac{4}{p^2}}\right) - \log\frac{p^2}{p^2+4}\right]$$

$$= \frac{1}{2}\left[\log 1 + \log\frac{p^2+4}{p^2}\right] = \frac{1}{2}\log\left(\frac{p^2+4}{p^2}\right).$$

3.20 *SUMMARY*

1. Laplace transform L{F(t)} = f(p) = $\int_0^\infty e^{-pt}F(t)dt$.

2. Laplace transform of some elementary functions

(i) L{1} = $\frac{1}{p}$, p > 0

(ii) L{t^n} = $\frac{n!}{p^{n+1}}$, where n is positive integer.

(iii) L{e^{at}} = $\frac{1}{p-a}$, p > a

(iv) L{$sinat$} = $\frac{a}{p^2+a^2}$, p > 0

3. Exitance theorem: If F(t) is piecewise continuous for t $\geq$ 0 and is of exponential order b, then L{F(t)} = f(p) exist for p > b.

4. Laplace transform of derivative:

L{F'(t)} = pL{F(t)} – F(0) = pf(p) – F(o) ,   if L{F(t)}= f(p)

5. Initial-value theorem: If F(t) is continuous for all t $\geq$ 0 and is of exponential order as t $\to \infty$ and if $F'(t)$ is of class A then

$\lim_{t\to 0}$F(t) = $\lim_{p\to\infty} pL$ {F(t)}.

6. Final-value theorem: If F(t) is continuous for all t $\geq$ 0 and is of exponential order as t $\to \infty$ and if $F'(t)$ is of class A then $\lim_{t\to\infty}$F(t) = $\lim_{p\to 0} pL$ {F(t)}.

7. If L{F(t)} = f(p), then $L\left\{\frac{1}{t}\ F(t)\right\} = \int_p^\infty f(p)dp$ provided the integral exists.

CHECK YOUR PROGRESS

True and False questions

Problem 1. $\lim_{t\to 0} F(t) = \lim_{p\to\infty} pL\{F(t)\}$ is Initial-value theorem.

Problem 2. Laplace transform is defined as

$$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt}F(t)dt\ .$$

Problem 3. $L\{t^n\} = \frac{n!}{p^{n+1}}$, where n is positive integer.

Problem 4. $L\{sinat\} = \frac{a}{p^2+a^2}$, p > 0

Problem 5. If L{F(t)} = f(p), then $L\left\{\frac{1}{t}\ F(t)\right\} = \int_p^\infty f(p)dp$ provided not the integral exists.

3.21 *GLOSSARY*

Discontinuous functions

Periodic Functions

Integration

Even, odd functions

Trigonometric functions

Integrations

3.22 *REFERENCES*

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3.23 *SUGGESTED READING*

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3.24 *TERMINAL AND MODEL QUESTIONS*

Q 1. Definition of Laplace transform.

Q 2. What is Initial-value theorem and Final value theorem.

Q 3 Find the Laplace transform of the following functions:

(i) t^5e^{3t} (ii) $e^{-2t}sin4t$

Q 4. Find the Laplace transform of the function $L\{sin^2t\}$.

Q 5. State and prove Existence theorem.

3.25 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. True

CYQ 5. False

TERMINAL QUESTIONS

TQ 3. (i) $\frac{120}{(p-3)^6}$ **(ii)** $\frac{4}{p^2+4p+20}$

TQ 4. $\frac{p^2+8}{p(p^2+16)}$

UNIT 4: Laplace Transform II

Contents

4.1 *INTRODUCTION*

In mathematics, the inverse Laplace transform of a function F(s) is the piecewise-continuous and exponentially-restricted real function f(t) which has the property: denotes the Laplace transform.

4.2 *OBJECTIVE*

At the end of this topic Lerner will be able to understand:

(i) Heaviside's Unit Step Function

(ii) inverse Laplace transform

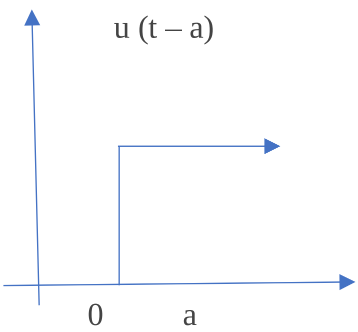
(iii) Heavisede Expansion Formula for Inverse Laplace Transform

(iv) Convolution theorem

4.3 *LAPLACE TRANSFORM OF SOME SPECIAL FUNCTIONS*

■ **(1) Unit Step Function (Or Heaviside's Unit Step Function)**

The unit step function u(t – a) is defined as



u(t – a) = $\begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$, where a ≥ 0.

As a particular case, u(t) = $\begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

The product F(t).u(t – a) = $\begin{cases} 0, & t < a \\ F(t), & t \geq a \end{cases}$

t

The function F(t – a). u(t – a) represents the graph of F(t) shifted through a distance 'a' to be right.

Laplace Transform of Unit Step function

L{u(t – a)} = $\int_0^\infty e^{-pt} u(t-a)dt$

$$= \int_0^a e^{-pt}.0\,dt + \int_a^\infty e^{-pt}.1\,dt$$

$$= 0 + \left[-\frac{e^{-pt}}{p}\right]_a^{\infty} = \frac{1}{p}\, e^{-at}$$

In particular, $L\{u(t)\} = \frac{1}{p}$.

Second Shifting Theorem

If L{F(t)} = f(p), then L{F(t – a).u(t-a)} = e^{-ap}f(p).

$$L\{F(t-a).u(t-a)\} = \int_0^{\infty} e^{-pt} F(t-a).\, u(t-a)\, dt$$

$$= \int_0^{\infty} e^{-p(u+a)} F(u)\, du, \text{ where } u = t - a$$

$$= e^{-ap} \int_0^{\infty} e^{-pu} F(u)\, du = e^{-ap}\, f(p)$$

Note: ■ if a = 0, L{F(t) u(t)} = f(p) = L{F(t)}.

ILLUSTRATIVE EXAMPLES

Example 1. Express the following functions in terms of Heaviside's unit step function:

(i) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$ **(ii)** $F(t) = \begin{cases} e^{-t} & , \ 0 < t < 3 \\ 0 & , \ x > 3 \end{cases}$

(iii) $F(t) = \begin{cases} \sin t & , \ t > \pi \\ \cos t & , \ 0 < t < \pi \end{cases}$.

Sol. (i) F(t) = sin t { u(t) – u(t – π)} + sin 2t{u(t – π) $- u(t - 2\pi)$}

+ sin3t u(t - 2 π)

= sin t u(t) + (sin 2t – sin t) u(t - π) + (sin 3t – sin 2t) u(t - 2 π).

(ii) F(t) = e^{-t} { u(t) – u(t – 3)} + 0 {u(t – 3)}

= e^{-t}{ u(t) – u(t – 3)}.

(iii) F(t) = sin t u(t - π) + cos t{u(t) - u(t - π)}

= cos t u(t) + (sin t – cos t) u(t - π)

Example 2. Find the Laplace transformation of the following functions

(i) $(t-1)^2\ u(t-1)$ **(ii)** sin t u(t - π)

(iii) e^{-3t}u (t – 2) **(iv)** e^{-t} { 1 – u(t – 2)}

Sol. (i) Comparing $(t-1)^2\ u(t-1)$ with F(t – a) u(t -a)

a= 1 and F(t) = $\frac{2}{p^3}$

∴ f(p) = L{F(t)} = $\frac{2}{p^3}$

∴ L{$(t-1)^2\ u(t-1)$ } = e^{-t} f(p)

$$= \frac{2e^{-p}}{p^3}.$$

(ii) Expressing sin t as a function of (t - π) , we have

Sin t = sin [(t - $\pi) + \pi$] = - sin(t - π)

Comparing - sin(t - $\pi)u$(t - π) with F(t – a) u(t - a), we get

a= π and F(t) = - sin t

∴ f(p) = L{F(t)} = $-\frac{1}{p^2+1}$

Now by second shifting property

∴ L[sin t u(t - π)] = $e^{-\pi p}$f(p)

$$= \frac{-e^{-\pi p}}{p^2+1}.$$

(iii) L{u(t – 2)} = $\frac{e^{-2p}}{p}$

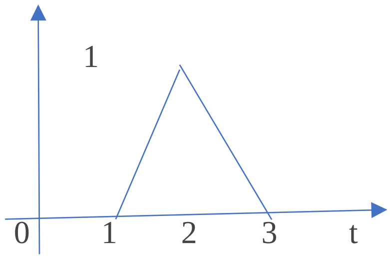
$\therefore$ L$\{e^{-3t}$u(t – 2)} = $\frac{e^{-2(p+3)}}{p+3}$ (using by first shifting property).

(iv) L{1 – u(t – 2)} = $\frac{1}{p}$ - $\frac{e^{-2p}}{p}$

$\therefore$ L[e^{-t}{1 – u(t – 2)}] = $\frac{1}{p+1}$ - $\frac{e^{-2(p+1)}}{p+1}$ (using by first shifting property).

Example 3. Express the function shown in the diagram in term of unit step function and obtain its Laplace transformation.

Sol. Here $f(t) = \begin{cases} t-1\ , & 1 < 2 < 3 \\ 3-t\ , & 2 < t < 3 \end{cases}$



$\therefore$ F(t) = (t – 1) {u(t – 1) – u (t – 2)} +

(3 – t) {u(t – 2) – u (t – 3)}

= (t – 3) u (t – 3) – 2 (t – 2) u (t – 2) + (t -1) u (t -1)

Hence, L{F(t)} = L{(t – 3) u (t – 3) – 2 (t – 2) u (t – 2) + (t -1) u (t -1)}

$$= \frac{e^{-3p}}{p^2} - \frac{2e^{-2p}}{p^2} + \frac{e^{-p}}{p^2}$$

$$= \frac{e^{-p}(1-e^{-p})^2}{p^2}\ .$$

■ **(2) Periodic Function.**

If f(t) is a periodic function with period T i.e. f(t + T) = f(t). then

$$L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)dt$$

Here, L{f(t)} = $\int_0^\infty e^{-pt} f(t)dt$

$$= \int_0^T e^{-pt} f(t)dt + \int_T^{2T} e^{-pt} f(t)dt + \int_{2T}^{3T} e^{-pt} f(t)dt + \ldots.$$

Putting t = u, t = u + T, t = u + 2T, ……. In the successive integrals

L{f(t)} = $\int_0^T e^{-pu} f(u)du + \int_0^T e^{-p(u+T)} f(u+T)du$

$+ \int_0^T e^{-p(u+2T)} f(u+2T))du + \ldots.$

Since f(u) = f(u + T) = f(u + 2T) = ……, we have

L{f(t)} = $\int_0^T e^{-pu} f(u)du + e^{-pT} \int_0^T e^{-pu} f(u)du$

$+ e^{-2pT} \int_0^T e^{-pu} f(u)du + \ldots.$

$= (1 + e^{-pT} + e^{-2pT} + \ldots.) \int_0^T e^{-pu} f(u)du$

$= \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)dt$.

ILLUSTRATIVE EXAMPLES

Example 1. Find the Laplace transform of the following periodic functions:

(i) f(t) = $\frac{t}{T}$, for 0< t < T (saw-tooth wave of period T)

(ii) f(t) = $\sin\left(\frac{\pi t}{a}\right)$ for 0 < t < a. (Rectified sine wave of period a)

Sol. (i) Here, L{f(t)} = $\frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)dt = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt}.\frac{t}{T}\, dt$

$= \frac{1}{T(1-e^{-pT})}\left[\left(\frac{te^{-pt}}{-p}\right)_0^T - \int_0^T 1.\frac{e^{-pt}}{T}\, dt\right]$

$= \frac{1}{1-e^{-pT}}\left[-\frac{e^{-pt}}{T} + \frac{1-e^{-pt}}{p^2T}\right] = \frac{1}{p^2T} - \frac{e^{-pT}}{p(1-e^{-pT})}$

(ii) L{f(t)} = $\frac{1}{1-e^{-ap}} \int_0^a e^{-pt} \sin\left(\frac{\pi t}{a}\right) dt$ ….. (1)

Let I = $\int_0^a e^{-pt}. \sin\left(\frac{\pi t}{a}\right) dt$

$= \left[\frac{e^{-pt}}{p^2+\frac{\pi^2}{a^2}}\left(-p\sin\left(\frac{\pi t}{a}\right) - \frac{\pi}{a}\cos\left(\frac{\pi t}{a}\right)\right)\right]_0^a$

$$= \left[-\frac{e^{-at}}{p^2+\frac{\pi^2}{a^2}}\left(\frac{\pi}{a}\right)\right] - \left[\frac{1}{p^2+\frac{\pi^2}{a^2}}\left(-\frac{\pi}{a}\right)\right] = \frac{(1+e^{-ap})a\pi}{a^2p^2+\pi^2}$$

$\therefore$ From (1), L{f(t)} = $\frac{(1+e^{-ap})a\pi}{(1-e^{-ap})(a^2p^2+\pi^2)}$

$$= \left(\frac{e^{\frac{ap}{2}}+e^{-\frac{ap}{2}}}{e^{\frac{ap}{2}}-e^{-\frac{ap}{2}}}\right)\left(\frac{a\pi}{a^2p^2+\pi^2}\right) = \frac{a\pi\ \cot h\frac{ap}{2}}{a^2p^2+\pi^2}.$$

Example 2. Draw the graph and find the Laplace transform of the triangular wave function of period 2c given by

$$\mathrm{f(t)} = \begin{cases} \mathrm{t} & ,\ 0 < \mathrm{t} \le \mathrm{c} \\ 2\mathrm{c} - \mathrm{t} & ,\ \mathrm{c} < \mathrm{t} < 2\mathrm{c} \end{cases}.$$

Sol. L{f(t)} = $\frac{1}{1-e^{-2cp}}\int_0^{2c} e^{-pt} f(t)dt$

$$= \frac{1}{1-e^{-2cp}}\left[\int_0^c e^{-pt}.t\,dt + \int_c^{2c} e^{-pt}(2c-t)dt\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\left\{t.\frac{e^{-pt}}{-p} - 1.\frac{e^{-pt}}{p^2}\right\}_0^c + \left\{(2c-t).\frac{e^{-pt}}{-p} - (-1).\frac{e^{-pt}}{p^2}\right\}_c^{2c}\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\left\{-\frac{ce^{-cp}}{p} - \frac{e^{-cp}}{p^2} + \frac{1}{p^2}\right\} + \left\{\frac{e^{-2cp}}{p^2} + \frac{ce^{-cp}}{p} - \frac{e^{-cp}}{p^2}\right\}\right]$$

$$= \frac{1}{1-e^{-2cp}}\left(\frac{1-2e^{-cp}+e^{-2cp}}{p^2}\right) = \frac{1}{p^2}\frac{(1-e^{-cp})^2}{(1+e^{-cp})(1-e^{-cp})} = \frac{1}{p^2}.\frac{1-e^{-cp}}{1+e^{-cp}}$$

$$= \frac{1}{p^2}\left(\frac{e^{\frac{cp}{2}}+e^{-\frac{cp}{2}}}{e^{\frac{cp}{2}}-e^{-\frac{cp}{2}}}\right) = \frac{1}{p^2}\tan\left(\frac{cp}{2}\right).$$

The graph of the given function is shown below:

Example 3. Find the Laplace transform of the rectified semi-wave function defined by

$$f(x) = \begin{cases} \sin \omega t & , \ 0 < t \leq \frac{\pi}{\omega} \\ 0 & , \ \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad \text{or}$$

Find the Laplace transform of the following periodic function

Sol. Here f(t) is a periodic function with period $\frac{2\pi}{\omega}$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi p}{\omega}}} \int_0^{\frac{2\pi p}{\omega}} e^{-pt} f(t)dt$$

$$= \frac{1}{1-e^{-\frac{2\pi p}{\omega}}} \left[\int_0^{\pi/\omega} e^{-pt} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-pt}.0 \, dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi p}{\omega}}} \left[\frac{e^{-pt}(-p \sin \omega t - \omega \cos \omega t)}{p^2+\omega^2}\right]_0^{\pi/\omega}$$

$$= \frac{\omega e^{-\frac{\pi p}{\omega}} + \omega}{\left(1- e^{-\frac{\pi p}{\omega}}\right)\left(1+ e^{-\frac{\pi p}{\omega}}\right)(p^2+\omega^2)} = \frac{\omega}{\left(1- e^{-\frac{\pi p}{\omega}}\right)(p^2+\omega^2)} .$$

4.4 *INVERSE LAPLACE TRANSFORM*

If L{F(t)} = f(p), then F(t) is called the inverse Laplace transform of f(p) and is defined as

$L^{-1}\{f(p)\}$ = F(t)

Here L^{-1} denotes the inverse Laplace transform operator

Example: Since $L\{e^{5t}\} = \frac{1}{p-5}$ $\therefore$ $L^{-1}\{\frac{1}{p-5}\} = e^{5t}$

The inverse Laplace transform given below at once from the results of Laplace transforms given earlier:

(1) $L^{-1}\{\frac{1}{p}\}$ = 1 (2) $L^{-1}\{\frac{1}{p-5}\}$ = e^{at}

(3) $L^{-1}\{\frac{1}{p^n}\}$ = $\frac{t^{n-1}}{(n-1)!}$, if n is a positive integer.

(4) $L^{-1}\{\frac{1}{(p-a)^n}\}$ = $e^{at}\frac{t^{n-1}}{(n-1)!}$ (5) $L^{-1}\{\frac{1}{p^2+a^2}\}$ = $\frac{1}{a}$ sin at

(6) $L^{-1}\{\frac{p}{p^2+a^2}\}$ = cos at

(7) $L^{-1}\{\frac{1}{p^2-a^2}\}$ = $\frac{1}{a}$ sinh at

(8) $L^{-1}\{\frac{p}{p^2-a^2}\}$ = cosh at

4.5 *LINEARITY PROPERTY*

If c_1 *and* c_2 are constants and L$\{F_1(t)\}$ = f_1(p) and L$\{F_2(t)\}$ = f_2(p), then

$L^{-1}\{c_1f_1(\text{p}) + c_2f_2(\text{p})\} = c_1L^{-1}\{f_1(\text{p})\} + c_2L^{-1}\{f_2(\text{p})\}$.

By definition,

$$\text{L}\{c_1f_1(\text{p}) + c_2f_2(\text{p})\} = \int_0^\infty e^{-pt}\{c_1F_1(\text{t}) + c_2F_2(\text{t})\}\,dt$$

$$= c_1\int_0^\infty e^{-pt}F_1(\text{t})dt + c_2\int_0^\infty e^{-pt}F_2(\text{t})dt$$

$$= c_1f_1(\text{p}) + c_2f_2(\text{p})$$

$\therefore L^{-1}\{c_1f_1(\text{p}) + c_2f_2(\text{p})\} = c_1F_1(\text{t}) + c_2F_2(\text{t}) = c_1L^{-1}\{f_1(\text{p})\} + c_2L^{-1}\{f_2(\text{p})\}$.

Note: ■ The above result can be extended to more than two functions.

4.6 *FIRST TRANSLATION OR SHIFTING PROPERTY*

■ **If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f(p-a)\}$ = e^{at}F(t).**

f(p) = $\int_0^\infty e^{-pt}.\mathrm{F(t)}dt$ (By definition)

∴ f(p - a) = $\int_0^\infty e^{-(p-a)t}.\mathrm{F(t)}dt$

$= \int_0^\infty e^{-pt}.e^{at}\mathrm{F(t)}dt = \mathrm{L}\{e^{at}\mathrm{F(t)}\}$

∴ $L^{-1}\{f(p-a)\} = e^{at}\mathrm{F(t)}$.

4.7 *SECOND TRANSLATION OR SHIFTING PROPERTY*

■ **If $L^{-1}\{f(p)\}$ = F(t), then**

$L^{-1}\{e^{-at}f(p)\}$ **= G(t) where** $G(t) = \begin{cases} F(t-a), & x > a \\ 0 & , t < a \end{cases}$.

f(p) = $\int_0^\infty e^{-pt}.\mathrm{F(t)}dt$ (By definition)

∴ $e^{-ap}\mathrm{f(p)} = \int_0^\infty e^{-ap}.e^{-pt}.\mathrm{F(t)}dt = \int_0^\infty e^{-p(t+a)t}.\mathrm{F(t)}dt$

Put t + a = u then dt = du

$= \int_a^\infty e^{-pu}.\mathrm{F(u-a)}du = \int_a^\infty e^{-pt}.\mathrm{F(t-a)}dt$

$= \int_0^a e^{-pt}.0dt + \int_a^\infty e^{-pt}.\mathrm{F(t-a)}dt = \int_0^\infty e^{-pt}.\mathrm{G(t)}dt = \mathrm{L}\{\mathrm{G(t)}\}$

Hence the result.

4.8 *CHANGE OF SCALE PROPERTY*

■ **If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f(ap)\}$ = $\frac{1}{a}$ F$\left(\frac{t}{a}\right)$.**

f(p) = $\int_0^\infty e^{-pt}.\mathrm{F(t)}dt$ (By definition)

$\therefore$ f(ap) = $\int_0^\infty e^{-apt}.\,F(t)dt$, now put at = u therefore dt = $\frac{du}{a}$

$$= \int_0^\infty e^{-apt}.\,F\left(\frac{u}{a}\right)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-pt}.\,F\left(\frac{t}{a}\right)dt$$

$$= \int_0^\infty e^{-pt}\left\{\frac{1}{a}\ F\left(\frac{t}{a}\right)\right\} dt = L\left\{\frac{1}{a}\ F\left(\frac{t}{a}\right)\right\}$$

$$\therefore \qquad L^{-1}\{f(ap)\} = \frac{1}{a}\,F\left(\frac{t}{a}\right).$$

Note:■ Whenever it is convenient to break an expression into partial fractions, it becomes easier to manipulate inversion term by term.

ILLUSTRATIVE EXAMPLES

Example 1. Find the inverse Laplace transform of

(i) $\frac{3(p^2-1)^2}{2p^5}$ **(ii)** $\frac{2p+1}{p^2-4}$

(iii) $\frac{4p+15}{16p^2-25}$ **(iv)** $\frac{p+1}{p^2+p+1}$

Sol. (i) $\frac{3(p^2-1)^2}{2p^5} = \frac{3p^4-6p^2+3}{2p^5} = \frac{3}{2}.\frac{1}{p} - 3.\frac{1}{p^3} + \frac{3}{2}.\frac{1}{p^5}$

$$\therefore L^{-1}\left\{\frac{3(p^2-1)^2}{2p^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{1}{p}\right\} - 3L^{-1}\left\{\frac{1}{p^3}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{p^5}\right\}$$

$$= \frac{3}{2}(1) - 3\left\{\frac{t^2}{2!}\right\} + \frac{3}{2}\left\{\frac{t^4}{4!}\right\} = \frac{3}{2} - \frac{3}{2}t^2 + \frac{1}{16}t^4.$$

(ii) $L^{-1}\left\{\frac{2p+1}{p^2-4}\right\} = 2L^{-1}\left\{\frac{p}{p^2-4}\right\} + \frac{1}{2}L^{-1}\left\{\frac{2}{p^2-4}\right\}$

$$= 2\cosh 2t + \frac{1}{2}\sinh 2t.$$

(iii) $\frac{4p+15}{16p^2-25} = \frac{4p+15}{16\left(p^2-\frac{25}{16}\right)} = \frac{1}{4}.\frac{p}{p^2-\left(\frac{5}{4}\right)^2} + \frac{15}{16}.\frac{1}{p^2-\left(\frac{5}{4}\right)^2}$

$$\therefore L^{-1}\left\{\frac{4p+15}{16p^2-25}\right\} = \frac{1}{4}L^{-1}\left\{\frac{p}{p^2-\left(\frac{5}{4}\right)^2}\right\} + \frac{15}{16}L^{-1}\left\{\frac{1}{p^2-\left(\frac{5}{4}\right)^2}\right\}$$

$$= \frac{1}{4}\cosh\left(\frac{5}{4}t\right) + \frac{15}{16}.\frac{1}{\frac{5}{4}}\sinh\left(\frac{5}{4}t\right)$$

$$= \frac{1}{4}\cosh\left(\frac{5}{4}t\right) + \frac{3}{4}\sinh\left(\frac{5}{4}t\right).$$

(iv) $\frac{p+1}{p^2+p+1} = \frac{\left(p+\frac{1}{2}\right)+\frac{1}{2}}{\left(p+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{p+\frac{1}{2}}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\cdot\frac{1}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$$\therefore L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\} = L^{-1}\left\{\frac{p+\frac{1}{2}}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\left(\frac{t}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}e^{-\left(\frac{t}{2}\right)}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$= \frac{1}{\sqrt{3}}e^{-\left(\frac{t}{2}\right)}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t\right).$$

Example 2. Find the inverse Laplace transform of

(i) $\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}$ **(ii)** $\frac{p^3}{p^4-a^4}$

Sol.(i) $L^{-1}\left\{\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$

$$= 3L^{-1}\left\{\frac{1}{p-\frac{3}{2}}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{p^2-\left(\frac{4}{3}\right)^2}\right\} - \frac{4}{9}L^{-1}\left\{\frac{1}{p^2-\left(\frac{4}{3}\right)^2}\right\}$$

$$+\frac{1}{2}L^{-1}\left\{\frac{1}{p^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$-\frac{3}{8}L^{-1}\left\{\frac{p}{p^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{3}.\frac{3}{4}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{1}{2}.\frac{4}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}.$$

(ii) $L^{-1}\left\{\frac{p^3}{p^4-a^4}\right\} = L^{-1}\left[p\left\{\frac{p^2}{(p^2-a^2)(p^2+a^2)}\right\}\right]$

$$= L^{-1}\left[\frac{p}{2}\left\{\frac{1}{p^2-a^2} + \frac{1}{p^2+a^2}\right\}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{p^2-a^2} + \frac{1}{p^2+a^2}\right)$$

$$= \frac{1}{2}\left(\cosh at + \cos at\right) .$$

Example 3. Find the inverse Laplace transform of

(i) $L^{-1}\left(\frac{e^{-2p}}{p^2}\right)$ **(ii)** $L^{-1}\left(\frac{e^{-p} - 3e^{-3p}}{p^2}\right)$

Sol. (i) we have

$$L^{-1}\left(\frac{1}{p^2}\right) = t = F(t)$$

$$\therefore \quad L^{-1}\left(e^{-2p}\frac{1}{p^2}\right) = \begin{cases} t-2, & t > 2 \\ 0, & t < 2 \end{cases} = (t-2)\, u\,(t-2) .$$

(ii) $L^{-1}\left(e^{-p}\frac{1}{p^2}\right) = \begin{cases} t-1, & t > 1 \\ 0, & t < 2 \end{cases} = (t-1)\, u\,(t-1)$

$$L^{-1}\left(e^{-3p}\frac{1}{p^2}\right) = \begin{cases} t-3, & t > 3 \\ 0, & t < 3 \end{cases} = (t-3)\, u\,(t-3)$$

By second shifting theorem

Hence $L^{-1}\left(\frac{e^{-p} - 3e^{-3p}}{p^2}\right) = \begin{cases} \cos 3\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

By second shifting theorem

$$= \cos 3tu\left(t - \frac{2\pi}{3}\right).$$

Example 4. Find the inverse Laplace transform of

(i) $\frac{p^2+2p-3}{p(p-3)(p+2)}$ (ii) $\frac{1+2p}{(p+2)^2(p-1)^2}$

Sol. (i) $\frac{p^2+2p-3}{p(p-3)(p+2)} = \frac{1}{2p} + \frac{4}{5(p-3)} - \frac{3}{10(p+2)}$

$$\therefore L^{-1}\left(\frac{p^2+2p-3}{p(p-3)(p+2)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{p}\right) + \frac{4}{5}L^{-1}\left(\frac{1}{p-3}\right) - \frac{3}{10}L^{-1}\left(\frac{1}{p+2}\right)$$

$$= \frac{1}{2} + \frac{4}{5} e^{3t} - \frac{3}{10} e^{-2t}.$$

(ii) $\frac{1+2p}{(p+2)^2(p-1)^2} = \frac{A}{p+2} + \frac{B}{(p+2)^2} + \frac{C}{p-1} + \frac{D}{(p-1)^2}$

$\therefore$ 1 + 2p = A (p + 2)$(p-1)^2$+ B $(p-1)^2 + C(p-1)(p+2)^2$

+ D $(p+2)^2$

= A $(p^3 - 3p + 2) + B(p^2 - 2p + 1) + C(p^3 + 3p^2 - 4)$

$+D(p^2 + 4p + 4)$

After comparing the coefficient

A + C = 0, B + 3C + D = 0, -3A – 2B + 4D = 2, 2A + B – 4C + 4D = 1

Solving these equation then we get A = 0, B = $-\frac{1}{3}$, C = 0, D = $\frac{1}{3}$

$\therefore \frac{1+2p}{(p+2)^2(p-1)^2} = \frac{-1}{3(p+2)^2} + \frac{1}{3(p-1)^2}$

$\therefore L^{-1}\left(\frac{1+2p}{(p+2)^2(p-1)^2}\right) = -\frac{1}{3} L^{-1}\left(\frac{1}{(p+2)^2}\right) + \frac{1}{3} L^{-1}\left(\frac{1}{(p-1)^2}\right)$

$$= -\frac{1}{3} e^{-2t}.t + \frac{1}{3} e^{t}.t = \frac{t}{3}(e^{t} - e^{-2t}) .$$

4.9 *INVERSE LAPLACE TRANSFORM OF DERIVATIVES*

■ **If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t)$.**

We have, L$\{t^n F(t)\} = (-1)^n \left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n f^n(p)$

$\therefore L^{-1}\{f^n(p)\} = (-1)^n t^n F(t).$

4.10 *MULTIPLICATION BY p*

■ **If $L^{-1}\{f(p)\}$ = F(t) and F(0) = 0, then $L^{-1}\{pf(p)\}$ = F'(t) .**

We have L{F'(t)} = pf(p) – F(0) = pf(p)

$\therefore L^{-1}\{pf(p)\} = F'(t)$ ($\because$ F(0) = 0)

Note:■ if $F(0) \neq 0$, then $L^{-1}\{pf(p) - F(0)\} = F'(t)$ or

$L^{-1}\{pf(p)\} = F'(t) + F(0)\delta(t)$ where $\delta(t)$ is the unit impulse function.

■ Generalizations to $L^{-1}\{p^n f(p)\}$ are possible for n = 2, 3, …

4.11 *DIVISION BY p*

■ If $L^{-1}\{f(p)\}$ = F(t) then

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)du$$

Also, $L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^t F(u)du\,du$ and

$$L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^t \int_0^t F(u)du\,du\,du \text{ and so on}$$

$$L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \int_0^t \int_0^t \ldots\ldots.. \int_{0(n\text{ times})}^t F(u)du \ldots\ldots.du \text{ (n times)}$$

4.12 *HEAVISIDE EXPANSION FORMULA FOR INVERSE LAPLACE TRANSFORM*

■ **If F(p) and G(p) are two polynomials in p and the degree of F(p) is less than the degree of G(p) and if G(p) = (p - α_1) (p - α_2) ….$(p - \alpha_n)$, where $\alpha_1, \alpha_2, \ldots \alpha_n$ are distinct constants, real or complex, then**

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^{n}\left(\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}\right)$$

Proof. By the Methods of partial fractions, let

$$\frac{F(p)}{G(p)} = \frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \cdots + \frac{A_r}{p-\alpha_r} + \cdots \frac{A_n}{p-\alpha_n}$$

Multiplying both sides by $p - \alpha_r$ and allowing $p \to \alpha_r$, we obtain,

$$A_r = \lim_{p\to\alpha_r} \frac{F(p)(p-\alpha_r)}{G(p)} = \lim_{p\to\alpha_r} F(p). \lim_{p\to\alpha_r} \frac{(p-\alpha_r)}{G(p)}$$

$$= \lim_{p\to\alpha_r} F(p). \lim_{p\to\alpha_r} \frac{1}{G'(p)} = \frac{F(\alpha_r)}{G'(\alpha_r)}$$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(\alpha_1)}{G'(\alpha_1)} \cdot \frac{1}{p-\alpha_1} + \ldots + \frac{F(\alpha_r)}{G'(\alpha_r)} \cdot \frac{1}{p-\alpha_r} + \ldots + \frac{F(\alpha_n)}{G'(\alpha_n)} \cdot \frac{1}{p-\alpha_n}$$

$$\therefore L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} e^{\alpha_1 t} + \ldots + \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} + \ldots + \frac{F(\alpha_n)}{G'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{r=1}^{n} \left(\frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}\right).$$

4.13 *CONVOLUTION THEOREM*

■ **If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, then**

$L^{-1}\{f(p)g(p)\} = F * G = \int_0^t F(u)\, G(t-u)\, du$

Proof. Let $\varphi(t) = \int_0^t F(u)\, G(t-u)\, du$ then

$$L\{\varphi(t)\} = \int_0^\infty e^{-pt} \left[\int_0^t F(u)\, G(t-u)\, du\right] dt$$

$$= \int_0^\infty \int_0^t e^{-pt} F(u)\, G(t-u)\, du\, dt$$

On changing the order of integration, we get

$$L\{\varphi(t)\} = \int_0^\infty \int_u^\infty e^{-pt} F(u)\, G(t-u)\, dt\, du$$

$$= \int_0^\infty e^{-pu} F(u) \left[\int_u^\infty e^{-p(t-u)} G(t-u)\, dt\right] du$$

$$= \int_0^\infty e^{-pu} F(u) \left[\int_0^\infty e^{-pv} G(v)\, dv\right] du \qquad (\text{ on putting } t - u = v)$$

$$= \int_0^\infty e^{-pu} F(u) g(p) du = g(p) \int_0^\infty e^{-pu} F(u) du$$

$$= g(p).f(p) = f(p)\, g(p)$$

$\Rightarrow L^{-1}\{f(p)g(p)\} = F * G = \int_0^t F(u)\, G(t-u)\, du$

We call F * G, the convolution of G and G and the theorem is called convolution theorem or the convolution property.

ILLUSTRATIVE EXAMPLES

Example 1. Find the inverse Laplace transform of

(i) $\log\left(1+\frac{1}{p^2}\right)$ **(ii)** $\log\left(\frac{p+1}{p-1}\right)$

Sol. (i) let $L^{-1}\{\log\left(1+\frac{1}{p^2}\right)\} = F(t)$

$\therefore \quad L^{-1}\left[\frac{d}{dp}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}\right] = - t\, F(t)$

$\Rightarrow \quad L^{-1}\left[\frac{1}{1+\frac{1}{p^2}}\left(\frac{-2}{p^3}\right)\right] = - t\, F(t)$

$\Rightarrow \quad L^{-1}\left[\frac{-2}{p(p^2+1)}\right] = - t\, F(t)$

$\Rightarrow \quad L^{-1}\left[\frac{1}{p} - \frac{p}{p^2+1}\right] = \frac{t}{2} F(t)$

$\Rightarrow \quad 1 - \cos t = \frac{t}{2} F(t)$

$\therefore \quad F(t) = \frac{2(1-\cos t)}{t}$.

(ii) Let $L^{-1}\{\log\left(\frac{p+1}{p-1}\right)\} = F(t)$

$\therefore \quad L^{-1}\left[\frac{d}{dp}\{\log(p+1) - \log(p-1)\}\right] = - t\, F(t)$

$\Rightarrow \quad L^{-1}\left[\frac{1}{p+1} - \frac{1}{p-1}\right] = - t\, F(t)$

$\Rightarrow \quad e^{-t} - e^{t} = - t\, F(t)$

$\therefore \quad F(t) = \frac{e^{-t}-e^{t}}{t} = \frac{2\sinh t}{t}$.

Example 2. Find the inverse Laplace transform of

(i) $\frac{p^2 - a^2}{(p^2 + a^2)^2}$ **(ii)** $\frac{2ap}{(p^2 + a^2)^2}$

Sol. (i) since $L^{-1}\left\{\frac{p}{p^2+a^2}\right\}$ = cos at

$\therefore$ $L^{-1}\left[\frac{d}{dp}\left\{\frac{p}{p^2+a^2}\right\}\right]$ = - t cos at

$\Longrightarrow$ $L^{-1}\left[\frac{a^2 - p^2}{(p^2 + a^2)^2}\right]$ = - t cos at

$\therefore$ $L^{-1}\left[\frac{p^2 - a^2}{(p^2 + a^2)^2}\right]$ = - t cos at .

$(\boldsymbol{ii})$ since $L^{-1}\left\{\frac{a}{p^2+a^2}\right\}$ = sin at

$\therefore$ $L^{-1}\left[\frac{d}{dp}\left\{\frac{a}{p^2+a^2}\right\}\right]$ = - t sin at

$\Longrightarrow$ $L^{-1}\left[\frac{-2ap}{(p^2 + a^2)^2}\right]$ = - t sin at

$\therefore$ $L^{-1}\left[\frac{2ap}{(p^2 + a^2)^2}\right]$ = t sin at.

Example 3. Apply Heaviside expression theorem to obtain

(i) $L^{-1}\left\{\frac{2p^2 + 5p - 4}{p^3 + p^2 - 2p}\right\}$ **(ii)** $L^{-1}\left\{\frac{3p + 1}{(p-1)(p^2+1)}\right\}$

Sol. (i) let F(p) = $2p^2 + 5p - 4$ and

G(p) = $p^3 + p^2 - 2p$ = p(p – 1)(p +2)

G(p) = 0 gives p = 0, 1, -2

$\therefore$ $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = -2$

$\Longrightarrow$ $F(\alpha_1) = -4, F(\alpha_2) = 3, F(\alpha_3) = -6$

$\Longrightarrow G'(p) = 3p^2 + 2p - 2 \Longrightarrow G'(\alpha_1) = -2, G'(\alpha_2) = 3, G'(\alpha_3) = 2$

$\therefore$ $L^{-1}\left\{\frac{2p^2 + 5p - 4}{p^3 + p^2 - 2p}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} e^{\alpha_1 t} + \frac{F(\alpha_2)}{G'(\alpha_2)} e^{\alpha_2 t} + \frac{F(\alpha_3)}{G'(\alpha_3)} e^{\alpha_3 t}$

$= \left(\frac{-4}{-2}\right) e^{0t} + \left(\frac{3}{3}\right) e^{t} + \left(\frac{-6}{6}\right) e^{-2t} = 2 + e^t - e^{-2t}$.

(ii) let F(p) = $3p + 1$ and

$$G(p) = (p-1)(p^2+1) = (p-1)(p+i)(p-i)$$

G(p) = 0 gives p = 1, i, -i

$\therefore$ $\alpha_1 = 1, \alpha_2 = i, \alpha_3 = -i$

$\Rightarrow$ $F(\alpha_1) = 4,\ F(\alpha_2) = 3i + 1,\ F(\alpha_3) = 1 - 3i$

$\Rightarrow G'(p) = (p-1).(2p) + p^2 + 1 = 3p^2 - 2p + 1$

$\Rightarrow G'(\alpha_1) = 2,\ G'(\alpha_2) = -2 - 2i,\ G'(\alpha_3) = 2i - 2$

$$\therefore\ L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)}e^{\alpha_1 t} + \frac{F(\alpha_2)}{G'(\alpha_2)}e^{\alpha_2 t} + \frac{F(\alpha_3)}{G'(\alpha_3)}e^{\alpha_3 t}$$

$$= \left(\frac{4}{2}\right)e^t + \left(\frac{3i+1}{-2-2i}\right)e^{it} + \left(\frac{1-3i}{2i-2}\right)e^{-it}$$

$$= 2e^t - \left(\frac{i}{2}+1\right)e^{it} + \left(\frac{i}{2}-1\right)e^{-it}$$

$$= 2e^t - \frac{i}{2}(e^{it} - e^{-it}) - (e^{it} + e^{-it})$$

$$= 2e^t - \frac{i}{2}\,.2i\sin t - 2\cos t$$

$$= 2e^t - 2\sin t - 2\cos t\ .$$

Example 4. Use convolution theorem to evaluate:

$$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$$

Sol. $\frac{p}{(p^2+4)^2} = \frac{1}{p^2+4} \cdot \frac{p}{p^2+4}$

Let, $f(p) = \frac{1}{p^2+4}$ and $g(p) = \frac{p}{p^2+4}$

$\therefore F(t) = L^{-1}\{f(p)\} = L^{-1}\left(\frac{1}{p^2+4}\right) = \frac{1}{2}sin2t$

And $G(t) = L^{-1}\{g(p)\} = L^{-1}\left(\frac{p}{p^2+4}\right) = cos2t$

Now, $F(u) = \frac{1}{2}sin2u$, $G(t-u) = \cos 2(t-u)$

$\therefore$ by convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\} = \int_0^t \frac{1}{2} sin2u.\cos 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t [sin2t + \sin(4u-2t)]du$$

$$= \frac{1}{4}\left[u \sin 2t - \frac{\cos(4u-2t)}{4}\right]_0^t = \frac{t}{4} \sin 2t.$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f(ap)\}$ = $\frac{1}{a}$ F$\left(\frac{t}{a}\right)$.

Problem 2. If $L^{-1}\{f(p)\}$ = F(t), then

$$L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t).$$

Problem 3. If $L^{-1}\{f(p)\}$ = F(t) and F(0) = 0, then $L^{-1}\{pf(p)\}$ = F'(t).

Problem 4. If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f(p-a)\}$ = e^{at}F(t).

Problem 5. $L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^{n+5} t^n F(t)$.

4.14 *SUMMARY*

1. Laplace Transform of Unit Step function

L{u(t – a)} = $\int_0^\infty e^{-pt} u(t-a)dt = \frac{1}{p} e^{-at}$

2. Second Shifting Theorem

if L{F(t)} = f(p), then L{F(t – a).u(t-a)} = e^{-ap}f(p).

3. If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f(ap)\}$ $= \frac{1}{a}$ F$\left(\frac{t}{a}\right)$.

4. Inverse Laplace Transform of Derivatives

If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f^n(p)\}$ $= L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t)$.

5. **Convolution theorem**

If $\mathrm{L}^{-1}\{\mathrm{f}(\mathrm{p})\}$ = F(t) and $\mathrm{L}^{-1}\{\mathrm{g}(\mathrm{p})\}$ = G(t), then

$\mathrm{L}^{-1}\{\mathrm{f}(\mathrm{p})\mathrm{g}(\mathrm{p})\}$ = F * G = $\int_0^{\mathrm{t}} \mathrm{F}(\mathrm{u})\, \mathrm{G}(\mathrm{t}-\mathrm{u})\, \mathrm{du}$

4.15 *GLOSSARY*

Discontinuous functions

Periodic Functions

Integration

Even, odd functions

Trigonometric functions

Differentiation

Integrations

4.16 *REFERENCES*

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4.18 TERMINAL AND MODEL QUESTIONS

Q 1. Use convolution theorem to find

(i) $L^{-1}\left\{\frac{1}{(p^2+4)(p+2)}\right\}$ (ii) $L^{-1}\left\{\frac{1}{(p^2+a^2)^3}\right\}$

(iii) $L^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\}$

Q2. Find the inverse Laplace transform of

(i) $\cot^{-1}\left(\frac{p}{a}\right)$ (ii) $\tan^{-1}\left(\frac{2}{p}\right)$ (iii) $\log\left(1+\frac{1}{p}\right)$ (iv) $\log\left(1-\frac{a^2}{p^2}\right)$

Q3. Apply Heaviside expression theorem to obtain

(i) $L^{-1}\left\{\frac{2p-1}{p(p-1)(p+1)}\right\}$ (ii) $L^{-1}\left\{\frac{3p+16}{p^2-p-6}\right\}$

Q4. Find the inverse Laplace transform of

(i) $\log\left(\frac{p+1}{(p+2)(p+3)}\right)$ (ii) $p\log\left(\frac{p-1}{p+1}\right)$

Q5. State and prove convolution theorem.

4.19 *ANSWERS*

TQ1 (i) $\frac{1}{8}(\sin 2t - \cos 2t + e^{-2t})$ (ii) $\frac{1}{8a^3}(\sin at - \text{at} \cos at)$

(iii) $\frac{1}{8}(e^{2t} - e^{-2t} - 4te^{-2t})$

TQ2 (i) $\frac{\sin at}{t}$ (ii) $\frac{\sin 2t}{t}$ (iii) $\frac{1-e^{-t}}{t}$ (iv) $\frac{2}{t}(1 - \cosh at)$

TQ3 (i) $1 + \frac{1}{2}e^{t} - \frac{3}{2}e^{-t}$ (ii) $5e^{3t} - 2e^{-2t}$

TQ4 (i) $\frac{-e^{-t} + e^{-2t} + e^{-3t}}{t}$ (ii) $\frac{2}{t^2}(\sinh t - t\cosh t)$

CYQ 1 True

CYQ 2 True

CYQ 3 True

CYQ 4 True

CYQ 5 False

UNIT 5: - APPLICATIONS OF LAPLACE TRANSFORM TO DIFFERENTIAL EQUATION

Contents

5.1 *INTRODUCTION*

In mathematics, the inverse Laplace transform of a function F(s) is the piecewise-continuous and exponentially-restricted real function f(t) which has the property: denotes the Laplace transform. The Laplace Transform can be used to solve differential equations using a four step process. Take the Laplace Transform of the differential equation using the derivative property (and, perhaps, others) as necessary. Put initial conditions into the resulting equation. Solve for the output variable.

5.2 *OBJECTIVE*

At the end of this topic Lerner will be able to understand:

(i) Solution of ordinary linear differential equations with constant Coefficients.

(ii) Solution of simultaneous ordinary differential equations.

(iii) Solution of ordinary differential equations with variable Coefficients.

5.3 *APPLICATIONS OF LAPLACE TRANSFORM TO DIFFERENTIAL*

5.3.1 Solution of ordinary linear differential equations with constant coefficients:

Laplace transform can be used to solve ordinary linear differential equations with constant coefficients. The advantage of this method is that it yield the particular solution directly without the necessity of first finding the general solution and then evaluating the arbitrary constants.

Steps: (a) Take Laplace transform on both sides of the given differential equation, using initial conditions. This gives an algebraic equation.

(b) Solve the algebraic equation to get $\bar{y}$ in term of p.

(c) Take inverse Laplace transform on both sides. This gives y as a function of t which is the desired solution.

Note: L$\{F^n(t)\}$ = p^nf(p) - $p^{n-1}F(0) - p^{n-2}F'(0) - \ldots\ldots$ - p$F^{n-2}(0) -$ $F^{n-1}(0)$; if L{F(t)} = f(p).

ILLUSTRATIVE EXAMPLES

Example 1. Solve the equation $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$, where y = 1,

$\frac{dy}{dt} = 2$, $\frac{d^2y}{dt^2} = 2$ at t = 0.

Sol. The given equation is $y''' + 2\,y'' - y' - 2y = 0$

Taking Laplace transform on both sides, we get

$$[p^3\bar{y} - p^2y(0) - py'(0) - y''(0)\,] + 2[p^2\bar{y} - py(0) - y'(0)]$$

$$- [p\bar{y} - y(0)] - 2\bar{y} = 0 \qquad \textbf{..... (1)}$$

Using the give conditions y(0) = 1, $y'(0) = 2$, $y''(0) = 2$, equation (1) reduces to

$$(p^3 + 2p^2 - p - 2)\bar{y} = p^2 + 4p + 5$$

$$\therefore \qquad \bar{y} = \frac{p^2 + 4p + 5}{p^3+2p^2-p-2} = \frac{p^2 + 4p + 5}{(p-1)(p+1)(p+2)}$$

$$= \frac{5}{3(p-1)} - \frac{1}{p+1} + \frac{1}{3(p+2)} \qquad \text{(using partial fractions)}$$

Taking the inverse Laplace transform of both sides, we get y = $\frac{5}{3}L^{-1}\left\{\frac{1}{p-1}\right\}$ - $L^{-1}\left\{\frac{1}{p+1}\right\}$ + $\frac{1}{3}L^{-1}\left\{\frac{1}{p+1}\right\}$ = $\frac{5}{3}e^{t}$ - $e^{-t} + \frac{1}{3}e^{-2t}$

y = $\frac{1}{3}(5e^{t} + e^{-2t}) - e^{-t}$.

Example 2. Solve the equation $\frac{d^2y}{dt^2}$ -3 $\frac{dy}{dt}$ + 2y = 4t + e^{3t}, where y(0) = 1 and $y'(0)$ = -1 .

Sol. The given equation is $y'' - 3\,y' + 2y = 4\text{t} + e^{3t}$

Taking Laplace transform on both sides, we get

$[p^2\bar{y} - py(0) - y'(0)]$ – 3 $[p\bar{y} - y(0)]$ + $2\bar{y}$ = $\frac{4}{p^2} + \frac{1}{p-3}$ …. (1)

Using the give conditions y(0) = 1 and $y'(0)$ = -1, equation (1) reduces to

($p^2 - 3p + 2)\,\bar{y}$ – p + 1 + 3 = $\frac{4}{p^2} + \frac{1}{p-3}$

∴ ($p^2 - 3p + 2)\,\bar{y}$ = $\frac{4}{p^2} + \frac{1}{p-3}$ + p – 4

∴ $\bar{y} = \frac{p^4 - 7p^3 + 13p^2 + 4p - 12}{p^2(p-1)(p-2)(p-3)}$

= $\frac{3}{p} + \frac{2}{p^2}$ - $\frac{1}{2(p-1)}$ - $\frac{2}{p-2} + \frac{1}{2(p-3)}$ (using partial fraction)

Taking the inverse Laplace transform of both sides, we get

y = $3L^{-1}\left\{\frac{1}{p}\right\}$ - $2L^{-1}\left\{\frac{1}{p^2}\right\}$ - $\frac{1}{2}L^{-1}\left\{\frac{1}{p-1}\right\}$ -$2L^{-1}\left\{\frac{1}{p-2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p-3}\right\}$

= 3 + 2t - $\frac{1}{2}e^{t}$ - $2e^{2t} + \frac{1}{2}e^{3t}$

= 3 + 2t + $\frac{1}{2}(e^{3t}$ - $e^{t}) - 2e^{2t}$.

Example 3. Using Laplace transformation, solve the equation

$\frac{d^2x}{dt^2}$ + 9x =cos 2t,

where x(0) = 1, x$\left(\frac{\pi}{2}\right)$= -1 .

Sol. The given equation is $x'' + 9x = \cos 2t$

Taking Laplace transform on both sides, we get

$L(x'') + 9L(x) = L(\cos 2t)$

$[p^2\bar{x} - px(0) - x'(0)] + 9\bar{x} = \frac{p}{p^2+4}$

$\Rightarrow (p^2 + 9)\bar{x} - p - A = \frac{p}{p^2+4}$ (where $x'(0) = A$)

$\Rightarrow \bar{x} = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{p^2+9} + \frac{A}{p^2+4}$

Taking the inverse Laplace transform of both side, we get

$x = \frac{1}{5}(\cos 2t - \cos 3t) + \cos 3t + \frac{A}{3}\sin 3t$, but $x\left(\frac{\pi}{2}\right) = -1$

$\Rightarrow -1 = \frac{1}{5}(-1 - 0) + 0 + \frac{A}{3}(-1) \Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \Rightarrow A = \frac{12}{5}$

$\therefore$ $x(t) = \frac{1}{5}(\cos 2t - \cos 3t) + \cos 3t + \frac{4}{5}\sin 3t$

$= \frac{1}{5}(\cos 2t + 4\cos 3t + 4\sin 3t)$.

Example 4. Using Laplace transformation, solve the equation

$$(D^2 + n^2)x = a\sin(nt + \alpha); x = Dx = 0 \; at \; t = 0$$

Sol. The given equation is $(D^2 + n^2)x = a\sin(nt + \alpha)$

Taking Laplace transform on both sides, we get

$L(x'') + n^2L(x) = L\{a\sin(nt + \alpha)\}$

$\Rightarrow [p^2\bar{x} - px(0) - x'(0)] + n^2\bar{x} = a\,cos\alpha \,.\frac{n}{p^2+n^2} + a\,sin\alpha\,.\frac{p}{p^2+n^2}$

$\Rightarrow \qquad (p^2 + n^2)\bar{x} = \frac{an\,cos\alpha}{(p^2+n^2)^2} + \frac{an\,sin\alpha}{(p^2+n^2)^2}$ ….. (1)

Taking the inverse Laplace transform of both side, we get

$x = (a\cos\alpha)\, L^{-1}\left[\frac{n}{(p^2+n^2)^2}\right] + (a\sin\alpha)\, L^{-1}\left[\frac{p}{(p^2+n^2)^2}\right]$ …. (2)

we know that $L^{-1}\left[\frac{n}{p^2+n^2}\right] = \frac{1}{n}$ sin nt …. (3)

$L^{-1}\left[\frac{1}{dn}\left\{\frac{1}{p^2+n^2}\right\}\right] = \frac{nt\cos nt - \sin nt}{n^2}$

$L^{-1}\left[\frac{-2n}{(p^2+n^2)^2}\right] = \frac{nt\cos nt - \sin nt}{n^2}$

$L^{-1}\left[\frac{n}{(p^2+n^2)^2}\right] = \frac{1}{2n^2}(\sin nt - nt\cos nt)$

Again, from (3), $L^{-1}\left[\frac{d}{dp}\left(\frac{1}{p^2+n^2}\right)\right]$ = - t. $\frac{1}{n}$ sin nt

$\Longrightarrow L^{-1}\left[\frac{-2p}{(p^2+n^2)^2}\right] = -\frac{t}{n}$ sin nt

$\Longrightarrow L^{-1}\left[\frac{p}{(p^2+n^2)^2}\right] = \frac{t}{2n}$ sin nt

∴ From (2), x = (a cosα) . $\frac{1}{2n^2}(\sin nt - nt\cos nt)$ + (a sin α). $\frac{t}{2n}$ sin nt

$= \frac{a}{2n^2}$ [cosα $\sin nt - nt\cos(\alpha + nt)$].

Example 5. Using Laplace transformation, find the general solution of the equation

$$(D^2 + k^2)y = 0.$$

Sol. The given equation is $(D^2 + k^2)y = 0$ ……. (1)

Taking Laplace Transform on both sides of (1),

L(y'') + $k^2 L(y)$ = 0

Or s^2 L{y} – s y(0) - $y'(0)$ + $k^2 L(y)$ = 0

Or $(s^2 + k^2)$ L{y} – As – B = 0, where y(0) = A and $y'(0) = B$, say

Or L{y} = $\frac{A\,s+B}{s^2+k^2}$ = A $\frac{s}{s^2+k^2}$ + B $\frac{1}{s^2+k^2}$ …… (2)

Taking inverse Laplace transform of both sides of (2), we get

y = A cos kt + $\frac{B}{k}$(sin kt) = A cos kt + C sin kt …… (3)

where C = B/k . (3) is the general solution of given equation and A and B are arbitrary constants.

5.3.2 Solution of simultaneous ordinary differential equations:

A simultaneous differential equation is one of the mathematical equations for an indefinite function of one or more than one variable that relate the values of the function. Differentiation of an equation in various orders. Differential equations play an important function in engineering, physics, economics, and other disciplines. This analysis concentrates on linear equations with Constant Coefficients.

Laplace transform technique can be also used in solving two or more simultaneous ordinary differential equations.

ILLUSTRATIVE EXAMPLES

Example 1. Solve the equation $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given x(0) = 1, y(0) = 0.

Sol. Taking Laplace transform of the given equations, we get

$[p\bar{x} - x(0)] - \bar{x} = \frac{1}{p-1}$

$\Rightarrow p\bar{x} - 1 - \bar{y} = \frac{1}{p-1}$ $(since\ x(0) = 1)$ $\Rightarrow p\bar{x} - \bar{y} = \frac{p}{p-1}$ …. (1)

And $[p\bar{y} - y(0)] + \bar{x} = \frac{1}{p^2+1}$

$\Rightarrow \bar{x} + p\bar{y} = \frac{1}{p^2+1}$ …. (2) [since y(0) = 0]

Solving equation (1) and (2) for $\bar{x}\ and\ \bar{y}$, we have

$$\bar{x} = \frac{p^2}{(p-1)(p^2+1)} + \frac{1}{(p^2+1)^2} = \frac{1}{2}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + \frac{1}{(p^2+1)^2}$$

And

$\bar{y} = \frac{1}{(p^2+1)^2} - \frac{p}{(p-1)(p^2+1)} = \frac{p}{(p^2+1)^2} - \frac{1}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$

Taking Inverse Laplace transform of both sides, we get

x = $\frac{1}{2}L^{-1}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + L^{-1}\left[\frac{1}{(p^2+1)^2}\right]$

$= \frac{1}{2}[e^t + \cos t + 2\sin t - t\cos t]$

y = $L^{-1}\left[\frac{p}{(p^2+1)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$

$= \frac{1}{2}\ t\sin t - \frac{1}{2}[e^t - \cos t + \sin t]$

$= \frac{1}{2}[t\sin t - e^t + \cos t - \sin t]$

Hence x = $\frac{1}{2}[e^t + \cos t + 2\sin t - t\cos t]$

And y = $\frac{1}{2}[t\sin t - e^t + \cos t - \sin t]$.

Example 2. Solve the simultaneous equations

$(D^2 - 3)x - 4y = 0$ and

x + $(D^2 + 1)y = 0$ for t > 0, given that x = y = $\frac{dy}{dt} = 0$

and $\frac{dx}{dt} = 2\ at\ t = 0$.

Sol. Taking Laplace transform of the given equations, we get

$p^2\bar{x} - px(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0$

$\Rightarrow$ $(p^2 - 3)\bar{x} - 4\bar{y} = 2$ …. (1) and

$\bar{x} + p^2\bar{y}$ - $py(0) - y'(0) + \bar{y} = 0$

i.e. $\bar{x} + (p^2 + 1)\bar{y} = 0$ ……. (2)

solving (1) and (2) for $\bar{x}\ and\ \bar{y}$, we get

$\bar{x} = \frac{2(p^2+1)}{(p^2-1)^2} = \frac{1}{(p-1)^2} + \frac{1}{(p+1)^2}$ and

$\bar{y} = -\frac{2}{(p^2-1)^2} = -\frac{1}{2}\left[\frac{1}{p+1} - \frac{1}{p-1} - \frac{1}{(p+1)^2} + \frac{1}{(p-1)^2}\right]$

Taking inverse Laplace transform on both sides, then we get

x = $L^{-1}\left[\frac{1}{(p-1)^2} + \frac{1}{(p+1)^2}\right]$ = $te^t + te^{-t}$ = $2t\left(\frac{e^t + e^{-t}}{2}\right)$ = 2t cosh t

and $y = -\frac{1}{2}L^{-1}\left[\frac{1}{p+1} - \frac{1}{p-1} - \frac{1}{(p+1)^2} + \frac{1}{(p-1)^2}\right]$

$= -\frac{1}{2}(e^{-t} - e^t - te^{-t} + te^t) = \frac{e^t - e^{-t}}{2} - t\left(\frac{e^t - e^{-t}}{2}\right)$

= (1 – t) sinh t

Therefore x = 2t cash t, y = (1 – t) sinh t.

Example 3. The co-ordinate (x, y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt} + 2x = \sin 2t$,

$\frac{dx}{dt} - 2y = \cos 2t$; (t > 0)

It is given that at t = 0, x = 1 and y = 0. Show using transforms that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Sol. The given equations are $\frac{dy}{dt} + 2x = \sin 2t$ ……… (1)

$\frac{dx}{dt} - 2y = \cos 2t$ …….. (2)

Above equation may be re-written as

2x + Dy = sin 2t

Dx – 2y = cos 2t, where D = $\frac{d}{dt}$

Taking Laplace transform of equation (1) on both sides, we get

$2\,\bar{x} + p\,\bar{y} - y(0) = \frac{2}{p^2+4}$, where $\bar{x}$ = L(x) and $\bar{y}$ = L(y)

Therefore $2\,\bar{x} + p\,\bar{y} = \frac{2}{p^2+4}$ ……. (3)

Again, taking Laplace transform of equation (2) on both sides, we get

$$p\,\bar{x} - x(0) - 2\bar{y} = \frac{p}{p^2+4} \quad \text{, where } \bar{x} = L(x) \text{ and } \bar{y} = L(y)$$

therefore $p\,\bar{x} + 2\,\bar{y} = \frac{p}{p^2+4} + 1$ …….. (4)

multiplying equation (3) by 2 and equation (4) by p and then adding, we get

$$4\,\bar{x} + p^2\,\bar{x} = \frac{4}{p^2+4} + \frac{p^2}{p^2+4} + p$$

Therefore $(4 + p^2)\bar{x} = 1 + p$

Therefore $\bar{x} = \frac{1+p}{p^2+4} = \frac{1}{p^2+4} + \frac{p}{p^2+4}$

Taking inverse Laplace transform, we get

$$x = \frac{1}{2}\sin 2t + \cos 2t \qquad \text{……. (5)}$$

Again, multiplying equation (3) by p and equation (4) by 2 then subtracting equation (4) from (3), we get

$$p^2\,\bar{y} + 4\bar{y} = \frac{2p}{p^2+4} - \frac{2p}{p^2+4} - 2$$

Therefore $\bar{y} = \frac{-2}{p^2+4}$

Taking inverse Laplace transform, we get $y = -\sin 2t$

Now, $4x^2 = 4\left[\frac{1}{4}sin^2 2t + cos^2 2t + sin2t\,cos2t\right]$

Therefore $5y^2 = 5\,sin^2 2t$ and

$$4xy = 4\left[(\frac{1}{2}sin2t + cos2t)(-sin2t)\right]$$

$$= -(2sin^2 2t + 4sin2t\,cos2t)$$

Therefore $4x^2 + 5y^2 + 4xy = 4\,sin^2 2t + 4\,cos^2 2t = 4$

Hence the result.

Example 4. Use Laplace transform to solve

$$\frac{dx}{dt} + y = sint, \frac{dy}{dt} + x = cos\ t \text{ given that x = 2, y = 0 at t= 0.}$$

Sol. Taking Laplace transform of the given equations, we get

$$\text{p } \bar{x} - \text{x}(0) + \bar{y} = \frac{1}{p^2+1}$$

therefore $\quad \text{p } \bar{x} + \bar{y} = \frac{1}{p^2+1} + 2 \quad$ ……. (1)

and $\quad \text{p } \bar{y} - \text{y}(0) + \bar{x} = \frac{p}{p^2+1}$

therefore $\quad \text{p } \bar{y} + \bar{x} = \frac{p}{p^2+1} \quad$ …… (2)

solving (1) and (2) for $\bar{x}$ and $\bar{y}$, we get

$\bar{x} = \frac{2p}{p^2-1}$ and $\bar{y} = \frac{1}{p^2+1} + \frac{2}{1-p^2}$

Therefore $\bar{x} = \frac{1}{p+1} + \frac{2}{p-1}$ and $\bar{y} = \frac{1}{p^2+1} + \frac{1}{p+1} - \frac{1}{p-1}$

Taking inverse Laplace transform on both sides, we get

$\text{x} = e^{-t} + e^{t}$ ……. (3)

and $\text{y} = \sin \text{t} + e^{-t} - e^{t}$ ……. (4)

equations (3) and (4), when takes together, give the complete solution.

Example 5. Solve the following simultaneous differential equations by Laplace transform

$$\frac{dx}{dt} + 4\frac{dy}{dt} - \text{y} = 0; \frac{dx}{dt} + 2\text{ y} = e^{-t}$$

With the conditions $\quad$ x(0) = y(0) = 0 .

Sol. The given conditions are

$$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0 \qquad \text{……... (1)}$$

And $$\frac{dx}{dt} + 2y = e^{-t} \qquad \text{……. (2)}$$

Taking Laplace transform on both sides of the equation (1), we get

$$L(x') + 4L(y') - L(y) = L(0)$$

Therefore $$p\,\bar{x} - x(0) + 4[p\bar{y} - y(0)] - \bar{y} = 0$$

Therefore $$p\,\bar{x} + (4p - 1)\,\bar{y} = 0 \qquad \text{……… (3)}$$

Similarly taking Laplace transform on both sides of equation (2), we get

$$L(x') + 2L(y) = L(e^{-t})$$

$$p\,\bar{x} - x(0) + 2\bar{y} = \frac{1}{p+1}$$

therefore $$p\,\bar{x} + 2\bar{y} = \frac{1}{p+1} \qquad \text{……… (4)}$$

subtracting (4) from (3), we get

$$(4p - 3)\,\bar{y} = -\frac{1}{p+1}$$

Therefore $$\bar{y} = -\frac{1}{(p+1)(4p-3)} = -\frac{1}{7}\left(-\frac{1}{p+1} + \frac{1}{p-3/4}\right)$$

$$= \frac{1}{7}\left(\frac{1}{p+1} + \frac{1}{p-3/4}\right)$$

Taking inverse Laplace transform on both sides, we get

$$y = \frac{1}{7}\left(e^{-t} - e^{\frac{3t}{4}}\right) \qquad \text{……… (5)}$$

substituting $\bar{y}$ in (4), we get

$$p\,\bar{x} + \frac{2}{7}\left(\frac{1}{p+1} - \frac{1}{p-3/4}\right) = \frac{1}{p+1}$$

therefore $$p\,\bar{x} = \frac{5}{7(p+1)} + \frac{2}{7(p-3/4)}$$

therefore $$\bar{x} = \frac{5}{7p(p+1)} + \frac{2}{7p(p-3/4)}$$

therefore $= \frac{5}{7}\left(\frac{1}{p} - \frac{1}{p+1}\right) + \frac{8}{21}\left(\frac{1}{p-3/4} - \frac{1}{p}\right)$

$$= \frac{1}{3p} - \frac{5}{7(p+1)} + \frac{8}{21(p-3/4)}$$

Taking inverse Laplace transform on both sides, we get

$$x = \frac{1}{3} - \frac{5}{7}e^{-t} + \frac{8}{21}e^{\frac{3t}{4}} \qquad \text{......... (6)}$$

equation (5) and (6), when taken together, give the complete solution.

5.3.3 Solution of ordinary differential equations with variable coefficients:

Given functions a_1, a_0, f: R → R, the differential equation in the unknown function y: R → R given by $y'' + a_1(t)\, y' + a_0(t)\, y = f(t)$ ....(1)

is called a second order linear differential equation with variable coefficients. The equation in (1) is called homogeneous iff for all t ∈ R holds f(t) = 0. The equation in (1) is called of constant coefficients iff

a_1 , a_0 , and f are constants.

The solution of the second-order linear differential equation with variable coefficients can be determined using the Laplace transform are as follows:

ILLUSTRATIVE EXAMPLES

Example 1. Solve the equation

$$\frac{d^2y(t)}{dt^2} + t\frac{dy(t)}{dt} - y(t) = 0 \ if \ y(0) = 0, \left(\frac{dy}{dt}\right)_{t=0} = 1.$$

Sol. Taking Laplace transform of both sides of the given equation, we get

$$L(y'') + L(ty') - L(y) = L(0)$$

$$\Rightarrow \{p^2\bar{y} - py(0) - y'(0)\} - \frac{d}{dp}L(y') - \bar{y} = 0$$

$\Rightarrow p^2\bar{y} - 1 - \frac{d}{dp}\{p\bar{y} - y(0)\} - \bar{y} = 0$

$\Rightarrow p^2\bar{y} - 1 - \frac{d}{dp}(p\bar{y}) - \bar{y} = 0$

$\Rightarrow -p\frac{d\bar{y}}{dp} + (p^2 - 2)\bar{y} = 1$

$\Rightarrow \frac{d\bar{y}}{dp} + \left(\frac{2}{p} - p\right)\bar{y} = -\frac{1}{p}$ which is linear differential equation .

I.F. $= e^{\int\left(\frac{2}{p}-p\right)dp} = p^2e^{-\frac{p^2}{2}}$

Solution of equation is given below

$$\bar{y}p^2e^{-\frac{p^2}{2}} = \int\left(-\frac{1}{p}\right)p^2e^{-\frac{p^2}{2}}dp + \text{c}$$

$$= -\int pe^{-\frac{p^2}{2}}dp + \text{c} = \text{c} + e^{-\frac{p^2}{2}}\text{, where c is a constant.}$$

c must vanish if $\bar{y}$ is transform since $\bar{y} \longrightarrow 0 \ as\ p \longrightarrow \infty$

therefore $\bar{y} = \frac{1}{p^2}$ or $y = L^{-1}\left(\frac{1}{p^2}\right) = t$.

Example 2. Solve the equation

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} - 4ty = 0\ if\ y = 3, \frac{dy}{dt} = 0\ when\ t = 0\ .$$

Sol. Taking Laplace transform of both sides of the given equation, we get

$\text{L}(ty'') + L(y') + 4\text{L}(\text{ty}) = L(0)$

$\Rightarrow -\frac{d}{dp}L(y'') + L(y') - 4\frac{d}{dp}L(y) = 0$

$\Rightarrow -\frac{d}{dp}\{p^2\bar{y} - py(0) - y'(0)\} + \{p\bar{y} - y(0)\} - 4\frac{d\bar{y}}{dp} = 0$

$\Rightarrow (p^2 + 4)\frac{d\bar{y}}{dp} + p\bar{y} = 0$ …… (1)

Separating the variables, we have

$$\frac{d\bar{y}}{\bar{y}} + \frac{p\,dp}{p^2 + 4} = 0 \qquad \text{…… (2)}$$

Taking Integration on both side then we get

$$\log \bar{y} + \frac{1}{2}\log(p^2 + 4) = \log c$$

$$\Rightarrow \qquad \bar{y} = \frac{c}{\sqrt{p^2 + 4}} \qquad \ldots\ldots (3)$$

Taking inverse Laplace transform, we get

$$y = c\, J_0(2t) \qquad \ldots\ldots. (4)$$

since y(0) = 3, from (4),

$$y = c\, J_0(0) = c$$

therefore c = 4

hence the required solution is $y = 3\, J_0(2t)$.

5.3.4 Solution of integral equations: An equation in which an unknown function

occur inside an integral is called an equation.

Thus an equation of the form $Y(t) = F(t) + \int_a^b Y(u)K(u,t)du$ …… (1)

in which F(t) and K(u, t) are known functions and Y(t) is unknown function is an integral equation. Here a and b are either constants or functions of t.

the function K(u, t) is often called the kernel of the integral equation.

If a and b are constants, equation (1) is called Fredholm integral equation. Is a is a constant while b = t, it is called a Volterra integral equation.

A special integral equation of convolution type is

$$Y(t) = F(t) + \int_0^t Y(u)G(t-u)du$$

The Laplace transform is an excellent tool for solving such integral equations of convolution type. The method is illustrated as follows:

ILLUSTRATIVE EXAMPLES

Example 1. Solve the integral equation

$$y(t) = t^2 + \int_0^t y(u).\sin(t-u)\,du\ .$$

Sol. We have $y(t) = t^2 + y(t)* \sin t$

Let $L\{y(t)\} = \bar{y}(p)$ then taking Laplace transform and using convolution theorem, we find that

$$\bar{y} = \frac{2}{p^3} + \bar{y}.\frac{1}{p^2+1}$$

Therefore $\bar{y}\,(1 - \frac{1}{p^2+1}) = \frac{2}{p^3}$ therefore $\bar{y}\,(\frac{p^2}{p^2+1}) = \frac{2}{p^3}$

therefore $\bar{y} = \frac{2(p^2+1)}{p^5} = \frac{2}{p^3} + \frac{2}{p^5}$

taking inverse Laplace transform, we get

$$y = t^2 + \frac{t^4}{12}\ .$$

Example 2. Solve the integral equation

$$y(t) = 1 + \int_0^t y(u).\cos(t-u)\,du\ .$$

Sol. We have $y(t) = 1 + (y(t)* \cos t)$

$$\bar{y} = \frac{2}{p} + \bar{y}.\frac{p}{p^2+1}$$

Therefore $\bar{y}\,(1 - \frac{p}{p^2+1}) = \frac{2}{p}$

Therefore $\bar{y} = \frac{p^2+1}{p^2-p+1} = \frac{1}{p} + \frac{1}{p^2-p+1}$

$$= \frac{1}{p} + \frac{1}{(p-\frac{1}{2})^2+\frac{3}{4}}$$

Taking inverse Laplace transform on both sides of (2), we get

$$y = 1 + e^{t/2}L^{-1}\left(\frac{1}{p^2+3/4}\right)$$

therefore $$y = 1 + \frac{2}{\sqrt{3}}e^{t/2}sin\,\frac{\sqrt{3}}{2}t.$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Inverse Laplace Transform of Derivatives

If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right]$

$$= (-1)^n t^n F(t).$$

Problem 2. If $L^{-1}\{f(p)\}$ = F(t) and $L^{-1}\{g(p)\}$ = G(t), then

$L^{-1}\{f(p)g(p)\} = F * G = \int_0^t F(u)\, G(t-u)\, du$ is called Convolution theorem.

Problem 3. Applications of Laplace transform to differential is not possible.

Problem 4. The value of $L^{-1}\left\{log\left(\frac{p+a}{p+b}\right)\right\}$ is 8.

Problem 5. The value of $L^{-1}\left\{log\left(\frac{p+1}{p-1}\right)\right\}$ is 9.

5.4 *SUMMARY*

1. $L\{F^n(t)\} = p^n f(p) - p^{n-1}F(0) - p^{n-2}F'(0) - \ldots\ldots - pF^{n-2}(0)$

$$- F^{n-1}(0);\ \text{if}\ \ L\{F(t)\} = f(p).$$

2. Solution of ordinary linear differential equations with constant Coefficients.

3. Solution of simultaneous ordinary differential equations.

4. Solution of ordinary differential equations with variable coefficients.

5. Inverse Laplace Transform of Derivatives

If $L^{-1}\{f(p)\}$ = F(t), then $L^{-1}\{f^n(p)\}$

$$= L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t).$$

6. Convolution theorem

If $\mathrm{L}^{-1}\{\mathrm{f(p)}\}$ = F(t) and $\mathrm{L}^{-1}\{\mathrm{g(p)}\}$ = G(t), then

$\mathrm{L}^{-1}\{\mathrm{f(p)g(p)}\}$ = F * G = $\int_0^t \mathrm{F(u)\,G(t-u)\,du}$

7. L$\{F^n(t)\}$ = p^nf(p) – $p^{n-1}F(0) - p^{n-2}F'(0) -$ …… - p$F^{n-2}(0)$

$- F^{n-1}(0)$; if L{F(t)} = f(p).

5.5 *GLOSSARY*

Differential equations

Integral equations

Discontinuous functions

Periodic Functions

Integration

Differentiation

Even, odd functions

5.6 *REFERENCES*

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5.8 *TERMINAL AND MODEL QUESTIONS*

Q 1. Using Laplace transform, solve the differential equation

$$y'' + 2ty' - y = t, when\ y(0) = 0\ and\ y'(0) = 1.$$

Q2. Solve the following equations by Laplace transform

$(D^2 - 2D + 2)x = 0, x = Dx = 1\ at\ t = 0.$

Q3. Solve the following equations by Laplace transform

$(D^2 - D - 2)x = 20 \sin 2t, x(0) = -1,\ x'(0) = 2.$

Q4. Solve the following equations by Laplace transform

(i) $y'' - 2y' - 8y = 0, when\ y(0) = 3\ and\ y'(0) = 6.$

(ii) $y'' - 8y' + 15y = 9te^{2t}, when\ y(0) = 5\ and\ y'(0) = 10.$

Q5. Solve the following simultaneous equations by Laplace transform

$$2\frac{dx}{dt} + \frac{dy}{dt} - x - y = e^{-t},\ \frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^{t};$$

$$y(0) = 1, x(0) = 2.$$

Q6. Solve the following simultaneous differential equations by Laplace transform

$$3\frac{dx}{dt} - y = 2t,\ \frac{dx}{dt} + \frac{dy}{dt} - y = 0\ with\ the\ condition$$

$$y(0) = x(0) = 0.$$

Q7. A function f(t) obeys the equation f(t) + 2 $\int_0^t f(t)dt$ = cosh 2t, find the Laplace transform of f(t).

5.9 *ANSWERS*

TQ1 y = $L^{-1}\left(\frac{1}{p^2}\right)$ = t.

TQ2 x = $e^t \cos t$

TQ3 x = $2e^{2t} - 4e^{-t} + \cos 2t - 3 \sin 2t$

TQ4 (i) y = $2e^{4t} + e^{-2t}$ (ii) y = $4e^{2t} + 3te^{2t} + 3e^{3t} - 2e^{5t}$

TQ5 x = 2 cos t + 8 sin t, y = cos t – 13 sin t + sinh t

TQ6 $x = \frac{t^2}{2} + \frac{t}{2} - \frac{3}{4}e^{\frac{2t}{3}} + \frac{3}{4}$

TQ7 $L\{f(t)\} = \frac{p^2}{(p+2)^2(p-2)}$

CHECK YOUR PROGRESS

CYQ1 True

CYQ2 True

CYQ3 False

CYQ4 False

CYQ5 False

BLOCK-III

UNIT 6: INTEGRAL EQUATION AND CONVERSION OF ORDINARY DIFFERENTIAL EQUATIONS INTO INTEGRAL EQUATIONS

Contents

6.1 *INTRODUCTION*

Integral equations are equations in which the unknown function appears inside a definite integral. They are closely related to differential equations. In this unit learner learn about Fredholm, Volterra integral equation and other types of integral equation, also learn about the conversion of initial and boundary value problem into integral equation. In 1823 Abel proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the first kind, and in 1837 Liouville showed that the determination of a particular solution of a linear differential equation. The solution of integral equation is much easier than the original boundary value problem or initial value problem.

6.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) integral equations

(ii) Volterra integral equation

(iii) Fredholm integral equation

(iv) Leibnitz's rule

(v) initial value problem

(vi) Boundary value problem

6.3 *INTEGRAL EQUATION*

An equation in which an unknown function appears under one or more integral sign is called in integral equation.

For Example: for a$\leq x \leq b, a \leq t \leq b$, the integral equations

$\int_a^b k(x,t)y(t)dt = f(x)$ …… (1)

y(x) - $\lambda \int_a^b k(x,t)y(t)dt = f(x)$ …… (2)

y(x) = $\int_a^b k(x,t)[y(t)]^2 dt$ …… (3)

where the function y(x), is unknown function while the functions f(x) and k(x, t) are known functions and λ, a and b are constants, are all integral equations.

6.4 LINEAR AND NON-LINEAR INTEGRAL EQUATION

An integral equation is called linear if only linear operations are performed in it upon the unknown function. An integral equation which is not linear is known as non-linear integral equation.

For Example: for a$\leq x \leq b, a \leq t \leq b$, the integral equations

$\int_a^b k(x,t)y(t)dt = f(x)$ …… (1)

y(x) - $\lambda \int_a^b k(x,t)y(t)dt = f(x)$ …… (2)

y(x) = $\int_a^b k(x,t)[y(t)]^2 dt$ …… (3)

Here equation (1), (2) are called linear and equation (3) is called non-linear.

Now the most general type of linear equation is of the form

g(x) y(x) = f(x) + $\lambda \int_{\mathrm{a}} k(x,t)y(t)dt$ …… (D)

where upper limit may be either variable or fixed. The functions f, g and k are known functions while y is to be determined, λ is a non-zero real or complex. The function k(x, t) is known as the kernel of the integral equation.

Note: ■ if g(x) $\neq$ 0, equation (D) is known as linear integral equation of the third kind. When g(x) = 0, (D) reduces to f(x) + $\lambda \int_{a} k(x,t)y(t)dt = 0$, is known as linear integral equation of the second kind. Again when g(x) = 1, (D) reduces to y(x) = f(x) + $\lambda \int_{a} k(x,t)y(t)dt$, which is known as linear integral equation of the second kind.

6.5 *FREDHOLM INTEGRAL EQUATION*

A linear integral equation is of the form

g(x) y(x) = f(x) + $\lambda \int_a^b k(x,t)y(t)dt$ …… (D)

where a and b are constants, f(x), g(x) and k(x, t) are known functions while y(x) is unknown function and λ is a non-zero real or complex parameter, is called Fredholm integral equation of third kind. The function k(x, t) is known as kernel of the integral equation.

The following special cases of (D) are as follows.

(i) Fredholm integral equation of First kind.

A linear equation of the form f(x) + $\lambda \int_a^b k(x,t)y(t)dt = 0$, is known as Fredholm integral equation of First kind.

(ii) Fredholm integral equation of Second kind.

A linear equation of the form y(x) = f(x) + $\lambda \int_a^b k(x,t)y(t)dt$ is known as Fredholm integral equation of second kind.

(iii) Homogeneous Fredholm integral equation of Second kind.

A linear equation of the form y(x) = $\lambda \int_a^b k(x,t)y(t)dt$ is known as Homogeneous Fredholm integral equation of second kind.

6.6 *VOLTERRA INTEGRAL EQUATION*

A linear integral equation is of the form

g(x) y(x) = f(x) + $\lambda \int_a^x k(x,t)y(t)dt$ …… (D)

where a is constants, f(x), g(x) and k(x, t) are known functions while y(x) is unknown function and λ is a non-zero real or complex parameter, is called Volterra integral equation of third kind. The function k(x, t) is known as kernel of the integral equation.

The following special cases of (D) are as follows.

(i) Volterra integral equation of First kind.

A linear equation of the form f(x) + $\lambda \int_a^x k(x,t)y(t)dt = 0$, is known as Volterra integral equation of First kind.

(ii) Volterra integral equation of Second kind.

A linear equation of the form y(x) = f(x) + $\lambda \int_a^x k(x,t)y(t)dt$ is known as Volterra integral equation of second kind.

(iii) Homogeneous Volterra integral equation of Second kind.

A linear equation of the form y(x) = $\lambda \int_a^x k(x,t)y(t)dt$ is known as Homogeneous Volterra integral equation of second kind.

6.7 *SPECIAL KINDS OF KERNELS*

(i) Symmetric Kernel.

A kernel k(x, t) is symmetric (or complex symmetric or Hermitian) if

k(x, t) = $\bar{k}$(t, x)

where the bar denotes the complex conjugate. A real kernel k(x, t) is said to be symmetric kernel if k(x, t) = k(t, x).

for example: sin(x + t), log(xt), $x^2t^2 + xt + 1$ etc. all are symmetric kernels. Again sin(2x + 3t) and $x^3t^3 + 1$ are not symmetric kernels.

(ii) Separable or degenerate Kernel.

A kernel k(x, t) is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only, that is

$$k(x, t) = \sum_{i=1}^{n} g_i(x)\, h_i(t).$$

6.8 *LEIBNITZ'S RULE OF DIFFERENTATION UNDER INTEGRAL SIGN*

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x\,,\ \xi)\, d\xi\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x} d\xi + F(x\,,\ \beta(x))\ \frac{d\beta(x)}{dx}\ - F(x\,,\ \alpha(x))\frac{d\alpha(x)}{dx}$$

In particular, we have

$$\frac{d}{dx}\left[\int_{a}^{x} K(x\,,\ \xi)u(\xi)\, d\xi\right] = \int_{a}^{x} \frac{\partial K}{\partial x} u(\xi)\, d\xi + K(x\,,\ x)\, u(x)\,.$$

Note: ■

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-2}} \int_a^{x_{n-1}} F(x_n)\, dx_n\ dx_{n-1} \ldots dx_1 = \frac{1}{n-1!} \int_a^x (x-\xi)^{n-1} f(\xi)\, d\xi\,.$$

6.9 *SOLUTION OF INTEGRAL EQUATION*

Consider the integral equations:

g(x) y(x) = f(x) + $\lambda \int_a^b k(x,t)y(t)dt$ …… (1) and

g(x) y(x) = f(x) + $\lambda \int_a^x k(x,t)y(t)dt$ …… (2)

a solution of the integral equation (1) and (2) is a function y(x), which when substituted into the equation, reduces it to an identity.

ILLUSTRATIVE EXAMPLES

Example 1. Show that the function y(x) = $(1+x^2)^{-3/2}$ is a solution of the Volterra integral equation y(x) = $\frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} y(t)dt$.

Sol. Given integral equation is y(x) = $\frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} y(t)dt$ …... (1)

Also given as y(x) = $(1+x^2)^{-3/2}$ …… (2)

From equation (2) y(t) = $(1+t^2)^{-3/2}$ …… (3)

Then, R.H.S. of (1) = $\frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}(1+t^2)^{-3/2}dt$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_0^{x^2}(1+u)^{-3/2}.\frac{1}{2}du$ (on putting $t^2 = u\ and\ du = 2tdt$)

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}.\frac{1}{2}.\left[\frac{(1+u)^{-3/2}}{-1/2}\right]_0^{x^2}$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\left[\frac{1}{(1+u)^{1/2}}\right]_0^{x^2}$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\left[\frac{1}{(1+x^2)^{1/2}} - 1\right]$

= $(1+x^2)^{-3/2}$ = y(x), by (2)

= L.H.S. of (1)

Hence (2) is a solution of given integral equation (1).

Example 2. Show that the function y(x) = xe^x is a solution of the Volterra integral equation y(x) = $\sin x + 2\int_0^x \cos(x-t)y(t)dt$.

Sol. Given integral equation is

y(x) = $\sin x + 2\int_0^x \cos(x-t)y(t)dt$ …… (1)

Also, given $\quad y(x) = xe^{x} \quad$ ……. (2)

From (1) $\quad y(t) = te^{t} \quad$ ……. (3)

Again we know that the following standard results:

$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - bcos(bx+c)] \quad \text{...... (4)}$$

And $\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + bsin(bx+c)]$

…… (5)

Then R.H.S. of (1)

$$= \sin x + 2\int_0^x \{\cos(x-t)\,te^{t}\}\,dt = sinx + 2\int_0^x t\{e^{t}\cos(t-x)\}dt$$

$$= \sin x + 2\left\{\left[t\frac{e^{t}}{2}\{\cos(t-x) + \sin(t-x)\}\right]_0^x - \int_0^x 1.\frac{e^{t}}{2}\{\cos(t-x) + \sin(t-x)\}dt\right\}$$

$$= \sin x + xe^{x} - \int_0^x e^{t}\cos(t-x)\,dt - \int_0^x e^{t}\sin(t-x)\,dt$$

$$= \sin x + xe^{x} - \left[\frac{e^{t}}{2}\{\cos(t-x) + \sin(t-x)\}\right]_0^x$$

$$- \left[\frac{e^{t}}{2}\{\sin(t-x) - \cos(t-x)\}\right]_0^x$$

$$= \sin x + xe^{x} - \left[\frac{e^{t}}{2} - \frac{1}{2}(cosx - \sin x)\right] - \left[-\frac{e^{t}}{2} - \frac{1}{2}(-\sin x - cosx)\right]$$

$$= xe^{x} = y(x)$$

Hence (2) is solution of (1).

Example 3. Show that y(x) = cos 2x is a solution of the integral equation

$$y(x) = \cos x + 3\int_0^{\pi} k(x,t)y(t)dt \text{ where } k(x, t)$$

$$= \begin{cases} sinx\ cost, & 0 \le x \le t \\ \cos x\ sint, & t \le x \le \pi \end{cases}.$$

Sol. Given integral equation is y(x) = cosx + $3\int_0^{\pi} k(x,t)y(t)dt$ …… (1)

Where k(x, t) = $\begin{cases} sinx\ cost, & 0 \le x \le t \\ \cos x\ sint, & t \le x \le \pi \end{cases}$ ……. (2)

Also given y(x) = cos 2x ……. (3)

From (3) y(t) = cos 2t ………. (4)

Then R.H.S. of (1)

$$= \text{cosx} + 3\left[\int_0^x k(x,t)y(t)dt + \int_x^{\pi} k(x,t)y(t)dt\right]$$

$$= \text{cosx} + 3\left[\int_0^x \cos x\ sint\ cos2t\ dt + \int_x^{\pi} \sin t \cos t \cos 2t\ dt\right],$$

by (2) and (4)

$$= \text{cosx} + 3cosx \int_0^x cos2t \sin t\ dt + 3sinx \int_x^{\pi} cos2t\ cost\ dt$$

$$= \text{cosx} + \frac{3}{2} cosx \int_0^x (sin3t - sint)dt + \frac{3}{2} sinx \int_x^{\pi} (cos3t + cost)\ dt$$

$$= \text{cosx} + \frac{3}{2} cosx \left[-\frac{1}{3} cos3t + cost\right]_0^x + \frac{3}{2} sinx \left[\frac{1}{3} sin3t + sint\right]_x^{\pi}$$

$$= \text{cosx} + \frac{3}{2} cosx \left[-\frac{1}{3} cos3x + cosx + \frac{1}{3} - 1\right] + \frac{3}{2} sinx \left[-\frac{1}{3} sin3x - sinx\right]$$

$$= \text{cosx} - \frac{1}{2}\ (cos3x\ cosx + sin3x\ sinx) + \frac{3}{2}(\cos^2 x - \sin^2 x) - cosx$$

$$= -\frac{1}{2} \cos\ (3x - x) + \frac{3}{2}\ cos2x = -\frac{1}{2} \cos 2x + \frac{3}{2}\ cos2x$$

= cos 2x = y(x)

Hence (3) is solution of (1).

6.10 *INITIAL VALUE PROBLEM*

While searching for the representation formula for the solution of an ordinary differential equation in such manner so as to include the boundary conditions or initial conditions.

When an ordinary differential equation is to be solved under conditions involving dependent variable and its derivative at the same value of the independent variable, then the problem under consideration is said to be an initial value problem.

For example: $\frac{d^2y}{dx^2} + y = x$, y(0) = 2, $y'(0) = 3$ …….. (1)

And $\frac{d^2y}{dx^2} + y = x$, y(1) = 2, $y'(1) = 3$ ……... (2)

Are both initial value problem.

Note: ■ Initial value problem is always converted into Volterra integral equation.

■ After converting the initial value problem into an integral equation, it can be solved by shorter methods of solving integral equations.

6.11 METHOD OF CONVERTING AN INITIAL VALUE PROBLEM INTO A VOLTERRA INTEGRAL EQUATION

This method is illustrated with the help of the following solved example.

ILLUSTRATIVE EXAMPLES

Example 1. Convert the following differential equation into integral equation:

$$y'' + y = 0 \text{ when } y(0) = y'(0) = 0.$$

Sol. Given $y''(x) + y(x) = 0$ ……. (1)

With initial Conditions y(0) = $y'(0) = 0$ ……..(2)

From (1) we get $y''(x) = -y(x)$ ……. (3)

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$\int_0^x y''(x)dx = -\int_0^x y(x)dx$ or $[y'(x)]_0^x = -\int_0^x y(x)dx$

or $y'(x) - y'(0) = -\int_0^x y(x)dx$ or $y'(x) = -\int_0^x y(x)dx$ …… (4)

integrating both sides of (4) w.r.t. x from 0 to x, we have

$\int_0^x y'(x)dx = -\int_0^x y(x)dx^2$ or $[y(x)]_0^x = -\int_0^x y(x)dx^2$

y(x) – y(0) = $-\int_0^x y(x)dx^2$ or y(x) = $-\int_0^x y(t)dt^2$ using (2)

or y(x) = $-\int_0^x (x-t)y(t)dt$ which is desired integral equation.

Example 2. Convert the following differential equation into integral equation:

$$y'' + \lambda xy = f(x) \text{ when } y(0) = 1 \text{ and } y'(0) = 0.$$

Sol. Given $y''(x) + \lambda xy(x) = f(x)$ ……. (1)

With initial Conditions y(0) = 1 , $y'(0) = 0$ ……..(2)

From (1) we get $y''(x) = f(x) - \lambda xy(x)$ ……. (3)

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$\int_0^x y''(x)dx = -\int_0^x [f(x) - \lambda xy(x)]dx$ or

$[y'(x)]_0^x = -\int_0^x [f(x) - \lambda xy(x)]dx$ or

$y'(x) - y'(0) = -\int_0^x [f(x) - \lambda x y(x)]dx$ or

$y'(x) = -\int_0^x [f(x) - \lambda x y(x)]dx$ …… (4)

Integrating both sides of (4) w.r.t. 'x' from 0 to x, we have

$\int_0^x y'(x)dx = \int_0^x [f(x) - \lambda x y(x)]dx^2$ or

$[y(x)]_0^x = -\int_0^x [f(x) - \lambda x y(x)]dx^2$

y(x) – y(0) = $-\int_0^x [f(x) - \lambda x y(x)]dx^2$ or

y(x) = 1 $+\int_0^x [f(t) - \lambda t y(t)]dt^2$ using (2)

y(x) = 1 $+\int_0^x (x-t)[f(t) - \lambda t y(t)]dt$

which is the required integral equation.

Example 3. The initial value problem corresponding to the integral equation

y(x) = $\int_0^x y(t)dt$ is

(a) $y' - y = 0, y(0) = 1$ (b) $y' + y = 0, y(0) = 0$

(c) $y' - y = 0, y(0) = 0$ (d) $y' + y = 0, y(0) = 1$.

Sol. Given y(x) = $\int_0^x y(t)dt$ is ……. (1)

Differentiating both sides of (1) with respect to x and using the Leibnitz's rule of

differentiation under the sign of integral, we obtain

$y'(x) = 0 + \int_0^x \frac{\partial y(t)}{\partial x} dt + y(x)\frac{dx}{dx} - y(0)\frac{d0}{dx}$ or

$y'(x) = y(x),$ i.e. $y' - y = 0$ …… (2)

From (1), y(0) = 1 + $\int_0^0 y(t)dt = 1,$ i.e. y(0) = 1 …….. (3)

(2) and (3) show that result (a) is true.

6.12 *BOUNDARY VALUE PROBLEM*

When an ordinary differential equation is to be solved under conditions involving dependent variable and its derivative at two different values of the independent variable, then the problem under consideration is said to be an initial value problem.

For example: $\frac{d^2y}{dx^2} + y = 0$, y(a) = y_1, $y(b) = y_2$ …….. (1)

> Note: ■ Boundary value problem is always converted into Fredholm integral equation.

ILLUSTRATIVE EXAMPLES

Example 1. Convert the following differential equation into integral equation:

$$y'' + \lambda y = 0 \text{ when } y(0) = 0,\ y(l) = 0.$$

Sol. Given $y''(x) + \lambda y(x) = 0$ ……… (1)

With boundary conditions y(0) = 0 …….. (2a)

And y(ℓ) = 0 …….. (2b)

From (1), $y''(x) = -\lambda\, y(x)$ …….(3)

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$\int_0^x y''(x)dx = -\lambda \int_0^x y(x)\, dx$ or

$[y'(x)]_0^x = -\lambda \int_0^x y(x)\, dx$ or

$y'(x) - y'(0) = -\lambda \int_0^x y(x)\, dx$ or ……. (4)

Let $y'(0) = C$, a constant ……. (5)

Using (5), (4) $y'(x) = C - \lambda \int_0^x y(x)\, dx$ ……… (6)

integrating both sides of (6) w.r.t. x from 0 to x, we have

$\int_0^x y'(x)dx = C\int_0^x dx - \lambda \int_0^x y(x)\, dx^2$ or $[y(x)]_0^x$ = Cx $-\lambda \int_0^x y(t)dt^2$

y(x) – y(0) = $Cx - \lambda \int_0^x (x-t)y(t)dt$ or

y(x) – 0 = $Cx - \lambda \int_0^x (x-t)y(t)dt$ or

y(x) = $Cx - \lambda \int_0^x (x-t)y(t)dt$ ……. (7)

putting x = ℓ in (7), we get

y(ℓ) = $C\ell - \lambda \int_0^\ell (\ell - t)y(t)dt$ or 0 = $C\ell - \lambda \int_0^\ell (\ell - t)y(t)dt$, using(2b)

or C = $\frac{\lambda}{\ell}\int_0^\ell (\ell - t)y(t)dt$ ……. (8)

using (8), (7) reduces to

y(x) = $\frac{\lambda}{\ell} x \int_0^\ell (\ell - t)y(t)dt$ - $\lambda \int_0^x (x-t)y(t)dt$ ……. (9)

or y(x) = $\int_0^\ell \frac{\lambda x(\ell - t)}{\ell} y(t)dt$ - $\lambda \int_0^x (x-t)y(t)dt$

or y(x) = $\int_0^x \frac{\lambda x(\ell - t)}{\ell} y(t)dt + \int_x^\ell \frac{\lambda x(\ell - t)}{\ell} y(t)dt$ - $\lambda \int_0^x (x-t)y(t)dt$

or y(x) = $\lambda \int_0^x \left[\frac{x(\ell - t)}{\ell} - (x-t)\right] y(t)dt + \lambda \int_x^\ell \frac{x(\ell - t)}{\ell} y(t)dt$

or y(x) = $\lambda \int_0^x \left[\frac{x(\ell - t) - \ell(x-t)}{\ell}\right] y(t)dt + \int_x^\ell \frac{x(\ell - t)}{\ell} y(t)dt$

or y(x) = $\lambda \left[\int_0^x \frac{t(\ell - t)}{\ell} y(t)dt + \int_x^\ell \frac{x(\ell - t)}{\ell} y(t)dt\right]$

or y(x) = $\lambda \int_0^\ell k(x,t)y(t)dt$ ……. (10)

where $\quad k(x, t) = \begin{cases} \frac{t(\ell-t)}{\ell}, & if\ 0 < t < x \\ \frac{x(\ell-t)}{\ell}, & if\ x < t < \ell \end{cases}$ …….. (11)

(10) is the required Fredholm integral equation, where k(x, t) is given by (11).

Example 2. Convert the following differential equation into integral equation:

$$y'' + xy = 1 \text{ when } y(0) = 0,\ y(1) = 0.$$

Sol. Given $\quad y''(x) + xy(x) = 1$ ……… (1)

With boundary conditions $\quad y(0) = 0$ …….. (2a)

And $\quad y(0) = 0$ …….. (2b)

From (1), $\quad y''(x) = 1 - x\,y(x)$ …….(3)

Integrating both sides of (3) w.r.t. ‘x’ from 0 to x, we have

$$\int_0^x y''(x)dx = \int_0^x dx - \int_0^x xy(x)dx \quad \text{or} \quad [y'(x)]_0^x = x - \int_0^x x\,y(x)dx$$

or $\quad y'(x) - y'(0) = x - \int_0^x x\,y(x)\,dx$ $\quad$ or ……. (4)

Let $\quad y'(0) = C$, a constant ……. (5)

Using (5), (4) $\quad y'(x) = C + x - \int_0^x x\,y(x)\,dx$ ……… (6)

integrating both sides of (6) w.r.t. x from 0 to x, we have

$$\int_0^x y'(x)dx = \int_0^x (c + x)dx - \lambda \int_0^x x\,y(x)\,dx^2 \quad \text{or}$$

$$[y(x)]_0^x = \left[Cx + \frac{1}{2}x^2\right]_0^x - \int_0^x t\,y(t)dt^2 \quad \text{or}$$

$$y(x) - y(0) = Cx + \frac{1}{2}x^2 - \int_0^x (x - t)y(t)dt \quad \text{or}$$

$$y(x) = Cx + \frac{1}{2}x^2 - \int_0^x (x - t)y(t)dt \quad \text{using (2a)} \quad ……. (7)$$

putting x = 1 in (7), we have

y(1) = c + $\frac{1}{2} - \int_0^1 (1-t)\, t\, y(t)dt$ or 1 = c + $\frac{1}{2} - \int_0^1 (1-t)\, t\, y(t)dt$

by (2b)

or c = $\frac{1}{2} + \int_0^1 (1-t)\, t\, y(t)dt$ …… (8)

using (8), (7) reduces to

y(x) = x$\left[\frac{1}{2} + \int_0^1 (1-t)\, t\, y(t)dt\right] + \frac{1}{2}x^2 - \int_0^x (x-t)y(t)dt$

or y(x) = $\frac{1}{2}$ x (1 + x) + $\int_0^1 xt(1-t)y(t)dt - \int_0^x t(x-t)y(t)dt$

or y(x) = $\frac{1}{2}$ x (1 + x) +$\int_0^x xt(1-t)y(t)dt$ + $\int_x^1 xt(1-t)y(t)dt$ - $\int_0^x t(x-t)y(t)dt$

or y(x) = $\frac{1}{2}$ x (1 + x) +$\int_0^x t\{x - xt - x + t\}y(t)dt$ + $\int_x^1 xt(1-t)y(t)dt$

or y(x) = $\frac{1}{2}$ x (1 + x) +$\int_0^x t^2(1-t)y(t)dt$ + $\int_x^1 xt(1-t)y(t)dt$

or y(x) = $\frac{1}{2}$ x (1 + x) +$\int_0^x k(x,t)y(t)dt$ …… (9)

where $k(x,t) = \begin{cases} t^2(1-t), & if\ t < x \\ xt(1-t), & if\ t > x \end{cases}$ …… (10)

(10) is the required Fredholm integral equation, where k(x, t) is given by (11).

<u>CHECK YOUR PROGRESS</u>

True or false / MCQ Questions

Problem 1. The integral equation

y(x) = $\int_0^x (x-t)y(t)dt - x\int_0^1 (1-t)y(t)dt$ is equivalent to:

(a) $y'' - y = 0, y(0) = 0, y(1) = 0$

(b) $y'' - y = 0, y(0) = 0, y'(0) = 0$

(c) $y' + y = 0, y(0) = 0, y(1) = 0$

(d) $y' + y = 0, y(0) = 1, y'(0) = 0$

Problem 2. Boundary value problem is always converted into Fredholm integral equation. True / false.

Problem 3. Initial value problem is always converted into Volterra integral equation. True / false.

Problem 4. A linear equation of the form y(x) = f(x) + $\lambda \int_a^b k(x,t)y(t)dt$ is known as Fredholm integral equation of third kind.

Problem 5. A linear equation of the form y(x) = $\lambda \int_a^x k(x,t)y(t)dt$ is known as Homogeneous Volterra integral equation of second kind.

6.13 *SUMMARY*

1. Fredholm integral equation of First kind.

A linear equation of the form f(x) + $\lambda \int_a^b k(x,t)y(t)dt = 0$, is known as Fredholm integral equation of First kind.

2. Fredholm integral equation of Second kind.

A linear equation of the form y(x) = f(x) + $\lambda \int_a^b k(x,t)y(t)dt$ is known as Fredholm integral equation of second kind.

3. Homogeneous Fredholm integral equation of Second kind.

A linear equation of the form y(x) = $\lambda \int_a^b k(x,t)y(t)dt$ is known as Homogeneous Fredholm integral equation of second kind.

4. Volterra integral equation of First kind.

A linear equation of the form f(x) + $\lambda \int_a^x k(x,t)y(t)dt = 0$, is known as Volterra integral equation of First kind.

5. Volterra integral equation of Second kind.

A linear equation of the form y(x) = f(x) + $\lambda \int_a^x k(x,t)y(t)dt$ is known as Volterra integral equation of second kind.

6. Homogeneous Volterra integral equation of Second kind.

A linear equation of the form y(x) = $\lambda \int_a^x k(x,t)y(t)dt$ is known as Homogeneous Volterra integral equation of second kind.

7. Symmetric Kernel. k(x, t) = $\bar{k}$(t, x).

8. Initial value problem. $\frac{d^2y}{dx^2} + y = x$, y(0) = 2, $y'(0) = 3$.

9. Boundary value problem. $\frac{d^2y}{dx^2} + y = 0$, y(a) = y_1, $y(b) = y_2$.

6.14 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

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6.17 *TERMINAL AND MODEL QUESTIONS*

Q 1. Verify that the given functions are solutions of the corresponding integral equations.

(i) $y(x) = 1 - x$; $\int_0^x e^{x-t} y(t)dt = x$.

(ii) $y(x) = \frac{1}{2}$; $\int_0^x \frac{y(t)}{\sqrt{x-t}} dt = \sqrt{x}$

(iii) $y(x) = 3$; $x^3 = \int_0^x (x-t)^2 y(t)dt$

(iv) y(x) = x - $\frac{x3}{6}$; y(x) = x - $\int_0^x \sinh(x-t)y(t)dt$

(v) y(x) = $e^x\left(2x - \frac{2}{3}\right)$; y(x) + 2$\int_0^t e^{x-t}y(t)dt$ = 2xe^x

Q2. Reduce the following initial value problem into an integral equation

$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, y(0) = 1, $y'(0) = 1$.

Q3. Convert $y''(x) - sinx\, y'(x)$ + $e^x y = x$ with initial conditions

y(0) = 1, $y'(0)$ = -1 to a Volterra integral equation of second kind.

Q4. Show that the solution of the Volterra equation y(x) = 1 + $\int_0^x (t-x)y(t)dt$ satisfying the differential equation $y''(x) + y'(x) = 0$ and the boundary conditions y(0) = 1, $y'(0) = 1$.

Q5. Convert the boundary value problem $y''(x) + y'(x) = 0\ with$ boundary condition y(0) = 1, $y'(1) = 0$, into an integral equation.

6.18 *ANSWERS*

TQ2 y(x) = 1 + x - $\int_0^t t\, y(t)dt$

TQ3 y(x) = $\frac{x^3}{6} - x + 1 + \int_0^x [sint - (x-t)(e^t + cost)\,]y(t)dt.$

TQ5 y(x) = 1 + $\int_0^1 k(x,t)\, y(t)dt$, where k(x, t) = $\begin{cases} t\,, t < x \\ x\,, t > x \end{cases}$.

CHECK YOUR PROGRESS

CYQ 1. (a)

CYQ 2. True

CYQ 3. True

CYQ 4. False

CYQ 5. True

UNIT 7: FREDHOLM INTEGRAL EQUATIONS OF SECOND KIND WITH SEPARABLE KERNELS

Contents

7.1 *INTRODUCTION*

Integral equations are equations in which the unknown function appears inside a definite integral. They are closely related to differential equations. In this unit learner learn about Fredholm, Volterra integral equation and other types of integral equation, also learn about the conversion of initial and boundary value problem into integral equation. In this unit learner learn about Solution of homogeneous Fredholm integral equation of second kind with separable kernel and Solution of non-homogeneous Fredholm integral equation of second kind with separable kernel.

7.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) integral equation

(ii) Fredholm integral equation

(iii) initial value problem

(vi) Boundary value problem

(v) Solution of homogeneous Fredholm integral equation of second kind with separable kernel.

(vi) Solution of non-homogeneous Fredholm integral equation of second kind with separable kernel.

7.3 *CHARACTERISTIC VALUES (EIGEN VALUE) OR CHARACTERISTIC FUNCTION (EIGEN FUNCTION)*

Consider the Homogeneous Fredholm integral equation of the second kind:

$$y(x) = \lambda \int_a^b k(x,t)y(t)dt \qquad \ldots\ldots\ldots (1)$$

then (1) has always the obvious solution y(x) = 0, which is known as zero or trivial solution of (1). The value of the parameter λ for which (1) has a non-zero (or non trivial) solution y(x) ≠ 0 are known as the eigenvalue of (1) or of the kernel k(x, t). Further if $\varphi(x)$ is continuous and $\varphi(x) \neq 0$ on the interval (a, b) and

$$\varphi(x) = \lambda \int_a^b k(x,t)\varphi(x)\, dt \qquad \ldots\ldots\ldots (2)$$

Then $\varphi(x)$ is known as an eigenfunction of (1) corresponding to the eigenvalue λ_0.

Note: ■ The number λ = 0 is not an eigenvalue since for $\lambda = 0$, (1) yield y(x) = 0, which is non-zero solution.

■ if the kernel k(x, t) is continuous in the rectangle R: a$\leq x \leq b$, a$\leq t \leq b$, and the numbers a and b are finite, then to every eigen value λ there exist a finite number of linearly independent eigenfunction; the number of such functions is known as the index of the eigenvalue. Different eigenvalue have different indices.

■ A Homogeneous Fredholm integral equation may, generally, have no eigenvalues and eigenfunctions or it may not have any real eigenvalue and eigenfunction.

7.4 *SOLUTION OF HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE KERNEL.*

Consider a homogeneous Fredholm integral equation of the second kind:

$$y(x) = \lambda \int_a^b k(x,t)y(t)dt \qquad \ldots\ldots\ldots (1)$$

since kernel k(x, t) is separable, we take

$$k(x, t) = \sum_{i=1}^{n} f_i(x)g_i(t) \qquad \ldots\ldots (2)$$

using (2), (1) reduces to

y(x) = $\lambda \int_a^b [\sum_{i=1}^{n} f_i(x) g_i(t)] y(t) dt$ or

y(x) = $\lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t) y(t) dt$ ……. (3)

let $\int_a^b g_i(t) y(t) dt$ = C_i , where I = 1, 2, 3, …, n. …… (4)

using (4), (3) reduces to y(x) = $\lambda \sum_{i=1}^{n} C_i f_i(x)$ …… (5)

where constants $C_i (i = 1, 2, 3, \dots, n)$ are to be determined in order to find solution of (1) in the form given by (5).

We now proceed to evaluate C_i's as follows:

Multiplying both sides of (5) successively by $g_1(x), g_2(x), \dots, g_n(x)$ and integrating over the interval (a, b), we have

$\int_a^b g_1(t) y(x) dx$ = $\lambda \sum_{i=1}^{n} C_i \int_a^b g_1(t) f_i(x) dx$ …… (A_1)

$\int_a^b g_2(t) y(x) dx$ = $\lambda \sum_{i=1}^{n} C_i \int_a^b g_2(t) f_i(x) dx$ …… (A_2)

…. …. …. …. …. ….

$\int_a^b g_n(t) y(x) dx$ = $\lambda \sum_{i=1}^{n} C_i \int_a^b g_n(t) f_i(x) dx$ …… (A_2)

Let α_{ij} = $\int_a^b g_j(t) f_i(x) dx$, where i, j = 1, 2, …, n.

Using (4) and (6), (A_1) reduces to

C_1 = $\lambda \sum_{i=1}^{n} C_i \alpha_{ij}$ or C_1 = $\lambda [C_1 \alpha_{11} + C_2 \alpha_{12} + \dots + C_n \alpha_{1n}]$

Or $(1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 - \dots - \lambda \alpha_{1n} C_n$ = 0 …… (B_1)

$-\lambda \alpha_{21} C_1 + (1 - \lambda \alpha_{22}) C_2 - \dots - \lambda \alpha_{2n} C_n$ = 0 …… (B_2)

… … … … … … …

… … … … … … …

$-\lambda \alpha_{n1} C_1 - -\lambda \alpha_{n2} C_2 - \dots + (1 - \lambda \alpha_{nn}) C_n$ = 0 …… (B_n)

The determinant D(λ) of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & \dots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & \dots & -\lambda\alpha_{1n} \\ \dots & \dots & \dots & \dots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \dots & 1-\lambda\alpha_{nn} \end{vmatrix} \quad \ldots\ldots (7)$$

D(λ) $\neq 0$, the system of equations $(B_1), (B_2), \dots, (B_n)$ has only trivial solution $C_1 = C_2 = \cdots = C_n = 0$ and hence from (5) we notice that (1) has only zero or trivial solution y(x) = 0. However, if D(λ) $= 0$, at least one of the $C_i{}'s$ can be assigned arbitrarily, and the remaining $C_i{}'s$ can be determined accordingly. Hence when D(λ) $= 0$, infinitely many solutions of the integral equation (1) exist.

Those value of λ for which D(λ) $= 0$ are called the eigenvalues, and any non-trivial solution of (1) is called a corresponding eigenfunction of (1).

The eigenvalue of (1) are given by D(λ) $= 0$, i.e.

$$\begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & \dots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & \dots & -\lambda\alpha_{1n} \\ \dots & \dots & \dots & \dots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \dots & 1-\lambda\alpha_{nn} \end{vmatrix} = 0 \quad \ldots\ldots (8)$$

So degree of equation (8) in λ is m$\leq n$. It follows that if integral equation (1) has separable kernel given by (2), then (1) has at the most n eigenvalues.

ILLUSTRATIVE EXAMPLES

Example 1. Solve the Homogeneous Fredholm equation

$$y(x) = \lambda \int_0^1 e^x e^t y(t)dt \ .$$

Sol. Given $\quad y(x) = \lambda \int_0^1 e^x e^t y(t)dt \quad$ or $\quad y(x) = \lambda e^x \int_0^1 e^t y(t)dt$ ….. (1)

Let $\quad c = \int_0^1 e^t y(t)dt \quad \ldots\ldots (2)$

Then (1) reduce to $\quad y(x) = \lambda c e^{x}$ …….. (3)

From (3), $\quad y(t) = \lambda c e^{t}$ ………(4)

Using (4), (2) becomes $\quad c = \int_0^1 e^t \lambda c e^t dt$ or

$c = \lambda c\left[\frac{e^{2t}}{2}\right]_0^1 = \frac{\lambda c}{2}(e^2 - 1)$ or $c\left[1 - \frac{\lambda}{2}(e^2 - 1)\right] = 0$ …… (5)

if c = 0 then (4) gives y(x) = 0.we, therefore, assume that for non-zero solution of (1), $c \neq 0$.

Then (5) gives

$1 - \frac{\lambda}{2}(e^2 - 1) = 0$ or $\lambda = \frac{2}{(e^2-1)}$ …….. (6)

Which is an eigenvalue of (1).

Putting the value of λ given by (6) in (3), the corresponding eigenfunction is given by

$$y(x) = \left\{\frac{2c}{(e^2-1)} e^x\right\}$$

Hence, corresponding to eigenvalue $\frac{2}{(e^2-1)}$ there corresponds the eigenfunction e^x.

Note:■ While writing eigenfunction the constant $\frac{2c}{(e^2-1)}$ is taken as unity.

Example 2. Show that the Homogeneous integral equation $y(x) - \lambda \int_0^1 (3x-2)\, t\, y(t) dt = 0$ has no characteristic number and eigenfunction.

Sol. Given $\quad y(x) = \lambda \int_0^1 (3x-2)\, t\, y(t) dt$ or $y(x) = \lambda(3x-2)\int_0^1 t\, y(t) dt$ …… (1)

Let $\quad c = \int_0^1 t\, y(t) dt$ …… (2)

Then (1) reduces to $\quad y(x) = \lambda c(3x - 2) \quad$ ……. (3)

From (3), $\quad y(t) = \lambda c(3t - 2) \quad$ ……. (4)

Using (4), (2) becomes

$$c = \int_0^1 t\,\lambda c(3t - 2)dt \quad \text{or} \quad c = \lambda c[t^3 - t^2]_0^1 \quad \text{or } c = 0.$$

Therefore from (3) y(x) ≡ 0, which is a zero solution of (1). Hence for any $\lambda, (1)\ has\ only$ zero solution y(x) ≡ 0. Therefore, (1) does not possess any eigenvalue or eigenfunction.

Note: ■ Note that the kernel k(x, t) = (3x – 2) t of the above example is not symmetric. Thus we shown that a kernel which is not symmetric does not necessarily have a characteristic constant.

Example 3. Find the eigenvalue and the corresponding eigenfunctions of the homogeneous integral equation y(x) = $\lambda \int_0^1 sin\pi x\ cos\pi x\ y(t)dt.$

Sol. Given y(x) = $\lambda \int_0^1 sin\pi x\ cos\pi x\ y(t)dt$ or y(x) = $\lambda sin\pi x \int_0^1 cos\pi x\ y(t)dt$ …….(1)

Let C = $\int_0^1 cos\pi x\ y(t)dt$ ……… (2)

Then (1) reduce to $\quad y(x) = \lambda\ C\ sin\pi x \quad$ …….. (3)

From (3), $\quad y(t) = \lambda\ C\ sin\pi t \quad$ ……… (4)

Using (4), (2) becomes

$$C = \int_0^1 cos\pi t(\lambda\ c\ sin\pi t)dt \text{ or } c = \frac{\lambda\ C}{2}\left[-\frac{cos2\pi t}{2\pi}\right]_0^1 = \frac{\lambda\ C}{2}\left[-\frac{1}{2\pi} + \frac{1}{2\pi}\right].$$

Hence C = 0 and so from (3), y(x) ≡ 0. Thus for any λ, (1) has only zero solution y(x) ≡ 0.

Therefore, (1) does not possess any characteristic number or eigenfunction.

Example 4. Find the eigenvalues and the corresponding eigenfunctions of the integral equation y(x) = $\lambda \int_0^1 (2xt - 4x^2)\, y(t)dt$.

Sol. Given y(x) = $\lambda \int_0^1 (2xt - 4x^2)\, y(t)dt$ or

y(x) = $2\lambda x \int_0^1 ty(t)dt - 4\lambda x^2 \int_0^1 y(t)\, dt$ ……… (1)

$C_1 = \int_0^1 t\, y(t)\, dt$ ……. (2) and $C_2 = \int_0^1 y(t)\, dt$ ……. (3)

Then (1) reduces to y(x) = $2\lambda C_1 x - 4\lambda C_2 x^2$ ……. (4)

y(t) = $2\lambda C_1 t - 4\lambda C_2 t^2$ ……. (5)

Using (5), (2) becomes

$C_1 = \int_0^1 t(2\lambda C_1 t - 4\lambda C_2 t^2)\, dt$ or $C_1\left[1 - 2\lambda \int_0^1 t^2\, dt\right] + 4\lambda C_2 \int_0^1 t^3\, dt = 0$

$C_1\left(1 - \frac{2\lambda}{3}\right) + \lambda C_2 = 0$ ……… (6)

Again, using (5), (3) becomes

$C_2 = \int_0^1 (2\lambda C_1 t - 4\lambda C_2 t^2)\, dt$ or $2\lambda C_1 \int_0^1 dt - C_2\left[1 + 4\lambda \int_0^1 t^2\, dt\right] = 0$

Or $\lambda C_1 - C_2\left(1 + \frac{4\lambda}{3}\right) = 0$ …… (7)

Thus, we have a system of homogeneous linear equations (6) and (7) for determining C_1 and C_2, for non-zero solution of this system of equations, we must have

$$\begin{vmatrix} \left(1 - \frac{2\lambda}{3}\right) & \lambda \\ \lambda & -\left(1 + \frac{4\lambda}{3}\right) \end{vmatrix} = 0 \quad \text{or} \quad -\left(1 - \frac{2\lambda}{3}\right)\left(1 + \frac{4\lambda}{3}\right) - \lambda^2 = 0$$

Or $\lambda^2 + 6\lambda + 9 = 0$ so that $\lambda = -3, -3$

Hence eigenvalue are $\lambda_1 = -3, \lambda_2 = -3$.

Putting $\lambda = \lambda_1 = -3$ in (6) and (7), we get

$3C_1 - 3C_2 = 0$ ...... (8) and $-3C_1 + 3C_2 = 0$ .... (9)

(8) or (9) give $C_1 = C_2$. Hence from (4), we have

$$y(x) = 2C_1\lambda_1(x - 2x^2) = -6C_1(x - 2x^2)$$

Taking $-3C_1 = 1$, the eigenfunction is $(x - 2x^2)$

Hence eigenfunction corresponding to eigenvalue $\lambda_1 = \lambda_2 = -3$

is $x - 2x^2$.

Example 5. Solve the Homogeneous Fredholm integral equation of the second kind:

$$y(x) = \lambda \int_0^{2\pi} \sin(x + t)y(t)dt \; .$$

Sol. Given $y(x) = \lambda \int_0^{2\pi} \sin(x + t)y(t)dt$

Or $y(x) = \lambda \int_0^{2\pi} (\sin x \cos t + \cos x \sin t)\, y(t)dt$

Or $y(x) = \lambda \sin x \int_0^{2\pi} \cos t\, y(t)dt + \lambda \cos x \int_0^{2\pi} \sin t\, y(t)dt$ .... (1)

Let $C_1 = \int_0^{2\pi} \cos t\, y(t)dt$ .... (2)

And $C_2 = \int_0^{2\pi} \sin t\, y(t)dt$ .... (3)

Then (1) reduces to $y(x) = \lambda C_1 \sin x + \lambda C_2 \cos x$ .... (4)

From (4) $y(t) = \lambda C_1 \sin t + \lambda C_2 \cos t$ .... (5)

Using (5), (2) becomes $C_1 = \int_0^{2\pi} \cos t(\lambda C_1 \sin t + \lambda C_2 \cos t)\, dt$

Or $C_1 = \frac{\lambda C_1}{2} \int_0^{2\pi} \sin 2t dt + \frac{\lambda C_2}{2} \int_0^{2\pi} (1 + \cos 2t)\, dt$

Or $C_1 = \frac{\lambda C_1}{2} \left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{\lambda C_2}{2} \left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$

Or $C_1 = 0 + \lambda C_2 \pi$ or $C_1 - \lambda \pi C_2 = 0$ .... (6)

Using (5), (3) becomes $C_2 = \int_0^{2\pi} \sin t\, (\lambda C_1 \sin t + \lambda C_2 \cos t)dt$

Or $C_2 = \frac{\lambda C_1}{2}\int_0^{2\pi}(1-\cos 2t)dt + \frac{\lambda C_2}{2}\int_0^{2\pi} sin2tdt$

or $C_2 = \frac{\lambda C_1}{2}\left[t - \frac{sin2t}{2}\right]_0^{2\pi} + \frac{\lambda C_2}{2}\left[-\frac{cos2t}{2}\right]_0^{2\pi}$

or $C_2 = \lambda C_1 \pi$ or $\lambda C_1 \pi - C_2 = 0$ ……. (7)

Thus, we have a system of homogeneous linear equations (6) and (7) for determining C_1 and C_2.

For non-zero solution of this system of equations, we must have

$\begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0$ or $-1 + \lambda^2\pi^2 = 0$ so that $\lambda = \pm\frac{1}{\pi}$.

Hence eigenvalue are $\lambda_1 = \frac{1}{\pi}, \lambda_2 = -\frac{1}{\pi}$. ……… (8)

To determine eigenvalues corresponding to $\boldsymbol{\lambda = \lambda_1 = \frac{1}{\pi}}$

Putting $\lambda = \lambda_1 = \frac{1}{\pi}$ in (6) and (7), we get

$C_1 - C_2 = 0$ …… (9) and $C_1 - C_2 = 0$ ……. (10)

Both (9) and (10) give $C_1 = C_2$. Hence from (4), we have

y(x) = $\frac{1}{\pi}C_1 sinx + \frac{1}{\pi}C_1 cosx$ or y(x) = $\frac{C_1}{\pi}(sinx + cosx)$

Taking $\frac{C_1}{\pi} = 1$, the required eigenfunction

$y_1(x) = sinx + cosx$. ……. (11)

To determine eigenvalues corresponding to $\boldsymbol{\lambda = \lambda_2 = \frac{1}{\pi}}$.

Putting $\lambda = \lambda_1 = \frac{1}{\pi}$ in (6) and (7), we get

$C_1 + C_2 = 0$ …… (12) and $C_1 + C_2 = 0$ ……. (13)

Both (12) and (13) give $C_2 = -C_1$. Hence from (4), we have

y(x) = $-\frac{1}{\pi}C_1 sinx - \frac{1}{\pi}(-C_1)cosx$ or y(x) = $\frac{-C_1}{\pi}(sinx - cosx)$

Taking $\frac{-C_1}{\pi} = 1$, the required eigenfunction

$y_2(x) = \ sinx - cosx$. ……. (14)

From (8), (11) and (14), the required eigenvalues and eigenfunctions are given by

$\lambda_1 = \frac{1}{\pi}$, $y_1(x) = \ sinx + cosx$ and $\lambda_1 = -\frac{1}{\pi}$, $y_2(x) = \ sinx - cosx.$

7.5 SOLUTION OF NON HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE KERNEL.

ILLUSTRATIVE EXAMPLES

Example 1. Solve the Fredholm integral equation

y(s) = s+ $\int_0^1 su^2 g(u)du$

Sol. Given y(s) = s+ $s\int_0^1 u^2 g(u)du$ …… (1)

Let C = $\int_0^1 u^2 g(u)du$ ……. (2)

Using (2), (1) yields g(s) = s + Cs = s(1 + C) …… (3)

From (3) g(u) = u(1 + C) ……. (4)

Using (4), (2) yields C = $\int_0^1 u^3 (1 + C)du$ = (1 + C) $\left[\frac{u^4}{4}\right]_0^1$

Or C = (1 + C) $\frac{1}{4}$ so that C = $\frac{1}{3}$

Hence, (3) ⟹ g(s) = s(1+ $\frac{1}{3}$) = $\frac{4s}{3}$.

Example 2. Solve the Fredholm integral equation

y(x) = e^x+ $\lambda \int_0^1 2e^x e^t y(t)dt$.

Sol. Given y(x) = $e^x + \lambda \int_0^1 2e^x e^t y(t)dt$ or

$$y(x) = e^x + 2\lambda e^x \int_0^1 e^t y(t)dt \quad \ldots\ldots (1)$$

Let $\quad C = \int_0^1 e^t y(t)dt \quad \ldots\ldots. (2)$

Using (2), (1) yields y(x) = e^x + 2Cλe^x = e^x (1 + 2Cλ) …… (3)

From (3) $\quad y(t) = e^t\,(1 + 2C\lambda) \quad \ldots\ldots. (4)$

Using (4), (2) yields C = $\int_0^1 [e^t . e^t (1 + 2C\lambda)]dt$

$$= (1 + 2C\lambda)\left[\frac{e^{2t}}{2}\right]_0^1$$

$$= (1 + 2C\lambda)\frac{1}{2}(e^2 - 1)$$

Or $\quad C[1 - \lambda(e^2 - 1)] = \frac{1}{2}(e^2 - 1)$

or C = $\frac{e^2-1}{2[1-\lambda(e^2-1)]}$, where $\lambda \neq \frac{1}{e^2-1}$

putting this value of C in equation (3), we get

y(x) = $e^x\left[1 + 2\lambda.\frac{e^2-1}{2\{1-\lambda(e^2-1)\}}\right]$ or

y(x) = $\frac{e^x}{1-\lambda(e^2-1)}$, where $\lambda \neq \frac{1}{e^2-1}$

which is the required solution of the given integral equation.

Example 2. Solve the Fredholm integral equation

$$y(x) = cosx + \lambda \int_0^\pi sinx\, y(t)dt .$$

Sol. Given $\quad$ y(x) = $cosx + \lambda \int_0^\pi sinx\, y(t)dt$ or

$$y(x) = cosx + \lambda sinx \int_0^\pi y(t)dt \quad \ldots\ldots. (1)$$

let $\quad C = \int_0^\pi y(t)dt \quad \ldots\ldots (2)$

Using (2), (1) reduces y(x) = cosx + λC sinx ……. (3)

From (3), y(t) = cost + λC sint ……. (4)

Using (4), (2) reduces to C = $\int_0^{\pi}$ (cosx + λC sinx)dt

C = $[sint]_0^{\pi}$ + $\lambda C[-cost]_0^{\pi}$

C = 0 + $\lambda C[-cos\pi + cos0]$

C = 2 λC or C(1 - 2λ) $= 0$

So that C = 0, if $\lambda \neq \frac{1}{2}$

Hence by (3), the required solution is y(x) = cosx, provided $\lambda \neq \frac{1}{2}$.

Example 3. Solve the Fredholm integral equation

y(x) = $(1 + x)^2 + \int_{-1}^{1}(xt + x^2t^2)y(t)dt$.

Sol. Given y(x) = $(1 + x)^2 + \int_{-1}^{1}(xt + x^2t^2)y(t)dt$

Or y(x) = $(1 + x)^2 + x\int_{-1}^{1}ty(t)dt$ + $x^2\int_{-1}^{1}t^2y(t)dt$ ……
(1)

Let C_1 = $\int_{-1}^{1}ty(t)dt$ ……. (2)

And C_2 = $\int_{-1}^{1}t^2y(t)dt$ …… (3)

Using (2) and (3) , (1) reduces to y(x) = $(1 + x)^2 +\ C_1x + C_2x^2$ …… (4)

From (4), y(t) = $(1 + t)^2 +\ C_1t + C_2t^2$ ……. (5)

Using (5), (2) reduces to

C_1 = $\int_{-1}^{1}t[(1 + t)^2 +\ C_1t + C_2t^2]dt$ or

C_1 = $\int_{-1}^{1}t[1 + (2 + C_1)t + (1 + C_2)t^2]dt$ or

$C_1 = \left[\frac{t^2}{2}\right]_{-1}^{1} + (2 + C_1)\left[\frac{t^3}{3}\right]_{-1}^{1} + (1 + C_2)\left[\frac{t^4}{4}\right]_{-1}^{1}$ or

$C_1 = \frac{2}{3}(2 + C_1)$ so that $C_1 = 4$ …… (6)

Using (5), (3) reduces to

$C_2 = \int_{-1}^{1} t^2[(1+t)^2 + C_1 t + C_2 t^2]dt$ or

$= \int_{-1}^{1} t^2[1 + (2 + C_1)t + (1 + C_2)t^2]dt$ or

$C_2 = \left[\frac{t^3}{3}\right]_{-1}^{1} + (2 + C_1)\left[\frac{t^4}{4}\right]_{-1}^{1} + (1 + C_2)\left[\frac{t^5}{5}\right]_{-1}^{1}$ or

$C_2 = \frac{2}{3} + (1 + C_2)\frac{2}{5}$ or $C_1 = \frac{16}{9}$ …… (7)

y(x) = $(1+x)^2$ + 4x + $\frac{16}{9}x^2$ or y(x) = 1 + 6x + $\frac{25}{9}x^2$.

Example 4. Show that the integral equation

y(x) = $f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)y(t)dt$ possesses no solution for f(x) = x, but that it possesses infinitely many solutions when f(x) = 1.

Sol. Given y(x) = $f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)y(t)dt$

Or y(x) = $f(x) + \frac{1}{\pi}\int_0^{2\pi}(\text{sinx cost} + \text{cosx sint})y(t)dt$

Or y(x) = $f(x) + \frac{\text{sinx}}{\pi}\int_0^{2\pi} \text{cost}\, y(t)dt + \frac{\text{cosx}}{\pi}\int_0^{2\pi} \text{sint}\, y(t)dt$ …… (1)

Let $C_1 = \int_0^{2\pi} \text{cost}\, y(t)dt$ …… (2)

And $C_2 = \int_0^{2\pi} \text{sint}\, y(t)dt$ …… (3)

Using (2) and (3), (1) reduces to y(x) = f(x) + $\frac{C_1}{\pi} sinx + \frac{C_1}{\pi} cosx$ …… (4)

We now discuss two particular cases as mentioned in the problem.

Case I. let f(x) = x. then (4) reduces to

$$y(x) = x + \frac{C_1}{\pi} sinx + \frac{C_1}{\pi} cosx \qquad \ldots\ldots (5)$$

from (5), $\quad y(t) = t + \frac{C_1}{\pi} sint + \frac{C_1}{\pi} cost \qquad \ldots\ldots. (6)$

using (6), (2) becomes

$$C_1 = \int_0^{2\pi} \text{cost} \left(t \;+\; \frac{C_1}{\pi} sint + \frac{C_1}{\pi} cost \right) dt$$

$$= \int_0^{2\pi} \text{t cost}\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \text{sin2t}\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} (1 + \text{cos2t})\, dt$$

Or $\quad C_1 = [tsint]_0^{2\pi} - \int_0^{2\pi} \text{sint}\, dt + \frac{C_1}{2\pi}\left[-\frac{cos2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{sin2t}{2}\right]_0^{2\pi}$

Or $\quad C_1 = [-cost]_0^{2\pi} + \frac{C_2}{2\pi}(2\pi + 0) \quad$ or $\quad C_1 - C_2 = 0 \quad \ldots\ldots (7)$

Again using (6), (3) becomes

$$C_2 = \int_0^{2\pi} \text{sint} \left(t \;+\; \frac{C_1}{\pi} sint + \frac{C_1}{\pi} cost \right) dt$$

$$= \int_0^{2\pi} \text{t sint}\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} (1 - \text{cos2t})\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \text{sin2t}\, dt$$

$$C_2 = [-tcost]_0^{2\pi} - \int_0^{2\pi} (-\text{cost})\, dt + \frac{C_1}{2\pi}\left[t - \frac{sin2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[-\frac{cos2t}{2}\right]_0^{2\pi}$$

$C_2 = -2\pi + [sint]_0^{2\pi} + \frac{C_1}{2\pi}(2\pi + 0) \quad$ or $\quad C_1 - C_2 = 2\pi \quad \ldots\ldots (8)$

The system of equation (7) and (8) is inconsistent and so it possesses no solution.

Hence $C_1 and\, C_2$ cannot be determined and so (5) shows that the given integral possesses no solution when f(x) = x.

Case II. let f(x) = 1. then (4) reduces to

$$y(x) = 1 + \frac{C_1}{\pi} sinx + \frac{C_1}{\pi} cosx \qquad \ldots\ldots (9)$$

from (9), $\quad y(t) = 1 + \frac{C_1}{\pi} sint + \frac{C_1}{\pi} cost \qquad \ldots\ldots. (10)$

using (6), (2) becomes

$$C_1 = \int_0^{2\pi} \text{cost} \left(1 + \frac{C_1}{\pi} sint + \frac{C_2}{\pi} cost\right) dt$$

$$= \int_0^{2\pi} \text{cost}\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \text{sin2t}\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} (1 + \text{cos2t})\, dt$$

$$= [sint]_0^{2\pi} + \frac{C_1}{2\pi}\left[-\frac{cos2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{sin2t}{2}\right]_0^{2\pi}$$

Or $\quad C_1 = 0 + 0 + \frac{C_2}{2\pi}(2\pi + 0)\quad$ or $\quad C_1 = C_2 \quad$ …… (11)

Again using (6), (3) becomes

$$C_2 = \int_0^{2\pi} \text{sint} \left(1 + \frac{C_1}{\pi} sint + \frac{C_2}{\pi} cost\right) dt$$

$$= \int_0^{2\pi} \text{sint}\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} (1 - \text{cos2t})\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \text{sin2t}\, dt$$

$$= [-cost]_0^{2\pi} + \frac{C_1}{2\pi}\left[t - \frac{sin2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[-\frac{cos2t}{2}\right]_0^{2\pi}$$

$C_2 = 0 + \frac{C_1}{2\pi}(2\pi + 0) + 0\quad$ or $\quad C_1 = C_2 \quad$ ……(12)

From (11) and (12), we see that $C_1 = C_2 = C'$(say). Here C' is an arbitrary constant. Thus, the system (11) – (12) has infinite number solutions $C_1 = C'$ and $C_2 = C'$. putting these value in (9), the required solution of the given integral equation is

y(x) = 1 + $\frac{C'}{\pi}(sinx + cosx)$ or y(x) = 1 + $C(sinx + cosx)$. Where C = $\frac{C'}{\pi}$ is another arbitrary constant. Since C is an arbitrary constant, we have infinitely many solution of (1) when

f(x) = 1.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Solution the following Homogeneous integral equations:

y(x) = - $\int_0^1 y(t)dt\ \ is$?.

Problem 2. The eigenvalue of the homogeneous integral equation:

y(x) = $\lambda\int_0^{\frac{\pi}{4}} sin^2x\ y(t)dt$ True / false.

Problem 3. Initial value problem is always converted into Fredholm integral equation. True / false.

Problem 4. Solution of homogeneous integral equation y(x) = $\frac{1}{2}\int_0^{\pi}$ sinx $y(t)dt$ is ?

Problem 5. Eigen function of y(x) = $cosx + \lambda\int_0^{\pi} sinx\ y(t)dt$ is?

Problem 6. The eigenvalue λ of the Fredholm integral equation y(x) = λ $\int_0^1 x^2ty(t)dt\ \ is$

7.6 *SUMMARY*

1. Eigen Value and eigen function.

Consider the Homogeneous Fredholm integral equation of the second kind:

y(x) = $\lambda\int_a^b k(x,t)y(t)dt$ ……… (1)

then (1) has always the obvious solution y(x) = 0, which is known as zero or trivial solution of (1). The value of the parameter λ for which (1) has a non-zero (or non trivial) solution y(x) ≠ 0 are known as the eigenvalue of (1) or of the kernel k(x, t). Further if $\varphi(x)$ is continuous and $\varphi(x)\ \neq 0$ on the interval (a, b) and

$$\varphi(x)\ = \lambda\int_a^b k(x,t)\varphi(x)\ dt \quad \text{……… (2)}$$

Then $\varphi(x)$ is known as an eigenfunction of (1) corresponding to the eigenvalue λ_0.

2. Solution of Homogeneous Fredholm integral equation of second kind with separable kernel.

3. Solution of non-Homogeneous Fredholm integral equation of second kind with separable kernel.

7.7 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

Solution of System of linear equations

7.8 *REFERENCES*

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7.9 *SUGGESTED READING*

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7.10 TERMINAL AND MODEL QUESTIONS

Q 1. Solve the following integral equations:

$$y(x) = \tan x + \int_{-1}^{1} e^{\sin^{-1} x} y(t)dt.$$

Q 2. Solve the following integral equations:

$$y(x) = \sin x + \lambda \int_0^{\frac{\pi}{2}} \sin x \cos t \, y(t)dt.$$

Q 3. Solve the following integral equations:

$$y(x) - \lambda \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan t \, y(t)dt = \cot x.$$

Q 4. Determine the characteristic values of λ and the characteristic functions of the integral equation $y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+t) y(t)dt.$

Q 5. Solve the following Homogeneous integral equations:

(i) $y(x) = -\int_0^1 y(t)dt$ (ii) $y(x) = \frac{1}{2}\int_0^{\pi} \sin x \, y(t)dt$

(iii) $y(x) = \frac{1}{50}\int_0^{10} t \, y(t)dt$ (iv) $\frac{1}{e^2-1}\int_0^1 2e^x e^t y(t)dt.$

Q 6. Determining the eigenvalue and eigen functions of the homogeneous integral equations:

(i) y(x) = $\lambda \int_0^{\frac{\pi}{4}} sin^2x \; y(t)dt$ (ii) y(x) = $\lambda \int_0^{2\pi} sinx \; cost \; y(t)dt$

(iii) y(x) = $\lambda \int_0^{2\pi} sinx \; sint \; y(t)dt$ (iv) y(x) = $\lambda \int_{-1}^{1}(5xt^3 + 4x^2t) \; y(t)dt.$

7.11 *ANSWERS*

TQ1 y(x) = tanx

TQ2 y(x) = $\left\{\frac{2}{2-\lambda}\right\} sinx, \lambda \neq 2.$

TQ3 y(x) = cotx + $\frac{\pi\lambda}{2}$.

TQ4 $\lambda_1 = \frac{1}{\pi}$, $y_1 = cosx$; $\lambda_2 = -\frac{1}{\pi}$, $y_2 = sinx$.

TQ5 (i) y(x) = 0 (ii) y(x) = 0 (iii) y(x) = 0 (iv) y(x) = 0

TQ6 (i) $\lambda = \frac{8}{\pi-2}$, y(x) = sin^2x

(ii) Eigen value and eigen function do not exist.

(iii) $\lambda = \frac{8}{\pi}$, y(x) = sinx

(iv) $\lambda = \frac{1}{2}$, y(x) = $\frac{5x}{2} + \frac{10x^2}{3}$.

CHECK YOUR PROGRESS

CYQ 1. y(x) = 0

CYQ 2. $\frac{8}{\pi-2}$

CYQ 3. False

CYQ 4. y(x) = 0 **CYQ 5.** y(x) = cosx

CYQ6. 4

UNIT 8: METHOD OF SUCCESSIVE APPROXIMATIONS

Contents

8.1 *INTRODUCTION*

In this unit, the solving of a class of both linear and nonlinear Volterra integral equations of the first kind is investigated. Here, by converting integral equation of the first kind to a linear equation of the second kind and the ordinary differential equation to integral equation we are going to solve the equation easily. The method of successive approximations (Neumann's series) is applied to solve linear and nonlinear Volterra integral equation of the second kind. Some examples are presented to illustrate methods.

8.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) integral equation

(ii) Fredholm integral equation

(iii) initial value problem

(vi) Boundary value problem

(v) Solution of Fredholm integral equation of second kind with separable kernel using Method of successive approximation.

8.3 *ITERATED KERNELS OR FUNCTIONS*

(i) Consider the Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt \qquad \text{....... (1)}$$

then the iterated kernels $k_n(x,t), n = 1, 2, 3, \ldots$ are defined as follows:

$$k_1(x,t) = k(x,t) \qquad \text{...... (2a)}$$

And $$\left.\begin{aligned} k_n(x,t) &= \int_a^b k(x,z)k_{n-1}(z,t)dz, n = 2,3,\ldots \\ k_n(x,t) &= \int_a^b k_{n-1}(x,z)k(z,t)dz, n = 2,3,\ldots \end{aligned}\right\} \quad \ldots\ldots (2b)$$

(ii) Consider the Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt \quad \ldots\ldots. (3)$$

then the iterated kernels $k_n(x,t), n = 1,2,3, \ldots$ are defined as follows:

$$k_1(x,t) = k(x,t) \quad \ldots\ldots (3a)$$

And $$\left.\begin{aligned} k_n(x,t) &= \int_t^x k(x,z)k_{n-1}(z,t)dz, n = 2,3,\ldots \\ k_n(x,t) &= \int_t^x k_{n-1}(x,z)k(z,t)dz, n = 2,3,\ldots \end{aligned}\right\} \quad \ldots\ldots (3b)$$

8.4 *RESOLVENT KERNELS OR RECIPROCAL KERNEL*

(i) Suppose solution of Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt \quad \ldots\ldots. (1)$$

take the form $$y(x) = f(x) + \lambda \int_a^b R(x,t;\ \lambda)f(t)dt \quad \ldots\ldots. (2a)$$

or $$y(x) = f(x) + \lambda \int_a^b \Gamma(x,t;\ \lambda)f(t)dt \quad \ldots\ldots. (2b)$$

then $R(x,t;\ \lambda)$ or $\Gamma(x,t;\ \lambda)$ is known as the resolvent kernel of (1).

(ii) Suppose solution of Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt \quad \ldots\ldots. (3)$$

take the form $$y(x) = f(x) + \lambda \int_a^x R(x,t;\ \lambda)f(t)dt \quad \ldots\ldots. (4a)$$

or $$y(x) = f(x) + \lambda \int_a^x \Gamma(x,t;\ \lambda)f(t)dt \quad \ldots\ldots. (4b)$$

then $R(x, t;\ \lambda)$ or $\Gamma(x, t;\ \lambda)$ is known as the resolvent kernel of (3).

Note:■ Resolvent kernel is also written as $R(x, t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x, t)$.

8.5 *SOME IMPORTANT THEOREMS*

Theorem 1. Let $R(x, t;\ \lambda)$ be a resolvent kernel of a Fredholm integral equation. y(x) = f(x) + $\lambda \int_a^b k(x, t) y(t) dt$, then the resolvent kernel satisfies the integral equation

$R(x, t;\ \lambda) = k(x, t) + \lambda \int_a^b k(x, z) R(z, t;\ \lambda) dz$.

Proof: We know that $R(x, t;\ \lambda)$ is given by

$$R(x, t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x, t) \quad \ldots\ldots (1)$$

Where iterated kernel are given by

$$k_1(x, t) = k(x, t) \quad \ldots\ldots (2a)$$

And $$k_m(x, t) = \int_a^b k(x, z) k_{m-1}(z, t) dz \quad \ldots\ldots (2b)$$

Now, from (1), we have

$$R(x, t;\ \lambda) = k_1(x, t) + \sum_{m=2}^{\infty} \lambda^{m-1} k_m(x, t)$$

$$= k(x, t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^b k(x, z) k_{m-1}(z, t) dz \text{ using (2a), (2b)}$$

$$= k(x, t) + \sum_{n=1}^{\infty} \lambda^{n} \int_a^b k(x, z) k_{n}(z, t) dz \quad (\text{ setting m - 1=n})$$

$$= k(x, t) + \sum_{m=1}^{\infty} \lambda^{m} \int_a^b k(x, z) k_{m}(z, t) dz$$

$$= k(x, t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b k(x, z) k_{m}(z, t) dz$$

$= \text{k(x, t)} + \lambda \int_a^b [\sum_{m=1}^{\infty} \lambda^{m-1} k_m(z,t)] k(x,z) dz$

(on changing the order of summation and integration)

$= \text{k(x, t)} + \lambda \int_a^b R(z,t;\, \lambda) k(x,z) dz$, using (1)

Therefore $R(x,t;\, \lambda) = \text{k(x,t)} + \lambda \int_a^b k(x,z) R(z,t;\, \lambda) dz$.

Note: ■ The series for the resolvent kernel $R(x,t;\, \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t)$ …. (1)

is absolutely and uniformly convergent for all values of x and t in the circle $|\lambda| < B^{-1}$.

8.6 SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BU SUCESSIVE APPROXIMATIONS

ILLUSTRATIVE EXAMPLES

Type 1. Determine the iterated kernels (or functions) for

$$\mathbf{y(x) = f(x)} + \boldsymbol{\lambda \int_a^b k(x,t) y(t) dt.}$$

Example1. Find the iterated kernel for the following kernel

$$\text{K(x, t) = sin(x – 2t)}, 0 \le x \le 2\pi, 0 \le t \le 2\pi.$$

Sol. Iterated kernel $k_n(x,t)$ are given by

$$k_1(x,t) = k(x,t) \quad ….. (1)$$

And $k_n(x,t) = \int_0^{2\pi} k(x,z) k_{n-1}(z,t) dz$, (n = 2, 3, …) ….. (2)

From (1), $k_1(x,t) = k(x,t) = \sin(x - 2t)$ ….. (3)

Putting n = 2 in (2), we have

$k_2(x,t) = \int_0^{2\pi} k(x,z)k_1(z,t)dz$

$= \int_0^{2\pi} \sin(x-2z)\sin(z-2t)dz$, using (3)

$= \frac{1}{2}\int_0^{2\pi} [\cos(x+2t-3z) - \cos(x-2t-z)]\, dz$

$= \frac{1}{2}\left[-\frac{1}{3}sin(x+2t-3z) + \sin(x-2t-z)\right]_0^{2\pi}$

= 0, on simplification

∴ $k_2(x,t) = 0$ …… (4)

Putting n = 3 in (2), we have

$k_3(x,t) = \int_0^{2\pi} k(x,z)k_2(z,t)dz = 0$ using (4)

Hence $k_1(x,t) = \sin(x-2t)$ and $k_n(x,t) = 0$ for n = 2, 3, 4, …

Example2. Find the iterated kernel for the following kernel

$$k(x, t) = e^x cost\ , 0 \le x \le 2\pi;\ a = 0,\ b = \pi.$$

Sol. Iterated kernel $k_n(x,t)$ are given by

$$k_1(x,t) = k(x,t) \quad ….. (1)$$

And $k_n(x,t) = \int_0^{2\pi} k(x,z)k_{n-1}(z,t)dz$, (n = 2, 3, …) ….. (2)

From (1), $k_1(x,t) = k(x,t) = e^x cost$ ….. (3)

Putting n = 2 in (2), we have

$k_2(x,t) = \int_0^{\pi} k(x,z)k_1(z,t)dz = \int_0^{\pi} e^x cosz\ e^z cost\ dz$, using (3)

$= e^x cost \int_0^{\pi} e^z cosz\ dz$

$= e^x cost \left[\frac{e^z}{1^2+1^2}(cosz + sinz)\right]_0^{\pi}$

$[\because \int e^{ax} cosbx\, dx = \frac{e^{ax}}{a^2+b^2}(a\, cosbx + b\, sinbx)]$

$= e^{x} cost$ { -(1/2) $e^{\pi} - (1/2)$} ...... (4)

$\therefore k_2(x,t) = (-1)^1 \frac{1+e^{\pi}}{2} e^{x} cost.$

Next putting n = 3 in (2), we have

$k_3(x,t) = \int_0^{\pi} k(x,z) k_2(z,t) dz = \int_0^{\pi} e^{x} cosz \left\{(-1)^1 \frac{1+e^{\pi}}{2} e^{z} cost\right\} dz,$
using (3) and (4)

$= -\frac{1+e^{\pi}}{2} e^{x} cost \int_0^{\pi} e^{z} cosz\, dz = -\frac{1+e^{\pi}}{2} e^{x} cost \left(-\frac{1+e^{\pi}}{2}\right)$, as before

$\therefore \quad k_3(x,t) = (-1)^2 \left(\frac{1+e^{\pi}}{2}\right)^2 e^{x} cost$ …… (5)

And so on noting (3) (4) and (5), we see that the iterated kernels are given by

$k_n(x,t) = (-1)^{n-1} \left(\frac{1+e^{\pi}}{2}\right)^{n-1} e^{x} cost$, n = 1, 2, 3, ….

Type 2. Determine the Resolvent kernels or Reciprocal kernel $\boldsymbol{R(x,t;\ \lambda)}$

If $\boldsymbol{k_n(x,t)}$ be iterated kernels then $\boldsymbol{R(x,t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t)}$.

Example 3. Find the Resolvent kernels for the Fredholm integral equation having kernel

k(x, t) = e^{x+t}; a = 0, b = 1.

Sol. Iterated kernel $k_m(x,t)$ are given by

$k_1(x,t) = k(x,t)$ ….. (1)

And $k_m(x,t) = \int_0^1 k(x,z) k_{m-1}(z,t) dz$, (n = 2, 3, …) ….. (2)

From (1), $k_1(x,t) = k(x,t) = e^{x+t}$ ….. (3)

Putting n = 2 in (2), we have

$k_2(x,t) = \int_0^1 k(x,z)k_1(z,t)dz = \int_0^1 e^{x+z}e^{z+t}\, dz$, using (3)

$= e^{x+t}\int_0^1 e^{2z}\, dz = e^{x+t}\left[\frac{e^{2z}}{2}\right]_0^1 = e^{x+t}\left(\frac{e^2-1}{2}\right)$. ….. (4)

Putting n = 3 in (2), we have

$k_3(x,t) = \int_0^1 k(x,z)k_2(z,t)dz = \int_0^1 e^{x+z}e^{z+t}\left(\frac{e^2-1}{2}\right) dz$

$= e^{x+t}\left(\frac{e^2-1}{2}\right)\int_0^1 e^{2z}\, dz = e^{x+t}\left(\frac{e^2-1}{2}\right)^2$. As before …… (5)

And so on, observing (3), (4) and (5), we may write

$k_m(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}$, m = 1, 2, 3, … …… (6)

Now the required resolvent kernel is given by

$R(x,t;\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}k_m(x,t)$

$= \sum_{m=1}^{\infty}\lambda^{m-1}e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1} = e^{x+t}\sum_{m=1}^{\infty}\left(\frac{\lambda(e^2-1)}{2}\right)^{m-1}$ ….. (7)

But $\sum_{m=1}^{\infty}\left(\frac{\lambda(e^2-1)}{2}\right)^{m-1} = 1 + \frac{\lambda(e^2-1)}{2} + \left(\frac{\lambda(e^2-1)}{2}\right)^2 + \ldots\ldots$

Which is an infinite geometric series with common ratio $\frac{\lambda(e^2-1)}{2}$.

Therefore $\sum_{m=1}^{\infty}\left(\frac{\lambda(e^2-1)}{2}\right)^{m-1} = \frac{1}{1-\frac{\lambda(e^2-1)}{2}} = \frac{1}{2-\lambda(e^2-1)}$,

Provided $\left|\frac{\lambda(e^2-1)}{2}\right| < 1$ or $|\lambda| < \frac{2}{e^2-1}$ …… (9)

Using (8) and (9), (7) reduces to

$R(x,t;\lambda) = \frac{2e^{x+t}}{2-\lambda(e^2-1)}$, provided $|\lambda| < \frac{2}{e^2-1}$.

Type 3. Solution of Fredholm integral equation with the help of the resolvent kernel.

Working rule: let $y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt$ **….. (1)**

be given Fredholm integral equation. Let $k_m(x,t)$ be the mth iterated kernel and let $R(x,t;\ \lambda)$ be the resolvent kernel of (1). Then we have

$$R(x,t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) \qquad \text{……. (2)}$$

Suppose the sum of infinite series (2) exist and so $R(x,t;\ \lambda)$ can be obtained in the closed form. Then the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_a^b R(x,t;\ \lambda) f(t)dt. \qquad \text{……. (3)}$$

Example 4. Find the Resolvent kernels for the Fredholm integral equation

$$y(x) = x + \int_0^{1/2} y(t)dt \quad \text{and also find the solution.}$$

Sol. Given $y(x) = x + \int_0^{1/2} y(t)dt$ …… (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^{1/2} k(x,t)y(t)dt$

We have $f(x) = x$, $\lambda = 1$, $k(x, t) = 1$ ……. (2)

Let $k_m(x,t)$ be the mth iterated kernel. Then we have

$k_1(x,t) = k(x,t)$ ….. (3)

And $k_m(x,t) = \int_0^{1/2} k(x,z)k_{m-1}(z,t)dz,$ …… (4)

From (3), $k_1(x,t) = k(x,t) = 1$ …… (5)

Putting m = 2 in (4), we have

$k_2(x,t) = \int_0^{1/2} k(x,z)k_1(z,t)dz = \int_0^{1/2} dz = [z]_0^{1/2} = \frac{1}{2}$ …… (6)

Putting m = 3 in (4), we have

$k_3(x,t) = \int_0^{1/2} k(x,z)k_2(z,t)dz$

$= \int_0^{1/2} \frac{1}{2}\, dz$, by (5) and (6)

$= \left(\frac{1}{2}\right)^2$ .... (7)

And so on. Observing (5) , (6) and (7), we find

$k_m(x,t) = \left(\frac{1}{2}\right)^{m-1}$ .... (8)

Now, the resolvent kernel $\mathrm{R}(x, t;\ \lambda)$ is given by

$\mathrm{R}(x, t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x, t)$

$= \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}$, using (2) and (8) ...... (9)

But $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\ldots$

Which is an infinite series with common ratio ½.

$\therefore \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} = \frac{1}{1-\frac{1}{2}} = 2.$

Substituting the above value in (9), we have $\mathbf{R(x, t;\ \lambda) = 2.}$

Finally, the required solution of (1) is given by

$y(x) = f(x) + \lambda \int_0^{1/2} \mathrm{R}(x, t;\ \lambda)\, f(t) dt$ or $y(x) = x + \lambda \int_0^{1/2} 2t\, dt$

therefore $y(x) = x + 2\left[\frac{t^2}{2}\right]_0^{1/2} = x + (1/4)$

hence the required solution is $y(x) = x + \frac{1}{2}$.

Example 5. Solve the following Fredholm integral equations by the method of successive approximations

$$y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\; y(t) dt.$$

Sol. Given $y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\; y(t) dt$ ...... (1)

Comparing (1) with y(x) = f(x) + λ $\int_0^1 k(x,t)y(t)dt$

We have f(x) = $\frac{5x}{6}$, $\lambda = \frac{1}{2}$, k(x, t) = xt ……. (2)

Let $k_m(x,t)$ be the mth iterated kernel. Then we have

$k_1(x,t) = k(x,t)$ ….. (3)

And $k_m(x,t) = \int_0^1 k(x,z)k_{m-1}(z,t)dz,$ …… (4)

From (3), $k_1(x,t) = k(x,t) = xt$ …… (5)

Putting m = 2 in (4), we have

$k_2(x,t) = \int_0^1 k(x,z)k_1(z,t)dz = \int_0^1 (xz)(zt)dz$

$= xt\int_0^1 z^2dz = \frac{1}{3}(xt)$ …… (6)

Putting m = 3 in (4), we have

$k_3(x,t) = \int_0^1 k(x,z)k_2(z,t)dz = \int_0^1 (xz)(\frac{1}{3}zt)dz$, by (5) and (6)

$= \frac{1}{3}xt\int_0^1 z^2dz = \left(\frac{1}{3}\right)^2 xt$

And so on. Observing (5), (6) and (7), we find

$k_m(x,t) = \left(\frac{1}{3}\right)^{m-1} xt$

Now the resolvent kernel is given by

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}\left(\frac{1}{3}\right)^{m-1} xt$, using (2) and (8)

$= xt\sum_{m=1}^{\infty}\left(\frac{1}{6}\right)^{m-1} = xt\left[1 + \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \cdots\right]$

$\mathbf{R(x,t;\ \lambda) = \frac{6}{5}\,xt}$ ……. (9)

Finally, the required solution of (1) is given by y(x) = f(x) + $\lambda \int_0^1$ R(x, t; λ) $f(t)dt$ or y(x) = $\frac{5x}{6} + \frac{1}{2}\int_0^1 \frac{6xt}{5}.\frac{5t}{6}\ dt$

therefore y(x) = $\frac{5x}{6} + \frac{1}{2}\left[\frac{t^3}{3}\right]_0^1 = \frac{5x}{6} + \frac{x}{6}$ = x

hence the required solution is y(x) = x.

8.7 SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BU SUCESSIVE APPROXIMATIONS

■ **An important theorem**

Theorem 1. Let $R(x, t;\ \lambda)$ be a resolvent kernel of a Volterra integral equation.

y(x) = f(x) + $\lambda \int_a^x k(x,t)y(t)dt$, then the resolvent kernel satisfies the integral equation $R(x, t;\ \lambda)$ = k(x, t) + $\lambda \int_t^x$ k(x, z)R$(z, t;\ \lambda)dz$.

Proof: We know that $R(x, t;\ \lambda)$ is given by

$$R(x, t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) \quad \ldots\ldots (1)$$

Where iterated kernel are given by

$$k_1(x,t) = \text{k (x, t)} \quad \ldots\ldots (2a)$$

And $$k_m(x,t) = \int_t^x k(x,z)k_{m-1}(z,t)dz \quad \ldots\ldots (2b)$$

Now, from (1), we have

$$R(x, t;\ \lambda) = k_1(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} k_m(x,t)$$

$$= \text{k(x, t)} + \sum_{m=1}^{\infty} \lambda^{m-1} \int_t^x k(x,z)k_{m-1}(z,t)dz \text{ using (2a), (2b)}$$

$$= \text{k(x, t)} + \sum_{m=1}^{\infty} \lambda^{n} \int_t^x k(x,z)k_{n}(z,t)dz \quad \text{(setting m - 1=n)}$$

$= k(x, t) + \sum_{m=1}^{\infty} \lambda^m \int_t^x k(x,z)k_m(z,t)dz$

$= k(x, t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_t^x k(x,z)k_m(z,t)dz$

$= k(x, t) + \lambda \int_t^x [\sum_{n=1}^{\infty} \lambda^{m-1}\, k_m(z,t)]k(x,z)dz$

(on changing the order of summation and integration)

$= k(x, t) + \lambda \int_t^x R(z,t;\, \lambda)k(x,z)dz,$ using (1)

Therefore $R(x,t;\, \lambda) = k(x,t) + \lambda \int_t^x k(x,z)R(z,t;\, \lambda)dz$.

ILLUSTRATIVE EXAMPLES

Type 1. Determine the resolvent kernels or reciprocal kernel for Volterra integral equation

$$\mathbf{y(x) = f(x) + \boldsymbol{\lambda} \int_a^x \boldsymbol{k(x,t)y(t)dt}.}$$

Example1. Find the resolvent kernel of the Volterra integral equation for the following kernel k(x, t) = 1.

Sol. Iterated kernel $k_n(x,t)$ are given by

$k_1(x,t) = k(x,t)$ ….. (1)

And $k_n(x,t) = \int_t^x k(x,z)k_{n-1}(z,t)dz$, (n = 1, 2, 3, …) ….. (2)

From (1), $k_1(x,t) = k(x,t) = 1$ ….. (3)

Putting n = 2 in (2), we have

$k_2(x,t) = \int_t^x k(x,z)k_1(z,t)dz = \int_t^x dz$, using (3)

$= [z]_t^x$

= x – t, on simplification

$\therefore$ $k_2(x,t) = x - t$ …… (4)

Putting n = 3 in (2), we have

$k_3(x,t) = \int_t^x k(x,z)k_2(z,t)dz = \int_t^x 1.(z-t)dz$ using (3), (4)

$= \left[\frac{(z-t)^2}{2}\right]_t^x = \frac{(x-t)^2}{2!}$ …… (5)

Putting n = 4 in (2), we have

$k_4(x,t) = \int_t^x k(x,z)k_3(z,t)dz = \int_t^x 1.\frac{(z-t)^2}{2!}dz$ using (3),(5)

$= \frac{1}{2}\left[\frac{(z-t)^3}{3}\right]_t^x = \frac{(x-t)^3}{3!}$ …… (6)

And so on. Observing (3), (4), (5) and (6) etc, we find by Mathematical induction, that

$k_n(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}$, n = 1, 2, 3, …

Now by the definition of resolvent kernel we have

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) = k_1(x,t) + \lambda\, k_2(x,t) + \lambda^2 k_3(x,t) +$ …..

$= 1 + \frac{\lambda(x-t)}{1!} + \frac{[\lambda(x-t)]^2}{2!} + \frac{[\lambda(x-t)]^3}{3!} + \ldots\ldots$

$= e^{\lambda(x-t)}$.

Example2. Find the resolvent kernel of the Volterra integral equation for the following kernel k(x, t) =e^{x-t}.

Sol. Iterated kernel $k_n(x,t)$ are given by

$k_1(x,t) = k(x,t)$ ….. (1)

And $k_n(x,t) = \int_t^x k(x,z)k_{n-1}(z,t)dz$, (n = 1, 2, 3, …) ….. (2)

From (1), $k_1(x,t) = k(x,t) = e^{x-t}$ ….. (3)

Putting n = 2 in (2), we have

$k_2(x,t) = \int_t^x k(x,z)k_1(z,t)dz = \int_t^x e^{x-z}e^{z-t}dz$, using (3)

$= e^{x-t}\int_t^x dz = e^{x-t}(x-t)$

$\therefore \quad k_2(x,t) = e^{x-t}(x-t) \quad \ldots\ldots (4)$

Putting n = 3 in (2), we have

$k_3(x,t) = \int_t^x k(x,z)k_2(z,t)dz = \int_t^x e^{x-z}(z-t)e^{z-t}dz$ using (3), (4)

$= e^{x-t}\int_t^x (z-t)dz$

$= e^{x-t}\left[\frac{(z-t)^2}{2}\right]_t^x = e^{x-t}\frac{(x-t)^2}{2!} \quad \ldots\ldots (5)$

Putting n = 4 in (2), we have

$k_4(x,t) = \int_t^x k(x,z)k_3(z,t)dz = \int_t^x e^{x-z}e^{z-t}\frac{(z-t)^2}{2!}dz$ using (3),(5)

$= \frac{e^{x-t}}{2!}\int_t^x (z-t)^2 dz$

$= \frac{e^{x-t}}{2!}\left[\frac{(z-t)^3}{3}\right]_t^x = e^{x-t}\frac{(x-t)^3}{3!} \quad \ldots\ldots (6)$

And so on. Observing (3), (4), (5) and (6) etc, we find by Mathematical induction, that

$k_n(x,t) = e^{x-t}\frac{(x-t)^{n-1}}{(n-1)!}$, n = 1, 2, 3, …

Now by the definition of resolvent kernel we have

$R(x,t;\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}k_m(x,t)$

$= k_1(x,t) + \lambda\, k_2(x,t) + \lambda^2 k_3(x,t) + \ldots..$

$= e^{x-t} + e^{x-t}\frac{\lambda(x-t)}{1!} + e^{x-t}\frac{[\lambda(x-t)]^2}{2!} + e^{x-t}\frac{[\lambda(x-t)]^3}{3!} + \ldots\ldots$

$= e^{\lambda(x-t)}\left[1 + \frac{\lambda(x-t)}{1!} + \frac{[\lambda(x-t)]^2}{2!} + \frac{[\lambda(x-t)]^3}{3!} + \ldots\right]$

$= e^{x-t}e^{\lambda(x-t)} = e^{(x-t)+\lambda(x-t)} = e^{(x-t)(1+\lambda)}$.

Type 2. Solution of Volterra integral equation with the help of the resolvent kernel.

Working rule: let y(x) = f(x) + $\lambda \int_a^x k(x,t)y(t)dt$ **…… (1)**

be given Volterra integral equation. Let $k_m(x,t)$ be the mth iterated kernel and let $R(x,t;\ \lambda)$ be the resolvent kernel of (1). Then we have

$$R(x,t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) \qquad \text{……. (2)}$$

Suppose the sum of infinite series (2) exist and so $R(x,t;\ \lambda)$ can be obtained in the closed form. Then the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_a^x R(x,t;\ \lambda) f(t)dt. \qquad \text{……. (3)}$$

Example 3. Find the Resolvent kernels for the Volterra integral equation

$$y(x) = 1 + \int_0^x y(t)dt \quad \text{and also find the solution.}$$

Sol. Given y(x) = 1 + $\int_0^x y(t)dt$ …… (1)

Comparing (1) with y(x) = f(x) + $\lambda \int_0^x k(x,t)y(t)dt$

We have f(x) = 1, $\lambda = 1$, k(x, t) = 1 ……. (2)

Proceeding as in Example 1, we have

$$R(x,t;\ \lambda) = e^{\lambda(x-t)} = e^{x-t} \quad \text{since } \lambda = 1, \text{ by (2)} \qquad \text{…… (3)}$$

Now the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^x R(x,t;\ \lambda) f(t)dt$$

or $\quad y(x) = 1 + \int_0^x e^{x-t} dt$, using (1)

$$= 1 + e^x \int_0^x e^{-t} dt = 1 + e^x \left[-e^{-t}\right]_0^x$$

$$= 1 + e^x[-e^{-x} + 1] = 1 - 1 + e^x$$

Hence y(x) = e^x.

Example 4. Solve the following integral equation by successive approximation

$$y(x) = f(x) + \lambda \int_0^x e^{x-t} y(t) dt .$$

Sol. Given y(x) = f(x) + $\lambda \int_0^x e^{x-t} y(t) dt$ …… (1)

Comparing (1) with y(x) = f(x) + $\lambda \int_0^x k(x,t) y(t) dt$

We have k(x, t) = e^{x-t} ……. (2)

Proceeding as in Example 2, we have

$$R(x, t;\ \lambda) = e^{(x-t)(1+\lambda)} \quad \text{...... (3)}$$

Now the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^x R(x, t;\ \lambda) f(t) dt$$

$$y(x) = f(x) + \lambda \int_0^x e^{(x-t)(1+\lambda)} f(t) dt, \text{ by (3)}$$

Type 3. Solution of Volterra integral equation of second kind with the help of the method of successive approximations:

$$y(x) = f(x) + \lambda \int_a^x k(x,t) y(t) dt \quad \textbf{...... (1)}$$

Working rule: let f(x) be a continuous in [0, a] and k(x, t) be continuous for $0 \leq x \leq a, 0 \leq t \leq x$.

We start with some function $y_0(x)$ continuous in [0, a]. replacing y(t) on R.H.S. of (1) by $y_0(x)$, we obtain

$$y_1(x) = f(x) + \lambda \int_a^x k(x,t) y_0(x) dt \quad \textbf{...... (2)}$$

$y_1(x)$ given by (2) is itself continuous in [0, a]. proceeding likewise we arrive at a sequence of functions $y_0(x)$, $y_1(x)$, … , $y_n(x)$, …, where

$$y_n(x) = \text{f(x)} + \lambda \int_a^x k(x,t) y_{n-1}(x) dt \quad \ldots\ldots (3)$$

In view of continuity of f(x) and k(x, t), the sequence $\{y_n(x)\}$ converges, as n⟶ ∞ to obtain the solution y(x) of the given integral equation (1).

Example 5. Solve the following integral equation by successive approximations

$$\text{y(x)} = 1 + \int_0^x y(t) dt \text{ , taking } y_0(x) = 0.$$

Sol. Given $\quad$ $y(x) = 1 + \int_0^x y(t) dt \quad \ldots\ldots (1)$

And $\quad$ $y_0(x) = 0 \quad \ldots\ldots (2)$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x k(x,t) y(t) dt$

Here $\quad$ f(x) = 1, $\quad \lambda = 1,$ $\quad$ k(x, t) = 1 $\quad$ …… (3)

The nth order approximation is given by

$$y_n(x) = \text{f(x)} + \lambda \int_0^x k(x,t) y_{n-1}(t) dt$$

Or $\quad$ $y_n(x) = 1 + \int_0^x y_{n-1}(t) dt$, using (3) $\quad$ …… (4)

Putting n = 1 in (4) and using (5), we have

$$y_1(x) = 1 + \int_0^x y_0(t) dt = 1 + \int_0^x 0 dt = 1$$

Next, Putting n = 2 in (4) and using (5), we have

$$y_2(x) = 1 + \int_0^x y_1(t) dt = 1 + \int_0^x dt = 1 = 1 + x \quad \ldots\ldots (6)$$

Next, Putting n = 3 in (4) and using (6), we have

$$y_3(x) = 1 + \int_0^x y_2(t) dt = 1 + \int_0^x (1+t) dt = \left[t + \frac{t^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2!} \quad \ldots\ldots (7)$$

Next, Putting n = 4 in (4) and using (7), we have

$$y_4(x) = 1 + \int_0^x y_3(t) dt = 1 + \int_0^x \left(1 + t + \frac{t^2}{2!}\right) dt$$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ …… (8)

And so on, observing (5), (6), (7), (8) etc, we find

$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}$ ……. (9)

Making $n \longrightarrow \infty$, we find the required solution is given by

$y(x) = \lim_{n\to\infty} y_n(x)$ or

$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.. or $y(x) = e^x$.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The iterated kernels for Fredholm integral equation is $k_n(x,t) = \int_a^b k(x,z)k_{n-1}(z,t)dz$. True/False.

Problem 2. The resolvent kernel of the Fredholm integral equation for the kernel k(x, t) = x^2t^2; a = -1, b =1 is $R(x, t;\ \lambda) = \frac{5x^2t^2}{5-2\lambda}$; $|\lambda| < 2$. True /False.

Problem 3. The resolvent kernel of the Volterra integral equation for the kernel k(x, t) = $2 - (x-t), taking\ \lambda = 1$ is $R(x, t;\ \lambda) = e^{x-t}(x - t + 2)$True /False.

Problem 4. The eigen value of homogeneous integral equation

$$y(x) = 2\int_0^\pi \sin x\, y(t)dt \text{ is}$$

(a) 1 (b) 2 (c) 3 (d) 4

Problem 5. Eigen function of $y(x) = cosx + \lambda \int_0^\pi sinx\ y(t)dt$ is?

Problem 6. Solution of the integral equation $y(x) = \frac{1}{2}x^3 - 2x - \int_0^x y(t)dt, y_0(x) = x^2$.

8.8 *SUMMARY*

1. Iterated kernel or functions

For Volterra integral equation

The iterated kernels $k_n(x,t), n = 1, 2, 3, ...$ are defined as follows:

$$k_1(x,t) = k(x,t)$$

And
$$\left.\begin{aligned} k_n(x,t) &= \int_a^b k(x,z)k_{n-1}(z,t)dz, n = 2, 3, ... \\ k_n(x,t) &= \int_a^b k_{n-1}(x,z)k(z,t)dz, n = 2, 3, ... \end{aligned}\right\}$$

For Volterra integral equation

The iterated kernels $k_n(x,t), n = 1, 2, 3, ...$ are defined as follows:

$$k_1(x,t) = k(x,t)$$

And
$$\left.\begin{aligned} k_n(x,t) &= \int_t^x k(x,z)k_{n-1}(z,t)dz, n = 2, 3, ... \\ k_n(x,t) &= \int_t^x k_{n-1}(x,z)k(z,t)dz, n = 2, 3, ... \end{aligned}\right\}$$

2. Resolvent kernel is written as $R(x,t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t)$.

3. The series for the resolvent kernel

$R(x,t;\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t)$ …. (1)

is absolutely and uniformly convergent for all values of x and t in the circle $|\lambda| < B^{-1}$.

4. Solution of Fredholm integral equation with the help of the resolvent kernel.

5. Solution of Volterra integral equation with the help of the resolvent kernel.

6. Solution of Volterra and Fredholm integral equation with the help of successive approximations.

8.9 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives
Expansions of function
Series

8.10 *REFERENCES*

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8.12 TERMINAL AND MODEL QUESTIONS

Q 1. Find the resolvent kernel for the Volterra integral equations with the following

kernel k(x, t) = 2x, taking $\lambda = 1$.

Q 2. Solve the following Volterra integral equation

y(x) = 1 + $\int_0^x xt\, y(t)dt$ by the method of successive Approximations.

Q 3. Using the method of successive approximations, solve the following integral equation with given value $y_0(x)$ of zero-order approximation: y(x) =1- $\int_0^x (x-t)\, y(t)dt\, , y_0(x) = 0$.

Q 4. Using the method of successive approximations, solve the following integral equation with given value $y_0(x)$ of zero-order approximation: y(x) =2x + 2 - $\int_0^x y(t)dt\, , y_0(x) = 1$.

Q 5. With the help of the resolvent kernel, find the solution of the integral equation

y(x) = 1 + x^2 + $\int_0^x \frac{1+x^2}{1+t^2} y(t)dt.$

8.13 *ANSWERS*

TQ1 $2xe^{x^2-t^2}$

TQ2 $y(x) = 1 + \frac{x^3}{2} + \frac{x^6}{2.5} + \frac{x^9}{2.5.8} + \frac{x^{12}}{2.5.8.11} + \ldots.$

TQ3 $y(x) = \cos x$

TQ4 $y(x) = 2$

TQ5 $y(x) = e^x(1 + x^2)$

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. (b)

CYQ 5. $y(x) = \cos x$

CYQ6. $Y(x) = x^2 - 2x.$

UNIT 9: APPLICATIONS OF INTEGRAL EQUATIONS AND GREEN'S FUNCTION TO ORDINARY DIFFERENTIAL EQUATIONS

Contents

9.1 *INTRODUCTION*

We have already learnt about the conversion of boundary value problem into integral equations. In this unit we shall consider the initial and boundary problems again in the different context. We shall introduce the concept of Green's function and utilize it in converting initial and boundary value problems into integral equations. Sometimes we shall be able to solve the given initial and boundary value problems completely with the help of Green's function.

9.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) integral equation

(ii) Green's function

(iii) initial value problem

(vi) Boundary value problem

(v) Solution of integral equation using Green's Function.

9.3 *GREEN'S FUNCTION*

Consider a linear homogeneous differential equation of order n:

$$L[y] = 0 \qquad \ldots\ldots (1)$$

Where L is the differential operator

$$L \equiv p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x) \qquad \ldots\ldots (2)$$

Where the functions $p_0(x), p_1(x), \ldots, p_n(x)$ are continuous on $[a, b]$, $p_0(x) \neq 0$ on $[a, b]$ and the boundary conditions are

$$V_k(y) = 0, \quad (k = 1, 2, 3, \ldots, n) \qquad \ldots\ldots (3)$$

Where $V_k(y) = \alpha_k y(a) + \ a_k{}^{(1)} y'(a) + \cdots + a_k{}^{(n-1)} y^{n-1}(a)$

$$+\beta_k y(b) + \ \beta_k{}^{(1)} y'(b) + \cdots + \beta_k{}^{(n-1)} y^{n-1}(b) \qquad \ldots\ldots (4)$$

Where the linear form $V_1, \ \ldots \ , V_n$ in y(a), $y'(a)$, $\ldots$, $y^{n-1}(a)$, $y(b), y'(b), \ldots, y^{n-1}(b)$ are linearly independent.

Suppose that the homogeneous boundary – value problem given by (1) to (4) has only a trivial solution y(x) ≡ 0. Then the Green's function of the boundary value problem (1) to (4) is the function G(x, t) constructed for any point t, a < t < b, and which has the following four properties:

(i) In each of the intervals $[a, t)$ and (t, b] the function G(x, t), considered as a function of x, is a solution of (1), that is,

$$L\,[G] = 0 \qquad \ldots\ldots (5)$$

(ii) G(x, t) is continuous and has continuous derivative with respect to x upto order (n - 2) inclusive for $a \leq x \leq b$.

(iii) (n – 1)th derivative of G(x, t) with respect to x at the point x = t has discontinuity of the first kind, * the jump being equal to $-\frac{1}{p_0(t)}$, $that$ is

$$\left(\frac{\partial^{n-1}G}{\partial x^{n-1}}\right)_{x\,=\,t\,+\,0} - \left(\frac{\partial^{n-1}G}{\partial x^{n-1}}\right)_{x\,=\,t-\,0} = -\frac{1}{p_0(t)} \qquad \ldots\ldots (6)$$

(iv) G(x, t) satisfies the boundary conditions (3), that is,

$$V_k(G) = 0. (k = 1, 2, \ldots, n) \qquad \ldots\ldots. (7)$$

Note:■ if the boundary value problem given by (1) and (4), has only trivial solution y(x) = 0, the operator L has a unique Green's function G(x, t).

■ if the boundary value problem (1) to (4) is self-adjoint, then Green's function is symmetric, that is, G(x, t) = G(t, x). the converse is also true.

■ if at one of the extremities of an interval [a, b] the coefficient of the highest derivative vanishes, for example $p_0(a)$ = 0, then the natural boundary condition for boundedness of the solution at x = a is imposed, at the other extremity the ordinary boundary condition is specified.

9.4 *CONVERSION OF A BOUNDARY VALUE PROBLEM INTO FREDHOLM INTEGRAL EQUATION*

We shall use the following notations:

$$L \equiv p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$$

And

$$V_k(y) \equiv \alpha_k y(a) + a_k{}^{(1)}y'(a) + \cdots + a_k{}^{(n-1)}y^{n-1}(a)$$

$$+\beta_k y(b) + \beta_k{}^{(1)}y'(b) + \cdots + \beta_k{}^{(n-1)}y^{n-1}(b)$$

Suppose G(x, t) is the Green's function of the boundary value problem

L[y] = 0 …… (1)

$V_k(y)$ = 0, k = 1, 2, 3, …. , n ….. (2)

Involving homogeneous boundary conditions (2) at the end points x = a and x = b of an interval $a \le x \le b$.

Result 1. Consider the boundary value problem

L[y] + φ(x) = 0 ……. (3)

$V_k(y)$ = 0, k = 1, 2, 3, …. , n ….. (4)

Involving the same homogeneous boundary conditions (2). Here φ(x) is a direct function of x.

*then solution of the boundary problem (3)-(4) is given by the formula

y(x) = $\int_a^b G(x,t)\varphi(t)dt$ …… (5)

Result 2. Consider the boundary value problem

$$L[y] + \varphi(x) = 0 \qquad \ldots\ldots (6)$$

$$V_k(y) = 0, \quad k = 1, 2, 3, \ldots, n \qquad \ldots\ldots (7)$$

Involving the same homogeneous boundary conditions (2).

In this result we assume that φ(x) is not a given direct function of x. however, φ(x) may also depend upon x indirectly by also involving the unknown function y(x), and so being expressible in the form $\varphi(x) = \varphi(x, y(x))$ ……. (8)

Then the boundary – value problem (6) – (7) can be reduced to the following integral equation

$$y(x) = \int_a^b G(x,t)\varphi(t, y(t))dt \qquad \ldots\ldots (9)$$

Particular case of result (2):

Let $\varphi(x) = \lambda\, r(x)y(x) - f(x)$, where λ is a parameter. Then, we see that the boundary value problem $L[y] + \lambda\, r(x)y(x) = f(x)$ …….. (10)

$$V_k(y) = 0, \quad k = 1, 2, 3, \ldots, n \qquad \ldots\ldots (11)$$

Reduces to the following integral equation

$$y(x) = \lambda \int_a^b G(x,t)r(t)y(t)dt - \int_a^b G(x,t)f(t)dt \qquad \ldots\ldots (12)$$

where G(x, t) is the relevant Green's function. In (12), G(x, t) r(t) is not symmetric unless the function r(t) is a constant. However, if we write

$$\{r(x)\}^{\frac{1}{2}}y(x) = Y(x)$$

Under the assumption that r(x) is non-negative over (a, b), as is usually the case in practice, the equation (12) can be written in the form

$$Y(x) = \lambda \int_a^b k^*(x,t)Y(t)dt - \int_a^b k^*(x,t)\frac{f(t)}{\{r(x)\}^{\frac{1}{2}}}dt,$$

Where $k^*(x,t) = G(x,t)\{r(x)r(t)\}^{\frac{1}{2}}$ is a symmetric kernel.

Result 3. When the prescribed end conditions are not homogeneous, we shall use a modified method as explained below:

In this case, Let G(x, t) denote the Green's function corresponding to the associated homogeneous end conditions. We now search for a function P(x) such the relation

$$y(x) = P(x) + \int_a^b G(x,t)\varphi(t)dt \qquad \ldots\ldots (13)$$

is equivalent to the differential equation $L(y) + \varphi(t) = 0$ …… (14)

together with the prescribed nonhomogeneous end conditions.

since $$L\left[\int_a^b G(x,t)\varphi(t)dt\right] = -\varphi(x) \qquad \ldots\ldots (15)$$

the requirement that (13) imply (14) leads us to

$$L[P(x)] = 0 \qquad \ldots\ldots (16)$$

Furthermore, since the second term in (13) satisfies the associated homogeneous end conditions, we conclude that function P(x) in (13) must be the solution of (16) which satisfies the prescribed nonhomogeneous end conditions. When G(x, t) exists, then P(x) always exists.

ILLUSTRATIVE EXAMPLES

Based on construction of Green's Function

Example 1. Find the Green's function of the boundary value problem $y'' = 0,\ y(0) = y(l) = 0$.

Sol. Given boundary value problem $y'' = 0$ ……. (1)

With the boundary conditions: y(l) = 0 ……. (2a)

And y(l) = 0 ……. (2b)

The general solution of (1) is y(x) = Ax + B …… (3)

Putting x = 0 in (3) and using (2a), we get B = 0 …… (4)

Next, putting x = l in (3) and using (2b), we get $0 = A + B\, l$ …… (5)

Solving (4) and (5), we get A = B = 0. Hence (3) yields only the trivial solution y(x) = 0 for the given boundary value problem. Therefore, the Green's function exists and is given by

$$G(x, t) = \begin{cases} a_1x + a_2, & 0 \le x < t \\ a_1x + a_2, & t < x \le l \end{cases} \quad \text{……. (6)}$$

In addition to the above property (6). The proposed Green's function must satisfy the following three properties:

(i) G(x, t) is continuous at x = t, that is

$$b_1t + b_2 = a_1t + a_2 \quad \text{or} \quad (b_1 - a_1) + (b_2 - a_2) = 0 \text{ ……. (7)}$$

(ii) the derivative of G has a discontinuity of magnitude $-\frac{1}{p_0(t)}$ at the point x = t, where $p_0(x)$ coefficient of the highest order derivative in (1) = 1.

Therefore $\left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1$ or $b_1 - a_1 = -1$ …… (8)

(iii) G(x, t) must satisfy the boundary conditions (2a) and (2b), that is

G(0, t) = 0 so that $a_2 = 0$ …… (9)

And G(l, t) = 0 so that $b_1l + b_2 = 0$ …… (10)

Using (8), (7) becomes $-t + b_2 - a_2 = 0$ ……. (11)

Solving (8), (9), (10) and 11, we have

$a_2 = 0$ $b_2 = t$, $b_1 = -t/l$, $a_1 = 1 - t/l$

Therefore $a_1x + a_2 = \left(1 - \frac{t}{l}\right)x = \frac{x}{l}(l - t)$ and

$$b_1x + b_2 = -\frac{t}{l}x + t = \frac{t}{l}(l - x).$$

Substituting the above value in (6), the required Green's function of the given boundary problem is given by

$$G(x, t) = \begin{cases} \left(\frac{x}{l}\right)(l - t), \ 0 \le x < t \\ \left(\frac{t}{l}\right)(l - t), \ t < x \le l \end{cases} \quad \text{........ (12)}$$

Example 2. Construct the Green's function for the differential equation

$xy'' + y' = 0$ *for the following* conditions. y(x) is bounded as

$x \to 0, y(1) = \alpha y'(1), \alpha \neq 0.$

Sol. Given boundary value problem is : $xy'' + y' = 0$ or $x^2y'' + xy' = 0$

Or $(x^2D^2 + xD)y = 0$, $D \equiv d/dx$ ...... (1)

With the boundary conditions: y(x) is bounded as $x \to 0$ .... (2a)

$y(1) = \alpha y'(1), \alpha \neq 0$ .... (2b)

To solve the linear homogeneous differential equation (1), we proceed by the usual method

Put $x = e^z$ so that $\log x = z$ .... (3)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$, where $D_1 = d/dz$....... (4)

Using (4),(1) reduces to $[D_1(D_1 - 1) + D_1]\, y = 0$ or ${D_1}^2 y = 0$....... (5)

The auxiliary equation of (5) is ${D_1}^2 = 0$ so that $D_1 = 0$, hence the solution is

$Y = Az + B$ or $y(x) = A\log x + B$, by (3) .... (6)

Now from (6), $y'(x) = A/x$ ...... (7)

From (6) and (7), $y(1) = B$ and $y'(1) = A$

Putting these value in (2b) we get $B = \alpha A$

In view of B.C. (2a), we must take A = 0 in (6). Then A = 0 and $B = \alpha A$ then B = 0.

Thus A = B = 0. Hence yield the trivial solution y(x) = 0. Therefore the Green's function exist and given by

$$G(x, t) = \begin{cases} a_1 logx + a_2, \; 0 \le x < t \\ b_1 logx + b_2, \; t \le x < 1 \end{cases} \qquad \text{........ (8)}$$

In addition to above property (8), the proposed Green's function must also satisfy the following three properties:

(i) G(x, t) is continuous at x = t, that is,

$b_1 logt + b_2 = a_1 logt + a_2$ or

$(b_1 - a_1)$ log t + $(b_2 - a_2)$ = 0 .... (9)

(ii) The derivative of G has a discontinuity of magnitude – $1/p_0(t)$ at the point x = t, where $p_0(x)$ = coefficient of the highest power of x in the given differential equation = x. thus we have

$\left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1/t$ or $b_1/t - a_1/t = -1/t$

Or $(b_1 - a_1)$ = -1 .... (10)

(iii) G(x, t) must satisfy the boundary conditions (2a) and (2b), that is

For (2a) , G(x, t) must be bounded as x → 0, i.e. $a_1 logx + a_2$ must be bounded at x → 0, which is possible only if we take

a_1 = 0 ...... (11)

For (2b) we must have G(1, t) = $\alpha G'(1, t)$

i.e. $b_1 log1 + b_2 = \alpha \left(\frac{b_1}{x}\right)_{x=1}$ or $b_2 = \alpha b_1$ ...... (12)

solving (9), (10), (11) and (12), we get

$a_2 = 0$, $b_1 = -1$, $b_2 = -\alpha$ and $a_2 = -\alpha - logt$

Substituting the above values in (8), the required Green's function is

$$G(x, t) = \begin{cases} -\alpha - logt \,, \; 0 \le x < t \\ -\alpha - logx, \; t < x \le 1 \end{cases} .$$

■ **Solved example based on result 1:**

Example 3. Use the Green's function solve the boundary value problem

$y'' + y = x$, y(0) = ($\pi/2$) = 0.

Sol. Given boundary value problem $y'' + y = x$ …… (1)

With boundary conditions: y(0) = y($\pi/2$) = 0 ……. (2)

Consider the associated boundary value problem

$y'' + y$ = 0 or $(D^2 + 1)y = 0, D = d/dx$ ……. (3)

Subject to boundary conditions y(0) = 0 ……. (4a)

And y($\pi/2$) = 0 …… (4b)

We first find the Green's of the above mentioned boundary value problem given by (3), (4a) and (4b).

The auxiliary equation of (3) is $D^2 + 1$ = 0 so that D = $\pm i$

Hence the general solution of (3) is y(x) = A cosx + B sinx …… (5)

Putting x = 0 in (5) and using B.C. (4a), we get A = 0 …… (6)

Putting x = 0 in (5) and using B.C. (4b), we get B = 0 …… (7)

From (6) and (7), A = B = 0. Hence (5) yield only trivial solution y(x) = 0. Therefore, Green's function exists for the boundary value problem given by (3), (4a) and (4b) and it is given by

$$G(x, t) = \begin{cases} a_1 cosx + a_2 sinx, & 0 \le x < t \\ b_1 cosx + b_2 sinx, & t < x \le \pi/2 \end{cases} \quad \ldots\ldots (8)$$

In addition to the above property (8), the proposed Green's functions must also satisfy the following three properties:

(i) G(x, t) is continuous at x = t, that is

$$b_1 cost + b_2 sint = a_1 cost + a_2 sint$$

or $(b_1 - a_1)cost + (b_2 - a_2)sint = 0$ …… (9)

(ii) the derivative of G has a discontinuity of magnitude – $1/p_0(t)$ at the point x = t, where $p_0(x)$ = coefficient of the highest order derivative in (3) = 1. thus we have

$$\left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1 \quad \text{or}$$

$-b_1 sint + b_2 cost - (-a_1 sint + a_2 cost)$ = -1 or

- $(b_1 - a_1)sint + (b_2 - a_2)cost$ = -1 …… (10)

(iii) G(x, t) must satisfy the boundary condition (4a) and (4b), that is

G(0, t) = 0 so that $a_1 = 0$ …… (11)

And G($\pi/2$, t) = 0 so that $b_2 = 0$ …… (12)

Let $b_1 - a_1 = C_1$ and $b_2 - a_2 = C_2$ ……. (13)

The (9) and (10) may be written as

C_1 cos t + C_2 $sint$ + 0 = 0 …… (14)

$-C_1$ sin t + $C_2 cost$ + 1 = 0 ……. (15)

Solving (14) and (15) by cross-multiplication method, we have

$\frac{C_1}{sint} = \frac{C_2}{-cost} = \frac{1}{cos^2t + sin^2t}$ hence $C_1 = sint$ $and\ C_2 = -cost$

Therefore $b_1 - a_1 = sint, by$ (13) ……. (16)

$b_2 - a_2 = -cost, by$ (13) ……. (17)

Solving (11), (12), (16) and (17) we have

$a_1 = 0$, b_2 = 0, $b_1 = sint,$ a_2 = cost

Substituting these values in (6) we have

$$G(x, t) = \begin{cases} cost\ sinx, & 0 \le x < t \\ sint\ cosx, & t < x \le \pi/2 \end{cases} \quad …… (18)$$

Then we known that the solution of the given boundary value problem (1)-(2) is given by

$$y(x) = \int_0^{\pi/2} G(x,t)\emptyset(t)dt \qquad \ldots\ldots (19)$$

where $\emptyset(x) = -x$ so that $\emptyset(t) = -t$, hence the required solution is given by

$$y(x) = -\int_0^{\pi/2} G(x,t)tdt = -\left[\int_0^x tG(x,t)dt + \int_x^{\pi/2} tG(x,t)dt\right]$$

$$= -\int_0^x t\,sint\,cosx\,dt - \int_x^{\pi/2} t\,cost\,sinx\,dt, \text{ using (18)}$$

$$= -\cos x\int_0^x t\,sint\;dt - sinx\int_x^{\pi/2} t\,cost\;dt$$

$$= -\cos x\left[[-t\cos t]_0^x - \int_0^x(-\cos t)dt\right] - \sin x\left[[t\sin t]_x^{\pi/2} - \int_x^{\pi/2}\sin t\,dt\right]$$

$$= -\cos x[-x\cos x + \sin x] - \sin x[\tfrac{\pi}{2} - x\sin x - \cos x]$$

Thus $y(x) = x - \frac{\pi}{2}\,sinx$.

■ **Solved example based on result 2:**

Example 4. Reduce the boundary value problem $y'' + \lambda y = x$,

$y(0) = y(\pi/2) = 0$ to the integral equation.

Sol. Given boundary-value problem is

$$y'' + \lambda y = x,\ y(0) = y(\pi/2) = 0 \quad \ldots\ldots (1)$$

We shall first find the Green's function of the following associated boundary value problem

$$y'' = 0 \quad \text{or} \quad D^2y = 0,\ D = d/dx \qquad \ldots\ldots (2)$$

With boundary conditions $\quad y(0) = 0 \quad \ldots\ldots (3)$

Or $\quad y(\pi/2) = 0 \quad \ldots\ldots. (4)$

So the general solution of (2) is $y(x) = Ax + B \quad \ldots\ldots. (5)$

Putting x = 0 is (5) and using B.C. (3), we get B = 0 ……. (6)

Next putting x = $\pi/2$ in (5) and using B.C. (4), we get

0 = A($\pi/2$) + B ……. (7)

From (6) and (7), A = B = 0. Hence (5) yields only trivial solution y(x) = 0. Therefore Green's function G(x, t) exists for the associated boundary value problem given by (2), (3) and (4) and given by

$$G(x, t) = \begin{cases} a_1x + a_2, & 0 \le x < t \\ b_1x + b_2, & t < x \le \pi/2 \end{cases} \quad \text{……} (8)$$

In addition to the above property (8), the proposed Green's functions must also satisfy the following three properties:

(i) G(x, t) is continuous at x = t, that is

$$b_1t + b_2 = a_1t + a_2$$

or $(b_1 - a_1)t + b_2 - a_2 = 0$ …… (9)

(ii) the derivative of G has a discontinuity of magnitude – $1/p_0(t)$ at the point x = t, where $p_0(x)$ = coefficient of the highest order derivative in (2) = 1. thus we have

$$\left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1 \quad \text{or}$$

$b_1 - a_1$= -1 …… (10)

G(x, t) must satisfy the boundary condition (2) and (4), that is

G(0, t) = 0 so that $a_2 = 0$ …… (11)

And G($\pi/2$, t) = 0 so that $\frac{b_2\pi}{2} + b_2 = 0$ …… (12)

Using (10) , (9) gives

$-t + b_2 - a_2$ + = 0 …… (13)

Solving (10), (11), (12) and (13), we have

$a_2 = 0$, b_2 = t, $b_1 = -(\frac{2t}{\pi})$, a_1 = 1 - $\frac{2t}{\pi}$

Therefore $a_1 x + a_2 = \left\{1 - \frac{2t}{\pi}\right\} x$ and $b_1 x + b_2 = \left\{1 - \frac{2x}{\pi}\right\} t$

Substituting these values in (8) we have

$$G(x, t) = \begin{cases} \left(1 - \frac{2t}{\pi}\right) x, & 0 \le x < t \\ \left(1 - \frac{2t}{\pi}\right) t, & t < x \le \pi/2 \end{cases} \qquad \text{...... (14)}$$

Comparing $y'' + \lambda y - x = 0$ with $y'' + \varphi(x) = 0, we\ have$

$\varphi(x) = \lambda y(x) - x$ so that $\varphi(t) = \lambda y(t) - t$ ...... (15)

Also, we know that , if G(x, t) is Green's function of the boundary value problem given by (2), (3),(4) then the boundary value problem (1) can be reduced to the following integral equation

$$y(x) = \int_0^{\pi/2} G(x,t)\emptyset(t)dt = \int_0^{\pi/2} G(x,t)[\lambda y(t) - t\,]dt$$

or $y(x) = \lambda \int_0^{\pi/2} G(x,t)y(t)dt$ - $\int_0^{\pi/2} t\, G(x,t)dt$ ...... (16)

Now, we have

$$\int_0^{\pi/2} t\, G(x,t)dt = \int_0^{x} t\, G(x,t)dt + \int_x^{\pi/2} t\, G(x,t)dt$$

$$= \int_0^x t^2\left(1 - \frac{2x}{\pi}\right) dt + \int_x^{\pi/2} t\, x\left(1 - \frac{2t}{\pi}\right) dt \text{ , using (14)}$$

$$= \left(1 - \frac{2x}{\pi}\right)\int_0^x t^2\, dt + x\int_x^{\pi/2}\left(t - \frac{2t^2}{\pi}\right) dt$$

$$= \left(1 - \frac{2x}{\pi}\right)\left[\frac{t^3}{3}\right]_0^x + x\left[\frac{t^2}{2} - \frac{2t^3}{3\pi}\right]_x^{\pi/2}$$

$$= -\frac{x^3}{6} + \frac{\pi^2 x}{24}$$

Substituting the above value in (16), we obtain the required integral equation

$y(x) = \lambda \int_0^{\pi/2} G(x,t)y(t)dt + \frac{x^3}{6} - \frac{\pi^2 x}{24}$, where G(x, t) is given by (14).

CHECK YOUR PROGRESS

True or false Questions

In the following boundary value problem examine whether a Green's function exists or not ?

Problem 1. $y'' = 0,\ y(0) = y(1),\ y'(0) = y'(1).$

Problem 2. $y'' = 0,\ y(0) = 0,\ y(1) = y'(1).$

Problem 3. if the boundary value problem is self-adjoint, then Green's function is symmetric. True/False.

Problem 4. if the boundary value problem has only trivial solution y(x) = 0, the operator L has two Green's function G(x, t).True/False

Problem 5. When the prescribed end conditions are not homogeneous, we shall use a modified method. True/False

9.5 *SUMMARY*

1. If the boundary value problem has only trivial solution y(x) = 0, the operator L has a unique Green's function G(x, t).

2. If the boundary value problem is self-adjoint, then Green's function is symmetric

3. When the prescribed end conditions are not homogeneous, we shall use a modified method.

9.6 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

Expansions of function

Series

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9.8 *SUGGESTED READING*

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9.9 TERMINAL AND MODEL QUESTIONS

Q 1. Find the Green's function for the boundary value problem

$\frac{d^2y}{dx^2} + \mu^2 y = 0$, y(0) = y(1) = 0.

Q 2. Find the Green's function for the boundary value problem

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, y(x) is bounded as x→ 0, y(1) = 0 .

Q 3. Using Green's function solve the boundary value problem

$y'' - y = x, y(0) = y(1) = 0.$

Q 4. Using Green's function solve the boundary value problem

$y'' - y = -2e^x, \quad y(0) = y'(0), \qquad y(l) + y'(l) = 0.$

Q 5. Reduce the following boundary – value problems to the integral

equations $y'' + \lambda y = e^x, \quad y(0) = y'(0), \qquad y(1) = y'(1).$

9.10 *ANSWERS*

TQ1 G(x, t) = $\begin{cases} -\frac{\sin\mu(t-1)sin\mu x}{\mu sin\mu}, & 0 \le x < t \\ -\frac{\sin\mu t sin\mu(x-1)}{\mu sin\mu}, & t < x \le 1 \end{cases}$

TQ2 G(x, t) = $\begin{cases} \frac{x}{2}\left\{\frac{1}{t^2} - 1\right\}, & 0 \le x < t \\ \frac{1}{2}\left\{\frac{1}{x} - x\right\}, & t < x \le 1 \end{cases}$

TQ3 y(x) = $\frac{\sinh x}{\sinh 1} - x$

TQ4 $y(x) = \sinh x + e^x(l - x)$

TQ5 y(x) = $e^x + \lambda \int_0^1 G(x,t)\, y(t) dt$,

where G(x, t) = $\begin{cases} -(1+x)t, & 0 \le x < t \\ -(1+t)x, & t < x \le 1 \end{cases}$

CHECK YOUR PROGRESS

CYQ 1. No

CYQ 2. No

CYQ 3. True

CYQ 4. False

CYQ 5. True

UNIT 10: MODIFIED GREEN'S FUNCTION AND ITS APPLICATIONS INTO INTEGRAL EQUATION

Contents

10.1 *INTRODUCTION*

The Modified Global Green's Function Method (MGGFM) is an integral technique that is characterized by good accuracy in the evaluation of boundary fluxes. This method uses only projections of the Green's Function for the solution of the discrete problem and this is the origin of the term 'Modified' of its name. We shall introduce the concept of Green's function and utilize it in converting initial and boundary value problems into integral equations. Sometimes we shall be able to solve the given initial and boundary value problems completely with the help of Green's function.

10.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) integral equation

(ii) Green's function

(iii) initial value problem

(vi) Boundary value problem

(v) Solution of integral equation using Green's Function.

(vi) Modified Green's function

(v) Wronskian

10.3 *GREEN'S FUNCTION APPROACH FOR CONVERTING AN INITIAL VALUE PROLEM INTO AN INTEGRAL EQUATION*

Consider the following initial value problem

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right)+qy=f(x) \qquad \ldots\ldots. (1)$$

y(a) = 0, $y'(a) = 0$ ……... (2)

let L = $\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$ …….. (3)

which is self - adjoint differential operator. Here the function p(x) is continuously differentiable and positive and q(x) and f(x) are continuous in a given interval (a, b).

the associated homogeneous second order equation

Ly = 0 i.e. $\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = 0$ ……. (4)

Has exactly two independent solutions u(x) and v(x) which are twice differential in the interval a < x < b. any other solution of (4) is a linear combination of u(x) and v(x) , i.e.

y(x) = $c_1u(x) + c_2v(x)$, where $c_1 and\ c_2$ are constants.

For the self-adjoint operator L, the Green's formula is given by

$\int_a^b (vLu - uLv)dx = [p(x)(vu' - uv']_a^b$ …….. (5)

In order to convert the initial value problem (1) – (2) into an integral equation, we consider the function w(x) given by w(x) = u(x)$\int_a^x v(t)f(t)dt -$ v(x)$\int_a^x u(t)f(t)dt$ ….. (6)

Differentiating both sides of (6) w.r.t. 'x' we have

$$w'(x) = u'(x)\int_a^x v(t)f(t)dt + u(x)\frac{d}{dx}\int_a^x v(t)f(t)dt$$
$$- v'(x)\int_a^x u(t)f(t)dt - v(x)\frac{d}{dx}\int_a^x u(t)f(t)dt$$

Or $w'(x) = u'(x)\int_a^x v(t)f(t)dt - v'(x)\int_a^x u(t)f(t)dt$ ……. (7)

From (6) and (7) we have w(a) = $w'(a)$ = 0 …… (8)

Now, u and v are the solution of (4)

Therefore $\frac{d}{dx}(pu') + qu = 0$ and $\frac{d}{dx}(pv') + qv = 0$

Therefore $\frac{d}{dx}(pu') - qu$ and $\frac{d}{dx}(pv') - qv$ ……. (9)

Using the value $w'(x)$ given by (7), we have

$$\frac{d}{dx}(pw') = \frac{d}{dx}[p(x)u'(x)\int_a^x v(t)f(t)dt - p(x)v'(x)\int_a^x u(t)f(t)dt]$$

$$= \frac{d}{dx}(pu')\int_a^x v(t)f(t)dt] + pu'\frac{d}{dx}\int_a^x v(t)f(t)dt]$$

$$-\frac{d}{dx}(pv')\int_a^x u(t)f(t)dt] + pv'\frac{d}{dx}\int_a^x u(t)f(t)dt]$$

$$= \frac{d}{dx}(pu')\int_a^x v(t)f(t)dt] + pu'(x)v(x)f(x)$$

$$-\frac{d}{dx}(pv')\int_a^x u(t)f(t)dt] - pv'(x)u(x)f(x),$$

by Leibnitz's rule

$$= -\text{qu}\int_a^x v(t)f(t)dt] + \text{qv}\int_a^x u(t)f(t)dt] + \text{p}(u'v - uv')f(x)$$

Therefore $\frac{d}{dx}(pw')$ = - q(x) w(x) + p$(u'v - uv')f(x)$, using (6)…… (10)

Now $\frac{d}{dx}[\text{p}(u'v - uv')] = \frac{d}{dx}[(pv')u - (pu')v]$ = (-qv)u – (-qu)v = 0, using (9)

Thus, $\frac{d}{dx}\{p(uv' - u'v)\} = 0$ so that p$(uv' - u'v) = A$ ….. (11)

Where A is a constant. (11) is Abel's formula

Also $uv' - u'v = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$ = W(u, v) …….. (12)

Where W(u, v) is the Wronskian of u and v. since u and v are linearly independent solution of (4), we have W(u, v) = $uv' - u'v \neq 0$ …….. (13)

Using (11) , (10) may be re-written as

$\frac{d}{dx}\left\{p\frac{dw}{dx}\right\} + qw$ = -A f(x) or $\frac{d}{dx}\left\{p\frac{d(-\frac{w}{A})}{dx}\right\} + q(-\frac{w}{A})$ = f(x) …... (14)

Where $\quad w(a) = w'(a) = 0 \quad$ by (8) $\quad$ …… (15)

Comparing (14) with (1) , we have y = -w/A so that w = -Ay. Substituting this value of w in (6), we obtain

$-A\, y = \int_a^x \{u(x)v(t) - v(x)u(t)\} f(t)dt$ or

$y(x) = \int_a^x \frac{u(x)v(t)-v(x)u(t)}{A}$ f(t) dt or $v(x) = \int_a^x R(x,t)f(t)dt$ …… (16)

where $\quad R(x, t) = (1/A)\ \{v(x)\ u(t) - u(x)\ v(t)\}$ ……. (17)

from (17) we find that $R(x, t) = -R(t, x)$ ……. (18)

one can easily verify that, for a fixed value of t, the function R(x, t) is completely characterized as the solution of initial value problem

$L\, R = \frac{d}{dx}\left\{p(x)\frac{dR}{dx}\right\} + q(x)\, R = \delta(x-t)$ …… (19)

$[R]_{x=t} = 0, \quad \left[\frac{dR}{dx}\right]_{x\,=\,t} = 1/p(t)$ …… (20)

$where\ \ \delta(x-t)$ is the Dirac delta function.

This function describes the influence on the value of y at x due to a concentrated disturbance at t. it is called the influence function. The function G(x, t) is defined as

$$G(x,t) = \begin{cases} 0, & x < t \\ R(x,t), & x > t \end{cases} \quad ……. (21)$$

is known as the causual Green's function.

Note: when the value of y(a) and y(b) are prescribed to be other than zero, then we simply add a suitable solution A u(x) + B v(x) of (4) to the integral equation (16) we get Volterra integral equation of second kind of the form

$y(x) = A\, u(x) + B\, v(x) + \int_a^x R(x,t)f(t)dt$ …... (22)

the constants A and B are evaluated by using the prescribed initial conditions.

ILLUSTRATIVE EXAMPLES

Example 1. Convert the initial value problem $y'' + y = f(x)$, 0 < x <1, y(0) = $y'(0) = 0$ into an integral equation.

Sol. Given $y'' + y = f(x)$, 0 < x <1 ……. (1)

With initial conditions y(0) = $y'(0) = 0$ ……. (2)

Comparing (1) with $\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = f(x)$ here p = q = 1

The associated homogeneous equation of (1) is

$y'' + y = 0$ or $(D^2 + 1)y = 0$ …… (3)

Its general solution is y = A cosx + B sinx

Let u = cosx and v = sinx …….. (4)

Where u and v are linearly independent solution of (3)

Now A = $p(uv' - u'v)$ = p[$\cos^2$x + $\sin^2$x] = p = 1

Therefore R(x, t) = (1/A){v(x) u(t) - u(x) v(t)}

= sinx cost – cosx sint = sin(x – t)

Hence the given initial value problem reduces to the integral equation

y(x) = $\int_0^x R(x,t)f(t)dt$ i.e. y(x) = $\int_0^x \sin(x-t)f(t)dt.$

Example 2. Convert the initial value problem $y'' + y = f(x)$, 0 < x <1, y(0) =1, $y'(0) = -1$ into an integral equation of second kind.

Sol. Here the values of y(0) and $y'(0)$ are prescribed to be other than zero, hence the given initial value problem will transform into Volterra integral equation of the second kind of the form

y(x) = A u(x) + B v(x) + $\int_0^x R(x,t)f(t)dt$ …... (1)

proceed as in Ex. 1 and show that u(x) = cosx,

v(x) = sinx and R(x, t) = sin(x – t). so (1) reduces to

y(x) = A cosx + B sinx + $\int_0^x \sin(x-t)f(t)dt$ …... (2)

putting y = 0 in (2) and using the condition y(0) = 1, we get A = 1.

Now, differentiating both sides of (2) w.r.t. 'x' and using Leibnitz's rule, we obtain

$y'(x)$ = - A sinx + B cosx + $\int_0^x \cos(x-t)f(t)dt$ ……. (3)

Putting x = 0 in (3) and using the given condition $y'(0) = -1$, we get B = -1.

Putting A = 1 and B = -1 in (2), the required Volterra integral equation is given by

y(x) = cosx – sinx + $\int_0^x \sin(x-t)f(t)dt$.

10.4 *WORKING RULE FOR CONSTRUCTION OF MODIFIED GREEN'S FUNCTION*

Given an inhomogeneous equation with boundary conditions:

L y = $\varphi(x),\ \alpha_1 y(a) + \beta_1 y'(a) = 0,\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \quad \ldots\ldots..(1)'$

Consider a linear homogeneous equation of order two

L y = 0, where L = p(x)$\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q(x)$ ……. (1)

Together with homogeneous boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = 0 \qquad \ldots\ldots.. (2a)$$

$$\alpha_2 y(b) + \beta_2 y'(b) = 0 \qquad \ldots\ldots... (2b)$$

With usual assumption that at least one of α_1 and β_1 and one of α_2 and β_2 are non-zero.

Suppose that the homogeneous boundary value problem given by (1) , (2a) and (2b) has a non-trivial solution y(x).

Then $\|y(x)\|$ = norm of y(x) = $\left\{\int_a^b [y(x)]^2 dx\right\}^{1/2}$ ……. (3)

Let w(x) = y(x)/ $\|y(x)\|$

So that w(x) is non-trivial normalized solution of the boundary value problem given by (1), (2a) and (2b). clearly by definition we have

$\|w(x)\|$ = 1 so that $\int_a^b [w(x)]^2 dx$ = 1 …… (4)

Then, by definition $G_M(x,t)$ is called the modified Green's function of the given boundary value problem if it satisfies the differential equation

$LG_M = \delta(x-t)$ – w(x) w(t) …….(5)$'$

For x $\neq t$, $(5)'$ reduces to LG_M = – w(x) w(t) …… (5)

For a given t, let $G_M(x,t) = \begin{cases} G_1(x,t), if\ a \leq x < t \\ G_2(x,t), if\ t < x \leq b \end{cases}$ ……. (6)

Where G_1 and G_2 are such that

(i) the function G_1 and G_2 satisfy the equation (5) in their respective intervals of definition,

That is LG_1 = w(x) w(t), $a \leq x < t$ ……. (7a)

LG_2 = – w(x) w(t), $t \leq x \leq b$ ……. (7b)

(ii) G_1 satisfies the boundary condition (2a) whereas G_2 satisfies the boundary condition (2b).

(iii) the function $G_M(x,t)$ is continuous at x = t,

i.e. $G_1(t,t) = G_2(t,t)$ ……. (8)

(iv) the derivative of $G_M(x,t)$ with respect to x at the point x = t has a discontinuity of the first kind, the jump being equal to 1/p(t). here p(x) is the coefficient of d^2y/dx^2 in (1).

Thus $\left(\frac{\partial G_M}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G_M}{\partial x}\right)_{x=t-0} = 1/p(t)$ ……. (9)

(v) in order that $G_M(x,t)$ may be symmetric, we must have

$\int_a^b G_M(x,t)w(x)dx = 0$ …….. (10)

Method of reducing the inhomogeneous differential equation $(1)'$ with prescribed homogeneous boundary condition into an integral equation.

The required integral equation is given by y(x) = $\int_a^b G_M(x,t)\varphi(t)dt$ + c w(x), where c is an arbitrary constant.

……. (11a)

(11a) may also be re-written in the form

y(x) = $\int_a^b G_M(x,t)\varphi(t)dt$ + w(x) $\int_a^b w(x)y(x)dx$ ……. (11b)

consistency condition for existence of the desired integral equation (11a) or (11b) is given by $\int_a^b \varphi(x)w(x)dx$ = 0. ……... (12)

ILLUSTRATIVE EXAMPLES

Example 1. Find the modified Green's function for the system

$$y'' + f(x) = 0, y'(0) = y'(l) = 0, \quad 0 < x < l.$$

And hence transform this boundary value problem into an integral equation.

Sol. Given -y''= f(x), $y'(0) = y'(l) = 0$, $0 \le x \le l$ …… (1)

Here – (d^2/dx^2) is self - adjoint operator

Consider the associated self- adjoint system -$y'' = 0$, $0 \le x \le l$…... (2)

With boundary condition $y'(0) = 0$ ……. (3a)

$y'(l) = 0$ …….. (3b)

The general solution of (2) is y(x) = Ax + B ……… (4)

From (4), $y'(x) = A$ ……… (5)

Putting x = 0 and x = l in (5) and using (3a) and (3b), we get A = 0. Hence the boundary value problem given by 92), (3a0 and (3b0 has a non-trivial solution y(x) = B, where B is an arbitrary constant.

Here $\|y(x)\|$ = norm of y(x) = $\left\{\int_0^l [y(x)]^2 dx\right\}^{1/2}$

$$= \left\{\int_0^l [B]^2 dx\right\}^{1/2} = B\sqrt{l}$$

Let w(x) = y(x)/ $\|y(x)\|$ = B/B$\sqrt{l}$ = 1/$\sqrt{l}$ ……. (6)

So that w(x) is non-zero normalized solution of the boundary value problem given by (2), (3a) and (3b). clearly by definition we have

$$\int_0^l [w(x)]^2 dx = 1 \quad \text{……. (7)}$$

Then, for x $\neq t$ the required modified Green's function $G_M(x,t)$ must satisfy the equation

-d^2G_M/dx^2 = -w(x) w(t) or d^2G_M/dx^2 = 1/l …….. (8)

Then general solution of (8) is of the form $G_M(x,t)$ = Ax + B + x^2/2l

Hence we take $G_M(x,t) = \begin{cases} a_1x + a_2 + \frac{x^2}{2l}, & if\ 0 \leq x < t \\ b_1x + b_2 + \frac{x^2}{2l}, & if\ t < x \leq l \end{cases}$ …... (9)

from (9), $\partial G_M/\partial x = \begin{cases} a_1 + \text{x}/l, if\ 0 \leq x < t \\ b_1 + \text{x}/2, if\ t < x \leq l \end{cases}$ …… (10)

in addition to the above property (9), the proposed modified Green's function must satisfy the following properties:

(i) since $G_M(x,t)$ must satisfy the boundary conditions (3a) and (3b), (10) gives

$(\partial G_M/\partial x)_{x=0}$ = 0 and $(\partial G_M/\partial x)_{x=l}$ = 0

Therefore $a_1 = 0$ and $b_1 + 1 = 0$ so that $a_1 = 0\ and\ b_1 = -1$…. (11)

(ii) $G_M(x,t)$ is continuous at x = t, that is

$a_1t + a_2 + t^2/2l$ = $b_1x + b_2 + t^2/2l$ so that

$(a_1 - b_1)t + a_2 - b_2 = 0$ ….. (12)

(iii) the derivative of $G_M(x,t)$ wit respect to x at the point x = t has a discontinuity of the first kind, the jump being 1/p(t), where p(x) is the coefficient of y'' in (1), i.e. p(x) = -1. Thus,

$$\left(\frac{\partial G_M}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G_M}{\partial x}\right)_{x=t-0} = 1/p(t)$$

i.e. $b_1 + t/l - (a_1 + t/l) = -1$ or $a_1 - b_1 = 1$ ……. (13)

from (12) and (13) $t + a_2 - b_2 = 0$ so that $b_2 = a_2 + t$ ……. (14)

substituting values of a_1, b_1 and b_2 from (11) and (14) in (9), we get

$$G_M(x,t) = \begin{cases} a_2 + x^2/2l, & if\ 0 \le x < t \\ a_2 - x + t + x^2/2l, & if\ t \le x \le l \end{cases} \quad …….. (15)$$

(iv) in order that $G_M(x,t)$ may be symmetric, we have

$\int_0^l G_M(x,t)w(x)dx = 0$ or $\int_0^l G_M(x,t)dx = 0,$ as $w(x) = \frac{1}{\sqrt{l}}$

Or $\int_0^t G_M(x,t)dx + \int_t^l G_M(x,t)dx = 0$

Or $\int_0^t \left(a_2 + \frac{x^2}{2l}\right)dx + \int_t^l \left(a_2 - x + t + \frac{x^2}{2l}\right)dx = 0$, using (15)

Or $[a_2x + x^3/6l]_0^t + \left[a_2x - \frac{x^2}{2} + tx + x^3/6l\right]_t^l = 0$

Or $a_2 = \frac{l}{3} - t + \frac{t^2}{2l}$

Substituting the above value of a_2 in (15), the symmetric modified Green's function $G_M(x,t)$ is given by

$$G_M(x,t) = \begin{cases} \frac{l}{3} - t + (x^2+t^2)/2l, & if\ 0 \le x < t \\ \frac{l}{3} - x + (x^2+t^2)/2l, & if\ t < x \le l \end{cases} \quad …….. (16)$$

Which may also be re-written as

$$G_M(x,t) = \frac{l}{3} + \frac{x^2+t^2}{2l} - \begin{cases} t, & if\ 0 \le x < t \\ x, & if\ t < x \le l \end{cases} \quad ……. (17)$$

The above result (16) could have been obtained by inspecting (15) and making a judicious choice of a_2.

Second part: transformation of the given boundary value problem into an integral equation.

The required integral equation is given by

y(x) = $\int_M^l G_M(x,t)f(t)dt + \frac{c}{\sqrt{l}}$ or y(x) = $c' + \int_0^l G_M(x,t)f(t)dt$

where $c' = \frac{c}{\sqrt{l}}$ is an arbitrary constant and $G_M(x,t)$ is given by (16) or (17).

Example 2. Find the modified Green's function for the system

$y'' = 0$, -1 < x < l. Subject to the conditions $y(-1) = y(1)$

and $y'(-1) = y'(1)$.

Sol. Given -y''= 0, -1 ≤ x ≤ l …… (1)

With boundary condition y(-1) = y(1) ……. (2a)

$y'(-1) = y'(1)$ ……… (2b)

The general solution of (1) is y(x) = Ax + B ……… (3)

From (3), $y'(x) = A$ ……… (4)

From (3) and (4),

y(-1) = -A + B , y(1) = A + B, $y'(-1) = y'(1) = A$ …….. (5)

From (2a) and (5) we ge - A + B = A + B so that A = 0 ……… (6)

Then (2b), (5) and (6) ⟹ A = A = 0

Hence the given boundary value problem has a non-trivial solution

y(x) = B, where B is an arbitrary constant.

Here $\|y(x)\|$ = norm of y(x) = $\left\{\int_0^l [y(x)]^2 dx\right\}^{1/2}$

$= \left\{\int_0^l [B]^2 dx\right\}^{1/2} = B\sqrt{2}$

Let w(x) = y(x)/ $\|y(x)\|$ = B/B$\sqrt{2}$ = 1/$\sqrt{2}$ ……. (7)

So that w(x) is non-zero normalized solution of the boundary value problem given by (2), (3a) and (3b). clearly by definition we have

$$\int_{-1}^{1} [w(x)]^2 dx = 1$$

Then, for x $\neq t$ the required modified Green's function $G_M(x, t)$ must satisfy the equation

-d^2G_M/dx^2 = -w(x) w(t) or $d^2G_M/dx^2 = 1/2$ …….. (8)

Then general solution of (8) is of the form $G_M(x, t)$ = Ax + B + $x^2/4$

Hence we take $G_M(x,t) = \begin{cases} a_1x + a_2 + \frac{x^2}{4}, & if -1 \leq x < t \\ b_1x + b_2 + \frac{x^2}{4}, & if\ t < x \leq 1 \end{cases}$ …... (9)

from (9), $\partial G_M/\partial x = \begin{cases} a_1 + \frac{x}{2}, if -1 \leq x < t \\ b_1 + x/2, if\ t < x \leq l \end{cases}$ …… (10)

in addition to the above property (9), the proposed modified Green's function must satisfy the following properties:

(i) $G_M(x, t)$ is continuous at x = t, that is

$a_1t + a_2 + t^2/4 = b_1x + b_2 + t^2/4$ so that

$(a_2 - b_2) = t(b_1 - a_1)$ ….... (11)

(ii) since $G_M(x, t)$ must satisfy the boundary conditions (2a) , we must have

$-a_1 + a_2 + 1/4 = b_1 + b_2 + 1/4$

so that a_2 - b_2 = $a_1 + b_1$, by (9) ……. (12)

Again $G_M(x,t)$ must satisfy the boundary conditions (2b) , we must have

b_1 + 1/2 = a_1 - 1/2 so that b_1 - a_1 =-1, by (10) ……. (13)

(iii) the derivative of $G_M(x,t)$ wit respect to x at the point x = t has a discontinuity of the first kind, the jump being 1/p(t), where p(x) is the coefficient of y'' in (1), i.e. p(x) = -1. Thus,

$$\left(\frac{\partial G_M}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G_M}{\partial x}\right)_{x=t-0} = 1/p(t)$$

i.e. $b_1 + t/2 - (a_1 + t/2)$ = -1 or $b_1 - a_1 = -1$

which is the same relation as (13). Thus, we see that the jump condition on $\frac{\partial G_M}{\partial x}$ is automatically satisfied.

from (11) and (13) $a_2 - b_2$ = -t so that $b_2 = a_2 + t$ ……. (14)

again from (13) $b_1 = a_1 - 1$ ……… (15)

substituting values of b_2, b_1 given by (14) and (15) respectively in (12) , we have

$a_2 - (a_2 + \text{t}) = a_1 + a_1 - 1$ so that $a_1 = (1-t)/2$ ……… (16)

Substituting the value of b_2, b_1 given by (14) and (15) respectively in (9), we have

$$G_M(x,t) = \begin{cases} a_1 x + a_2 + \frac{x^2}{4}, & if -1 \le x < t \\ (a_1 - 1)x + a_2 + t + \frac{x^2}{4}, & if\ t < x \le 1 \end{cases} \quad \text{……. (17)}$$

(iv) in order that $G_M(x,t)$ may be symmetric, we have

$\int_{-1}^{1} G_M(x,t) w(x) dx = 0$ or $\int_{-1}^{1} G_M(x,t) dx = 0,$ as w(x) = $\frac{1}{\sqrt{2}}$

Or $\int_{-1}^{t} G_M(x,t) dx + \int_{t}^{1} G_M(x,t) dx = 0$

Or $\int_{-1}^{t} \left(a_1 x + a_2 + \frac{x^2}{4}\right) dx + \int_{t}^{1} \left(a_2 + t + a_1 x - x + \frac{x^2}{4}\right) dx = 0,$ by (17)

After integrating we get $a_2 = \frac{t^2}{4} - \frac{t}{2} + 1/6$ ………. (18)

Substituting the value of a_1 and a_2 given by (16) and (18) respectively in (17), the symmetric Green's function is given by

$$G_M(x,t) = \begin{cases} \frac{(1-t)x}{2} + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} + \frac{x^2}{4}, & if -1 \le x < t \\ \frac{-(1+t)x}{2} + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} + \frac{x^2}{4}, & if\ t < x \le 1 \end{cases} \quad \text{or}$$

$$G_M(x,t) = \begin{cases} \frac{t^2}{4} + \frac{x^2}{4} - \frac{xt}{2} + \frac{(x-t)}{2} + \frac{1}{6}, & if -1 \le x < t \\ \frac{t^2}{4} + \frac{x^2}{4} - \frac{xt}{2} - \frac{(x-t)}{2} + \frac{1}{6}, & if\ t < x \le 1 \end{cases} \quad \text{or}$$

$$G_M(x,t) = \begin{cases} \frac{(x-t)^2}{4} + \frac{(x-t)}{2} + \frac{1}{6}, & if -1 \le x < t \\ \frac{(x-t)^2}{4} - \frac{(x-t)}{2} + \frac{1}{6}, & if\ t < x \le 1 \end{cases}$$

Which can be also re-written as

$$G_M(x,t) = \frac{(x-t)^2}{4} - \frac{1}{2}|x-t| + 1/6 .$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. $\left(\frac{\partial G_M}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G_M}{\partial x}\right)_{x=t-0} = \frac{1}{p(t)}$. True/False

In the following boundary value problem examine whether a Green's function exists or not ?

Problem 2. $y'' = 0,\ y(0) = 0,\ y(1) = y'(1)$.

Problem 3. $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$ = W(u, v) is Wronskian of u and v. True/False.

Problem 4. if the boundary value problem has only trivial solution y(x) = 0, the operator L has two Green's function

G(x, t).True/False

Problem 5. The general solution of $y'' = 0$ is $y(x) = Ax + B$.
True/False

10.5 *SUMMARY*

1. If the boundary value problem has only trivial solution $y(x) = 0$, the operator L has a unique Green's function $G(x, t)$.

2. If the boundary value problem is self-adjoint, then Green's function is Symmetric.

3. When the prescribed end conditions are not homogeneous, we shall use a modified method.

4. $\left(\frac{\partial G_M}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G_M}{\partial x}\right)_{x=t-0} = \frac{1}{p(t)}$.

5. Suppose that the homogeneous boundary value problem has a non-trivial solution $y(x)$.

Then $\|y(x)\|$ = norm of $y(x) = \left\{\int_a^b [y(x)]^2 dx\right\}^{1/2}$.

6. $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = W(u, v)$ is Wronskian of u and v.

10.6 *GLOSSARY*

Integration
Even, odd functions
Trigonometric functions
Differentiation
First order derivatives
Second order derivatives

Wronskian

Linearly independent functions

10.7 *REFERENCES*

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10.8 *SUGGESTED READING*

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10.9 TERMINAL AND MODEL QUESTIONS

Q 1. Find Modified Green's function for the system $y'' = 0$, -1 < x < t subject to the boundary conditions y(-1) = y(1) and $y'(-1) = y'(1)$.

Q 2. Find Modified Green's function for the system $ky'' + f(x) = 0$,

- $l \leq$ x $\leq l$ subject to the boundary conditions y(-l) = y(l) and

$y'(-l) = y'(l)$.

Q 3. Transform boundary value problem into respective integral equation $ky'' + f(x) = 0$, - $l \leq$ x $\leq l$ subject to the boundary conditions y(-l) = y(l) and $y'(-l) = y'(l)$.

Q 4. Using Green's function solve the boundary value problem

$$y'' - y = -2e^x, \quad y(0) = y'(0), \qquad y(l) + y'(l) = 0.$$

Q 5. Reduce the following boundary – value problems to the integral equations

$$y'' + \lambda y = e^x, \quad y(0) = y'(0), \qquad y(1) = y'(1).$$

Q 6. Develop the theory of modified Green's function in case of self-adjoint system where the completely homogeneous system has two linearly independent solutions $w_1(x)$ and $w_2(x)$.

10.10 *ANSWERS*

TQ1 $G_M(x,t) = \begin{cases} \frac{(x-t)^2}{4} + \frac{(x-t)}{2} + \frac{1}{6}, & -1 \leq x < t \\ \frac{(x-t)^2}{4} - \frac{(x-t)}{2} + \frac{1}{6}, & t < x \leq 1 \end{cases}$

TQ2 $G_M(x,t) = \frac{1}{6k} + \frac{(x-t)^2}{4kl} + \frac{|x-t|}{2k}$

TQ3 y(x) = c $+ \int_{-l}^{l} G_M(x,t)) \, f(t) dt$

TQ4 $y(x) = \sinh x + e^x(l - x)$

TQ5 y(x) = $e^x + \lambda \int_0^1 G(x,t)\, y(t)dt$, where

G(x, t) = $\begin{cases} -(1+x)t, & 0 \le x < t \\ -(1+t)x, & t < x \le 1 \end{cases}.$

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. No

CYQ 3. True

CYQ 4. False

CYQ 5. True

BLOCK-IV

UNIT 11: VARIATIONAL PROBLEM WITH FIXED BOUNDARIES

Contents

11.1 *INTRODUCTION*

The calculus of variation has its origin in the generalization of the elementary theory of maxima and minima of the function of a single or more variable. Its object is to find extreme or stationary values of functionals. The aims of the calculus of variations are:

a) To explore methods for finding the maximum or minimum of a function defined over a class of functions.
b) **Geodesic Curve**: To find a geodesic curve on the surface which means finding the shortest curve joining two points on the surface.
c) **Brachistochrone Problem:** If a smooth body is allowed to slide down a smooth curve from point A to B under gravity, then determine the curve along which the time taken will be the least.
d) **Minimal Surface:** To determine which curve will yield the least area of the surface of revolution.
e) **Iso-perimetric Problem:** In this problem, we aim to find what a curve of a given perimeter will enclose the maximum area.

11.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) Functional

(ii) extremum

(iii) Euler's equation

(vi) Isoperimetric Problems

11.3 *FUNCTIONAL*

A function whose values are determined by one or several functions is called functional. Also, we can say a function is a function of function.

Note 1: If it is required to find the curve $y = y(x)$ where $y(x_0) = y_0$

and $y(x_1) = y_1$ such that the given function $F(x, y, y')$, the definite integral

$$I[y(x)] = \int_{x_0}^{x_1} F\ (x, y, y'\ )\ dx$$

is maximum or minimum i.e., extremum.

This integral is known as functional.

the calculus of variation deals with the problems of maxima or minima of functionals.

11.4 *EXTREMAL*

A function y = y(x) which extremizes a functional is called extremal or extremizing function.

11.5 *EULER'S EQUATION*

■ **Euler's Equation**

(a necessary condition for the existence of extremal):

The necessary condition for functional

$$I[y(x)] = \int_{x_0}^{x_1} F\ (x, y, y'\ )\ dx$$

to be maximum or minimum is that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

prescribed.

Proof: Consider the functional

$$I[y(x)] = \int_{x_0}^{x_1} F\ (x, y, y'\ )\ dx \qquad \text{....... (1)}$$

Let y = y(x) be extremal of functional and $\bar{y}$(x) be neighbourhood of y(x) such that

$$\bar{y}(x) = y(x) + \varepsilon\eta(x) \qquad \text{.......(2)}$$

Where ε is small parameter and η(x) be an arbitrary function such that

$$\eta(x_0) = \eta(x_1) = 0 \qquad \text{……..(3)}$$

From (1),

$$I[\bar{y}\ (x)] = \int_{x_0}^{x_1} F\ (x, \bar{y}, \bar{y}')\ dx$$

Using (2) we get

$$I[\bar{y}\ (x)] = \int_{x_0}^{x_1} F\ (x,\ y + \varepsilon\eta,\ y' + \varepsilon\eta)\ dx$$

Which is function of ε.

$\therefore$
$$I[\varepsilon] = \int_{x_0}^{x_1} F\ (x,\ y + \varepsilon\eta,\ y' + \varepsilon\eta)\ dx$$

Or
$$I[\varepsilon] = \int_{x_0}^{x_1} \left[F\ (x,\ y + \varepsilon\eta,\ y' + \varepsilon\eta) + \ \varepsilon\eta\frac{\partial F}{\partial y} + \varepsilon\eta'\frac{\partial F}{\partial y'} + 0(\ \varepsilon^2)\right] dx$$

where $0(\varepsilon^2)$ is term containing ε^2 and higher power of ε^2, it is called

Bi Oh [by Tylor's theorem of function of several variable]

$\therefore$
$$\frac{dI}{d\varepsilon} = \int_{x_0}^{x_1} \left[0 +\ \eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'} + 0(\varepsilon)\right] dx$$

Or,
$$\frac{dI}{d\varepsilon} = \int_{x_0}^{x_1} \left[\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'} + 0(\varepsilon)\right] dx$$

The necessary condition for existence of extremal is

$$\left(\frac{dI}{d\varepsilon}\right)_{\varepsilon=0} = 0$$

$$\int_{x_0}^{x_1} \left[\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'}\right] dx = 0$$

$$\int_{x_0}^{x_1} \left[\eta\frac{\partial F}{\partial y}\right] dx + \int_{x_0}^{x_1} \left[\eta'\frac{\partial F}{\partial y'}\right] dx\ = 0$$

$$\int_{x_0}^{x_1} \eta\ \frac{\partial F}{\partial y} dx + \left[\eta(x)\frac{\partial F}{\partial y}\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x)\ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx\ = 0$$

$$\int_{x_0}^{x_1} \eta\ \frac{\partial F}{\partial y} dx + 0 - \int_{x_0}^{x_1} \eta(x)\ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx\ = 0 \quad [\because \eta(x_0) = \eta(x_1) = 0\ ]$$

$$\int_{x_0}^{x_1} \eta\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] dx = 0$$

Since η(x) is an arbitrary. Therefore, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$$

Or $$F_y - \frac{d}{dx}(F_{y\prime}) = 0$$

which is required Euler's equation. This equation is also known as Euler-Lagrange's equation.

11.6 *OTHER FORMS OF EULER'S EQUATION*

1. $$\frac{d}{dx} F(x, y, y') = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y\prime}\frac{dy\prime}{dx}$$

Or $$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y\prime}y'' \qquad \text{...... (1)}$$

But $$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y\prime}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) + \frac{\partial F}{\partial y\prime}y'' \qquad \text{....... (2)}$$

On subtracting (2) from (1), we have

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y\prime}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right)$$

Or $$\frac{d}{dx}\left[F - y'\left(\frac{\partial F}{\partial y\prime}\right)\right] - \frac{\partial F}{\partial x} = y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right)\right]$$

$$= y(0) \qquad \text{[by Euler's equation]}$$

$$= 0$$

Hence, $$\frac{d}{dx}\left[F - y'\left(\frac{\partial F}{\partial y\prime}\right)\right] - \frac{\partial F}{\partial x} = 0$$

Which is another form of Euler's equation.

2. we know that $\frac{\partial F}{\partial y}$ is also known a function of *x, y, y'* say φ (x, y, y')

∴ $$\frac{\partial F}{\partial y} = \varphi(x, y, y')$$

Differentiate w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = \frac{d}{dx}\varphi(x, y, y')$$

$$= \frac{\partial \varphi}{\partial x}\frac{dx}{dx} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} + \frac{\partial \varphi}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}y' + \frac{\partial \varphi}{\partial y'}y''$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)y' + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)y''$$

$$= \frac{\partial^2 F}{\partial x \partial y'} + y'\frac{\partial^2 F}{\partial y \partial y'} + y''\frac{\partial^2 F}{\partial y'^2}$$

Hence, Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \text{ becomes}$$

$$\boxed{\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - y'\frac{\partial^2 F}{\partial y \partial y'} - y''\frac{\partial^2 F}{\partial y'^2} = 0}$$

Or $$F_y - F_{xy'} - y'F_{yy'} - y''F_{y'^2} = 0$$

Note: ■ Every solution of Euler's equation which satisfies the boundary conditions is called an Extremal or a stationary function of the problem.

11.7 *PARTICULAR CASES OF EULER'S EQUATION*

Case I: When $F = F(x, y)$

$\therefore$ $\frac{\partial F}{\partial y'} = 0.$

Then Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \text{ becomes}$$

This implies, $$\frac{\partial F}{\partial y} - \frac{d}{dx}(0) = 0$$

$$\boxed{\frac{\partial F}{\partial y} = 0}$$

ILLUSTRATIVE EXAMPLES

Example 1. Extremize

$$\mathbf{I[y(x)]} = \int_0^1 (\boldsymbol{x\sin y + \cos y})\ \boldsymbol{dx}$$

$$\boldsymbol{y(0) = 0,\ y(1) = \pi / 2}$$

Sol. Here $F(x, y, y') = x\sin y + \cos y$

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

This implies, $x\cos - \sin y - \frac{d}{dx}(0) = 0$

This implies, $x\cos - \sin y = 0$

This implies, $x\cos = \sin y$

$\therefore$ $\tan y = x$

This implies, $\boxed{y = \tan^{-1} x}$

$\therefore$ $y(0) = \tan^{-1} 0 = 0, \quad y(1) = \tan^{-1}\frac{\pi}{2} = 1.$

This is required extremal which satisfying given boundary conditions.

Case II: When $F = M(x, y) + y'N(x, y)$

$$\frac{\partial F}{\partial y} = \frac{\partial M}{\partial y} + y'\frac{\partial N}{\partial y}$$

$$\frac{\partial F}{\partial y'} = N(x, y)$$

Hence Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \text{ becomes}$$

$$\frac{\partial M}{\partial y} + y'\frac{\partial N}{\partial y} - \frac{d}{dx}[N(x, y)] = 0$$

This implies, $\frac{\partial M}{\partial y} + y'\frac{\partial N}{\partial y} - \left[\frac{\partial N}{\partial x} + y'\frac{\partial N}{\partial y}\right] = 0$

This implies, $\frac{\partial M}{\partial y} + y'\frac{\partial N}{\partial y}$ - $\frac{\partial N}{\partial x}$ - $y'\frac{\partial N}{\partial y} = 0$

This implies, $\frac{\partial M}{\partial y}$ - $\frac{\partial N}{\partial x} = 0$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

This is condition of exactness of differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

Example 2. Extremize

$$\mathbf{I[y(x)]} = \int_0^1 (\boldsymbol{y^2 + y'x^2})\boldsymbol{dx}$$

$$\boldsymbol{y(0) = 0,\ \ y(1) = 1}$$

Sol. Here $F = y^2 + y'x^2$

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$$

This implies, $2y - \frac{d}{dx}(x^2) = 0$

This implies, $2y - 2x = 0$

∴ $\boxed{y = x}$

This is required extremal which satisfying given boundary conditions.

Case III: When $F = F(x, y)$

∴ $\frac{\partial F}{\partial y} = 0.$

Then Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0 \text{ becomes}$$

This implies, $0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$

This implies, $\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$

On integrating, we get

$$\boxed{\frac{\partial F}{\partial y'} = \text{Constant}}$$

$$F_{y'} = \text{constant}$$

Example 3. Extremize

$$\mathbf{I[y(x)]} = \int_0^1 \left(\sqrt{1+y'^2}\right) dx$$

$$y(0) = 0, \; y(1) = 1$$

Sol. Here $F = \sqrt{1+y'^2}$

$\therefore$ $\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y'} = \frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{1+y'^2}}$

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

This implies, $0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$

This implies, $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$

On integrating, we get

$$\frac{y'}{\sqrt{1+y'^2}} = \text{constant} = c_1$$

$\therefore$ $y' = c_1\sqrt{1+y'^2}$

Squaring, we get

$$y'^2 = c_1^2(1+y'^2)$$

$$(1 - c_1^2)\, y'^2 = c_1^2$$

$$y'^2 = \frac{c_1^2}{1-c_1^2}$$

$$y' = \frac{c_1}{\sqrt{1-c_1^2}} = c \text{ (constant)}$$

$\therefore$ $\frac{dy}{dx} = \text{c}$

$\therefore$ $\boxed{y = cx + d}$

This is required extremal which satisfying given boundary conditions.

Case IV: When $F = F(x, y')$

$\therefore \qquad \frac{\partial F}{\partial y} = 0.$

Then Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0 \text{ becomes}$$

This implies, $\quad 0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$

This implies, $\quad \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$

On integrating, we get

$$\boxed{\frac{\partial F}{\partial y\prime} = \text{Constant}}$$

$$F_{y\prime} = \text{constant}$$

Example 4. Find the curve, the time taken along which the least, when velocity at any point of it is v = x.

Sol. Consider the functional

$$\mathrm{I}[\mathrm{y(x)}] = \int_{x_0}^{x_1} \frac{dS}{v} = \int_{x_0}^{x_1} \frac{\sqrt{1+y\prime^2}}{x}\, dx$$

$$[\because \mathrm{ds} = \sqrt{1+y'^2}dx\ ]$$

Here $\quad F = F(x, y') = \frac{\sqrt{1+y\prime^2}}{x}$

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$$

This implies, $\quad 0 - \frac{d}{dx}\left(\frac{y\prime}{x\sqrt{1+y\prime^2}}\right) = 0$

This implies, $\quad \frac{d}{dx}\left(\frac{y\prime}{x\sqrt{1+y\prime^2}}\right) = 0$

On integrating, we get

$$\frac{y\prime}{x\sqrt{1+y\prime^2}} = \text{constant} = c_1{}' \qquad \text{……. (1)}$$

Put $\quad y' = \tan t$, i.e., $\frac{dy}{dx} = \tan t \qquad$ ……. (2)

We get $\quad \frac{\tan t}{x \sec t} = c_1{}'$

This implies, $\quad x = \frac{1}{c_1{}'}\sin t$

This implies, $x = c_1 \sin t$, where $c_1 = \frac{1}{c_1{}'}$ … (3)

Now since $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= \tan t \; c_1 \cos t$

i.e., $\frac{dy}{dt} = c_1 \sin t$

Integrating, we get

$$y = -c_1 \cos t + c_2$$

Or $y - c_2 = -c_1 \cos t$
… (4)

From (3) and (4),

$$x^2 + (y - c_2)^2 = c_1^2 sin^2 t + c_1^2 cos^2 t$$

$$= c_1^2 \,(sin^2 t + cos^2 t\,)$$

$$= c_1^2$$

$\therefore$ $\boxed{x^2 + (y - c_2)^2 = c_1^2}$

This is required extremal which represent circles.

Case V: When $F = F(y, y')$

Now, $\frac{dF}{dx} = \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'} \cdot \frac{\partial y'}{\partial x}$

or $\frac{dF}{dx} = y' \frac{\partial F}{\partial y} + y'' \frac{\partial F}{\partial y'}$

$$\frac{d}{dx}(F) = \left(y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'}\right) F$$

$\therefore$ $\frac{d}{dx} = y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'}$ ….…. (1)

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

or $F_y - \left(y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'}\right) F_{y'} = 0$ [By (1)]

or $F_y - y' \frac{\partial}{\partial y}(F_{y'}) - y'' \frac{\partial}{\partial y'}(F_{y'}) = 0$

or $F_y - y' F_{yy'} - y'' F_{y'y'} = 0$

Now, multiplying both sides by y', we get

$$y' F_y - y'^2 F_{yy'} - y' y'' F_{y'y'} = 0$$

or $$\frac{d}{dx}[F - y'F_{y'}] = 0$$

Integrating, we get

$$\boxed{\mathrm{F} - y'F_{y'} = \text{ Constant}}$$

This is required condition for necessary condition of existence of extremal.

Example 5. Show that the general solution of the Euler equation for the functional

$$\boldsymbol{\int_a^b \frac{1}{y}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx}$$

is $$\boldsymbol{(x-h)^2 + y^2 = k^2}$$

Sol. Given the functional is

$$I[y(x)] = \int_a^b \frac{1}{y}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_a^b \frac{\sqrt{1+y'^2}}{y}\ dx.$$

Here $$F = F(x, y') = \frac{\sqrt{1+y'^2}}{y}$$

The necessary condition of existence of extremal is

$$\mathrm{F} - y'F_{y'} = \text{ Constant}$$

i.e., $$\frac{\sqrt{1+y'^2}}{y} - y'\frac{y'}{y\sqrt{1+y'^2}} = \text{constant}$$

or $$\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}} = \text{constant}$$

or $$\frac{\left(1+y'^2\right) - y'^2}{y\sqrt{1+y'^2}} = \text{constant}$$

or $$\frac{1}{y\sqrt{1+y'^2}} = \text{constant} = c_1{}'$$

… (1)

Put $$y' = \tan t, \text{ i.e., } \frac{dy}{dx} = \tan t$$

… (2)

We get $$\frac{1}{y\sec t} = c_1$$

This implies, $$y = \frac{1}{c_1}\cos t$$

This implies, $$y = k\cos t, \quad \text{where } k = \frac{1}{c_1} \qquad \text{… (3)}$$

Now since $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$

$= -\cot t \text{ k} \sin t$ [By (2) and (3)]

i.e., $\frac{dx}{dt} = -k \cos t$

Integrating, we get

$x = -k \sin \mathrm{t} + h$ where h = constant

or $x - h = -k \sin \mathrm{t}$ … (4)

From (3) and (4), we get

$$(x-h)^2 + y^2 = k^2 sin^2 t + k^2 cos^2 t$$

$$= k^2 (sin^2 t + cos^2 t)$$

$$= k^2$$

∴ $$\boxed{(x-h)^2 + y^2 = k^2}$$

This is required extremal which represent circles.

11.8 *FUNCTIONAL DEPENDENT ON HIGHER DERIVATIVES*

■ **Euler-Poisson Equation**

Consider the functional

$$\text{I[y(x)]} = \int_{x_0}^{x_1} F\,(x, y, y', y'', \dots, y^n)\, dx \quad \text{……. (1)}$$

Where values of $x_0,\ x_1,\ y(x_0),\ y(x_1),\ y'(x_0),\ y'(x_1), \dots, y^{n-1}(x_0)$, $y^{n-1}(x_1)$ are prescribed.

Let y(x) be extremal of (1) and $\bar{y}$(x) be neighbourhood of y(x) such that

$$\bar{y}(\mathrm{x}) = \mathrm{y(x)} + \varepsilon\eta(\mathrm{x}) \quad \text{……. (2)}$$

Where ε is small parameter and η(x) be an arbitrary function such that

$$\left.\begin{aligned} \eta(x_0) &= \eta(x_1) = 0 \\ \eta'(x_0) &= \eta'(x_1) = 0 \\ &\vdots \\ \eta^{n-1}(x_0) &= \eta^{n-1}(x_1) = 0 \end{aligned}\right\} \quad \text{……. (3)}$$

Now from (1),

$$\text{I}[\bar{y}\,(\text{x})] = \int_{x_0}^{x_1} F\,(x, \bar{y}, \bar{y}', \bar{y}'', \dots, \bar{y}^n)\, dx$$

Using (2) we get

$$I[\bar{y}(x)] = \int_{x_0}^{x_1} F\,(x,\ y+\varepsilon\eta,\ y'+\varepsilon\eta', y+\varepsilon\eta, \ldots, y^n+\varepsilon\eta^n)\,dx$$

Using (3), we get

$$\int_{x_0}^{x_1} \eta(x)\,\frac{\partial F}{\partial y}dx - \int_{x_0}^{x_1}\eta(x)\,\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right)dx - \int_{x_0}^{x_1}\eta'(x)\,\frac{d}{dx}\left(\frac{\partial F}{\partial y"}\right)dx + \ldots$$

$$-\int_{x_0}^{x_1}\eta^{n-1}(x)\,\frac{d}{dx}\left(\frac{\partial F}{\partial y^n}\right)dx = 0$$

Or $\int_{x_0}^{x_1} \eta(x)\,\frac{\partial F}{\partial y}dx - \int_{x_0}^{x_1}\eta(x)\,\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right)dx - \left[\eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y"}\right)\right]_{x_0}^{x_1} +$

$$\int_{x_0}^{x_1}\eta(x)\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y"}\right)dx + \ldots - \left[\eta^{n-2}(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y^n}\right)\right]_{x_0}^{x_1} +$$

$$\int_{x_0}^{x_1}\eta^{n-2}(x)\,\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)dx = 0$$

Again using (2) and continuing the process, we get

$$\int_{x_0}^{x_1} \eta(x)\,\frac{\partial F}{\partial y}dx - \int_{x_0}^{x_1}\eta(x)\,\frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right)dx + \int_{x_0}^{x_1}\eta(x)\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y"}\right)dx + \ldots$$

$$(-1)^n + \int_{x_0}^{x_1}\eta(x)\,\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)dx = 0$$

This implies, $\int_{x_0}^{x_1}\eta(x)\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y"}\right) + \cdots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)\right]dx = 0$

Since η(x) is an arbitrary. Therefore, we have

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y"}\right) + \cdots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0}$$

Or $$F_y - \frac{d}{dx}(F_{y\prime}) + \frac{d^2}{dx^2}(F_{y"}) + \cdots + \frac{d^n}{dx^n}(F_{y^n}) = 0$$

This is necessary condition for existence of extremal of higher derivatives.

This equation is known as Euler- Poisson equation.

Particular Case

1. If $I[y(x)] = \int_{x_0}^{x_1} F\,(x, y, y', y")\,dx$

Then necessary condition for existence of extremal is

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y"}\right) = 0}$$

2. If $I[y(x)] = \int_{x_0}^{x_1} F\,(x, y, y', y", y''')\,dx$

Then necessary condition for existence of extremal is

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) = 0}$$

Example 6. Find the extremals of the functional

$$\text{I[y(x)]} = \int_a^b F[(y'')^2 - 2(y')^2 + y^2]\, dx$$

Solution: Here $F = (y'')^2 - 2(y')^2 + y^2$

The necessary condition for existence of extremal is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

This implies, $2y - \frac{d}{dx}(-4y') + \frac{d^2}{dx^2}(2y'') = 0$

Or $y + 2\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right) = 0$

Or $y + 2\frac{d^2y}{dx^2} + \frac{d^4y}{dx^4} = 0$

Or $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$

Auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$

This implies, $(m^2 + 1)^2 = 0$

This implies, $m^2 + 1 = 0,\ m^2 + 1 = 0$

This implies, $m = \pm i,\ \pm i.$

Hence, the solution is

$$\boxed{y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x}$$

11.9 *FUNCTIONAL FOR SEVERAL DEPENDENT VARIABLE*

Theorem: The necessary condition for

$$\mathbf{I} = \int_{x_0}^{x_1} F\,(x,\ y_1,\ y_2, \ldots, y_n,\ y_1',\ y_2', \ldots, y_n')\, dx$$

To be extremum is that

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0; \quad \mathbf{i = 0,1,2,3,\ldots,n.}$$

Particular case

1. If $I = \int_{x_0}^{x_1} F\,(x,\ y_1,\ y_2,\ y_1',\ y_2')\,dx$

Then necessary condition for existence of extremal is

$$\boxed{\begin{aligned} \frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) &= 0 \\ \frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) &= 0 \end{aligned}}$$

1. If $I = \int_{x_0}^{x_1} F\,(x,\ y_1,\ y_2,\ y_3,\ y_1',\ y_2',\ y_3')\,dx$

Then necessary condition for existence of extremal is

$$\boxed{\begin{aligned} \frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) &= 0 \\ \frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) &= 0 \\ \frac{\partial F}{\partial y_3} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_3'}\right) &= 0 \end{aligned}}$$

Example 7. Find the extremal of the functional

$$\mathbf{I[y(x),\ z(x)]} = \int_a^b (2zy - 2y^2 + y'^2 - z'^2)\,dx$$

Sol. Here $F = 2zy - 2y^2 + y'^2 - z'^2$ …..… (1)

Where y and z are two dependent variables.

The necessary condition for existence of extremal is

$$\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) = 0$$
$$\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) = 0$$

From (1), it becomes

$$2z - 4y - \frac{d}{dx}(2y') = 0$$

and $$2y - \frac{d}{dx}(-2z') = 0$$

Or $$\frac{d}{dx}\left(\frac{dy}{dx}\right) + 2y = z$$

and $$\frac{d}{dx}\left(\frac{dz}{dx}\right) + y = 0$$

Or $$\frac{d^2y}{dx^2} + 2y = z \quad \text{……(2)}$$

and $$\frac{d^2z}{dx^2} + y = 0 \qquad \text{........ (3)}$$

Now differentiate (2) twice w.r.t. x, we get

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}$$

Using (3), we get

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} = -y$$

i.e, $$(D^4 + 2D^2 + 1)y = 0$$

Auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$

This implies, $$(m^2 + 1)^2 = 0$$

This implies, $$m^2 + 1 = 0, \ \ m^2 + 1 = 0$$

This implies, $$m = \pm i, \ \pm i.$$

Hence, the solution is

$$\boxed{y = (c_1 + c_2x)\, cos\, x + (c_3 + c_4x)\, sin\, x}$$

Now,

$$\frac{dy}{dx} = -(c_1 + c_2x) sin\, x + c_2\, cos\, x + (c_3 + c_4x) cos\, x + c_4\, sin\, x$$

$$\frac{d^2y}{dx^2} = -(c_1 + c_2x) cos - c_2\, sin\, x - c_2\, sin\, x - (c_3 + c_4x) sin\, x + c_4\, cos\, x + c_4\, cos\, x$$

Or $$\frac{d^2y}{dx^2} = -(c_1 + c_2x) cos - (c_3 + c_4x) sin\, x - 2c_2\, sin\, x + 2c_4\, cos\, x$$

Therefore, from (2) , we have

$$z = \frac{d^2y}{dx^2} + 2y$$

$$z = -2c_2\, sin\, x + 2c_4\, cos\, x$$

Hence, the required extremals are

$$y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$$

and $$z = -2c_2\, sin\, x + 2c_4\, cos\, x\ .$$

11.10 *FUNCTIONAL FOR SEVERAL INDEPENDENT VARIABLES*

Theorem: The necessary condition for existence of extremals for functional

$$\mathbf{I} = \iint F\,(x,\ y,\ u,\ u_x,\ u_y)\, dx\, dy \qquad \dots (1)$$

Is $$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) = 0$$

Or $$F_u - \frac{\partial F}{\partial x}(F_{u_x}) - \frac{\partial F}{\partial y}(F_{u_y}) = 0 \qquad \dots (2)$$

Where u(x, y) is continuous and has continuous derivatives upto the second order and is prescribed on the region of integration D.

Equation (2) is known as Euler-Ostrogradsky Equation.

Example 8. Dirchlet's Problem: Find the Euler-Ostrogradsky equation for

$$\mathbf{I[u(x, y)]} = \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy$$

Where the values of u are prescribed on the boundary C of the domain D.

Sol. Here $F = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u_x^2 + u_y^2$ ….. (1)

By Euler-Ostrogradsky Equation

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

From (1)

This implies, $$0 - \frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) = 0$$

Or $$\frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = 0$$

Or $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 0$$

Or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is Laplace equation and its solution gives the required extremal u (x, y).

11.11 *ISOPERIMETRIC PROBLEM*

■**conditional Extremum**

There are some problems in which one has to find geometric figure i.e., extremum under the given condition, such problems is called isoperimetric problems. Such problems is solved by Lagrange's multiplier's method.

To find extremals of the functional

$$I[y(x)] = \int_{x_0}^{x_1} f\ (x, y, y'\ )\ dx \qquad \dots (1)$$

Subject the condition (constraint)

$$J[y(x)] = \int_{x_0}^{x_1} g\ (x, y, y'\ )\ dx = \text{constant} \qquad \dots (2)$$

Consider $\qquad F = f + \lambda g$

Where λ is called Lagrange's multiplier.

Then by Euler equation, the necessary condition for existence of extremal is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$$

This gives required extremal of functional (1) under the condition (2).

Example 9. Find the extremal of the functional

$$\mathbf{I} = \int_0^{\pi} (y'^2 - y^2)\ dx$$

Under the conditions y(0) = 0, y(π) = 1 and subject to constraint

$$\int_0^{\pi} y\ dx = 1$$

Sol. Let $\qquad I = \int_0^{\pi} (y'^2 - y^2)\ dx$

And $\qquad J = \int_0^{\pi} y\ dx = 1$

Here $\qquad f = y'^2 - y^2, \quad g = y$

Consider $\qquad F = f + \lambda g$

i.e., $\qquad F = y'^2 - y^2 + \lambda y$
… (1)

Where λ is called Lagrange's multiplier.

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$$

From (1),

$$-2y + \lambda - \frac{d}{dx}(2y') = 0$$

or $$-2y + \lambda - 2\,\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

or $$2\frac{d^2y}{dx^2} = \lambda - 2y$$

or $$\frac{d^2y}{dx^2} + y = \frac{\lambda}{2}$$

or $$(D^2 + 1)\,y = \frac{\lambda}{2} \qquad \ldots (2)$$

Auxillary equation is

$$m^2 + 1 = 0$$

This implies, m = ± i

$$C.F. = c_1 \cos x + c_2 \sin x$$

Now, $$P.I. = \frac{1}{D^2+1}\,\frac{\lambda}{2}$$

$$= \frac{\lambda}{2}\frac{1}{D^2+1}\,e^{ax}$$

$$= \frac{\lambda}{2}\frac{1}{0+1}\,e^{ax}$$

$$= \frac{\lambda}{2}$$

Hence solution of (2) is

y = C.F. + P.I.

i.e., $$y = c_1 \cos x + c_2 \sin x + \frac{\lambda}{2} \qquad \ldots (3)$$

Now, y(0) = 0, This implies, $c_1 + \frac{\lambda}{2} = 0$

This implies, $$c_1 = -\frac{\lambda}{2} \qquad \ldots (4)$$

y(π) = 1, This implies, $-c_1 + \frac{\lambda}{2} = 1$

This implies, $$c_1 = \frac{\lambda}{2} - 1 \qquad \ldots (5)$$

Solving (4) and (5), we get

$$c_1 = -\frac{1}{2}, \qquad \lambda = 1 \qquad \ldots (6)$$

Therefore, (3) becomes

$$y = -\frac{1}{2}\cos x + c_2 \sin x + \frac{1}{2} \qquad \ldots\ldots (7)$$

Now, from (2), we have

$$\int_0^{\pi} y\, dx = 1$$

Using (7), we get

$$\int_0^{\pi}\left[-\frac{1}{2}\cos x + c_2 \sin x + \frac{1}{2}\right] dx = 1$$

This implies, $\left[-\frac{1}{2}\sin x - c_2 \cos x + \frac{1}{2}\,\mathrm{x}\right]_0^{\pi} = 1$

This implies, $2c_2 + \frac{\pi}{2} = 1$

This implies, $c_2 = \frac{1}{2} - \frac{\pi}{4}$

Putting value of c_2 in (7), we get

$$y = -\frac{1}{2}\cos x + \left(\frac{1}{2} - \frac{\pi}{4}\right)\sin x + \frac{1}{2}$$

or $$\boxed{y = \frac{1}{2}(1 - \cos x) + \frac{1}{4}(2 - \pi)\sin x}$$

This is required extremal.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The value of $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = ?$

Problem 2. The value of Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ?$

Problem 3. Every solution of Euler's equation which satisfies the boundary conditions is called an Extremal or a stationary function of the problem. True/False.

Problem 4. The necessary condition to be extremum is that

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i{}'}\right) = 0; \quad \mathrm{i} = 0,1,2,3,\ldots,\mathrm{n}.$$

Problem 5. The general solution of $\frac{d^4 y}{dx^4} + 2\frac{d^2 y}{dx^2} + y = 0$ is ?

11.12 *SUMMARY*

1. Euler's equation: The necessary condition for functional

$$I[y(x)] = \int_{x_0}^{x_1} F\,(x, y, y')\,dx$$

to be maximum or minimum is that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0 \text{ prescribed.}$$

2. Dirichlet's Problem: Find the Euler-Ostrogradsky equation for

$$I[u(x, y)] = \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy$$

Where the values of u are prescribed on the boundary C of the domainD.

3. **Theorem:** The necessary condition for

$$I = \int_{x_0}^{x_1} F\,(x,\ y_1,\ y_2, \ldots,\ y_n,\ y_1{}',\ y_2{}', \ldots, y_n{}'\,)\,dx$$

To be extremum is that

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i{}'}\right) = 0;\ \ i = 0,1,2,3,\ldots,n.$$

11.13 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

Expansions of function

Series

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11.16 TERMINAL AND MODEL QUESTIONS

Q 1. Find the extremal of the functional

$I[y(x)] = \int_a^b (y^2 + y'^2 - 2ye^x)\, dx$

Q 2. Find the extremal of the functional

$I[y(x)] = \int_0^1 (y'^2 + 12xy)\, dx$

y(0) = 0, y(1) = 1

Q 3. Show that the variational problem of extremizing the functional

$I[y(x)] = \int_1^3 (y\,(3x - y)\, dx$ y(3) = $4\frac{1}{2}$, y(1) = 1 has no solution.

Q.4 Find the curves on which the functional

$$\int_1^2 \frac{x^3}{y'^2}\, dx$$

With y(1) = 0 and y(2) = 3 can be extremized.

Q 5. A light travel in a medium from one point to another point so that the time of travel given by $\int \frac{ds}{v(x,y)}$ where s is arc length and v(x, y) is the velocity of the light in the medium, is maximum, show that path of travel is

$$v\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{\partial v}{\partial x} - \frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{\partial v}{\partial x} = 0$$

Q 6. Find the extremal of the functional

$I[y(x)] = \int_0^{\pi/2} ((y'')^2 - y^2 - x^2)\, dx$

Under conditions y(0) = 0, y'(0) = 0, y(π/2) = 0 and y'(π/2) = -1.

Q 7. Find the extremals of the functional

$I[y(x), z(x)] = \int_0^{\pi/2} y'^2 + z'^2 + 2yz)\, dx$

With y(0) = 0, y(π/2) = 1, z(0) = 0, z (π/2) = -1.

Q 8. Find the extremizing function for

$I[u(x, y)] = \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2u\, f(x, y)\right] dx\, dy$

Where f(x, y) is known function.

Q 9. Find the extremal of the functional I = $\int_0^2 y'^2\, dx$ Under the conditions y(0) = 0, y(2) = 1 and subject to constraint $\int_0^2 y\, dx$ = 1

Q 10. Prove that the isometric problem I = $\int_1^4 y'^2\, dx$

Under the conditions y(1) = 3, y(4) = 24 and

subject to constraint $\int_1^4 y\, dx$ = 36 is parabola.

11.17 *ANSWERS*

1. $y = c_1 e^x + c_2 e^{-x} \frac{x}{2} e^x$

2. y = x^3

4. y = x^2-1

6. y = cos x

7. y = sin x, z = -sin x

8. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = f(x, y)$ gives the required extremal u(x, y)

9. y = $\frac{1}{2}$ x

CHECK YOUR PROGRESS

CYQ 1. 0

CYQ 2. 0

CYQ 3. True

CYQ 4. True

CYQ 5. $y = (c_1 + c_2 x)\, cos\, x\ + (c_3 + c_4 x)\, sin\, x$

UNIT 12: VARIATIONAL PROBLEMS WITH MOVING BOUNDARIES

Contents

12.1 *INTRODUCTION*

Consider the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

We have already learned if the boundary is fixed then the necessary condition for the existence of extremal is given by Euler's equation.

i.e., $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

or $$F_y - \frac{d}{dx}\left(F_{y'}\right) = 0$$

In this chapter, we consider the case when one or both the boundary points can move along the curve.

1.e., if the boundary point (x_1, y_1) moves along the curve $y = \Psi(x)$ and boundary point (x_2, y_2) move along the curve $y = \phi(x)$.

Then, such a problem is known as a variational problem with moving or free boundaries. Then our aim is to find the necessary condition for the existence of extremal of such problem.

12.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) Transversality Conditions

(ii) Orthogonality Conditions

(iii) One Sided Variations

12.3 *TRANSVERSALITY CONDITIONS*

Consider the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx \qquad \text{.......(1)}$$

For the sake of simplicity, let us assume that one of the boundary point (x_1, y_1) is fixed while the other boundary point (x_2, y_2) can move and varies from (x_2, y_2) to $(x_2 + \delta x_2,\ y_2 + \delta y_2)$ on the curve $y = f(x)$.

And let $\delta y(a)$ be variation in y as right end points vary.

Then total variation in I is given by

$$\triangle I = I[y(x) + \delta y(x)] - I[y(x)]$$

$$= \int_{x_1}^{x_2+\delta x_2} F(x, y + \delta y, y' + \delta y')dx - \int_{x_1}^{x_2} F(x, y, y')dx$$

$$= \int_{x_1}^{x_2} F(x, y + \delta y, y' + \delta y')dx + \int_{x_2}^{x_2+\delta x_2} F(x, y + \delta y, y' + \delta y')dx - \int_{x_1}^{x_2} F(x, y, y')dx$$

$$\triangle I = \int_{x_1}^{x_2} [F(x, y + \delta y, y' + \delta y') - F(x, y, y')]dx + \int_{x_2}^{x_2+\delta x_2} F(x, y + \delta y, y' + \delta y')dx \quad \ldots (2)$$

Now, by mean value theorem of integral calculus, we have

$$\int_{x_2}^{x_2+\delta x_2} F(x, y + \delta y, y' + \delta y')dx = [F(x, y, y')]_{x_2+\theta\delta x_2}^{x_2} \delta x_2 \quad \ldots (3)$$

Where $0 < \theta < 1$.

By the virtue of continuity of F, we may write

$$[F]_{x_2+\theta\delta x_2} = [F]_{x_2} + \epsilon \quad \ldots (4)$$

Where $\epsilon \to 0$ as $\delta x_2 \to 0$ and $\delta y_2 \to 0$.

The by (3) and (4), we have

$$\int_{x_2}^{x_2+\delta x_2} F(x, y + \delta y, y' + \delta y')dx = [F]_{x=x_2} \delta x_2. \quad \ldots (5)$$

Now consider,

$$\int_{x_1}^{x_2} [F(x, y + \delta y, y' + \delta y') - F(x, y, y')]dx$$

$$= \int_{x_1}^{x_2} \left[\left\{F(x, y, y') + \delta y \frac{\partial F}{\partial y} + \delta y' \frac{\partial F}{\partial y'} + \cdots\right\} - F(x, y, y')\right] dx$$

[By Tylor's theorem]

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx \qquad \text{[Neglecting higher term]}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y\right) dx + \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y'}\delta y'\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y\right) dx + \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\delta y\right)\right\} dx$$

$$= \int_{x_1}^{x_2} \left[\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}\delta y\right] dx + \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_1}^{x_2} \qquad \ldots (6)$$

Combining (5) and (6), we have

$$\triangle I = \int_{x_1}^{x_2} \left[\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}\delta y\right] dx + \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_1}^{x_2} + [F]_{x=x_2}\delta x_2 \qquad \ldots (7)$$

For extremum, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Therefore, (7) becomes

$$\triangle I = \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_1}^{x_2} + [F]_{x=x_2}\delta x_2 \qquad \ldots (8)$$

Since point (x_1, y_1) is fixed.

$\therefore$ $$\delta y(x_1) = 0$$

and it is clear from figure 2.1 that

$$BD = (\delta y)_{x_2} \quad \text{and} \quad FC = \delta y_2$$

Further $$EC = y'(x_2)\delta x_2$$

And hence BD = FC - EC gives

$$(\delta y)_{x_2} = \delta y_2 - y'(x_2)\delta x_2 \qquad \ldots (9)$$

Therefore, (8) becomes

$$\triangle I = \left[\frac{\partial F}{\partial y'}\delta y\right]_{x=x_2} + [F]_{x=x_2}\delta x_2$$

$$= \left[F_{y'}\right]_{x=x_2}[\delta y]_{x=x_2} + [F]_{x=x_2}\delta x_2$$

$$= \left[F_{y'}\right]_{x=x_2}\{\delta y_2 - y'(x_2)\delta x_2\} + [F]_{x=x_2}\delta x_2 \qquad [\text{ By } (9)\ ]$$

$$= \left[F_{y'}\right]_{x=x_2}\delta y_2 - \left[F_{y'}y'\right]_{x=x_2}\delta x_2 + [F]_{x=x_2}\delta x_2$$

$$\triangle I = \left[F - y'F_{y'}\right]_{x=x_2}\delta x_2 + \left[F_{y'}\right]_{x=x_2}\delta y_2 \qquad \ldots (10)$$

The necessary condition for the extremum is

$$\triangle I = 0$$

$$\Longrightarrow \qquad \left[F - y'F_{y'}\right]_{x=x_2}\delta x_2 + \left[F_{y'}\right]_{x=x_2}\delta y_2 = 0$$

Since δx_2 and δy_2 are not independent.

$$\Longrightarrow \qquad \begin{aligned} \left[F - y'F_{y'}\right]_{x=x_2} &= 0 \\ \left[F_{y'}\right]_{x=x_2} &= 0 \end{aligned} \qquad \ldots (11)$$

For example, if the boundary point (x_2, y_2) moves along the curve

$y = \phi(x)$

$$\therefore \qquad y_2 = \phi(x_2)\delta x_2 \qquad \ldots (12)$$

Then $\qquad \delta y_2 = \phi'(x_2)\,\delta x_2$

Thus, from (10), we get

$$\triangle I = \left[F - y'F_{y'}\right]_{x=x_2}\delta x_2 + \left[F_{y'}\right]_{x=x_2}\phi'(x_2)\,\delta x_2$$

$$= \left[F + (\phi' - y')F_{y'}\right]_{x=x_2}\delta x_2.$$

$\therefore$ The necessary condition for extremal is

$$\triangle I = 0$$

$$\Longrightarrow \qquad \left[F + (\phi' - y')F_{y'}\right]_{x=x_2}\delta x_2 = 0$$

Since δx_2 is arbitrary.

$$\boxed{\left[F + (\phi' - y')F_{y'}\right]_{x=x_2} = 0}$$

Which is the required condition at the free boundary.

This is known as transversality condition.

Particular case: If boundary point (x_1, y_1) moves along the curve

$y = \Psi(x)$ and boundary point (x_2, y_2) move along the curve $y = \phi(x)$.

Then transversality condition is

$$\left[F + (\Psi' - y')F_{y'}\right]_{x=x_1} = 0$$
$$\left[F + (\phi' - y')F_{y'}\right]_{x=x_2} = 0$$

This gives required extremal of functional.

12.4 *ORTHOGONALITY CONITIONS*

When F in (1) is given by

$$A(x,y)(1+y'^2)^{\frac{1}{2}}$$

i.e., $\quad F = A(x,y)(1+y'^2)^{\frac{1}{2}}$

where $A(x,y)$ does not vanish at the movable point x_2.

In this case (13) reduces to

$$A(x,y).\frac{(1+\phi' y')}{\sqrt{1+y'^2}} = 0 \text{ at } x = x_2$$

Since $A(x,y) \neq 0$ at $x = x_2$, we have

$$\frac{(1+\phi' y')}{\sqrt{1+y'^2}} = 0 \text{ at } x = x_2$$

Or $\quad \boxed{y' = -\frac{1}{\phi'} \quad \text{at} \quad x = x_2} \quad$ i.e., $\quad \phi' y' = -1$

Which is the orthogonality condition.

ILLUSTRATIVE EXAMPLES

Example 1. Find the shortest distance between the parabola $y = x^2$ and the straight line $x - y = 5$.

Sol. The problem is to find extremal of the function

$$I[y(x)] = \int_{x_1}^{x_2} ds$$

or $\quad I[y(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx \qquad \ldots (1)$

Here $$F = \sqrt{1 + y'^2}$$

Subject to condition that the point (x_1, y_1) moves along the curve

$$y = x^2$$

And point $(x_2, y_2\ )$ along the curve

$$x - y = 5 \quad i.e., \quad y = x - 5$$

Let $$\Psi(x) = x^2, \quad \phi(x) = x - 5$$

By transversality condition, we have

$$\left[F + (\Psi' - y')F_{y'}\right]_{x=x_1} = 0$$
$$\left[F + (\phi' - y')F_{y'}\right]_{x=x_2} = 0$$

Since $$F = \sqrt{1 + y'^2}, \quad \Psi(x) = x^2, \quad \phi(x) = x - 5$$

$$\therefore \qquad \left[\sqrt{1 + y'^2} + (2x - y')\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_1} = 0 \qquad \ldots (2)$$

And $$\left[\sqrt{1 + y'^2} + (1 - y')\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0 \qquad \ldots (3)$$

Now, since $$F = \sqrt{1 + y'^2}$$

Then by Euler's equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \qquad 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

Or $$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)$$

On integrating, we get

$$\frac{y'}{\sqrt{1+y'^2}} = \text{constant} = c_1'$$

Or $$y' = c_1'\sqrt{1 + y'^2}$$

Squaring on both sides, we get

$$y'^2 = c_1'\ (1 + y'^2)$$

Or $$\left(1 - {c_1'}^2\right) y'^2 = c_1'^2$$

Or $$y'^2 = \frac{c_1'^2}{(1-c_1'^2)}$$

Or $$y\ \frac{c_1'}{\sqrt{1-c_1'^2}}$$

∴ $$\frac{dy}{dx} = c_1$$

On integrating, we get required extremal is:

$$y = c_1 x + c_2 \qquad \ldots (4)$$

∴ $$y' = c_1 \qquad \ldots (5)$$

Since both end point (x_1, y_1) and (x_2, y_2) lies on the extremal (4).

∴ $$c_1 x_1 + c_2 = y_1$$

$$\Rightarrow \qquad c_1 x_1 + c_2 = x_1^2 \qquad [\therefore \quad y = x^2] \qquad \ldots (6)$$

And $$c_1 x_2 + c_2 = y_2$$

$$\Rightarrow \qquad c_1 x_2 + c_2 = x_2 - 5 \qquad [\therefore \quad y = x - 5] \qquad \ldots (7)$$

Now, put $y' = c_1$ in (2), we get

$$\sqrt{(1 + c_1^2)} + (2x_1 - c_1)\ \frac{c_1}{\sqrt{1+c_1^2}} = 0$$

Or $$(1 + c_1^2) + 2x_1 c_1 - c_1^2 = 0$$

Or $$1 + 2x_1 c_1 = 0 \qquad \ldots (8)$$

Similarly, put $y' = c_1$ in (3), we get

$$\sqrt{(1 + c_1^2)} + (1 - c_1)\ \frac{c_1}{\sqrt{1+c_1^2}} = 0$$

Or $$(1 + c_1^2) + (c_1 - c_1^2) = 0$$

Or $$1 + c_1 = 0$$

$$\Rightarrow \qquad c_1 = -1$$

Put $c_1 = -1$ in (8), we get

$$1 - 2x_1 = 0$$

$$\Rightarrow \qquad x_1 = \frac{1}{2}$$

Putting value of c_1 and x_1 in (6), we get

$$-\frac{1}{2}+c_2=\frac{1}{4}$$

$$\Rightarrow \qquad c_2=\frac{3}{4}$$

Putting value of c_1 and c_1 in (7), we get

$$-x_2+\frac{3}{4}=x_2-5$$

$$\Rightarrow \qquad 2x_2=\frac{3}{4}+5=\frac{23}{4}$$

$$\Rightarrow \qquad x_2=\frac{23}{8}$$

Hence, we get

$$c_1=-1, \quad c_2=\frac{3}{4}, \quad x_1=\frac{1}{2}, \quad x_2=\frac{23}{8}$$

Therefore, from (4), required extremal is

$$y=-x+\frac{3}{4}$$

And shortest distance between parabola and straight line is

$$I=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$$

$$=\int_{\frac{1}{2}}^{\frac{23}{8}}\sqrt{1+1}dx \qquad [\because \quad y'=c_1=-1]$$

$$=\sqrt{2}\,[x]_{\frac{1}{2}}^{\frac{23}{8}}$$

$$=\sqrt{2}\left[\frac{23}{8}-\frac{1}{2}\right]=\frac{19}{8}\sqrt{2}.$$

Example 2. Using only the basic necessary condition $\delta I=0$. Find the curve on which an extremum of the functional

$$\boldsymbol{I[y(x)]=\int_0^{x_1}\frac{(1+y')^{\frac{1}{2}}}{y}dx, \quad y(0)=0}$$

Can be achieved if the second boundary point (x_1, y_1) can move along the circumference

$$\boldsymbol{(x-9)^2+y^2=9}$$

Sol. Given functional is

$$I[y(x)] = \int_0^{x_1} \frac{(1+y')^{\frac{1}{2}}}{y} dx \qquad \ldots (1)$$

$$y(0) = 0$$

Here, $$F = F(y, y') = \frac{(1+y'^2)^{\frac{1}{2}}}{y}$$

Therefore, the necessary condition for existence of extremal is

$$F - y'F_{y'} = \text{constant}$$

$$\frac{\sqrt{1+y'^2}}{y} - y'\frac{y'}{y\sqrt{1+y'^2}} = constant$$

Or $$\frac{1+y'^2-y'^2}{y\sqrt{1+y'^2}} = \text{constant} = c_1'$$

Or $$\frac{1}{y\sqrt{1+y'^2}} = c_1'$$

Put $$y' = tan\, t \qquad \ldots (2)$$

∴ $$\frac{1}{y\, sec\, t} = c_1'$$

⟹ $$y = \frac{1}{c_1'} cos\, t$$

⟹ $$y = c_1 \cdot\ cos\, t, \quad \text{where } c_1 = \frac{1}{c_1'} \qquad \ldots (3)$$

Now, $$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dx}{dt} = cot\, t\,(-c_1\, sin\, t)$$

$$\frac{dx}{dt} = -c_1\, cos\, t$$

On integrating, we get

$$x = -c_1\, sin\, t + c_2$$

Or $$x - c_2 = -c_1\, sin\, t \qquad \ldots (4)$$

Now, squaring and adding (3) and (4), we get

$$(x - c_2)^2 + y^2 = c_1^2 \qquad \ldots (5)$$

Now, since $$y(0) = 0$$

Then from (5), we get

$$c_2^2 = c_1^2$$

⟹ $$c_1 = c_2$$

Therefore, equation (5) becomes

$$(x-c_2)^2+y^2=c_1^2 \qquad \dots (6)$$

Or $$x^2+c_1^2-2c_1x+y^2=c_1^2$$

Or $$x^2-2c_1x+y^2=0 \qquad \dots (7)$$

Now, since (x_1, y_1) lies on extremal (7) and given curve

$$(x-9)^2+y^2=9$$

Therefore, we have

$$x_1^2+y_1^2-2c_1x_1=0 \qquad \dots (8)$$

And $$(x_1-9)^2+{y_1}^2=9$$

Or $${x_1}^2+{y_1}^2-18x_1=-72$$

Now subtracting (8) and (9), we get

$$-2c_1x_1+18x_1=72$$

or $$x_1(9-c_1)=36$$

Or $$x_1(c_1 - 9)=-36$$

Now tangent at (x_1, y_1) to given circle $(x-9)^2+y^2=9$ and extremal circle (6) are orthogonal to each other.

∴ $$m_1 \cdot m_2=-1$$

i.e., $$\left(\frac{x_1-c_1}{y_1}\right)\left(\frac{x_1-9}{y_1}\right)=-1$$

or $$(x_1-c_1)(x_1-9)=-y_1^2$$

or $${x_1}^2-9x_1-c_1x_1+9c_1=-{y_1}^2$$

or $${x_1}^2+{y_1}^2-9x_1-c_1x_1+9c_1=0$$

or $$2c_1x_1-9x_1-c_1x_1+9c_1=0 \qquad [\text{ from(8) }]$$

or $$c_1x_1-9x_1+9c_1=0$$

or $$x_1(c_1-9)+9c_1=0$$

or $$-36+9c_1=0 \qquad [\text{ from(10) }]$$

or $$9c_1 = 36$$

or $$c_1 = 4$$

Putting value of c_1 in (7), we get

$$x^2 + y^2 - 8x = 0$$

This is required extremal.

Example 3. Find the shortest distance between the point $(1, 0)$ and the ellipse $4x^2 + 9y^2 = 36$.

Sol. We have to find shortest distance between $A(1,0)$ and $B(x_2, y_2)$ where B lies on the ellipse.

$$4x^2 + 9y^2 = 36 \qquad \ldots (1)$$

The arc length AB of the minimizing curve $y = f(x)$ is given by

$$I[y(x)] = \int_{x_1}^{x_2} ds$$

or $$I[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx \qquad \ldots (2)$$

where the end point $A(1,0)$ is fixed and the other end $B(x_2, y_2)$ lies on (1)

Here $$F = \sqrt{1 + y'^2}$$

By the Euler's equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$\Rightarrow$ $$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

Or $$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)$$

On integrating, we get

$$\frac{y'}{\sqrt{1+y'^2}} = \text{constant} = c_1'$$

Or $$y' = c_1'\sqrt{1 + y'^2}$$

Squaring both the sides, we get

$$y'^2 = c_1'(1 + y'^2)$$

Or $$\left(1 - {c_1'}^2\right)y'^2 = c_1'^2$$

Or $$y'^2 = \frac{c_1'^2}{(1-c_1'^2)}$$

Or $$y = \frac{c_1'}{\sqrt{1-c_1'^2}}$$

∴ $$\frac{dy}{dx} = c_1$$

On integrating, we get

$$y = c_1 x + c_2 \qquad \dots (3)$$

Which is straight line along the required shortest distance is attained.

Now since (3) passes through $A(1,0)$

∴ $$c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$$

Then (3) becomes

$$y = c_1 x - c_1$$

or $$y = c_1(x-1) \qquad \dots (4)$$

Also, it passes through (x_2, y_2)

∴ $$y_2 = c_1(x_2 - 1) \qquad \dots (5)$$

Now from equation (1), we get

$$y = \frac{2}{3}\sqrt{9-x^2} = \Psi(x) \qquad \dots (6)$$

By transversality condition for $\Psi(x)$, we have

$$\left[F + (\Psi' - y')F_{y'}\right]_{x=x_2} = 0$$

$$\Rightarrow \quad \left[\sqrt{1+y'^2} + \left(\frac{2}{3}\cdot\frac{1}{2}\frac{-2x}{\sqrt{9-x^2}} - y'\right)\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0$$

$$\Rightarrow \quad \sqrt{1+c_1^2} - \frac{2}{3}\cdot\frac{x_2}{\sqrt{9-x^2}}\frac{c_1}{\sqrt{1+c_1^2}} - \frac{c_1^2}{\sqrt{1+c_1^2}} = 0$$

$$\Rightarrow \quad 1 + c_1^2 - \frac{2}{3}\cdot\frac{x_2 c_1}{\sqrt{9-{x_2}^2}} - c_1^2 = 0$$

$$\Rightarrow \quad 1 - \frac{2}{3}\cdot\frac{x_2 c_1}{\sqrt{9-{x_2}^2}} = 0$$

$$\Rightarrow \quad 3\sqrt{9-x_2^2} = 2x_2 c_1$$

Squaring both sides, we get

$$9(9-x_2^2) = 4x_2^2 c_1^2 \qquad \dots (7)$$

Now since the point (x_2, y_2) lies on ellipse (1)

$$4x_2^2 + 9y_2^2 = 36 \qquad \dots (8)$$

Now, from (5) and (8), we get

$$4x_2^2 + 9c_1^2(x_2 - 1)^2 = 36$$

$$\Rightarrow \qquad 4(9 - x_2^2) = 9c_1^2(x_2 - 1)^2$$

From (7), we get

$$\Rightarrow \qquad 4 \cdot \frac{4x_2^2 c_1^2}{9} = 9c_1^2(x_2 - 1)^2$$

$$\Rightarrow \qquad 16x_2^2 = 81(x_2 - 1)^2$$

$$\Rightarrow \qquad \left(\frac{x_2}{x_2 - 1}\right)^2 = \frac{81}{16}$$

$$\Rightarrow \qquad \frac{x_2}{x_2 - 1} = \frac{9}{4}$$

$$\Rightarrow \qquad 9x_2 - 9 = 4x_2$$

$$\Rightarrow \qquad 5x_2 = 9$$

$$\Rightarrow \qquad x_2 = \frac{9}{5}$$

Now from equation (7), we get

$$9\left(9 - \frac{81}{25}\right) = 4 \times \frac{81}{25} c_1^2$$

$$\Rightarrow \qquad 9 \times \frac{144}{25} = \frac{4 \times 81}{25} c_1^2$$

$$\Rightarrow \qquad c_1^2 = 4$$

$$\Rightarrow \qquad c_1 = 2$$

Now putting value of c_1 and x_2 in (5), we get

$$y_2 = 2\left(\frac{9}{5} - 1\right)$$

$$\Rightarrow \qquad y_2 = \frac{8}{5}$$

Hence, we get the point

$$B(x_2, y_2) = B\left(\frac{9}{5}, \frac{8}{5}\right)$$

∴ The required shortest distance AB i.e., the distance between $A(1, 0)$ and $B\left(\frac{9}{5}, \frac{8}{5}\right)$ is

$$= \sqrt{\left(\frac{9}{5} - 1\right)^2 + \left(\frac{8}{5} - 0\right)^2}$$

$$= \sqrt{\frac{16}{25} + \frac{64}{25}}$$

$$= \sqrt{\frac{80}{25}} = \frac{4\sqrt{5}}{5}$$

12.5 *VARIATION PROBLEM WITH A MOVING BOUNDARY FOR A FUNCTIONAL DEPENDENT ON TWO FUNCTIONS*

In many problems arising in mathematics, physics, engineering, economics, and other sciences, it is necessary to minimize amounts of a certain functional. Because of the important role of this subject, considerable attention has been devoted to these kinds of problems. Such problems are called variational problems.

Consider the functional

$$I[y(x), z(x)] = \int_{x_1}^{x_2} F\big(x, y(x), z(x), y'(x), z'(x)\big)dx \qquad \ldots (1)$$

Where the lower point $A(x_1, y_1, z_1)$ be fixed and upper point $B(x_2, y_2, z_2)$ move in an arbitrary manner, or along a given curve or surface.

It is clear that extremum of (1) can be obtained by Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

and

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

The general solution of these equations four arbitrary constant.

Since the boundary point $A(x_1, y_1, z_1)$ is fixed, it is possible to eliminate two arbitrary constants. The other two constant can be determined from the necessary condition $\delta I = 0$ for extremum, where δI is the variation of I.

Hence, $\delta I = 0$ gives

Therefore $$\left(F - y'F_{y'} - z'F_{z'}\right)_{x=x_2} \cdot \delta x_2 + \left(F_{y'}\right)_{x=x_2} \cdot \delta y_2 + (F_{z'})_{x=x_2}\delta z_2 = 0 \qquad \ldots (2)$$

If $\delta x_2, \delta y_2$ and δz_2 are independent.

Then, $$\left[F - y'F_{y'} - z'F_{z'}\right]_{x=x_2} = 0$$

$$\left[F_{y'}\right]_{x=x_2} = 0 \quad \text{and} \quad [F_{z'}]_{x=x_2} = 0 \qquad \ldots (3)$$

If the boundary point (x_2, y_2, z_2) moves along some curve

$$y_2 = \phi(x_2), \quad z_2 = \Psi(x_2\}$$

Then $$\delta y_2 = \phi'(x_2)\delta x_2$$

And $$\delta z_2 = \Psi'(x_2)\delta x_2$$

Then, we get

$$\left[F + (\phi' - y')F_{y'} + (\Psi' - z')F_{z'}\right]_{x=x_2} \delta x_2 = 0$$

Since δx_2 is arbitrary, we have

$$\left[F + (\phi' - y')F_{y'} + (\Psi' - z')F_{z'}\right]_{x=x_2} = 0 \quad \ldots (4)$$

This is transversality condition in the problem of extremum of (1).

Along with the equation $y_2 = \phi(x_2), \quad z_2 = \Psi(x_2)$ the condition (4) gives the equations necessary for determining the two arbitrary constants in the general solution of Euler's equation.

Note: If the boundary point $B(x_2, y_2, z_2)$ moves alonga given surface $z_2 = \phi(x_2, y_2)$ then $\delta z_2 = \frac{\partial \phi}{\partial x_2}\delta x_2 + \frac{\partial \phi}{\partial y_2}\delta y_2$ such that the variation δx_2 and δy_2 are arbitrary.

In this case (4) reduces to

$$\left[F - y'F_{y'} + (\phi_x - z')\, F_{z'}\right]_{x=x_2} \delta x_2 + \left[F_{y'} + \phi_y F_{z'}\right]_{x=x_2} \delta y_2 = 0$$

Since δx_2 and δy_2 are independent, we get

$$\left[F - y'F_{y'} + (\phi_x - z')\, F_{z'}\right]_{x=x_2} = 0$$

$$\left[F_{y'} + \phi_y F_{z'}\right]_{x=x_2} = 0$$

These two conditions together with $z_2 = \phi(x_2, y_2)$ enable us to determine two arbitrary constants in the general solution of Euler's equation.

Example 4. Find the extremum of the functional

$$I = \int_{x_1}^{x_2} (\boldsymbol{y'^2 + z'^2 + 2yz})\,\boldsymbol{dx}$$

With y(0) = 0, z(0) = 0 and the point $(\boldsymbol{x_2, y_2, z_2})$ moves over the fixed plane $\boldsymbol{x = x_2}$.

Sol. Here,

$$F = y'^2 + z'^2 + 2yz \qquad \ldots (1)$$

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

and $$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

From (1), we have

$\Rightarrow$ $$2z - \frac{d}{dx}(2y') = 0$$

and $$2y - \frac{d}{dx}(2z') = 0$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} - z = 0 \qquad \ldots (2)$$

and $$\frac{d^2z}{dx^2} - y = 0 \qquad \ldots (3)$$

Now differentiating (2) with respect to x, we get

$$\frac{d^4y}{dx^4} - \frac{d^2z}{dx^2} = 0$$

Or $$\frac{d^4y}{dx^4} - y = 0 \qquad [\text{ By (3) }]$$

Or $$(D^4 - 1)y = 0$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$\Rightarrow \qquad (m^2-1)(m^2+1)=0$$

$$\Rightarrow \qquad m=\pm 1, \pm i$$

Therefore, solution is

$$y=c_1 \cos h\, x+c_2 \sin h\, x+c_3 \cos x+c_4 \sin x \qquad \ldots (4)$$

From (3),

$$z=\frac{d^2y}{dx^2}$$

i.e., $\quad z= c_1 \cos h\, x+c_2 \sin h\, x-c_3 \cos x-c_4 \sin x \quad \ldots (5)$

Now, $\qquad y(0)=0, \quad z(0)=0$

$$\Rightarrow \qquad c_1=c_3=0$$

Now since x_2 is fixed therefore, by condition of moving boundary point (x_2, y_2, z_2).

$$[F_{y'}]_{x=x_2}=0 \quad \text{and} \quad [F_{z'}]_{x=x_2}=0$$

$$\Rightarrow \qquad y'(x_2)=0, \quad z'(x_2)=0$$

Then equation (4) and (5) gives

$$c_2 \cos h\, x_2+c_4 \cos x_2 = 0$$

$$c_2 \cos h\, x_2 - c_4 \cos x_2 = 0$$

If $\cos h\, x_2 \neq 0$ then $c_2=c_4=0$

And therefore, an extremum is attained on $y=0, \quad z=0$.

But if $\quad \cos x_2=0$

Then $c_2=0$ and c_4 remains arbitrary.

Hence, in this case extremum is

$$y=c_4 \sin x$$

$$z=-c_4 \sin x.$$

12.6 *ONE SIDED VARIATIONS*

In some problems in minima of double integrals the surface over which the integral is taken is restricted to lie in a given closed region R. Then it may happen that there is no extremal surface bounded by a previously given space curve which lies entirely in R, but that there is a surface bounded by the given curve, consisting of an extremal surface and a part of the boundary of R, which minimizes the given integral.

Consider the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx \qquad \text{....... (1)}$$

Earlier, we have discussed that the extremal curve passes through end point (x_1, y_1) and (x_2, y_2).

But in this case, suppose that a restriction is imposed on the class of permissible curve in such a way that the curve cannot pass through the point of certain R bounded by the curve $\Psi(x, y) = 0$.

In such a problem that extremizing curve C either passes through a region which is completely outside R or C consists of arcs lying outside R and also consists of parts of the boundary of the region R.

Since on these parts two-sided variation (unaffected by the region R) is possible. We now derive conditions at the points of transition M, N, P and Q.

Now, if co-ordinate of M be $(\bar{x}, \bar{y})$ then the functional can be written as

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

$$= \int_{x_1}^{\bar{x}} F(x, y, y')\, dx + \int_{\bar{x}}^{x_2} F(x, y, y')\, dx$$

$$I = I_1 + I_2 \qquad \text{... (2)}$$

Where $\quad I_1 = \int_{x_1}^{\bar{x}} F(x, y, y')\, dx$

and $\quad I_2 = \int_{\bar{x}}^{x_2} F(x, y, y')\, dx$

If the point $M(\bar{x}, \bar{y})$ moves to neighbouring point $\overline{M}(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y})$ on the boundary of region R and if $y = \phi(x)$ be equation of boundary then

$$\delta I_1 = \left[F + (\phi' - y')F_{y'}\right]_{x=\bar{x}} \delta\bar{x} = 0 \qquad \text{... (3)}$$

Now, $\quad \delta I_2 = I_2(\bar{y} + \delta\bar{y}) - I_2(\bar{y})$

$$= \int_{\bar{x}+\delta\bar{x}}^{x_2} F(x, y, y')dx - \int_{\bar{x}}^{x_2} F(x, y, y')dx$$

$$\Delta I_2 = -\int_{\bar{x}}^{\bar{x}+\delta\bar{x}} F(x, y, y')dx$$

$$\Delta I_2 = -\int_{\bar{x}}^{\bar{x}+\delta\bar{x}} F(x, \phi(x), \phi'(x))dx \qquad [\because\ y = \phi(x)] \qquad \dots (4)$$

Using mean value theorem of integral calculus, we have

$$\Delta I_2 = -[F(x, \phi, \phi')]_{x=\bar{x}} \cdot \Delta\bar{x} + \alpha\,\Delta\bar{x}$$

Where $\alpha \to 0$ as $\Delta\bar{x} \to 0$.

Therefore, this gives

$$\delta I_2 = -[F(x, \phi, \phi')]_{x=\bar{x}} \cdot \Delta\bar{x}$$

… (5)

Combining (3) and (5), we find that

$$\delta I = \delta I_1 + \delta I_2$$

$$\delta I = \left[F(x, y, y') - F(x, y, \phi') - (y' - \phi')F_{y'}(x, y, y')\right]_{x=\bar{x}} \cdot \delta\bar{x}$$

With $y(\bar{x}) = \phi(\bar{x})$

Since $\delta\bar{x}$ is arbitrary.

Then the necessary condition $\delta I = 0$ for an extremum reduces to

$$\left[F(x, y, y') - F(x, y, \phi') - (y' - \phi')F_{y'}(x, y, y')\right]_{x=\bar{x}} = 0 \qquad \dots (6)$$

Applying the mean value theorem to this equation, we get

$$(y' - \phi')\left[F_{y'}(x, y, q) - F_{y'}(x, y, y')\right]_{x=\bar{x}} = 0$$

... (7)

Where $\bar{q}$ lies between q and $y'(\bar{x})$.

Assume $\qquad F_{y'y'}(x, y, q) \neq 0$

In this case $y'(\bar{x}) = \phi'(\bar{x})$ because $q = y'$ only when $y'(\bar{x}) = \phi'(\bar{x})$.

Hence, we conclude that at the point M, the extremal AM meets the boundary curve MN tangentially.

Example 5. Find the shortest path from the point $A(-2, 3)$ to the point $B(2, 3$ located in the region $y \leq x^2$.

Sol. Here we find the extremum of the functional

$$I[y(x)] = \int_{-2}^{2} \sqrt{1 + y'^2} dx \qquad \text{... (1)}$$

Subject to condition that

$$y \leq x^2, \quad y(-2) = 3, \quad y(2) = 3$$

Now, here $\qquad F = \sqrt{1 + y'} \qquad$ … (2)

∴ By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \qquad 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \qquad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

On integrating, we get

$$\frac{y'}{\sqrt{1+y'^2}} = \text{constant} = c_1'$$

$$\Rightarrow \qquad y'^2 = c_1'^2(1 + y'^2)$$

$$\Rightarrow \qquad (1 - c_1'^2)y'^2 = c_1'^2$$

$$\Rightarrow \qquad y'^2 = \frac{c_1'^2}{1-c_1'^2}$$

Or $\qquad y' = \frac{c_1'}{\sqrt{1-c_1'^2}} = c_2$

Or $\qquad \frac{dy}{dx} = c_2$

On integrating, we get

$$y = c_1 + c_2 x \qquad \text{… (3)}$$

This is required extremal curve where c_1 and c_2 are constant.

Now, $$F_{y'y'} = [1 + y'(x)]^{\frac{3}{2}} \neq 0$$

Thus, the required extremal will consist not proportion of the straight-line AP and QB both tangent to the parabola $y = x^2$ and the position POQ at the parabola.

Let $-\bar{x}$ and $\bar{x}$ be the abscissae of P and Q respectively.

$$\left.\begin{aligned} c_1 + c_2\bar{x} = \overline{x^2} \\ c_2 = 2\bar{x} \end{aligned}\right\} \qquad \dots (4)$$

Since tangent QB passes through $(2,3)$.

$\therefore$ $$c_1 + 2c_2 = 3 \qquad \dots (5)$$

Solving (4) and (5), we get two values of $\bar{x}$

i.e., $$\bar{x} = 1 \quad \text{and} \quad \bar{x} = 3$$

The second value is clearly not possible.

$\therefore$ $$\bar{x} = 1$$

Therefore, (4) becomes

$$c_1 + c_2 = 1, \quad c_2 = 2$$

$\Rightarrow$ $$c_1 = -1, \quad c_2 = 2$$

Hence, the required extremal to

$$\begin{cases} -2x - 2, & \text{if} \quad -2 \leq x \leq -1 \\ x^2, & \text{if} \quad -1 \leq x \leq 1 \\ 2x - 1, & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

This, clearly minimize the functional.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The distance between the curve $y_1(x) = x$ and $y_2(x) = x^2$ on the interval [0, 1] is $\frac{1}{4}$ True/False.

Problem 2. The shortest distance between the point $A(-1,5)$ and the parabola $y^2 = x$ is $\sqrt{20}$ True/False.

Problem 3. Ttransversality conditions

$$\left[F + (\Psi' - y')F_{y'}\right]_{x=x_1} = 0$$
$$\left[F + (\phi' - y')F_{y'}\right]_{x=x_2} = 0$$
True/False.

Problem 4. The shortest distance between the point $A(-1,3)$ and the straight line $y = 1 - 3x\ is\ \frac{1}{\sqrt{110}}$ True/False.

Problem 5. When the prescribed end conditions are homogeneous, we shall use a modified method. True/False

Problem 6. The functional $\int_{x_1}^{x_2}\left(y'^2 + x^2\right)dx$ with y(1) = 1 achieves its:

(a) Weak maximum on all its extremals

(b) Weak minimum on all its extremals

(c) Weak maximum on some, but not on all of its extremals

(d) Weak minimum on some but not all of its extremals

Problem 7. The shortest distance between the circle $x^2 + y^2 = 4$ and the straight line $2\ x\ + y\ =\ 6\ is\ \sqrt{5}\left(\frac{6}{5} - \frac{2}{\sqrt{5}}\right)$.

Problem 8. Extremals of the functional

$\int_0^{\pi/2}\left(y''^2 - y^2 + x^2\right)dx$

(a) one parameter family of curves

(b) two parameter family of curves

(c) three parameter family of curves

(d) four parameter family of curves

12.7 *SUMMARY*

1. If the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

Such that the boundary point (x_1, y_1) is fixed and other boundary point (x_2, y_2) is moving along curve $y = \phi(x)$.

Then Euler's equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

And transversality condition.

$$\left[F + (\phi' - y')F_{y'}\right]_{x=x_2} = 0$$

gives extremal.

2. If the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

Such that the boundary point (x_1, y_1) moves along curve $y = \Psi(x)$ and other boundary point (x_2, y_2) is moves along curve $y = \phi(x)$.

Then Euler's equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

And transversality condition.

$$\left[F + (\Psi' - y')F_{y'}\right]_{x=x_1} = 0$$
$$\left[F + (\phi' - y')F_{y'}\right]_{x=x_2} = 0$$

gives the extremal of functional.

3. If the functional

$$I[y(x), z(x)] = \int_{x_1}^{x_2} F\big(x, y(x), z(x), y'(x), z'(x)\big) dx$$

Such that the point (x_1, y_1, z_1) be fixed and point (x_2, y_2, z_2) moves in an arbitrary manner, or along a given curve or surface.

$$y = \phi(x), \quad z = \Psi(x)$$

Then Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

and

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

and transversality condition

$$\left[F + (\phi' - y')F_{y'} + (\Psi' - z')F_{z'}\right]_{x=x_2} = 0$$

Gives extremal.

12.8 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

Expansions of function

Series

12.9 *REFERENCES*

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12.10 *SUGGESTED READING*

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12.11 TERMINAL AND MODEL QUESTIONS

TQ 1. Use the calculus of variation to find the shortest distance between the line $y = x$ and the parabola $y^2 = x - 1$.

TQ 2. If l is not prescribed show that the extremals corresponding to the problem $\delta \int_0^1 y'^2 dx = 0$, $y(0)$, $y(l) = \sin l$ are of the form

$y(x) = 2 + 2x \cos l$ where l satisfies the transcendental equation

$2 + 2l \cos l - \sin l = 0.$

TQ 3. If l is not prescribed show that the extremals of the problem

$$\delta \int_0^1 [y'^2 + 4(y - l)]dx = 0 \; y(0) = 2, \quad y(l) = l^2$$

Are of the form $y(x) = x^2 + 2 - \frac{2x}{l}$. Where l is root of equation

$2l^4 - 2l^3 + l = 0.$

TQ 4. Find the shortest distance between the point $A(-1, 5)$ and the parabola $y^2 = x$.

TQ 5. Find the shortest distance between the point $A(-1, 3)$ and the straight line $y = 1 - 3x$.

TQ 6. Find the shortest distance between the circle $x^2 + y^2 = 1$ and the straight line $x + y = 4$.

TQ 7. Find the shortest distance between the circle $x^2 + y^2 = 4$ and the straight line $2x + y = 6$.

TQ 8. Find the shortest distance between the parabola $y^2 = 4x$ and the circle $(x - 9)^2 + y^2 = 4$.

12.12 *ANSWERS*

TQ1 $\frac{3\sqrt{2}}{8}$

TQ4 $\sqrt{20}$

TQ5 $\frac{1}{\sqrt{10}}$

TQ6 $2\sqrt{2}-1$

TQ7 $\sqrt{5}\left(\frac{6}{5}-\frac{2}{\sqrt{5}}\right)$

TQ8 $\sqrt{8}\left(2-\frac{1}{\sqrt{5}}\right)$

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. False

CYQ 5. False

CYQ 6. (b)

CYQ 7. True

CYQ 8. (d)

UNIT 13: SUFFICIENT CONDITIONS FOR AN EXTREMUM

Contents

13.1 *INTRODUCTION*

The sufficient conditions in the calculus of variations have recently received a great deal of attention and it would seem fitting that attempts be made to simplify their discussion whenever possible, and to render the agreement more exact between the known necessary and the known sufficient conditions. Such is the purpose of this paper, which also seeks to present the sufficient conditions in compact form. The work will to a large extent follow lectures delivered at Göttingen by Professor Hubert, 1899-1901. In mechanics, Hamilton's principle and Lagrange's equation can be derived very easily with the help of calculus of variation. In this unit learner learnt about the sufficient condition of Legendre to find out the nature of extremal.

13.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) Proper Field

(ii) Legendre condition

(iii) Weak and Strong Extremum

(vi) Hamilton's principle

(v) Lagrangian of a system

13.3 *PROPER FIELD*

A family of curve y = y(x, c) where c is a parameter is said to form a proper field in a given region D of the xy-plane if one and only one curve of the family passes through every point of the region D.

■ **Jacobi Condition:** consider one parameter family of plane curves $\emptyset(x, y, c) = 0$ where c is parameter.

For this family c – discriminant is the locus of point of intersection of

$$\emptyset(x, y, c) = 0 \text{ and } \frac{\partial \emptyset}{\partial c} = 0$$

Which include envelope of the family, the locus of cuts and locus of nodal points too.

If we have a pencil of curves with centre at A(x_1, y_1) then A(x_1, y_1) also belongs to this locus.

Suppose a pencil of extremals passing from A(x_1, y_1) such that the $\emptyset(x, y) = 0$ is the c – discriminant.

Then the envelope Γ of this pencil of extremals will belongs to

$\emptyset(x, y) = 0$.

Every extremals of the family will touch this envelope Γ . the point A_1 where the extremal

y = y(x) touches the envelope is called the conjugate point of A. if B(x_2, y_2) be a point which lies in between A and A_1 then the extremals of the pencil close to AB do not intersect. Hence, it follows that extremal close to AB from a central field including the arc AB.

Now for the extremal AB_1 it follows that the conjugate point A_2 of A lies in between A and B_1 and the curves of the curves of the pencil closed to AB_1 intersect.

Therefore, the extremal AB_1 cannot be embedded in a central field.

Hence, to embed an arc AB of the extremal in a central field of extremals it is sufficient that the conjugate point of A does not lie on the curve.

This is known as **Jacobi condition.**

■ **Mathematical Definition:** let y = y(x, c) be the equation of pencil of extremals with c is parameter and A Centre. The parameter c is regarded as slope $y' = \frac{dy}{dx}$ of the extremals at A.

The c–discriminant is given by y = y(x, c) and $\frac{\partial y}{\partial c}$ = 0.

Let u = $\frac{\partial y(x,c)}{\partial c}$ which is a function of x along for every fixed curve of family.

For the extremum of

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad \text{........ (1)}$$

y = y(x, c) is a solution of Euler's equation.

Therefore $F_y\left[x, y(x, y);\ y'_x(x, c)\right] - \frac{d}{dx}F'_y\left[x, y(x, c);\ y'_x(x, c)\right] = 0$

Differentiating w.r.t. c, we get

$$(F_{yy} - \frac{d}{dx}F_{yy'})u - \frac{d}{dx}(F_{y'y'}\, u') = 0 \qquad \text{....... (2)}$$

This is Jacobi equation.

Let y(x) is a solution of Euler's equation with equation with c = c_0 for the extremal AB.

Further, if the solution u = $\frac{\partial y}{\partial c}$ vanishes at A(x_1, y_1) then Centre of the pencil belongs to

the c – discriminant curve, also vanish at some point of the internal

$x_1 < x < x_2$, then the point conjugate to A given by

$$y = y(x, c_0) \text{ and } \left(\frac{\partial y}{\partial c}\right)_{c=c_0} = 0$$

lies on the arc AB of the extremal with B at the point (x_2, y_2).

If there exist a solution of (2) which vanishes for x = x_1 and does not vanish at any point in

$x_1 \leq x \leq x_2$ then are no points conjugate to A lying on arc AB.

Thus, the Jacobi condition is satisfied and the arc at the extremal can be embedded in a central field of the extremals with Centre at A.

13.4 *SUFFICIENT CONDITION FOR EXTREMUM (LEGENDRE CONDITION)*

Legendre condition is sufficient condition to find out the nature of extremal.

Consider the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad \ldots\ldots\ldots (1)$$

With $\qquad y(x_1) = y_1$, $y(x_2) = y_2$

Let C be the extremal curve of functional (1) and $\bar{C}$ be neighboring curve of C.

Therefore, consider

$$I_1 = \int_C F(x, y, y')dx \text{ for extremal curve C and}$$

$$I_2 = \int_{\bar{C}} F(x, y, y')dx \text{ for extremal curve } \bar{C}$$

Let p = $\frac{dy}{dx}$ on C

Consider the auxiliary functional

$$\int_{\bar{C}} \left[F(x, y, p) + \left(\frac{dy}{dx} - p\right) F_p(x, y, p)\right] dx$$

The integral in this integration is an exact differentiation of function. Therefore

it is independent of path

Therefore $\int_{\bar{C}} \left[F(x, y, p) + \left(\frac{dy}{dx} - p\right) F_p(x, y, p)\right] dx$

$$= \int_C F(x, y, y')dx \qquad \ldots\ldots (2)$$

Now $\Delta I = \int_{\bar{C}} F(x, y, y')dx - \int_C F(x, y, y')dx$

$$= \int_{\bar{C}} F(x,y,y')dx - \int_{\bar{C}} \left[F(x,y,p) + \left(\frac{dy}{dx} - p\right) F_p(x,y,p)\right] dx$$

$$= \int_{\bar{C}} [F(x,y,y')dx - F(x,y,p) - \left(\frac{dy}{dx} - p\right) F_p(x,y,p)] dx$$

Or $\Delta I = \int_{\bar{C}} E(x,y,y')dx$

Where E (x, y, p, y') $= F(x,y,y') - F(x,y,p) - (y' - p)F_p(x,y,p)$

This E (x, y, p, y') is called **Weirstrass function.**

Now if E ≤ 0, then extremal is maximum. And if E ≥ 0, then extremal is minimum.

This is required **Legendre condition.**

ILLUSTRATIVE EXAMPLES

Example 1. Find the Weirstrass function and test the extremal of the functional

$$I[y(x)] = \int_0^a y'^2 dx \text{ and } y(0) = 0,\ y(a) = b \text{ where } a > 0,\ b > 0.$$

Sol. Here F = y'^2 ……. (1)

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Therefore $0 - \frac{d}{dx}\left(y'^2\right) = 0$

Therefore $\frac{d}{dx}\left(y'^2\right) = 0$

On integration, we get

$$y'^2 = constant$$

Or $\left(\frac{dy}{dx}\right)^2$ = constant

Or $\frac{dy}{dx}$ = constant = c_1.

Again integrating we get

$$y = c_1 x + c_2 \qquad \text{.........} \ (2)$$

Now, $y(0) = 0 \ \Rightarrow c_2 = 0$

$y(a) = b \ \Rightarrow c_1 a = b \ \Rightarrow c_1 = \frac{b}{a}$

putting value of c_1 and c_2 in (2), we get

$$y = \frac{b}{a}x \qquad \text{.......} \ (3)$$

this is required extremal.

Weirstrass Function

The Weirstrass function is

$E(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)$

$E(x, y, p, y') = y'^3 - p^3 - (y' - p).3p^2$

$= y'^3 - p^3 - 3p^2y' + 3p^3$

$= y'^3 + 2p^3 - 3p^2y'$

Therefore $E(x, y, p, y') = (y' - p)^2(y' + 2p)$

This is required Weirstrass function.

Now, since $E(x, y, p, y') = (y' - p)^2(y' + 2p) \geq 0$

Therefore, extremal is maxima.

13.5 *WEAK AND STRONG EXTEMUM*

Consider the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad \text{........} \ (1)$$

With $y(x_1) = y_1$, $y(x_2) = y_2$

Let C be the extremal curve of the given functional.

Also, assume that the extremal of curve C is included in a field of extremals.

Ther Legendre condition for weak extremum and strong extremum are:

■ **Weak Extremum**

1. The curve C is extremal satisfying the boundary condition.

2. Jacobi condition must Satisfied.

3. The Weirstrass function E does not change sign at any point (x, y) close to the curve C and for arbitrary values of y' close to p(x, y) on the extremals.

4. For weak minimum E ≥ 0 or $F_{y'y'} > 0$ on C and for weak maximum $F \leq 0$ or $F_{y'y'} < 0$ on C.

■ **Strong Extremum**

1. The curve C is extremal satisfying the boundary condition.

2. The extremal C is embedded in a field of extremals.

3. At a point (x, y) closed to the curve C and for arbitrary value of y', the Weirstrass function E does not change sign.

4. For strong minimum E ≥ 0 or $F_{y'y'} > 0$ at point close to C and also arbitrary value of y' and for strong maximum E ≤ 0 or $F_{y'y'} < 0$ at points closed curve C and also for arbitrary value of y'.

ILLUSTRATIVE EXAMPLES

Example 1. Test for the extremal of the functional

$$I[y(x)] = \int_0^2 (e^{y'} + 3)dx \text{ and } y(0) = 0, y(2) = 1.$$

Sol. Here F(x, y, y') $= e^{y'} + 3$ ……. (1)

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Therefore $0 - \frac{d}{dx}\left(e^{y'}\right) = 0$

Therefore $\frac{d}{dx}\left(e^{y'}\right) = 0$

Therefore $e^{y'} y'' = 0$

Therefore $y'' = 0 \quad (\because e^{y'} \neq 0)$

On integrating, we get y = $c_1 x$ + c_2 …… (2)

Hence the extremal of the given functional is attained only on the straight line.

Now, from (2)

y(0) = 0 $\Rightarrow c_2 = 0$

y(2) = 1 $\Rightarrow 2c_1 = 1$

therefore $c_1 = \frac{1}{2}$, $c_2 = 0$

then (1) becomes y = $\frac{1}{2}x$

hence the extremal satisfying the boundary condition is y = $\frac{x}{2}$ which is including in the central field of extremals y = $c_1 x$.

Now, F(x, y, y') $= e^{y'} + 3$

Therefore $F_{y'} = e^{y'}$ and $F_{y'y'} = e^{y'} > 0$ for any value of y'.

Therefore, by Legendre condition, the given functional is strong minimum on extremal y = $\frac{x}{2}$.

Example 2. Test for the extremal of the functional

$$I[y(x)] = \int_0^a (y'^2 - y^2)dx \text{ and } y(0) = 0,\ y(a) = 0,\ a > 0.$$

Sol. Here $F(x, y, y') = y'^2 - y^2$ ……. (1)

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Therefore $-2y - \frac{d}{dx}(2y') = 0$

Therefore $\frac{d}{dx}\left(\frac{dy}{dx}\right) + y = 0 \Rightarrow \frac{d^2y}{dx^2} + y = 0$

Auxiliary equation $m^2 + 1 = 0$

$\Rightarrow$ m = i, - i

Therefore the general solution is

$$y = c_1 \cos x + c_2 \sin x \quad \text{……… (2)}$$

Now, $y(0) = 0 \Rightarrow c_1 = 0$ and $y(a) = 0 \Rightarrow c_2 \sin a = 0$

Now if $a \neq n\pi$, i.e. $\sin a \neq 0$

hence we get $c_1 = 0$ and $c_2 = 0$

hence if $a \neq n\pi$, the extremum is attained only on the straight line y = 0.

Now, if $a < \pi$, then pencil at extremals $y = c_2 \sin x$ with centre (0, 0) for the central field.

And now science $F(x, y, y') = y'^2 - y^2$

Therefore $F_{y'} = 2y'$ and $F_{y'y'} = 2 > 0$ for all y'

Therefore, a strong minimum is attained on y = 0 for $a < \pi$.

For $a > \pi$, extremals $y = c_2 \sin x$ neither form a proper field nor form a central field.

Hence, for $a > \pi$, minimum is not attained on y = 0.

Example 3. Investigate for the extremal of the functional

$$I[y(x)] = \int_0^a (x + 2y - \frac{1}{2}{y'}^2)dx \text{ and } y(0) = 0,\ y(1) = 0.$$

Sol. Here F(x, y, y') $= x + 2y - \frac{1}{2}{y'}^2$ .... (1)

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Therefore $2 - \frac{d}{dx}\left(\frac{1}{2}2y'\right) = 0$

$\Rightarrow$ $2 - \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0 \Rightarrow \frac{d^2y}{dx^2} = 2$

On integrating, we get $y = x^2 + c_1x + c_2$ .... (2)

Now, $y(0) = 0 \Rightarrow c_2 = 0$ and

$y(1) = 0 \Rightarrow 1 + c_1 + c_2 = 0$

$\Rightarrow$ $c_1 = -1$, $c_2 = 0$

Therefore (2) becomes $y = x^2 - x$

Hence the extremal satisfying the boundary condition is $y = x^2 - x$ which is included in the central field of extremals $y = {x_1}^2 + c_1x$. Whose centre at (0, 0)

Now, since $F = x + 2y + \frac{1}{2}{y'}^2$

Therefore $F_{y'} = \frac{1}{2}.2y' = y'$

Therefore $F_{y'y'} = 1 > 0$

Therefore, by Legendre condition the given functional is strong minimum on

extremal $y = x^2 - x$.

13.6 *APPLICATION OF THE CALCULUS OF VARIATION*

The calculus of variation is widely applied in mechanics, mechanical engineering control theory etc. and used in solving some important problem in economics.

In mechanics, Hamilton's principle and Lagrange's equation can be derived very easily with the help of calculus of variation.

13.7 *HAMILTION'S PRINCIPLE*

A particle moves in a conservative field in such a way that

$\int_{t_1}^{t_2}(T-V)dt$ is extremum, actually a minimum, where T is kinetic energy and V is potential energy of the system.

Proof. Let there be n particles of mass $m_i; i = 1, 2, 3, \dots, n$ and their position vectors are

$r_i; i = 1, 2, 3, \dots, n$ relative to co-ordinate system. Let $F_i; i = 1, 2, 3, \dots, n$ be applied force acting on the i^{th} particle.

Then, the equation of the motion of particle are

$$m_i \frac{d^2 r_i}{dt^2} = F_i; i = 1, 2, 3, \dots, n \qquad \dots\dots (1)$$

And from these equations, we can determine the path $c_i; i = 1, 2, 3, \dots, n$ transversed by n particles.

Next, assume that the path of i^{th} particle has been varied without changing the end points.

If the variation of the path be δr_i , sometimes called virtual displacement.

Then from (1), we get $\left(m_i \frac{d^2 r_i}{dt^2}\right) \partial r_i = F_i \delta r_i \qquad \dots\dots (2)$

Summing for all particles, we get

$$\sum_{i=1}^{n} m_i \frac{d^2 r_i}{dt^2} \delta r_i = \sum_{i=1}^{n} F_i \delta r_i \qquad \dots\dots (3)$$

Where right hand side indicates the total work done δW under the displacement of the path, then $\delta W = \sum_{i=1}^{n} F_i \delta r_i$ or $\delta W = \sum_{i=1}^{n} m_i \frac{d^2 r_i}{dt^2} \delta r_i$ …… (4)

Now, the kinetic energy of the system is

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\frac{dr_i}{dt}\right)^2 \quad …….. (5)$$

Therefore $\delta T = \sum_{i=1}^{n} m_i \frac{dr_i}{dt} \delta(\frac{dr_i}{dt}) = \sum_{i=1}^{n} m_i \frac{dr_i}{dt} . \frac{d}{dt} (\delta r_i)$ …… (6)

But $\frac{d}{dt} (\frac{dr_i}{dt} \delta r_i) = \frac{d^2 r_i}{dt^2} . \delta r_i + \frac{dr_i}{dt} . \frac{d}{dt} (\delta r_i)$

Multiplying both side by m_i and summing from I = 1 to n, we get

$$\sum_{i=1}^{n} m_i \frac{d}{dt} [\frac{dr_i}{dt} \delta r_i] = \sum_{i=1}^{n} m_i \frac{d^2 r_i}{dt^2} . \delta r_i + \sum_{i=1}^{n} m_i \frac{dr_i}{dt} . \frac{d}{dt} (\delta r_i)$$

$$= \delta T + \delta W \quad \text{by (4) and (6)}$$

Hence, $\delta T + \delta W = \sum_{i=1}^{n} m_i \frac{d}{dt} [\frac{dr_i}{dt} \delta r_i]$

Integrate w.r.t. t from t_1 to t_2 , we get

$\int_{t_1}^{t_2} (\delta T + \delta W) dt = \left[m_i \frac{dr_i}{dt} \delta r_i\right]_{t_1}^{t_2} = 0$ because $\delta r_i = 0$ at t_1 and t_2.

Therefore $\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0$ ……. (7)

Now, if force is conservative then W = -V

Where V = potential function.

Therefore (7) becomes $\int_{t_1}^{t_2} (\delta T + \delta(-V)) dt = 0$

Or $\delta \int_{t_1}^{t_2} (T - V) dt = 0$

From this it can be stated that in a conservative field, a system moves from t_1 to t_2 in such a way that $\int_{t_1}^{t_2} (T - V) dt$ is an extremum, actually a minimum.

Hence, we conclude that in a conservative field, a system moves from t_1 to t_2 in such a way that $\int_{t_1}^{t_2}(T - V)dt$ is minimum.

13.8 *LAGRANGIAN OF A SYSTEM*

The expression L = T – V where T is kinetic energy and V is potential energy of the system is called Lagrangian.

Note: ■ Lagrangian L is function of the co-ordinate of particle i.e. generalized co-ordinate (q_i), their velocities $(\dot{q_i})$ and time (t).

i.e. L = L$(q_1, q_2, \ldots \ldots q_n, \dot{q_1}, \dot{q_2}, \ldots.., \dot{q_n},$ t) hence, Hamilton's principle states that for a conservation system. $\int_{t_1}^{t_2} \delta L \, dt = 0$.

13.9 *LAGRANGE'S EQUATION*

Consider Lagrangian L = L$(q_1, q_2, \ldots \ldots q_n, \dot{q_1}, \dot{q_2}, \ldots.., \dot{q_n})$ ….. (1)

Where $q_1, q_2, \ldots \ldots q_n$ are generalized co-ordinate of system of particles.

Now, from (1) $\delta L = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j$ ……. (2)

By Hamilton's principle $\int_{t_1}^{t_2} \delta L \, dt = 0$

Using (2), we get $\int_{t_1}^{t_2} \left[\sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right] dt = 0$

Therefore $\int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j dt + \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \, dt = 0$

Therefore $\int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j dt + \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} (\delta \dot{q_j}) \, dt = 0$

Therefore

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j dt + \left[\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_j}) \delta q_j dt = 0$$

Since δq_j = 0 at t_1 and t_2

Therefore $\int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j dt - \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{d}{dt} (\frac{\partial L}{\partial \dot{q_j}}) \delta q_j dt = 0$

Therefore $\int_{t_1}^{t_2} \left[\sum_{j=1}^{n} \frac{\partial L}{\partial q_j} - \frac{d}{dt} (\frac{\partial L}{\partial \dot{q_j}})\right] \delta q_j dt = 0$

Therefore $\sum_{j=1}^{n} \left\{\frac{\partial L}{\partial q_j} - \frac{d}{dt} (\frac{\partial L}{\partial \dot{q_j}})\right\} \delta q_j = 0$

Since δq_j are arbitrary and independent to each other.

Therefore $\frac{\partial L}{\partial q_j} - \frac{d}{dt} (\frac{\partial L}{\partial \dot{q_j}}) = 0$ this is Lagrange's equation.

ILLUSTRATIVE EXAMPLES

Example 1. Using Hamilton's principle, find the equation of one dimensional harmonic oscillator.

Sol. A system executing harmonic motion may be referred as harmonic oscillator.

e.g. A simple pendulum when the displacement of the motion is small is an example of harmonic oscillator.

Now, the kinetic energy of harmonic oscillator $= \frac{1}{2} m\dot{x}^2$

Potential energy of harmonic oscillator

V = - $\int F\, dx = \int kx\, dx = \frac{1}{2} kx^2$

Therefore the Lagrangian L = T – V

$$= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

By Hamilton's principle $\delta \int_{t_1}^{t_2} L\, dt = 0$

Therefore $\delta \int_{t_1}^{t_2} (\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2)\, dt = 0$

Therefore $\int_{t_1}^{t_2} \delta(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2)\, dt = 0$

Therefore $\int_{t_1}^{t_2} (m\dot{x}\delta\dot{x} \ - kx\ \delta x)\ dt = 0$

Or $\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x)dt \ - \int_{t_1}^{t_2} kx\ \delta x\ dt = 0$

Therefore $\left[mx\dot{\delta}x\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\frac{d}{dt}(\dot{x})\delta x dt \ - \int_{t_1}^{t_2} kx\ \delta x\ dt = 0$

Therefore 0 - $\int_{t_1}^{t_2} m\frac{d}{dt}(\dot{x})\delta x dt \ - \int_{t_1}^{t_2} kx\ \delta x\ dt = 0$ [since δx = 0 at t_1 and t_2]

Or $\int_{t_1}^{t_2} (m\ddot{x} + kx)\ dt = 0$

Since δx is an arbitrary

Therefore $m\ddot{x} + kx$ = 0.

This is required equation of motion of one dimensional harmonic oscillator.

<u>CHECK YOUR PROGRESS</u>

Problem 1. The function E (x, y, p, y') $= F(x,y,y') - F(x,y,p) - (y'-p)F_p(x,y,p)$ is called …….

Problem 2. Extremal is maximum if E $\leq$?

Problem 3 Extremal is minimum if E ≥ 0. True/False.

Problem 4. A central field is called a field of extremals, if it is not formed by a family of extremals. True/False

Problem 5. Legendre condition is sufficient condition to find an extremal of the functional. True/False

13.10 *SUMMARY*

1. Any family of curve y = y(x, c) in a given region D of xy – plane is said to be form Proper Field if one and only one curve of family posses through every point of the region D.

2. A family of curve y = y(x, c) is said to be form central field over domain D if:

(i) Curves cover D without self intersection.

(ii) All curve passes through single point (x_0, y_0).

3. Any family of curve y = y(x, c) passes through a single point (x_0, y_0) which is not in domain D. then point (x_0, y_0) is called centre of pencil of curves.

4. A central field is called a field of extremals, if it is formed by a family of extremals.

5. Legendre condition is sufficient condition to find an extremal of the functional.

6. The function E (x, y, p, y')

$= F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)$ is called Weierstrass function.

7. Extremal is maximum if E ≤ 0 and Extremal is called minimum if E$\geq$ 0. This is Legendre condition.

13.11 *GLOSSARY*

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

Expansions of function

Limits

13.12 *REFERENCES*

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13.13 *SUGGESTED READING*

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13.14 TERMINAL AND MODEL QUESTIONS

Q 1. Define Hamilton's principle.

Q 2. Define Legendre sufficient condition for extremal.

Q 3. Prove that $\frac{\partial L}{\partial q_j} - \frac{d}{dt}(\frac{\partial L}{\partial q_j}) = 0$.

Q 4. Define proper field and Jacobi Condition.

Q 5. Show that the Jacobi condition for the Central field of extremals for

I[y(x)] = $\int_0^{\pi/2} \left(xyy' + y^2 - \frac{y'^2}{2}\right) dx$, u(0) = 0 where u = δy is fulfilled. Also Show that for extremals the functional is maximum.

Q 5. Show that the Jacobi condition satisfied for the extremal of the functional

I[y(x)] = $\int_0^{a} \left(y'^2 + y^2 + x^2\right) dx$ which passes through (0, 0) and (a, 0).

Q 6. Investigate for the extremal of the functional

I[y(x)] = $\int_0^{a} (x + 2y - \frac{1}{2}y'^2) dx$ and y(0) = 0, y(1) = 0.

Q 7. Test for the extremal of the functional

I[y(x)] = $\int_0^{2} (e^{y'} + 3) dx$ and y(0) = 0, y(2) = 1.

13.15 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. Weierstrass function

CYQ 2. 0

CYQ 3. True

CYQ 4. False

CYQ 5. True

UNIT 14: Variational Method for Boundary Value Problems

(Ordinary and partial differential equation)

Contents

14.1 *INTRODUCTION*

The solution of Euler's equation with boundary conditions gives extremal of functional. This approach gives a method of solving a boundary value problem approximately by assuming a trivial solution satisfying the given boundary conditions and extremizing the integral whose integrated is found from the given differential equation. The sufficient conditions in the calculus of variations have recently received a great deal of attention and it would seem fitting that attempts be made to simplify their discussion whenever possible, and to render the agreement more exact between the known necessary and the known sufficient conditions.

14.2 *OBJECTIVE*

At the end of this topic learner will be able to understand:

(i) Rayleigh – Ritz Method for Ordinary differential equation

(ii) Rayleigh – Ritz Method for partial differential equation

(iii) Galerkin's method

(vi) Kantorovich Method

14.3 *RAYLEIGH RITZ METHOD*

(For ordinary Differential equation)

The Rayleigh–Ritz method is a variational method to solve the eigenvalue problem for elliptic differential operators, that is, to compute their eigenvalues and the corresponding eigenfunctions. It is the direct counterpart of the Ritz method for the solution of the assigned boundary value problems. The Rayleigh–Ritz method has the advantage of being based on minimal, very general assumptions and produces optimal solutions in terms of the approximation properties of the underlying trial spaces. The theory of the Rayleigh–Ritz method has to a large extent been developed in the context of finite element methods The Rayleigh Ritz method utilize the principle of minimizing total potential energy in a system and calculus of variations. It employs the use of trial functions that satisfy specific conditions, including boundary conditions, to solve boundary value problems.

Consider an ordinary Differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=f(x) \qquad \text{……. (1)}$$

With boundary condition $y(x_1)=y_1$, $y(x_2)=y_2$

In order to solve differential equation (1) by variational method, first we construct F(x, y, y') in such way that Euler's equation of functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad \text{……… (2)}$$

Becomes given differential equation (1).

Next, the proper choice of $y_0(x)$, $y_1(x)$, $y_2(x)$, …., $y_n(x)$

We get $y(x) = y_0(x)+c_1\, y_1(x)+ c_2y_2(x)+ \dots + c_ny_n(x)$ …… (3)

Out of these functions $y_0(x)$ is the simplest function which satisfy given boundary condition

i.e. $y_0(x_1) = y(x_1) = y_1$

and $y_0(x_2) = y(x_2) = y_2$

and other function $y_1(x)$, $y_2(x)$, …., $y_n(x)$ are linearly independent function satisfy homogeneous boundary condition

$y_k(x_1) = y_k(x_2) = 0$; k 1, 2, 3, ….

i.e. $y_1(x_1) = y_1(x_2) = 0$

$y_2(x_1) = y_2(x_2) = 0$

…. …. …..

…. …. …..

Now putting the value of y(x) from (3) in (2), we get

I = I $(c_1, c_2, \ldots, c_n)$

These constants $c_1, c_2, \ldots, c_n$ are choose in such a way that I is extremum.

Therefore by necessary condition for existence of extremal is

$$\frac{\partial I}{\partial c_1} = \frac{\partial I}{\partial c_2} = \cdots = \frac{\partial I}{\partial c_n} = 0.$$

Solving these simultaneous equations, we will to get $c_1, c_2, \ldots, c_n$ and putting these value in (3) we get required solution of differential equation (1).

Remark: The selection of approximate solution may be done from solution

$$y(x) = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1})(x - x^2)$$

one term approximation y(x) = $c_1(x - x^2)$ and

two term approximation y(x) = $(c_1 + c_2 x)(x - x^2)$ which gives better approximation.

Obviously, approximate solution of ordinary differential equation are linearly independent.

ILLUSTRATIVE EXAMPLES

Example 1. Solve the boundary value problem $y'' = 1$ subject to boundary conditions

y(0) = 0, y(1) = 0 by Rayleigh-Ritz method.

Sol. The given differential equation is

$$y'' = 1 \qquad \ldots\ldots\ldots\ldots (1)$$

$$y(0) = 0, \; y(1) = 0$$

First, we construct F(x, y, y') in such a way that the Euler's equation of functional

$$I = \int_0^1 F(x, y, y')\,dx$$

Becomes given differential equation $y'' = 1$, we choose

$$F = {y'}^2 + 2y \text{ for which } y'' = 1 \text{ is Euler's equation}$$

By Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Therefore $\quad 2 - \frac{d}{dx}(2y') = 0$

Therefore $\quad 1 - y'' = 0 \quad \Rightarrow y'' = 1$

Hence the required variational problem is

$$I = \int_0^1 \left({y'}^2 + 2y\right)dx \qquad \text{......... (2)}$$

Next we assume the trivial solution

$$y(x) = c_0 + c_1 x + c_2 x^2$$

therefore $\quad y(0) = 0 \Rightarrow c_0 = 0 \qquad$ .... (3)

$$y(1) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

therefore trivial solution become

$$y(x) = c_1(x - x^2) \qquad \text{....... (4)}$$

therefore $\quad y'(x) = c_1(x - 2x)$

putting value of y and y' in (2), we get

$$I = \int_0^1 [{c_1}^2 (1-2x)^2 + 2c_1(x - x^2)]dx$$

Therefore $\quad \frac{dI}{dc_1} = \frac{d}{dc_1}\int_0^1 [{c_1}^2 (1-2x)^2 + 2c_1(x - x^2)]dx$

$$= \int_0^1 \frac{\partial}{\partial c_1}[{c_1}^2 (1-2x)^2 + 2c_1(x - x^2)]dx$$

$$= \int_0^1 [2c_1 (1-2x)^2 + 2(x - x^2)]dx$$

$$= 2c_1\left[-\frac{1}{2}\frac{(1-2x)^3}{3}\right]_0^1 + 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2c_1\left(\frac{1}{6} + \frac{1}{6}\right) + 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

Therefore $\quad \frac{dI}{dc_1} = \frac{(2c_1+1)}{3}$.

The necessary condition for existence of extremal is

$$\frac{dI}{dc_1} = 0$$

$\Rightarrow \quad \frac{2c_1+1}{3} = 0 \quad \Rightarrow c_1 = -\frac{1}{2}$

Therefore (4) becomes

$$y(x) = -\frac{1}{2}(x - x^2) \qquad \text{or} \quad y(x) = \frac{1}{2}(x^2 - x).$$

This is exact solution of the given boundary value problem.

Note:■ if we solve $\frac{d^2y}{dx^2} = 1$, y(0) = 0, y(1) =0.

Integrate, we get $\quad y(x) = \frac{x^2}{2} + c_1 x + c_2$

$y(0) = 0 \Rightarrow c_2 = 0$

$y(1) = 0 \Rightarrow \frac{1}{2} + c_1 + c_2 = 0$

$\Rightarrow c_1 = -\frac{1}{2}$

Therefore $\quad y(x) = \frac{x^2}{2} - \frac{1}{2}x \quad$ or $\quad y(x) = \frac{1}{2}(x^2 - x)$

Hence $y(x) = \frac{1}{2}(x^2 - x)$ is exact solution.

Example 2. Solve the boundary value problem $y'' - y + x = 0$

$(0 \le x \le 1)$ subject to boundary conditions y(0) = 0, y(1) = 0 by Rayleigh-Ritz method.

Sol. The given differential equation is

$$y'' - y + x = 0 \qquad \text{………..(1)}$$

with $\qquad y(0) = 0, y(1) = 0$

First, we construct F(x, y, y') in such a way that the Euler's equation of functional

$$I = \int_0^1 F(x, y, y')dx$$

Where $\qquad F(x, y, y') = 2xy - y^2 - y'^2.$

Hence the required variational problem is

Therefore $\quad I[y(x)] = \int_0^1 (2xy - y^2 - y'^2)dx \quad$ ……. (2)

Next, assume the trivial solution is

$$y(x) = c_0 + c_1 x + c_2 x^2 \quad \text{……. (3)}$$

therefore $\quad y(0) = 0 \Rightarrow c_0 = 0$

$$y(1) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

Therefore (3) become

$$y(x) = c_1(x - x^2) \quad \text{……. (4)}$$

therefore $\quad y'(x) = c_1(1 - 2x)$.

putting value of y and y' in (2), we get

$$I = \int_0^1 \left[c_1 2x^2 (1-x) - x^2 {c_1}^2 (1-x)^2 - {c_1}^2(1-2x)^2\right] dx$$

$$= \int_0^1 [2c_1 (x^2 - x^3) - {c_1}^2(x^2 + x^4 - 2x^3) - {c_1}^2(1 + 4x^2 - 4x)]dx$$

$$= \left[2c_1\left(\frac{x^3}{3} - \frac{x^4}{4}\right) - {c_1}^2\left(\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^4}{2} + x + \frac{4x^3}{3} - 2x^2\right)\right]_0^1$$

$$= \frac{1}{6}c_1 - \frac{11}{30}{c_1}^2.$$

Therefore the necessary condition for existence of extremal is

$$\frac{dI}{dc_1} = 0$$

$$\Rightarrow \quad \frac{1}{6} - \frac{11}{30}2c_1 = 0$$

$$\Rightarrow \quad c_1 = \frac{5}{22}$$

Put in (4), we get $\quad y(x) = \frac{5}{22}(x - x^2)$

This is the required approximate solution.

Example 3. Solve the boundary value problem

$(1 - x^2)y'' - 2xy' + 2y = 0$ subject to boundary conditions y(0) = 0, y(1) = 1 by Rayleigh-Ritz method.

Sol. The given differential equation is

$$(1 - x^2)y'' - 2xy' + 2y = 0 \quad \text{……….. (1)}$$

with $y(0) = 0, y(1) = 1$

First, we construct $F(x, y, y')$ in such a way that the Euler's equation of functional

$$I = \int_0^1 F(x, y, y')dx$$

Where $F(x, y, y') = (1- x^2)y'^2 - 2y^2$

Hence the required variational problem is

Therefore $I[y(x)] = \int_0^1 [(1 - x^2)y'^2 - 2y^2]dx$ ……. (2)

Next, assume the trivial solution is

$$y(x) = x + c(x - x^2) \quad \text{……. (3)}$$

be approximate solution which satisfying the given boundary conditions.

Therefore $y'(x) = 1 + c\,(1 - 2x)$.

Putting value of y and y' in (2), we get

$$I = \int_0^1 [(1 - x^2)\{1 + c - 2xc\}^2 - 2\{x + cx - cx^2\}^2]dx$$

Therefore, for existence of extremal $\frac{dI}{dc} = 0$ gives c = 0.

Putting value of c in (3), we get

$$y(x) = x$$

this is required solution of given differential equation.

Note: y(x) = 0 is exact solution of given differential equation.

14.4 *RAYLEIGH RITZ METHOD*

(For Partial Differential equation)

The German mathematician W. Ritz gave variational approach to solve boundary value problem for ordinary and differential equation in 1908.

The Rayleigh Ritz method utilize the principle of minimizing total potential energy in a system and calculus of variations. It employs the use of trial functions that satisfy specific conditions, including boundary conditions, to solve boundary value problems.

This is also known as Rayleigh-Ritz method.

Consider the boundary value problem

$$u_{xx} + u_{yy} + qu = r$$

In the region R bounded by the curve C.

Let u(x) be prescribed on C.

Then, the corresponding variation problem is to extremize the functional is

$$\int\int_{\mathrm{C}}\left(u_x{}^2 + u_y{}^2 - qu^2 + 2ru\right)dx\, dy$$

For which we take the trivial function such that it Satisfy the given boundary conditions.

ILLUSTRATIVE EXAMPLES

Example 4. Solve the Poisson's equation $u_{xx} + u_{yy} = -1$

In a square defined by $|x| \leq 1$, $|y| \leq 1$ and u = 0 when x = ± 1, y = ± 1.

Sol. Given that $u_{xx} + u_{yy} = -1$ …….. (1)

Compare with $u_{xx} + u_{yy} + qu = r$

We get q = 0, r = -1

Hence, the corresponding variational problem is to extremize the functional is

$$\mathrm{I} = \int\int_{\mathrm{R}}\left(u_x{}^2 + u_y{}^2 - qu^2 + 2ru\right)dx\, dy$$

i.e. $\mathrm{I} = \int\int_{\mathrm{R}}\left(u_x{}^2 + u_y{}^2 + 2u\right)dx\, dy$ …… (2)

where R is square defined by $|x| \leq 1$, $|y| \leq 1$

let u(x, y) = c (1 - x^2) (1 - y^2) …… (3)

be trivial function which satisfies the given boundary conditions.

Therefore u_x = - 2cx (1 - y^2); u_y = - 2cy (1 - x^2).

Putting value of u, u_x and u_y in (2), we get

$$I = \int_{-1}^{1}\int_{-1}^{1}[4\,c^2x^2(1 - y^2) + 4c^2x^2(1 - x^2)^2 - 2c(1 - x^2)(1 - y^2)]\text{dxdy}$$

$$= \frac{32}{45}(8c^2 - 5c)$$

Now, for existence of extremal

$$\frac{\text{dI}}{\text{dc}} = 0 \qquad \Rightarrow \frac{32}{45}(16c - 5) = 0$$

$$\Rightarrow (16c - 5) = 0 \qquad \Rightarrow c = \frac{5}{16}$$

Putting value of c in (3), we get

$$u(x, y) = \frac{5}{16}(1 - x^2)(1 - y^2)$$

this is the required approximate solution.

14.5 *GALERKIN'S METHOD*

The Galerkin's method of weighted residuals, the most common method of calculating the global stiffness matrix in the finite element method, the boundary element method for solving integral equations, Krylov subspace methods.

Galerkin's method is another method to solve any type of boundary value problems either linear or non-linear.

In this method, the boundary conditions are taken homogeneous i.e., of the forms

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$$

$$y(x_1) = 0 \text{ and } y(x_2) = 0 \qquad \dots (1)$$

If the boundary conditions are not homogeneous, it can be made homogeneous by choosing an approximate transformation

$$u = y - y - 0 - \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

With conditions that

$$u_r(x_1) = 0 \text{ and } u_r(x_2) = 0; \quad r = 1,2,\dots,n$$

Further let

$$y(x) = c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) \quad \dots (2)$$

Be an approximate solution.

To find the constants $c_i's$; $i = 1, 2, \dots, n$

We use the residue function defined as

$$R(x, c_r) = \left[a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)\right]\sum_{r=1}^{n} c_r u_r(x) - f(x) \quad \textbf{… (3)}$$

When $\int_{x_1}^{x_2} R(x, c_r)u_r(x)dx = 0; \quad r = 1,2,\dots,n$ **… (4)**

We can find the value of all constants $c_i's$ using (3) and then putting value of these constants in (1), we get required solution.

ILLUSTRATIVE EXAMPLES

Example 5. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + y = e^x$$

$$y(0) = y\left(\frac{\pi}{2}\right) = 0$$

By Galerkin's method.

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + y = e^x$$

With $\quad y(0) = y\left(\frac{\pi}{2}\right) = 0 \qquad \dots (1)$

Here given boundary conditions are homogeneous.

Let approximate solution is

$$y = c_1 u_1(x) = c_1 \sin 2x \qquad \dots (2)$$

Which satisfy given boundary condition.

Now we find constant c_1.

For consider, corresponding residue function by

$$R(x, c_r) = \left(\frac{d^2}{dx^2} + 1\right) c_1 \sin 2x - e^x$$

$$= -4c_1 \sin 2x + c_1 \sin 2x - e^x$$

Where $\quad \int_0^{\frac{\pi}{2}} (-4c_1 \sin 2x + c_1 \sin 2x - e^x)\ \sin 2x\, dx = 0$

On simplification, we get

$$c_1 = -\frac{8}{15\pi}\left(e^{\frac{\pi}{2}} + 1\right)$$

Put in (2), we get

$$y(x) = -\frac{8}{15\pi}\left(e^{\frac{\pi}{2}} + 1\right) \sin 2x$$

This is required approximation solution.

14.6 *KANTOROVICH METHOD*

In his method to reduce a partial differential equation to a system of ordinary differential equations, Kantorovich uses a cartesian coordinate as an independent variable. For partial differential equations arising from variational problems, an alternate formulation is presented, wherein an arbitrary function takes the role of the independent variable. This procedure should allow the subspace approximating the solution to be adapted to the problem at hand. The differential equations are put in a form to minimize regularity conditions on the base functions, e.g., for a second order differential equation, piecewise linear base functions will be admitted. The set of admissible base functions will be dependent on the boundary conditions of the problem. Iterative methods to solve the corresponding two-point boundary value problem are discussed. In order to solve partial differential equation, Kantorovich method is another more efficient method in comparison of Rayleigh-Ritz method.

In this method, a trivial solution

$$u(x,y) = f_1(x)u_1(x,y) + f_2u_2(x,y) + \cdots + f_nu_n(x,y)$$
$$= \sum_{i=1}^{n} f_i(x)u_i(x,y)$$

Is taken.

Here, $u_i(x,y), \forall_i$ satisfying the given boundary condition and $f_i(x)$ are unknown function of x.

ILLUSTRATIVE EXAMPLES

Example 6. Solve $u_{xx} + u_{yy} = 0$ in a square defined by $|x| \leq 1$, $|y| \leq 1$ where u$(\pm1, y) = 1 - y^2$ and u$(x, \pm1) = 0$.

Sol. Here, the corresponding variational problem is to extremize the functional

I = $\int\int_R (u_x{}^2 + u_y{}^2)dx\,dy$ …… (1)

where R is square defined by $|x| \leq 1$, $|y| \leq 1$

let u(x, y) = (1 - y^2)$f(x)$ …… (2)

where f(x) is unknown function to be determined such that f$(\pm1) = 0$.

Therefore $u_x =$ (1 - y^2)$f'(x)$, $u_y =$ - 2y f(x).

Putting value of u, u_x , u_y in (2) we get

$$I = \int_{-1}^{1}\int_{-1}^{1}[(1 - y^2)\{f'(x)\}^2 + 4y^2\{f(x)\}^2]dxdy$$

On simplification, we get

$$I = \frac{8}{3}\int_{-1}^{1}\left(\frac{2}{5}f'^2 + f^2\right)dx$$

And the corresponding Euler's equation is

$$-\frac{4}{5}f'' + 2f = 0 \quad \text{or} \quad f'' - \frac{5}{2}f = 0 \qquad \text{......... (3)}$$

Auxiliary equation is $m^2 - \frac{5}{2} = 0$

$\Rightarrow \quad m^2 = \frac{5}{2} \qquad \Rightarrow \quad m = \pm\sqrt{\frac{5}{2}}$

Hence solution of (2) is

$$f(x) = A\cosh\left(\sqrt{\frac{5}{2}}\right)x + B\sinh\left(\sqrt{\frac{5}{2}}\right)x$$

using boundary condition $f(\pm 1) = 1$

we get $f(x) = \frac{\cosh\left(\sqrt{\frac{5}{2}}\right)x}{\cosh\left(\sqrt{\frac{5}{2}}\right)}$

therefore (2) becomes

$$u(x, y) = (1 - y^2)\frac{\cosh\left(\sqrt{\frac{5}{2}}\right)x}{\cosh\left(\sqrt{\frac{5}{2}}\right)}$$

this is the required solution.

Example 7. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ in a square defined by

$|x| \le 1, |y| \le 1$, when x = 0 on $|x| = 1$ and u = 1 - x^2 on $|y| = 1$.

Sol. The given partial differential equation is $u_{xx} + u_{yy} = 0$

Compare with $u_{xx} + u_{yy} + qu = r$, we get q = 0, r = 0

Here, the corresponding variational problem is to extremize the functional is

I = $\int\int_R \left(u_x{}^2 + u_y{}^2 - qu^2 + 2ru\right)dx\,dy$

i.e. I = $\int\int_R \left(u_x{}^2 + u_y{}^2\right)dx\,dy$ …… (1)

Let u(x, y) = (1 - x^2)$[1 + c(1 - x^2)]$ …… (2)

Be trivial function, which satisfy the given boundary condition.

Therefore u_x = - 2x$[1 + c(1 - y^2)]$

Therefore u_y = - 2yc (1 - x^2)

Therefore (1) becomes

I = $\int_{-1}^{1}\int_{-1}^{1}[\,4x^2\{1 + 2c(1 - y^2) + c^2(1 - y^2)^2\} + 4y^2c^2(1 - 2x^2 + x^4]dxdy$

Now $\frac{dI}{dc} = 0$

Therefore $\int_{-1}^{1}\int_{-1}^{1}[\,8x^2(1 - y^2) + 8x^2c(1 - y^2)^2 + 8y^2c(1 - 2x^2 + x^4]dxdy$ = 0

Evaluating the integral, we get c = $\frac{-5}{8}$

Putting value of c in (1), we get

u(x, y) = $(1 - x^2)[1 - \frac{5}{8}(1 - y^2)]$

this is required approximate solution.

Example 8. Find the estimate of the least eigen value of $u_{xx} + u_{yy} + \lambda u = 0$ in the region R bounded by the circle x^2 +$y^2 = 1$ given that u = 0 on the boundary.

Sol. Here, the corresponding Variational problem is to extremize the functional is

$$I = \int\int_R \left(u_x{}^2 + u_y{}^2 - \lambda u^2\right) dx\, dy \qquad \ldots\ldots (1)$$

Where R is region bounded by $x^2 + y^2 \leq 1$

Let u(x, y) = c (1 - x^2 - y^2) be trivial solution which satisfy the boundary condition

i.e. u = 0 at $x^2 + y^2 = 1$.

Therefore $u_x = -2cx, u_y = -2cy$

Putting value of u, u_x , u_y in (1) we get

$I = \int\int_R [4c^2(x^2 + y^2) - \lambda c^2(1 - x^2 - y^2)^2 dx\, dy$

Therefore $\frac{dI}{dc} = 0$

Therefore $\int\int_R [8c(x^2 + y^2) - 2\lambda c(1 - x^2 - y^2)^2] dx\, dy = 0$

Therefore $\lambda = \frac{4\int\int_R (x^2 + y^2) dx\, dy}{\int\int_R (1 - x^2 - y^2)^2 dx\, dy}$

Putting x = r cosθ , y = r sinθ

Therefore dx dy = r dθdr

Therefore $\lambda = \frac{4\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2 r d\theta dr}{\int_{r=0}^{1}\int_{\theta=0}^{2\pi} (1 - r^2)^2 d\theta dr}$

Therefore $\lambda = \frac{2\pi}{2\pi(1/6)} = 6$

Hence $\lambda = 6$ is the required least eigen value.

CHECK YOUR PROGRESS

MCQ/True False Questions

Problem 1. Extremals of the functional $\int_{x_1}^{x_2} y\sqrt{1+y'^2}$ is attained on the: (a) Catenary (b) Parabola (c) Circle (d) Ellipse

Problem 2. The shortest distance between two points in a plane is:

(a) circle (b) parabola (c) Ellipse (d) straight line

Problem 3. Euler's equation for the functional

$$\int_{x_1}^{x_2} \left[a(x)y'^2 + 2b(x)y' + c(x)y^2\right]dx:$$

(a) First order linear differential equation.

(b) second order linear differential equation.

(c) second order non- linear differential equation.

(d) a linear differential equation of order more that two.

Problem 4. Extremal y = y(x) for the variational problem

I = $\int_0^1 (1 + y''^2)dx$ satisfies the ordinary differential equation is:

(a) Homogeneous linear differential equation of fourth order.

(b) Non-homogeneous linear differential equation of fourth order.

(c) Homogeneous non-linear differential equation of fourth order.

(d) Homogeneous linear differential equation of more than fourth order.

Problem 5. Necessary condition for existence of extremal is

$$\frac{\partial \mathrm{I}}{\partial \mathrm{c}_1} = \frac{\partial \mathrm{I}}{\partial \mathrm{c}_2} = \cdots = \frac{\partial \mathrm{I}}{\partial \mathrm{c}_\mathrm{n}} = 0.\text{True/False}$$

Problem 6. In which situation the variational problem $\int_{x_1}^{x_2} F(x, y, y')dx$,

y$(x_1) = y_1$, y$(x_2) = y_2$ becomes meaningless:

(a) When Euler's equation reduces into identity.

(b) When y(x) exists but it is not satisfy the given boundary condition.

(c) When f$(x, y, y') = M(x, y) + N(x, y)$.

(d) All of the above.

14.7 *SUMMARY*

1. Necessary condition for existence of extremal is

$$\frac{\partial \mathrm{I}}{\partial \mathrm{c}_1} = \frac{\partial \mathrm{I}}{\partial \mathrm{c}_2} = \cdots = \frac{\partial \mathrm{I}}{\partial \mathrm{c}_\mathrm{n}} = 0.$$

2. RAYLEIGH RITZ METHOD:

The Rayleigh–Ritz method is a variational method to solve the eigenvalue problem for elliptic differential operators, that is, to compute their eigenvalues and the corresponding eigenfunctions. It is the direct counterpart of the Ritz method for the solution of the assigned boundary value problems. The Rayleigh–Ritz method has the advantage of being based on minimal, very general assumptions and produces optimal solutions in terms of the approximation properties of the underlying trial spaces. The theory of the Rayleigh–Ritz method has to a large extent been developed in the context of finite element methods Consider the boundary value problem

$$u_{xx} + u_{yy} + qu = r$$

In the region R bounded by the curve C.

Let u(x) be prescribed on C.

Then, the corresponding variation problem is to extremize the functional is

$$\int \int_{C} \left({u_x}^2 + {u_y}^2 - qu^2 + 2ru \right) dx\, dy$$

For which we take the trivial function such that it Satisfy the given boundary conditions.

3. Galerkin's method is another method to solve any type of boundary value problems either linear or non-linear.

4. Kantorovich method:

In this method, a trivial solution

$$u(x, y) = f_1(x)u_1(x, y) + f_2u_2(x, y) + \cdots + f_nu_n(x, y)$$

$$= \sum_{i=1}^{n} f_i(x)u_i(x, y)$$

5. Kantorovich method is more efficient method to solve boundary value problem of partial differential equation.

6. The German mathematician W. Ritz gave variational approach to solve boundary value problem for ordinary and differential equation in 1908.

This is also known as Rayleigh-Ritz method.

7. The variational problem corresponding to given ordinary differential equation is

I = $\int_{x_1}^{x_2} F(x, y, y')dx$ where, we construct $F(x, y, y')$ in say that the corresponding Euler's equation becomes original differential equation.

8. Note that Poisson's Equation is a partial differential equation, and therefore can be solved using well-known techniques already established for such equations. In fact, Poisson's Equation is an inhomogeneous differential equation.

14.8 GLOSSARY

Integration

Even, odd functions

Trigonometric functions

Differentiation

First order derivatives

Second order derivatives

Expansions of function

Series

Functional

14.9 *REFERENCES*

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14.10 *SUGGESTED READING*

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14.11 *TERMINAL AND MODEL QUESTIONS*

TQ 1. Find the estimate of the least eigen value of

$$u_{xx} + u_{yy} + \lambda u = 0$$

In the region R bounded by $|x| \leq 1,\ |y| \leq 1$ and $u = 0$ at $x = \pm 1$ and at $y = \pm 1$.

TQ 2. Solve the boundary value problem

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 2x^2$$

$$y(0) = 0, \quad y(1) = 1$$

By Galerkin's method.

TQ 3. Solve the Poisson's equation in a circle $u_{xx} + u_{yy} = -1$, $x^2 + y^2 \leq 1$ when u = 0

on $x^2 + y^2 = 1$.

TQ 4. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ in a square defined by $|x| \leq 1,\ |y| \leq 1$ when $x = 0$ on $|x| = 1$ and u = 1 - x^2 on $|y| = 1$.

14.12 *ANSWERS*

TQ1 $\lambda = 5$

TQ2 y = $\frac{2}{3}x(x-1)+x$

TQ3 u(x, y) = $\frac{1}{4}(1-x^2-y^2)$

TQ4 u(x, y) = $(1-x^2)[1-\frac{5}{8}(1-y^2)]$

CHECK YOUR PROGRESS

CQ1 (a)

CQ2 (b)

CQ3 (b)

CQ4 (a)

CQ5 True

CQ6 (d)

Teen Pani Bypass Road, Transport Nagar
Uttarakhand Open University,
Haldwani, Nainital-263139
Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232,

E-mail:info@uou.ac.in
Website: https://www.uou.ac.in/