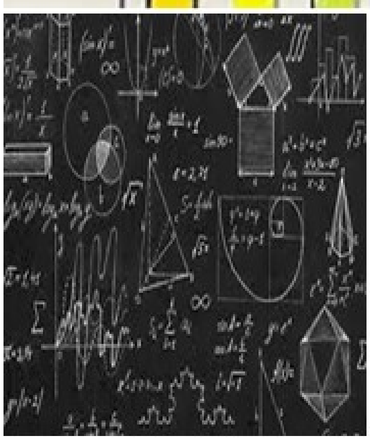
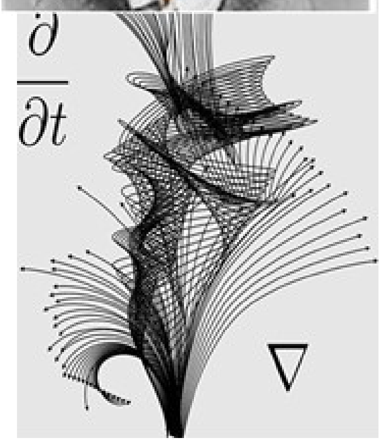
Master of Science

MATHEMATICS

Second Semester

MAT 507
MEASURE THEORY



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UTTARAKHAND OPEN UNIVERSITY
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CONTENTS
MAT-507

COURSE INFORMATION

The present self learning material **“Measure Theory”** has been designed for M.Sc. (Third Semester) learners of Uttarkhand Open University, Haldwani.

This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

First block is **Sets and Lebesgue Measure.** Which is composed by Sets and Cardinality, Boolean Algebra, Measure Space and Lebesgue Measure.

Second block is **Measurable functions and Convergence theorem.** In this block Measurable Functions, Lebesgue Integral of a Function, General Convergence Theorem and Differentiation of an integral explained.

Third block is **L_p Space and Weierstrass approximation theorem** third block is composition of TheL_p space, Theorem in Lebesgue integration and Weierstrass approximation theorem.

Fourth block is **Signed measure and Product measure** which is a collection of Signed measures, Product measure and Relation between Riemann and Lebesgue.

The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the leaner’s to grasp the subject easily.

BLOCK I:

SETS AND

LEBESGUE MEASURE

UNIT 1: SETS AND CARDINALITY

CONTENTS:

1.1 INTRODUCTION

In the branch of set theory it is always significant to know the number of elements in a set. For finite set it is always possible to count number of elements in a set, but for infinite set or countably infinite it is not possible to count number of elements. **Georg Cantor(1845-1918),** gave the generalized definition of cardinality which based on bijective map between two sets. Later he had given a very important theorem, Cantors theorem, that associate cardinality of set and its power set.

In this unit we explain about cardinality of sets, countable and uncountable sets, with illustrative examples.

1.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of Cardinality of a set.

ii. Describe the different between countable and uncountable set.

iii. Find the applications of countable and uncountable set.

1.3 SET

The concept of "Set Theory" was discovered by German mathematician Georg Cantor (1845-1918) . He was inspired by working on "Problems on Trigonometric Series." It has many applications in other topics like relations, functions, probability, sequences, geometry etc.

Definition 1: A setis a collection of well defined objects

- **i.** $A = \{1,2,3\}$ is a set with 3 elements
- **ii.** A set with no element is called null set or empty set
- **iii.** $A$is subset of B , denoted by $A \subset B$, if all the elements of A are also the element of B.
- **iv.** Union of sets: $AUB = \{x: x \in A \; or \; x \in B\}$
- **v.** Union of sets: A⍰$B = \{x: x \in A \; or \; x \in B\}$
- **vi.** Power Set: A set of all the subsets of set is called power set, denoted by $P(A) = \{x: x \subset A\}$

Any well-defined collection of objects or numbers are referred to as a set. The number, letter or any other object contained in a set are called elements of the set. The sets are denoted by capital letters e.g. X, Y, Zor . The elements are denoted by lower case letters *a*, *b*, *c*, …., *x*, *y*, *z*. To indicate that '*a*' is an element of the set *X* we use the notation $a \in X$. This read as "*a* is in *X*" or "a belongs to *X*". For example $A = \{1, 3, 5, 7, 11, 13, 17, 20\}$.

INTERVAL:

An open interval does not contain its endpoints, and is indicated with parentheses.

$$(a, b) =]a, b[= \{x \epsilon \mathbb{R}: a < x < b\}.$$

A closed interval is an interval which contains all its limit points, and is expressed with square brackets.

$$[a, b] = [a, b] = \{x \epsilon \mathbb{R}: a \leq x \leq b\}.$$

A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals

$.(a, b] =]a, b] = \{x \epsilon \mathbb{R}: a < x \leq b\}. [a, b) = [a, b[= \{x \epsilon \mathbb{R}: a \leq x < b\}.$

ORDERED PAIR :

An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a,b) \neq (b,a)$ unless $a = b$.

In this respect (a,b) differs from the set $\{a,b\}$.Again $(a,b)=(c,d) \Leftrightarrow a=c$ and $b=d$ .If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x \in X$ and $y \in Y$ is called Cartesian product of X and Y. It is denoted by $X \times Y$.

RELATION

A subset R of $X \times Y$ is called relation of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of R, then y is called image of x.A relation in which each element of X has a single image is called a function.

If $X = \{1,2,3,4\}$and $Y = \{a, b, c\}$then,

$$X \times Y = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c),(4,a),(4,b),(4,c)\}$$
$$R_1 = \{(1,a),(2,b),(3,c),(4,b)\}$$

- $R_1$is a relation as well as a function
- while $R_2 = \{(1,a),(2,b),(2,c),\ (3,c)\}$ is a relation but not a function (since 2 has two images).

FUNCTION:

The equation $y = x^2$ gives a rule which determines for each number x, a corresponding number y.The set of all such pairs of numbers (x, y) determines a function.

Definition.

Let X and Y are two sets and suppose that to each element x of X corresponds, by some rule, a single element y of Y. Then the set of all ordered pairs (x, y) is called function.The set X is called the domain of the function. The element y, which corresponds to the element x is called the value of function at x. It is denoted by $f(x)$, read as "f of x". The set of all the values of the function is called the range of the function.The term mapping is also used for a function and we say that the set X maps into the set Y under the mapping f. We write as $f : X \to Y$ and read as " the function f which maps X into Y ". We shall also use the notation $f : y = f(x)$ to denote "the function f defined by the rule $y = f(x)$". Basically function is a rule which binds one set X to another set Y.The rule is that for all elements of X their should be unique image in Y.

A symbol such as x or y, used to represent an arbitrary element of a set is called a variable. For example $y = f(x)$.The symbol x which represents an element in the domain is called the independent variable, and the symbol y which represent the element corresponding to x is called the dependent variable. This is based on the fact that value of x can be arbitrary chosen, then y has a value which depends upon the chosen value of x.

Real Numbers:

Numbers initiate with Natural Numbers. The natural numbers are the standard numbers, $1, 2, 3, ..$. with which humans count. Natural numbers were discovered by Pythagoras ($582 - 500\ BC$) and Archimedes ($287 - 212\ BC$) (both are Greek philosophers and mathematicians). After Natural Number the integer was introduced in the year 1563 when Arbermouth Holst was busy with his bunnies and elephants experiment. He stored count of the amount of bunnies in the cage and after

6 months he saw that the amount of bunnies increased. Then he concludes the addition and multiplication of a number system then rational number is defined. In arithmetic, a number that can be considered as the quotient p/q of two integers such that $q \neq 0$. In addition to all the fractions, the set of rational numbers added all the integers, each of which can be written as a quotient with the integer as the numerator and 1 as the denominator. Rational numbers were discovered in the sixth century BCE by Pythagoras. Later this Irrational numbers are the numbers that cannot be considered as a simple fraction. It cannot be considered in the form of a ratio, such as p/q, where p and q are integers, q≠0. It is a contradiction of rational numbers. The Greek mathematician Hippasus of Metapontum is the person who invented irrational numbers in the 5th century B.C., according to an article from the University of Cambridge. Subsequently real number introduced in the 16th century, Simon Stevin designed the basis for modern decimal notation, and asserted that there is no difference between rational and irrational numbers in this regard. In the 17th century, Descartes invented the term "real" to describe roots of a polynomial, distinguishing them from "imaginary" ones. Mathematician Richard Dedekind quarried these problems 159 years ago at ETH Zurich, and became the first person to characterize the real numbers. Bob sinclar defined the whole numbers in 1968. Whole Numbers is the subset of the number system that includes of all positive integers contained zero. In mathematics, a real number is a value of a continuous amount that can act for a distance along a line (or alternatively, a number that can be summarised as an infinite decimal expansion. The set of real numbers is expressed using the symbol R or $\mathbb{R}$. Real numbers can be consider of as points on an infinitely long line called the number line or real line, where the points interrelated to integers are equally spaced. Any real number can be resolved by a possibly infinite decimal representation. We can write the set of real numbersin the form of rational and irrational number as, $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$.

The main properties of real numbers are as follows:

i. **Closure Property:** If $a, b \epsilon \mathbb{R}, a + b \epsilon \mathbb{R}$ and $ab \epsilon \mathbb{R}$. It shows that sum and product of two real numbers is always a real number.

ii. **Associative Property:** If $a, b, c \epsilon \mathbb{R}$, $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$. It follows that sum or product of any three real numbers remains the same even when the grouping of numbers is changed.

iii. **Commutative Property:** If $a, b \epsilon \mathbb{R}, a + b = b + a$ and $a \times b = b \times a$. It means that the sum and the product of two real numbers remain the same even after interchanging the order of the numbers

iv. **Distributive Property:** Real numbers satisfy the distributive property. If $a, b, c \epsilon \mathbb{R}$.

- $a \times (b + c) = (a \times b) + (a \times c)$ is the distributive property of multiplication over addition.
- $a \times (b - c) = (a \times b) - (a \times c)$ is the distributive property of multiplication over subtraction.

If a and b are real numbers, we say that

i. $a > b$ if $a - b$ is a positive number,

ii. $a < b$ if $a - b$ is a negative number.

A relation involving $>$ or $<$ is known as an inequality. The following useful laws of inequalities can be easily obtained from the definition.

iii. If $a > b$, then $b < a$.

iv. If $a > b$ and $b > c$, then $a > c$.

v. If $a > b$and $c > d$, then $a + c > b + d$. (addition of inequalities).

vi. If $a > b$, then $a + c > b + c$.

vii. If $a > b$ and c is a positive number, $ac > bc$.

viii. If $a > b$ and c is a negative number, $ac < bc$.

(i) If $a + c > b$, then $a > b$-c (transposition of a term). A particular case of transposition is:

If $a > b$, then $-b > -a$.

The distance between two points x and a on the real line is denoted by $|x - a|$, and define as follows :

$$|x - a| = x - a \text{if} x \geq a,$$

$$|x - a| = x - a \text{if} x < a.$$

It is the numerical difference between the numbers x and a.

The absolute value$|x|$ of a real number x is defined by

i. $|x| = x$if $x \geq 0$.

ii. $|x| = -x$ if $x < 0$.

- In particular, $(-\infty, +\infty)$ denotes the set of all ordinary real numbers.
- $|x| \geq 0$.
- $|-x| = |x|$.
- $|x|$ =max$(x, -x)$.
- $-|x|$ =min$(x, -x)$.
- If $x, y \in \mathbb{R}$, then (i) $|x|^2 = x^2 = |-x|^2$.(ii) $|xy| = |x|.|y|$ (iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ provided $y \neq 0$.

Let A be a nonempty subset of $\mathbb{R}$.

i. The set is said to be bounded above if three exists a number $u\epsilon\mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such number u is called an upper bound of S.

ii. The set is said to be bounded below if three exists a number $w \epsilon \mathbb{R}$ such that $w \leq s$ for all $s \in S$. Each such number w is called an lower bound of S.

iii. A set is said to be bounded if it is both bounded above and bounded below. A set is said to be unbounded if it is not bounded.

iv. If A is bounded above, then a number u is said to be supremum (or a least upper bound) of A if it satisfies the conditions:

v. If A is bounded below, then a number w is said to be infimum (or a greatest lower bound) of A if it satisfies the conditions:w is an upper bound of A, and

- **a)** u is an upper bound of A, and
- **b)** If v is any upper bound of A, then $u \leq v$.

vi. If t is any upper bound of A, then $t \leq w$.

- The least upper bound or the greatest lower bound may not belong to the set A.1 is least upper bound of the sets $\{x: 0 < x < 1\}, \{x: 0 \leq x \leq 1\}$and $\left\{1 - \frac{1}{n}: n \in \mathbb{N}\right\}$.

- **Completeness Property Of Real Number System:** Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

ARCHIMEDEAN PROPERTY:

If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.

1.4 CARDINALITY OF SETS

In a set it is always important to know the number of elements in the set. For finite set it is always possible but for infinite set we need some well posed definition, which is given in next section.

1.4.1 DEFINITION

CARDINALITY OF A SET

Cardinality of a finite set S is the number of element in the set, denoted by $|S|$.

For example Cardinality of set $S = \{1,2,3\}$ is 3.

The above definition and notion of cardinality will work only for finite sets however Cantor in 1880, gave the extended definition of cardinality in the following way

COMPARISON OF CARDINALITY OF TWO SETS

Let A, B are two sets, then these two sets are said to have same cardinality written as $|A| = |B|$, if there exists a bijective map from A to B

We write $|A| \leq |B|$, if there is an injective map from A to B.

1.4.2 SOME EXAMPLES

Example 1: Show that $|2N \cup \{0\}| = |N|$, where N is set of natural numbers.

Solution: It is sufficient to show that there exists a bijection from N to $2N$ .Since we have $f: N \to 2N$ by $f(n) = 2n - 2$, which is a bijective map.

Example 2: Cardinality of a finite set $A = \{1,2,3,4\}$ is 4.

Example 3: The set R of real numbers is uncountable.

Proof. The set R is clearly infinite. Suppose it is countable. Then, there exists an infinite sequence $f_1, f_2, \ldots$ that contains every element of R. We obtain a contradiction by finding x in R such that x is not f_n for any positive integer n.

For each positive integer i, suppose that the part of the decimal expansion of f_i following the decimal point is $.d_{i,1}d_{i,2}d_{i,3}\ldots$, where each $d_{i,j}$ is one of 0, 1, . . . , 9. The real number $x = 0.x_1x_2x_3\ldots$ is defined by

$$x_i = 5, \;\; if \;\; d_{ii} = 6$$
$$x_i = 6, \;\; otherwise$$

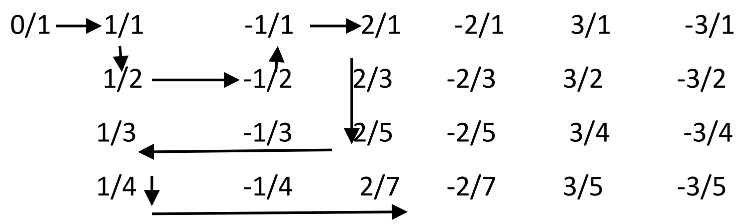
for each positive integer i.

We claim that x cannot be f_n for any positive integer n. First of all, x differs from f_n in the then-th digit after the decimal point (i.e., $x_i = d_{i,i}$). And, secondly, x is a number with only one infinite decimal expansion so it can not simply be a second representation of a number that is f_n for some n. This proves the claim, and contradicts the existence of the sequence $f_1, f_2, \ldots$

Therefore, R is uncountable. ∎

Example: 4: The set Q of rational numbers is countably infinite.

Proof: To show this, we just need to arrange the elements of the set Q in list form. This is done by arranging the first row consists of the rational numbers with denominator 1, the second row consists of those with denominator 2, and so on. In each row, the numerators appear in the order 0, −1, 1, −2, 2, ...Only the reduced form of rational numbers has been considered.



It is obvious that starting arrowing with 0 in the above manner we get all the rational numbers in its path. Hence it shows that set of rational numbers is countable.

1.4.3 SOME PROPERTIES

We have the following properties of cardinality :

- **i.** If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$
- **ii.** If $|A| = |B|$, then both $|A| \leq |B|$ and $|A| \geq |B|$ holds
- **iii.** If $|A| \leq |B|$ and $|B| \leq |C|$then $|A| \leq |C|$.
- **iv.** $|A \cup B| = |A| + |B|,\ where A \cap B = \emptyset$
- **v.** $|A \times B| = |A||B|$
- **vi.** $|P(A)| = 2^{|A|}$

1.4.4 CARDINALITY OF SET AND POWER SET

According to the Cantors theorem, for any set A, we have $|A| < |P(A)|$.Illustration: If $A = \{1,2\}$, then obviously $|A| = 2$. Also since $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, we have $|P(A)| = 4 > 2 = |A|$.

CHECK YOUR PROGRESS

1. What is the cardinality of set $S = \{1,2,3,4\}$? Explain your answer.

…………………………………………………………………………

…………………………………………………………………………

………………………………………………………………………….

2. Show that set of even and set of odd non negative integers have same cardinality…………………………………………………………………

…………………………………………………………………………

…….

3. Empty set is finite set …….True/False

1.5 COUNTABLE AND UNCOUNTABLE SETS

A set which is either finite or has the same cardinality as the set of natural numbers is called **countable set** . A set having same cardinality as the set of natural numbers is also called **Countably infinite** set. A set which is not **countable** is called **uncountable** set.

1.6. SOME PROPERTIES AND EXAMPLES

Example1: The set $A = \{a, b, c\}$ is countable set

Example2: The set of all integers $\mathbb{Z}$ is countable set

Verification: Since we have a function $f: N \to \mathbb{Z}$ defined by $(n) =$
$\begin{cases} \frac{n+1}{2}, & if n is odd \\ \frac{2-n}{2}, & if n is even \end{cases}$, which is a bijective map.

Example3: The sets of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$ and set of rational numbers $\mathbb{Q}$ are **Countably infinite** sets.

Example4: The sets of real numbers $\mathbb{R}$ **uncountable or infinite** set.

Properties of countable and uncountable sets

1. Subset of a countable set is a countable set
2. Union and intersection of two countable sets are countable sets
3. The power set of natural numbers, $2^{\mathbb{N}}$ is an uncountable set (In view of Cantors theorem)
4. Set of rational number is countable set

1.7 SUMMARY

This unit is an explanation of

i. Definition of Cardinality of sets.

ii. Definition and examples of countable and uncountable set

1.8 GLOSSARY

i. Set.

ii. Relation

iii. Function.

iv. Number System and its properties.

v. Cardinality

vi. Countable and uncountable set

1.9 REFERENCES

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1.10 SUGGESTED READINGS

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3. Lawrence Craig Evans, Ronald F. Gariepy, Measure Theory and Fine Properties of Functions (1st edition), Chapman and Hall/CRC, 2015.

1.10 *TERMINAL QUESTIONS*

1. Show that the following two sets have equal cardinality

$$\boldsymbol{A} = \{\boldsymbol{3x}: \boldsymbol{x} \in \mathbb{Z}\}, \qquad \boldsymbol{B} = \{\boldsymbol{7x}: \boldsymbol{x} \in \mathbb{Z}\}$$

2. Show that the sets $(0,1), (1,\infty)$ are two infinite sets have same cardinality.
3. Which of the following sets is countable

a) $(1,8)$

b) Set of rational numbers

c) Set of irrational numbers

4. If cardinality of power set of A is 16 then cardinality of A is

a) 6

b) 4

c) 2

d) 8

1.12 ANSWERS

CHECK YOUR PROGRESS:

1. There are four element in the set so cardinality of set is 4
2. Both sets have same cardinality. Define appropriate bijective map as $f(n) = n - 1$ n is even positive integer
3. True

TERMINAL QUESTIONS:

1. Define suitable bijective map
2. Since we have map $f: (0,1), \rightarrow (1, \infty)$, by $f(x) = \frac{1}{x}$, which is bijective and hence the sets $(0,1) and (1, \infty)$ have same cardinality.
3. (b)
4. (b)

UNIT 2: BOOLEAN ALGEBRA

CONTENTS:

2.1 INTRODUCTION

The topic of **Boolean algebra** is a branch of algebra first introduced by George Boole that involves mathematical logic. The motivation to study Boolean algebras comes from an interest in set theory. In previous unit we have defined the set, countable set and discussed about cardinality of set. In this unit we will discuss about Boolean ring of sets, Boolean algebra of sets and σ-algebra of sets.

2.2 OBJECTIVES

After completion of this unit learners will have the deep understanding of

- **i.** Boolean ring of sets
- **ii.** σ-ring of sets
- **iii.** Boolean algebra of sets
- **iv.** σ-algebra of sets
- **v.** Set function

2.3 BOOLEAN RING (RING OF SETS)

Let X be any set. Let $\mathscr{B}$ be a non-empty class of subsets of X then $\mathscr{B}$ is said to be Boolean ring or ring of sets, if

- **i.** $A, B \in \mathscr{B} \Rightarrow A \cup B \in \mathscr{B}$
- **ii.** $A, B \in \mathscr{B} \Rightarrow A - B \in \mathscr{B}$

Note: From the above definition it is clear that if $\mathscr{B}$ is Boolean ring and $A, B \in \mathscr{B}$ then

- i. $\varphi \in \mathscr{B}$, since $A - A = \varphi$
- ii. $A \cap B$ and $A \Delta B$ also lies in $\mathscr{B}$, Since $A \cap B = A - (A - B)$.

Therefore $A \cap B \in \mathscr{B}$ as $A \in \mathscr{B}$ and $A - B \in \mathscr{B}$.

Again $A \Delta B = (A - B) \cup (B - A)$. Therefore, $A \Delta B \in \mathscr{B}$.

Also by induction it can be shown that $\bigcup_{i=1}^{n} A_i$, $\bigcap_{i=1}^{n} A_i \in \mathscr{B}$, for all $A_i \in \mathscr{B}$.

Example 2.1 The class $\{\varphi\}$ is trivial example of ring of sets.

Example 2.2 Let X be any set then the class of all subsets of X (i.e., power set of X) are also trivial example of Boolean ring of sets.

Example 2.3 Let X=$\mathbb{R}$ **.** Let $\mathscr{B}$ be a set of all finite subsets of $\mathbb{R}$ then $\mathscr{B}$ forms a Boolean ring of sets (ring of sets).

Let A,B $\in\mathscr{B}\Longrightarrow$ A,B are finite subset of $\mathbb{R}$.

Therefore A $\cup$B is also a finite subset of $\mathbb{R}$ $\Longrightarrow$ A $\cup$B $\in\mathscr{B}$.

Also, A$-$B is finite$\Longrightarrow$ A$-$B $\in\mathscr{B}$

Hence $\mathscr{B}$ is Boolean ring of sets.

2.4 σ-RING

A Boolean ring $\mathscr{B}$ is called σ-ring, if it is closed under the formation of countable union i.e.,

A_i (i=1,2,3…)$\in\mathscr{B}\Longrightarrow\bigcup_{i=1}^{\infty}A_i\in\mathscr{B}$.

We can also define σ-ring as follow:

Let X be any set. Let $\mathscr{B}$ be a non-empty class of subsets of X then $\mathscr{B}$ is said to be σ-ring, if

- **i.** A_i (i=1,2,3…)$\in\mathscr{B}\Longrightarrow\bigcup_{i=1}^{\infty}A_i\in\mathscr{B}$.
- **ii.** A, B $\in\mathscr{B}\Longrightarrow$A$-$B $\in\mathscr{B}$

Remark: Every σ-ring is a Boolean ring.

Example 2.4 Let X=$\mathbb{R}$ **.** Let $\mathscr{B}$ be a set of all countable subsets of $\mathbb{R}$ then $\mathscr{B}$ forms a σ-ring.

Let A_i (i=1,2,3…)$\in\mathscr{B}\Longrightarrow A_i$ (i=1,2,3…) are countable subset of $\mathbb{R}$.

Since we know that countable union of countable sets is countable.

Therefore $\bigcup_{i=1}^{\infty}A_i$is also a countable subset of $\mathbb{R}$ $\Longrightarrow\bigcup_{i=1}^{\infty}A_i\in\mathscr{B}$.

Also, for any A,B$\in\mathscr{B}$, A$-$B is countable$\Longrightarrow$ A$-$B $\in\mathscr{B}$

Hence $\mathscr{B}$ a σ-ring of sets.

Note: Every Boolean ring need not be a σ-ring.

For counter example we can see the example 2.3, which forms Boolean ring but not σ-ring.

Take $A_n=\{n\}$; n=1,2,3…

Each A_n is finite having exactly one element hence $A_n \in \mathscr{B}$; n=1,2,3…

But $\bigcup_{i=1}^{\infty} A_n=\mathbb{N}$, which is not finite $\Rightarrow \mathscr{B}$ is not closed under countable union.

Hence $\mathscr{B}$ is not a σ-ring.

Theorem 1. Intersection of two Boolean ring is a Boolean ring.

Let $\mathcal{B}_1,\mathcal{B}_2$ be any two Boolean ring and A, B $\in \mathcal{B}_1 \cap \mathcal{B}_2$,

Then $A\in \mathcal{B}_1 \cap \mathcal{B}_2 \Rightarrow A\in \mathcal{B}_1$ and $A\in \mathcal{B}_2$, **...(i)**

$B\in \mathcal{B}_1 \cap \mathcal{B}_2 \Rightarrow B\in \mathcal{B}_1$ and $B\in \mathcal{B}_2$. **…(ii)**

Thus A, B $\in \mathcal{B}_1$ and A, B $\in \mathcal{B}_2$. But $\mathcal{B}_1$ and $\mathcal{B}_2$ are Boolean rings.

Combining **(i)** and **(ii),** we have

$A-B\in \mathcal{B}_1 \cap \mathcal{B}_2$ and $A \cup B \in \mathcal{B}_1 \cap \mathcal{B}_2 \Rightarrow \mathcal{B}_1 \cap \mathcal{B}_2$ forms a Boolean ring.

2.5 BOOLEAN ALGEBRA (OR ALGEBRA OF SETS OR A FIELD)

Let X be any set. Let $\mathscr{B}$ be class of subsets of X then $\mathscr{B}$ is said to be Boolean algebra of subsets of X, if

i. $\emptyset\in\mathscr{B}$

ii. $A,B \in\mathscr{B} \Rightarrow A \cup B \in \mathscr{B}$

iii. $A\in\mathscr{B} \Rightarrow A^c \in \mathscr{B}$; Where $A^c=X-A$.

Note: From the above definition it is clear that if $\mathscr{B}$ is Boolean algebra of subsets of X and let A,B$\in\mathscr{B}$ then

(i) X $\in\mathscr{B}$

(ii) If $A_1,A_2 ...,A_n\in$ F then $A_1\cup\cdots\cup A_n\in\mathscr{B}$

(iii) If $A_1 ..., A_n\in\mathscr{B}$ then$A_1\cap\cdots\cap A_n\in\mathscr{B}$

(iv) If A, B $\in\mathscr{B}$ then A − B $\in\mathscr{B}$

Since $\emptyset\in\mathscr{B}$ and $X = \emptyset^c$, it follows that X $\in\mathscr{B}$.

For (ii) we have $A_1\cup\cdot\ \cup A_n = A_1\cup(A_2\cdots\cup A_n)\in\mathscr{B}$ (by induction)

Then (iii) follows by complementation: $A_1\cap\cdots\cap A_n = (A_1^c\cup\cdots\cup A_n^c)^c$ which is in $\mathscr{B}$ because each $A_i^c\in\mathscr{B}$.

For (iv) we have $A - B = A\cap B^c$is in $\mathscr{B}$, because $A,B^c\in\mathscr{B}$.

Example 2.5 The collection $\{\emptyset, X\}$ is a trivial example of algebra of subsets of X.

Example 2.6 The set P(X) of all subsets of X is a algebra.

Example 2.7 Let X be an infinite set, and $\mathscr{B}$ be the collection of all subsets of X which are finite or have finite complement. Then $\mathscr{B}$ is an algebra of sets.

Note: Every Boolean algebra is a Boolean ring.

Hint: Use $A\Delta B=(A-B)\cup(B-A)$ and $(A-B)=A\cap B^c$, $(B-A)= B\cap A^c$.

2.6 σ-ALGEBRA OF SETS

Let X be an arbitrary set. A collection $\mathscr{B}$ of subsets of X is called a σ-algebra (sometimes also called a σ-field) on X if the following conditions are satisfied simultaneously:

(i) $X \in \mathscr{B}$.

(ii) For each $A \in \mathscr{B} \Longrightarrow A^c \in \mathscr{B}$

(iii) For each infinite sequence $\{A_i\}_{i=1}^{\infty}$ of sets that belong to $\mathscr{B} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathscr{B}$.

Note: condition (iii) is equivalent to the following condition:

For each infinite sequence $\{A_i\}_{i=1}^{\infty}$of sets that belong to $\mathscr{B} \Longrightarrow \cap_{i=1}^{\infty} A_i \in \mathscr{B}$….(check by Reader)

Remark:

(i) Condition (i) in both the definitions of algebra and σ-algebra can be replaced by saying that $\mathscr{B}$ is non-empty.

(ii) Each σ-algebra is an algebra on X. But the converse is NOT true in general.

(iii) The empty set belongs to any algebra or σ-algebra.

Example 2.8 Let X be any set and $\mathscr{B}$ be the power set P(X) of X. Then $\mathscr{B}$ is a σ-algebra on X

Example 2.9 Let X be any set and $\mathscr{B} = \{\emptyset, X\}$. Then $\mathscr{B}$ is a σ-algebra on X.

Example 2.10 Let X be an infinite set and $\mathscr{B}$ be the collection of all finite subsets of X. Then $\mathscr{B}$ is NOT an algebra (because X does not belongs to $\mathscr{B}$) and hence NOT a σ-algebra.

Example 2.11 Let X be an infinite set and $\mathscr{B}$ be the collection of all subsets A of X such that either A or $A^c$is finite. Check that $\mathscr{B}$ is an algebra but is NOT a σ-algebra. This is an exercise

Example 2.12 Let X be any set and $\mathscr{B}$ be the collection of all subsets A of X such that either A orA^c is countable. Check that A is a σ-algebra.

Example 2.13 Let X be a uncountable set. Let $\mathscr{B}$ be the collection of all countable subsets of X. Then A is NOT an algebra (because X does not belongs to $\mathscr{B}$**)** and hence NOT a σ-algebra.

2.6.1 CONSTRUCTION OF σ-ALGEBRAS

In this subsection, we will see how to construct a σ-algebra out of a given (arbitrary) collection $\mathcal{F}$ of subsets of a given set X.

Theorem 2: Let X be a set and $\mathcal{F}$ be an arbitrary collection of subsets of X. Then there exists a smallest σ-algebra $\mathscr{B}$ on X that contains $\mathcal{F}$.

Proof: Let C be the collection of all σ-algebras on X that contain $\mathcal{F}$. Clearly then C is non-empty since it contain the σ-algebra P(X) which is the power set of X. Take the intersection of all σ-algebras in C, this intersection will be a σ-algebra (verify by reader), call it $\mathscr{B}$. It is now easy to check that $\mathscr{B}$ has the required properties.

Definition: Given a set X and an arbitrary collection $\mathcal{F}$ of subsets of X. The smallest σ-algebra on X that contains $\mathcal{F}$ is unique (the proof of uniqueness follows from the proof of theorem 2 above) and is called the **σ-algebra generated by $\mathcal{F}$**, often denoted by σ($\mathcal{F}$). By the phrase 'smallest σ-algebra on X that contains $\mathcal{F}$', we mean a σ-algebra on X that includes $\mathcal{F}$ and every σ-algebra on X that includes $\mathcal{F}$ also includes it.

A natural question that arises after looking at the statement of exercise 2.4 is that if we replace the term 'intersection' by the term 'union' in the statement of exercise 2.4, does the new statement hold true? The answer to this is NO in general. As an example, take X = {1, 2, 3}, C_1 = {X, ∅, {1}, {2, 3}} and C_2 = {X, ∅, {2}, {1, 3}}. Then check that both C_1 and C_2 are σ-algebras but $C_1 \cup C_2$ is NOT.

2.6.2 THE BORELσ-ALGEBRA

In this subsection, we will discuss an important example of a σ-algebra on ℝ.

Definition: Let $\mathcal{F}(\mathbb{R})$ be the collection of all open subsets of ℝ.Let $\mathscr{B}(\mathbb{R}) = \sigma(\mathcal{F}(\mathbb{R}))$. We call $\mathscr{B}(\mathbb{R})$ the **Borel σ-algebra on** ℝ. Elements of$\mathscr{B}(\mathbb{R})$ are called the Borel subsets of ℝ.. We denote this σ-algebra by $\mathscr{B}(\mathbb{R})$.

2.7 SET FUNCTIONS

Let X be a set. Let $\mathscr{B}$ be a non-empty class of subsets of X then a set function on $\mathscr{B}$ is a function μ with domain $\mathscr{B}$ and codomain $[-\infty, \infty]$ i.e., $\mu: \mathscr{B} \to [-\infty, \infty]$.

Remark: It is typically assumed that μ(E)+μ(F) is always well defined for all E,F∈$\mathscr{B}$, or equivalently, thatμ does not take on both $-\infty$ and ∞ as valves.

Note: A set F∈$\mathscr{B}$ is called null set (with respect to μ) if μ(F)=0. Whenever μ is not identically equal to either $-\infty$ or $+\infty$ then it is typically also assume that μ(∅)=0 if ∅∈$\mathscr{B}$.

COMMON PROPERTIES OF SET FUNCTIONS:

A set functionμ: $\mathscr{B} \longrightarrow [-\infty, \infty]$ is said to be

(i) Non-negative: if μ takes valves in $[0, \infty]$.

(ii) Finitely additive: We say that μ is finitely additive if, for any family $A_1, ..., A_n \in \mathscr{B}$ of mutually disjoint sets such that $\cup_{i=1}^{n} A_i \in \mathscr{B}$,we have$\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

(iii) Countably additive or σ-additive: We say that μ is σ–additive if, for any sequence $(A_n) \subset \mathscr{B}$ of mutually disjoint sets such that $\cup_{i=1}^{\infty} A_i \in \mathscr{B}$, we have $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Remark:

(i) Any σ–additive set function on $\mathscr{B}$ is also finitely additive.

(ii) If μ is additive and $A, B \in \mathscr{B}$, and $A \supset B$,
then $\mu(A) = \mu(B) + \mu(A \setminus B)$. Therefore, $\mu(A) \geq \mu(B)$.
It is also called μ is monotone with respect to inclusion, i.e. $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

(iii) For any sequence $(A_n) \subset \mathscr{B}$ of mutually disjoint sets such that $\cup_{i=1}^{\infty} A_i \in \mathscr{B}$, then for set function μ we have $\mu(\cup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \mu(A_i)$.

Definition: Let X be a set. Let $\mathscr{B}$ be a σ-algebra of subsets of X then a set function μ: $\mathscr{B} \longrightarrow [-\infty, \infty]$ is said to be **measure** if following condition holds:

(i) $\mu(\emptyset)=0$.

(ii) Non-negativity: For all $A \in \mathscr{B}$, $\mu(A) \geq 0$

(iii) Countably additive or σ-additive: For any sequence $(A_n) \subset \mathscr{B}$ of mutually disjoint sets, we have $\mu(\cup_{i=1}^{\infty} A_i)=\sum_{i=1}^{\infty} \mu(A_i)$.

Remark: If the condition for non-negativity is dropped, and μ takes on at most one of the valves of $\pm\infty$, then μ is called a signed measure.

Example 2.14: Let X be a nonempty set and $x \in X$. Define, for every $A \in P(X)$,

$$\delta_x(A)= \begin{cases} 1; \text{ if } x \in A \\ 0; \text{ if } x \notin A \end{cases}$$

Then δ_x is a measure in X, called the Dirac measure.

Definition:

(i) A σ-additive set function $\mu : \mathscr{B} \to [0, \infty]$ is called **finite**, if $\mu(X) < \infty$.

(ii) A σ-additive set function $\mu : \mathscr{B} \to [0, \infty]$ is called **σ-finite**, if there exists a sequence $(A_n) \subseteq \mathscr{B}$ such that $\cup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty \ \forall n \in \mathbb{N}$.

2.8 SOLVED PROBLEM

Example 2.15 Let X be an infinite set and $\mathscr{B}$ be the collection of all subsets A of X such that either A or A^c isfinite. Show that $\mathscr{B}$ is an algebra but is NOT a σ-algebra.

Sol. $\mathscr{B}$ forms an algebra of subsets of X since

(i) $\emptyset \in \mathscr{B}$, as $\emptyset$ is empty hence it is finite.

(ii) Let A,B $\in\mathscr{B}$. Then there are four cases:

Case(i) If A,B both are finite subsets of X then A∪B is finite $\Rightarrow A\cup B\in\mathscr{B}$.

Case(ii) If A is finite and $B^c$is finite subsets of X then $(A\cup B)^c=A^c\cap B^c$ is finite (since B^c is finite)$\Rightarrow(A\cup B)^c\in\mathscr{B}$.

Case(iii) If $A^c$is finite and B is finite subsets of X then $(A\cup B)^c=A^c\cap B^c$ is finite (since A^c is finite) $\Rightarrow(A\cup B)^c\in\mathscr{B}$.

Case(iv) If $A^c$is finite and $B^c$is finite subsets of X then $A^c\cup B^c$ is finite.

From above four cases,

we have either (A∪B) or $(A\cup B)^c$ is finite $\Rightarrow A\cup B\in\mathscr{B}$.

(iii) Let $A\in\mathscr{B}\Rightarrow$ either A or $A^c$is finite $\Rightarrow A^c\in\mathscr{B}$.

Hence $\mathscr{B}$ forms a algebra of subsets of X.

If we take $A_n=\{n\}$; n=1,2,3… then $\cup_{n=1}^{\infty} A_n=\mathbb{N}$, which is infinite$\Rightarrow\mathscr{B}$ is not a σ-algebra.

Example 2.16 Let X =ℕ, consider $\mathscr{B}=\{A\in P(X)\mid A \text{ is finite, or } A^c \text{ is finite}\}$.

Define μ $:\mathscr{B}\rightarrow[-\infty,\infty]$ defined as $\mu(A)=\begin{cases} (A); & \text{if A is finite} \\ \infty; & \text{if } A^c \text{ is finite}\end{cases}$

forms a set function. Then μ forms a set function.

Where #(A) represent number of element in A.

Sol. It is clear that μ : $\mathscr{B} \to [-\infty,\infty]$ forms a function and μ(∅)=0.

Example 2.17 Let X={a, b, c, d} and let $\mathscr{B}$={∅, {a, b}, {c, d}, {a, b, c, d}} then $\mathscr{B}$ forms a Boolean algebra.

Example 2.18 Let X be a set and A is subset of X then {∅, A, X\A, X} is a simple σ-algebra generated by the subset A.

Example 2.19 Let X = R, $\mathscr{B}$ = {A ⊂ R|A is a union of finitely many intervals of the type (a, b], (a, ∞) or (−∞, b]}. It is easy to check that each set that belongs to $\mathscr{B}$ is the union of a finite disjoint collection intervals of the 3 types mentioned above. Check that A is an algebra over X, but it is NOT a σ-algebra on X (because intervals of the type (c, d) are unions of sequences of sets belonging to $\mathscr{B}$, but do not themselves belong to $\mathscr{B}$).

Example 2.20 In X = [0, 1), the class $\mathscr{B}$ consisting of ∅, and of all finite unions A = $\bigcup_{i=1}^{n}[ai\,, bi)$ with $0 \le a_i < b_i \le a_{i+1} \le 1$, is an algebra.

Sol. For A = $\bigcup_{i=1}^{n}[ai\,, bi)$ we have $A^c = [0, a_1) \cup [b_1, a_2) \cup \cdots \cup [b_n, 1) \in \mathscr{B}$**,** Moreover, in order to show that $\mathscr{B}$ is stable under finite union, it suffices to observe that the union of two (not necessarily disjoint) intervals [a, b) and [c, d) in [0, 1) belongs to $\mathscr{B}$.

CHECKYOUR PROGRESS

1. Every σ-ring is a Boolean ring. True/False
2. Let $\mathscr{B}$be a set of all finite subsets of ℝ then $\mathscr{B}$ does not forms a Boolean ring of sets. True/False
3. Every Boolean ring is a σ-ring. True/False
4. Every Boolean algebra is a Boolean ring. True/False

5. Let $\mathscr{B}$ be a an algebra of subsets of X then if A, B∈ $\mathscr{B}$ then A∆B∈ $\mathscr{B}$. True/False
6. LetX be a uncountable set. Let $\mathscr{B}$ be the collection of all countable subsets of X. Then A is an algebra of subset of X. True/False
7. The intersection of an arbitrary collection of σ-algebras on X is not a σ-algebra on X. True/False
8. Any σ–additive set function on $\mathscr{B}$ is also finitely additive. True/False

2.9 SUMMARY

This unit is complete combination of

- **i.** Definition of generalisation of Boolean ring and it's properties.
- **ii.** σ-ring and it's properties
- **iii.** Concept of Boolean algebra and it's related results.
- **iv.** Definition of σ-algebra of sets and it's properties.
- **v.** Definition of set function and it's properties.

2.10 GLOSSARY

1. Set.
2. Subset.
3. Ring of sets.
4. σ-ring.
5. Boolean algebra.
6. σ-algebra.
7. set function.

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2.13TERMINAL QUESTIONS

1. Let $\mathscr{B}$ be a Boolean ring of sets. Then prove that

(i) $\mathscr{B}$ is closed under finite intersection.

(ii) $\mathscr{B}$ is closed under finite union.

2. Let X be any set. Let $\mathscr{B}$ be a non-empty class of subsets of X then show that $\mathscr{B}$ is a Boolean algebra of subsets of X iff

(i) $\emptyset \in \mathscr{B}$

(ii) $A, B \in \mathscr{B} \Rightarrow A \cap B \in \mathscr{B}$

(iii) $A \in \mathscr{B} \Rightarrow A^c \in \mathscr{B}$; Where $A^c = X - A$.

3. Let $\mathscr{B}$ be a an algebra of subsets of X then show that if $A, B \in \mathscr{B}$ then $A \Delta B \in \mathscr{B}$.

4. Let X be a set. Then show that intersection of an arbitrary collection of σ-algebras on X is a σ-algebra on X.

6. Let X be a set and A, B ⊂ P(X) then Show that, if $\mathscr{B}$ is a σ–algebra, then $\sigma(\mathscr{B}) = \mathscr{B}$.

2.14 ANSWERS

CHECK YOUR PROGRESS

1. True
2. False
3. False
4. True
5. True
6. True
7. False
8. True

UNIT 3: MEASURE SPACE

CONTENTS:

3.1 INTRODUCTION

In mathematics, the concept of a measure is a generalization and formalization of geometrical measures (length, area, volume) and other common notions, such as magnitude, mass, and probability of events. In this unit we will discuss about measure, outer measure, measurable sets, example of non-measurable sets and some properties of measure.

3.2 OBJECTIVES

After completion of this unit learners will have the deep understanding of

- **i.** Outer Measure
- **ii.** Measurable sets
- **iii.** Non measurable sets
- **iv.** Example of Non measurable sets.

3.3 OUTER MEASURE

Definition: Let X be a set and P(X) be the power set of X. An outer measure on X is a set function

μ^* : P(X) → [0, ∞] which satisfies all the three properties mentioned below:

a) $\mu^*(\emptyset) = 0$.

b) If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$. (monotonicity of μ^*)

c) If $\{A_n\}$ is an infinite sequence of subsets of X, then

$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. (countable subadditivity of μ^*).

Remark:

1. The domain of an outer measure is not any arbitrary σ-algebra, it is always the power set σ-algebra.
2. For any set X, a measure on P(X) is always an outer measure, but the converse is not true in general. We will soon see some example (In fact, an outer measure can fail to be a measure because the countable additivity might fail to hold!).

Example 3.1: Let X be any set and μ^* : P(X) → [0, ∞] be given by:

$$\mu^* (A) = \begin{cases} 0; & \text{if A is countable} \\ 1; & \text{otherwise} \end{cases}.$$

Check that $\mu *$ is an outer measure.

Definition: The **Lebesgue outer measure m^* on $\mathbb{R}$**:

Let I be a nonempty interval of real numbers. We define its length, $\ell(I)$, to be∞if I is unbounded and otherwise define its length to be the difference of its endpoints. For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A, that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms.

We define the **Lebesgue outer measure** of A, m*(A), to be the infimum of all such sums, that is

$$m^*(A) = \operatorname{Inf}\{\textstyle\sum_{k=1}^{\infty} \ell(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k\}.$$

It follows immediately from the definition of outer measure that

$$m^*(\emptyset) = 0.$$

Moreover, since any cover of a set B is also a cover of any subset of B, outer measure is monotone in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Theorem 3.1: A countable set has Lebesgue outer measure zero.

Let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$.

Let$\epsilon \geq 0$. For each natural number k, define $I_k = (c_k - \epsilon/2^{k+1},\ c_k + \epsilon/2^{k+1})$.

The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$covers C.

Therefore $0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k)) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon$.

This inequality holds for each$\epsilon > 0$. Hence m* (C) = 0.

Remark: TheLebesgue outer measure of an interval is its length (verify by learner).

Theorem 3.1 The Lebesgue Outer measure is translation invariant, that is, for any set A and number y,

$$m^*(A + y) = m^*(A).$$

Proof: Observe that if$\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then $\{I_k\}_{k=1}^{\infty}$covers A if and only if $\{I_k + y\}_{k=1}^{\infty}$covers A + y.
Moreover, if each I_k is an open interval, then each I_k+ y is an open interval of the same length and so $\sum_{k=1}^{\infty} \ell(I_k)=\sum_{k=1}^{\infty} \ell(I_k + y)$.

The conclusion follows from these two observations.

3.4 MEASURABLE SETS

Definition: Let X be a set and μ^* be an **outer measure** on X. A subset B of X is said to be μ^* -measurable (or measurable set) if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

holds for each subset A of X.

Note(i)it follows from the defining properties of an outer measure that

$$\mu^*(A) \le \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

holds for any two subsets A, B of X. Thus to check for the μ^*-measurability of a subset B of X, we only need to check that $\mu^*(A) \ge \mu^*(A \cap B) + \mu^*(A \cap B^c)$………. **(3.1)**

holds for each subset A of X. But if $\mu^*(A) = \infty$, the inequality 3.1 anyways holds true.

So the μ^* -measurability of a subset B of X can be verified by checking that inequality **3.1** holds true for each $A \subseteq X$ which satisfies $\mu * (A) < \infty$.

(ii) A Lebesgue measurable subset of $\mathbb{R}$ is one that is measurable with respect to the Lebesgue outer measure m^*.

Theorem 3.2 Let X be a set and μ^* be an outer measure on X. Then each subset B of X which satisfies $\mu^*(B) = 0$ or $\mu^*(B^c) = 0$ is μ^* -measurable i.e., Any set of outer measure zero is measurable.

Proof :

Case(i): if $\mu^*(B) = 0$.

Then by the property of outer measure we have,

$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c)$ **…..(3.2)**

Also;$\mu^*(A \cap B) + \mu^*(A \cap B^c) \leq \mu^*(A) + \mu^*(B)$

$\Longrightarrow \mu^*(A \cap B) + \mu^*(A \cap B^c) \leq \mu^*(A)$ (as $\mu^*(B) = 0$)**….(3.3)**

Now using equation **3.2** and **3.3** we have,

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

holds for any subsets A of X.

$\Longrightarrow$B is measurable with respect to outer measure μ^*on X.

Case(ii) : if $\mu^*(B^c) = 0$

Then by the property of outer measure we have,

$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c)$

$= \mu^*(A \cap B^c) + \mu^*(A \cap (B^c)^c)$…..**(3.2)**

Also,

$\mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A \cap B^c) + \mu^*(A \cap (B^c)^c)$

$\leq \mu^*(B^c) + \mu^*(A) = \mu^*(A)$ (as $\mu^*(B^c) = 0$)

$\Longrightarrow \mu^*(A \cap B) + \mu^*(A \cap B^c) \leq \mu^*(A)$….**(3.3)**

Now using equation **3.2** and **3.3** we have,

$$\mu^* (A) = \mu^* (A \cap B) + \mu^* (A \cap B^c)$$

holds for any subsets A of X⟹B is measurable with respect to outer measure μ^*on X.

Notation:

Let X be a set and μ^* be an outer measure on X. Then the $\mathcal{M}_{\mu^*}$denotes the collection of sets that are measurable with respect to outer measure μ^*on X.

Recall the Lebesgue outer measurem^* on ℝ. We denote by $\mathcal{M}_{m^*}$ the collection of all Lebesgue measurable subsets of R, i.e., all subsets of R which are measurable with respect to the outer measure m^*.

Exercise: Let X be a set and μ^* be an outer measure on X. Let $\mathcal{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets of X. Then $\mathcal{M}_{\mu^*}$ is a σ-algebra.

Theorem 3.3: The union of a finite collection of measurable sets is measurable.

Proof: As a first step in the proof, we show that the union of two measurable sets E_1 and E_2 is measurable. Let A be any set. First using the measurability of E_1, then the measurability of E_2, we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap {E_1}^c)$$
$$m^*(A \cap E_1) + m^*([A \cap {E_1}^c] \cap E_2) + m^*([A \cap {E_1}^c] \cap {E_2}^c).$$

There are the following set identities:

$$[A \cap {E_1}^c] \cap {E_2}^c = A \cap [E_1 \cup E_2]^c.$$

and

$$[A \cap E_1] \text{U} [A \cap {E_1}^c \cap E_2)) = A \cap [E_1 \cup E_2].$$

We infer from these identities and the finite sub additivity of outer measure that

$m^*(A) = m^*(A\cap E_1) + m^*([A\cap {E_1}^c] \cap E_2)+m^* ([A\cap {E_1}^c] \cap {E_2}^c)$

$= m^*(A\cap E_1) + m^*([A\cap {E_1}^c] \cap E_2)+m^*(A \cap [E_1 \cup E_2]^c)$

$\geq m^*(A\cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^c)$.

Thus $E_1 \cup E_2$ is measurable.

Now let $\{E_k\}_{k=1}^{n}$ be any finite collection of measurable sets. We prove the measurability of the union $\bigcup_{k=1}^{n} E_k$, for general n, by induction. This is trivial for n = 1. Suppose it is true for $n - 1$.

Thus, since

$$\bigcup_{k=1}^{n} E_k = \{\bigcup_{k=1}^{n-1} E_k\} \cup E_n,$$

and we have established the measurability of the union of two measurable sets, the set $\bigcup_{k=1}^{n} E_k$is measurable.

3.5NONMEASURABLE SETS

We have defined what it means for a set to be measurable and studied properties of the collection of measurable sets.

The set that fail to be measurable is called **Non measurable set**.

We know that if a set E has outer measure zero, then it is measurable, and since any subset of E also has outer measure zero, every subset of E is measurable. This is the best that can be said regarding the inheritance of measurability through the relation of set inclusion: we now show that if E is any set of real numbers with positive outer measure, then there are subsets of E that fail to be measurable. For this we need some important result.

Theorem 3.4: Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda \in \Lambda}$, is disjoint. Then$m^*(E) = 0$.

Proof: The translate of a measurable set is measurable.

Thus, by the countable additivity of measure over countable disjoint unions of measurable sets,

$$m^* [\cup_{\lambda \in \Lambda}(\lambda + E)] = \sum_{\lambda \in \Lambda} m^*(\lambda + E) \ldots\ldots(3.4)$$

Since both E andΛ are bounded sets, the set $\cup_{\lambda \in \Lambda}(\lambda + E)$ also is bounded and therefore has finite measure.

Thus the left-hand side of (3.4) is finite.

However, since measure is translation invariant, $m^* (\lambda + E) = m^*(E) > 0$ for each$\lambda \in \Lambda$.

Thus, since the set Λ is countably infinite and the right-hand sum in (3.4) is finite, we must have $m^*(E) = 0$.

Definition: For any nonempty set E of real numbers, we define two points in E to be **rationally equivalent** provided their difference belongs to Q, the set of rational numbers.

- It is easy to see that this is an equivalence relation, that is, it is reflexive, symmetric, and transitive. We call it the rational equivalence relation on E. For this relation, there is the disjoint decomposition of E into the collection of equivalence classes. By a choice set for the rational equivalence relation on E we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class. We infer from the Axiom of Choicethat there are such choice sets. A choice set $\mathcal{C}_E$ is characterized by the following two properties:

i. the difference of two points in $\mathcal{C}_E$ is not rational;

ii. for each point x in E, there is a point c in $\mathcal{C}_E$ for which x = c + q, with q rational. This first characteristic property of $\mathcal{C}_E$ may be conveniently reformulated as follows:

For any set $\Lambda \subseteq Q$, $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ is disjoint**........(3.5)**

3.6 EXAMPLE OF NONMEASURABLE SETS

Example 3.2: Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof : By the countable sub additivity of outer measure, we may suppose E is bounded.

Let $\mathcal{C}_E$ be any choice set for the rational equivalence relation on E.

We claim that $\mathcal{C}_E$ is not measurable.

To verify this claim, we assume $\mathcal{C}_E$is measurable and derive a contradiction.

Let Λ be any bounded, countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable, and, by **(3.5),** the collection of translates of $\mathcal{C}_E$by members of Ao is disjoint, it follows from the theorem **3.4** that m($\mathcal{C}_E$) = 0.

Hence, again using the translation invariance and the countable additivity of measure over countable disjoint unions of measurable sets,

$$m^* [\cup_{\lambda \in \Lambda} (\lambda + \mathcal{C}_E)] = \sum_{\lambda \in \Lambda} m^*(\lambda + \mathcal{C}_E)$$

To obtain a contradiction we make a special choice of Λ.

Because E is bounded it is contained in some interval [-b, b]. We choose

$$\Lambda = [-2b, 2b] \cap Q.$$

Then Λ is bounded, and is countably infinite since the rationals are countable and dense.

We claim that

$$E \subseteq \bigcup_{\lambda \in [-2b, 2b] \cap Q} (\lambda + \mathcal{C}_E) \text{......} \mathbf{(3.6)}$$

Indeed, by the second characteristic property of $\mathcal{C}_E$, if x belongs to E, there is a number c in the choice set $\mathcal{C}_E$ for which x = c + q with q rational.

But x and c belong to [-b, b], so that q belongs to [-2b, 2b].Thus the inclusion (3.6) holds.

This is a contradiction because E, a set of positive outer measure, is not a subset of a set of measure zero.

The assumption that $\mathcal{C}_E$ is measurable has led to a contradiction and thus it must fail to be measurable.

Theorem 3.5: There are disjoint sets of real numbers A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof: We prove this by contradiction.

Assume m*(A U B) = m*(A) + m*(B) for every disjoint pair of sets A and B.

Then, by the very definition of measurable set, every set must be measurable.

This contradicts the preceding example.

3.7 SOLVED PROBLEM

Example 3.3 Let X be any set and $\mu^*: P(X) \to [0, \infty]$ be given by:

$$\mu^*(A) = \begin{cases} 0; & \text{if } A = \emptyset \\ 1; & \text{otherwise} \end{cases}.$$

Check that μ^* is an outer measure.

Example 3.4: The sets $\emptyset$ and X are measurable with respect to every outer measure μ^* on X.

Sol.

Since $\mu^*(A) = \mu^*(A \cap \emptyset) + \mu^*(A \cap \emptyset^c)$

$= \mu^*(\emptyset) + \mu^*(A \cap X)$

$= \mu^*(\emptyset) + \mu^*(A)$

$= 0 + \mu^*(A) = \mu^*(A)$

holds for any subsets A of $X \Longrightarrow \emptyset$ is measurable with respect to every outer measure μ^* on X.

Similarly,

$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \cap X^c)$

$= \mu^*(A) + \mu^*(A \cap \emptyset)$

$= \mu^*(A) + \mu^*(\emptyset)$

$= \mu^*(A) + 0 = \mu^*(A)$

holds for any subsets A of $X \Longrightarrow X$ is measurable with respect to every outer measure μ^* on X.

Example 3.5: Let X be any infinite set and $\mu^*: P(X) \to [0, \infty]$ be given by:

$$\mu^*(A) = \begin{cases} 0; & \text{if } A \text{ is finite} \\ 1; & \text{otherwise} \end{cases}.$$

Check that μ^* is NOT an outer measure.

Sol.

Let X= $\mathbb{N}$. Take A_n={n}; for n=1,2,3…

Then $\mu^*(A_n)$=0 for n=1,2,3… (since each A_n is finite).

Also $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.

Now,

$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \mu^*(\mathbb{N})=1$

And $\sum_{n=1}^{\infty} \mu^*(A_n)=0$; (since each $\mu^*(A_n)=0$)

Hence countable subadditivity of μ^*does not hold for this.

Therefore μ^* is not outer measure.

CHECK YOUR PROGRESS

1. The domain of an outer measure is any arbitrary σ-algebra. True/False
2. For any set X, a measure on P(X) is always an outer measure. True/False
3. A countable set has Lebesgue outer measure non-zero. True/False
4. The union of a finite collection of measurable sets is measurable. True/False
5. A countable set has Lebesgue outer measure zero **.** True/False
6. If m*(A) =0, then m*(AUB) =m*(B). True/False
7. If a set E has outer measure zero, then it is not measurable. True/False
8. The union of a countable collection of measurable sets is measurable. True/False

3.8 SUMMARY

This unit is complete combination of

- **i.** Definition and example of outer measure and it's properties.
- **ii.** Measurable sets and it's properties
- **iii.** Definition of Non measurable set.
- **iv.** Example of Non measurable sets.

3.9 GLOSSARY

1. Outer Measure.
2. Measurable set.
3. Non measurable sets.
4. σ-algebra.
5. Lebesgue outer measure

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3.12 TERMINAL QUESTIONS

1. By using properties of outer measure, prove that the interval [0, 1] is not countable.
2. Let A be the set of irrational numbers in the interval [0, 1). Prove that m*(A) = 1
3. Prove that if m*(A) =0, then m*(AUB) =m*(B).
4. Prove that the outer measure of an interval is its length
5. The union of a countable collection of measurable sets is measurable.
6. Prove that Every interval is measurable.

7. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.
8. Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set O for which $F \subseteq E \subseteq O$ and $m^*(O-F) < \epsilon$.

3.13 ANSWERS

CHECK YOUR PROGRESS

1. False
2. True
3. False
4. True
5. True
6. True
7. False
8. True

UNIT 4: LEBESGUE MEASURE

CONTENTS:

4.1 INTRODUCTION

In measure theory, a branch of mathematics, the Lebesgue measure, named after French mathematician Henri Lebesgue, is the standard way of assigning a measure to subsets of higher dimensional Euclidean n-spaces. For lower dimensions n=1, 2 or 3 it coincides with the standard measure of length, area or volume. In general, it is also called n-dimensional volume or simply volume. It is used throughout real analysis, in particular to define Lebesgue integration sets that can be assigned a Lebesgue measure of the Lebesgue- measurable set A is here denoted by $l(A)$.Henri Lebesgue described this measure in the year **190**1 which, a year after was followed up by his description of the Lebesgue Integral.

Bernhard Riemann (17-09- 1826 to 20 -07- 1866)

Henri Léon Lebesgue (28-06-1875 to 26-07-1941)

Ref 4.1

https://www.wikipedia.org/

The length $l(I)$ of an interval I is defined to be the difference of the endpoints of I if I is bounded, and ∞ if I is unbounded. Length is an example of a set function, that is, a function that associates an extended real number to each set in a collection of sets. In the case of length, the domain is the collection of all intervals. In this chapter we extend the set function length to a large collection of sets of real numbers. For instance, the "length" of an open set will be the sum of the lengths of the countable number of open intervals of which it is composed. However, the collection of sets consisting of intervals and open sets is still too limited for our purposes. We construct a collection of sets called Lebesgue measurable sets, and a set function of this collection called Lebesgue measure which is denoted by m.

4.2 OBJECTIVE

After Completion of this unit learners will be able to

i. Define Measure of a set.

ii. Define the concept of Lebesgue Measure.

iii. Evaluate the different type of Lebesgue Measure with example.

iv. Numerical Problems on Lebesgue Measure.

4.3 LEBESGUE MEASURE

Definition : The restriction of the set function outer measure to the class of measurable sets is called Lebesgue Measure.

It is denoted by m, so that E is a measurable set , its Lebesgue measure , $m(E)$ is defined by

$$m(E) = m^*(E).$$

Note - The countable collection of the sets $\{E_k\}_{k=1}^{\infty}$ is said to be ascending provided for each k , $E_k \subseteq E_{k+1}$, and said to be descending provided for each k , $E_{k+1} \subseteq E_k$.

4.4 DEFINITIONS

4.4.1 LEBESGUE MEASURE

Existence of Lebesgue Measure-There exist a collection M of subsets of R (the measurable sets) and a function $m: M \to [0, \infty)$ satisfying the following conditions-

- **i.** Every interval $I \subseteq R$ is measurable , with $m(I) = l(I)$.
- **ii.** If $E \subseteq R$ is a measurable set, then the complement $E^c = R - E$ is also measurable.
- **iii.** For each sequence $\{E_n\}$ of measurablesets in R, the union $\bigcup_{n\in N} E_n$ is also measurable. Moreover, if the sets $\{E_n\}$ are pairwise disjoint , then $m(\bigcup_{n\in N} E_n) = \sum_{n\in N} m(E_n)$.

4.4.2 LEBESGUE OUTER MEASURE

A subset E of R is said to be Lebesgue measurable if for each set A we have-

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

In this case, the outer measure $m^*(E)$ of E is called the Lebesgue Measure of E, and denoted by $m(E)$.

The arbitrary subset A of R that appears is the criterion is known as a test set.

Note that

$$m^*(A \cap E) + m^*(A \cap E^C) \geq m^*(A)$$

Automatically since m^* is subadditive. Thus a set E is Lebesgue measurable if and only if

$$m^*(A \cap E) + m^*(A \cap E^c) \le m^*(A)$$

For every test set A.

4.4.3 PROPERTIES OF LEBESGUE MEASURE

Proposition (1): Union of two measurable sets – If E and F are measurable subset of $\boldsymbol{R}$, then $\boldsymbol{E \cup F}$ is also measurable.

Proof- Let $A \subset R$ be a test set. Since E is measurable we know that

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \qquad (\boldsymbol{i})$$

Also if we used $A \cap (E \cup F)$ as a test set, we find that

$$m^*\big(A \cap (E \cup F)\big) = m^*(A \cap E) + m^*(A \cap E^c \cap F) \qquad (\boldsymbol{ii})$$

Finally, Since F is measurable , we know that

$$m^*(A \cap E^c) = m^*(A \cap E^c \cap F) + m^*(A \cap E^c \cap F^c) \qquad (\boldsymbol{iii})$$

Combining equation (1),(2) and (3) together yields

$$m^*(A) = m^*(A \cap (E \cup F) + m^*(A \cap E^c \cap F^c)$$

Since $E^c \cap F^c = (E \cup F^c)$, this proves that $E \cup F$ is measurable.

Corollary : Intersection of two Measurable Sets. If E and F are measurable subset of R , then $\boldsymbol{E \cap F}$ is also measurable.

Proof: Since E and F are measurable, their Complements E^c and F^c is also measurable. It follows that the union $E^c \cup F^c$ is measurable and the complement of this is $E \cap F$.

Proposition (2):Countable Additivity-

Let $\{E_k\}$ be a sequence of Pairwise disjoint measurable subset of R . then the union $\bigcup_{k \in N} E_k$ is measurable , and $m(\bigcup_{k \in N} E_k) = \sum_{k \in N} m(E_k)$.

Proof: Let$A \subseteq R$ be a test set , and let $\cup = \bigcup_{k \in N} E_k$.

We wish to show that $m^*(A) \ge m^*(A \cap U) + m^*(A \cap U^c)$.

For each $n \in N$, Let $\cup_n = \cup_{k=1}^{n} E_k$ by the proposition (2), each $\cup_n$ is measurable, so $m^*(A) = m^*(A \cap U_n) + m^*(A \cap U_n^c)$.

But each $U_n \subseteq U$, so $T \cap U^c \subseteq A \cap U_n^c$, and hence

$$m^*(A) \geq m^*(A \cap U_n) + m^*(A \cap U^C).$$

Thus it suffices to show that $m^*(A \cap U_n) \to m^*(A \cap U)$ as $n \to \infty$.

To prove this claim, observe first that

$$m^*(A \cap U_k) = m^*(A \cap U_k \cap E_k) + m^*(A \cap U_k \cap E_k^c)$$
$$= m^*(A \cap E_k) + m^*(A \cap U_{k-1}).$$

For each k By induction, it follows that

$$m^*(A \cap U_n) = \sum_{k-1}^{n} m^*(A \cap E_k)$$

For each n. Then

$$\sum_{k=1}^{n} m^*(A \cap E_k) = m^*(A \cap U_n) \leq m^*(A \cap U) \leq \sum_{k \in N} m^*(A \cap E_k),$$

Where the last inequality follows from the Countable subadditivity of m^*.

By the squeeze theorem, we conclude that

$$\lim_{n \to \infty} m^*(A \cap U_n) = m^*(A \cap U) = \sum_{k \in N} m^*(A \cap E_k),$$

Which proves that U is measurable. Moreover, in the case where $A = R$, the last equation gives

$$m(U) = \sum_{k \in N} m(E_K).$$

The continuity of Measure-Lebesgue Measure possesses the following continuity properties:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets then.

$$\left(m \bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} m(A_k). \qquad \textbf{(1)}$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then

$$m(\bigcap_{k=1}^{\infty} B_k) = \lim_{k\to\infty} m(A_k).(2)$$

Proof: **(i)** If there is an index K_0 for which $m(A_{k_o}) = \infty$, then by the monotonicity of measure , $m(\bigcup_{k=1}^{\infty} A_k) = \infty$ and $m(A_k) = \infty$ for all $K \geq K_0$. Therefore (1) holds Since each side equals ∞. It remains to consider the case that $m(A_k) < \infty$ for all k.

Define $A_0 = \phi$ and then define $C_k = A_k \sim A_{k-1}$ for each $k \geq 1$. By construction, since the sequence $\{A_k\}_{k=1}^{\infty}$ is ascending,

$\{C_k\}_{k=1}^{\infty}$ is disjoint and $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k$.

By the countable additivity of m,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) \qquad (\mathbf{3})$$

Since, $\{A_k\}_{k=1}^{\infty}$ is ascending , we infer from the excision property of measure that

$$\sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) = \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] \qquad (\mathbf{4})$$

$$= \lim_{n\to\infty} [m(A_n) - m(A_0)].$$

Since $m(A_0) = m(\phi) = 0$, (1) follows from (3) and (4).

(ii)- we define $D_k = B_1 \sim B_k$ for each k . Since the sequence $\{B_k\}_{k=1}^{\infty}$ is descending, the sequence $\{D_k\}_{k=1}^{\infty}$ is ascending . By part (i),

$$m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k).$$

According to De Morgan's Identities,

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} [B_1 \sim B_k] = B_1 \sim \bigcap_{k=1}^{\infty} B_k.$$

On the other hand , by the excision properties of measure, for each k, since $m(B_k) < \infty, m(D_k) = m(B_1) - m(B_k)$. Therefore

$$m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty} [m(B_1) - m(B_n)].$$

Once more using excision we obtain the equality (2).

For a measurable set E, we say that a property holds **almost everywhere on** E, or it holds for almost all $x \in E$, provided there is a subset E_0 of E for which $m(E_0) = 0$ and the property holds for all $x \in E \sim E_0$.

A set X of real numbers is said to have (Lebesgue) measure zero. Measure is a mathematical precise generalization of length/area/volume. So in this sense a measure zero set is one with volume zero, quasi with no interior. It is just too fine-grained, to thin, or too flat (depending on the dimension) to have any positive value as its length/area/volume. The cantor set are uncountable set has measure zero.

The Lebesgue measure of any countable set (no matter whether finite or infinite) is 0.

The set **Q** of rational numbers is countably infinite and, therefore, countable. Any subset of a countable set is countable and has measure 0; therefore, [0; 1] ∩ **Q** is countable with measure 0.

4.5 THE BOREL - CANTELLI LEMMA

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Then almost all x ∈ **R** belong to at most finitely many of the E_k's.

Proof For each n, by the countable subadditivity of m,

$$m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) < \infty.$$

Hence, by the continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

Therefore almost all $x \in \boldsymbol{R}$ fail to belong to $\bigcap_{n=1}^{\infty}[\bigcup_{k=n}^{\infty} E_k]$ and therefore belong to at most finitely many $E_k's$.

The set function Lebesgue measure inherits the properties possessed by Lebesgue outer measure.

For future reference we name some of these properties.

(**Finite Additivity**) For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k).$$

Monotonicity) If A and B are measurable sets and $A \subseteq B$, then

$$m(A) \leq m(B).$$

(**Excision**) If, moreover, $A \subseteq B$ and $m(A) < \infty$, then

$$m(B \sim A) = m(B) - m(A),$$

So that if $m(A) = 0$, then

$$m(B \sim A) = m(B).$$

(**Countable Monotonicity**) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E.

$$m(E) \leq \sum_{k=1}^{\infty} m(E_k).$$

Countable monotonicity is an amalgamation of the monotonicity and countable subadditivity properties of measure that is often invoked.

Remark- In our forthcoming study of Lebesgue integration it will be apparent that it is the countable additivity of Lebesgue measure that provides the Lebesgue integral with its decisive advantage over the Riemann integral.

4.6 THE CANTOR SET AND THE CANTOR LEBESGUE FUNCTION

We have shown that a countable set has measure zero and a Borel set is Lebesgue measurable. These two assertions prompt the following two questions.

Question 1 If a set has measure Zero, is it also countable?

Question 2 If a set is measurable, is it also Borel?

The answer to each of these questions is negative. In this section we construct a set called the Cantor set and a function called the Cantor-Lebesgue function .

Consider the closed, Bounded interval $I = [0,1]$. The first step in the construction of the Cantor set is to subdivide I into three intervals of equal length $1/3$ and remove the interior of the middle interval, that is, we remove the interval $(1/3, 2/3)$ from the interval $[0,1]$ to obtain the closed setC_1, which is the union of two disjoint closed intervals , each of length $1/3$:

$$C_1 = [1, 1/3] \bigcup [2/3, 1].$$

We now repeat this "open middle one-third removal" on each of the two intervals in C_1 to obtain a closed set C_2, which is the union of 2^2 closed intervals, each of length $1/3^2$:

$$C_2 = [0,1/9] \bigcup [2/9, 1/3] \bigcup [2/3, 7/9] \bigcup [8/9,1].$$

We now repeat this "open middle one-third removal" on each of the four intervals in $C_2$to obtain a closed set C_3, which is the union of 2^3 closed intervals, each of length $1/3^3$. We continue this removal operation countably many times to obtain the countable collection of sets $\{C_k\}_{k=1}^{\infty}$.

We define the Cantor set **C** by

$$\boldsymbol{C} = \bigcap_{k=1}^{\infty} C_k.$$

The collection $\{C_k\}_{k=1}^{\infty}$ possesses the following two properties:

- **i.** $\{C_k\}_{k=1}^{\infty}$ is a descending sequence of closed sets;
- **ii.** For each k, $C_k$is the disjoint union of 2^k closed intervals, each of length $1/3^k$.

Proposition 4 The cantor set **C** is a closed, uncountable set of measure zero.

Proof-The intersection of any collection of closed sets is closed. Therefore $\boldsymbol{C}$ is closed.

Each closed set is measurable so that each C_k and $\boldsymbol{C}$ itself is measurable.

Now each C_k is the disjoint union of 2^k intervals, each of length $1/3^k$, so that by the finite additivity of Lebegue measure,

$$m(C_k) = (2/3)^k.$$

By the monotonicity of measure,

since $m(\boldsymbol{C}) \leq m(C_k) = (2/3)^k$, for all $k, m(\boldsymbol{C}) = 0$.

It remains to show that $\boldsymbol{C}$is uncountable.

To do so we argue by contradiction.

Suppose **C** is countable.

Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\boldsymbol{C}$.

One of the two disjoint Cantor intervals whose union is C_1 fails to contain the point c_1; denote it by F_1.

One of the two disjoint Cantor intervals in C_2 whose union is F_1 fails to contain the point c_2; denote it by F_2.

One of the two disjoint cantor intervals in C_2 whose union is F_1 fails to contain the point c_2; denote it by F_2.

Continuing in this way, we construct a countable collection of sets $\{F_k\}_{k=1}^{\infty}$, which , for each k, possesses

the following three properties:

(i) F_k is closed and $F_{k+1} \subseteq F_k$;

(ii) (ii) $F_k \subseteq C_k$; and

(iii) $c_k \notin F_k$.

Form **(i)** and the Nested Set Theorem we conclude that the intersection $\bigcap_{k=1}^{\infty} F_k$ is nonempty.

Let the point x belong to this intersection. By property **(ii),**

$$\bigcap_{k=1}^{\infty} F_k \subseteq \bigcap_{k=1}^{\infty} C_k = \boldsymbol{C},$$

And therefore the point x belongs to $\boldsymbol{C}$.However, $\{c_k\}_{k=1}^{\infty}$ is an enumeration of **C** so that$x = c_n$ for some index n.

Thus $c_n = x \in \bigcap_{k=1}^{\infty} F_k \subseteq F_n$.

This contradicts property **(iii).**

Hence $\boldsymbol{C}$ must be uncountable.

A real-valued function f that is defined on a set of real numbers is said to **increasing,** provided $f(u) \leq f(v)$ whenever $u \leq v$

and said to be **strictly increasing,** provided $f(u) < f(v)$ whenever $u < v$.

We now define the Cantor – Lebesgue function, a continuous, increasing function φ defined on [0,1] which has the remarkable property that, despite the fact that $\varphi(1) > \varphi(0)$, its derivative exists and is zero on a set of measure 1.

For each k, let $\mathcal{O}_k$ be the union of the $2^k - 1$ intervals which have been removed during the first k stages of the Cantor deletion process. Thus $C_k = [0,1] \sim \mathcal{O}$.

Define $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$.

Then, by De Morgan's Identities, $\boldsymbol{C} = [0,1] \sim \mathcal{O}$.

We begin by defining φ on $\mathcal{O}$ and then we define it on $\boldsymbol{C}$.

Fix a natural number k. Define φ on $\mathcal{O}_k$ to be the increasing function on $\mathcal{O}_k$ which is constant on each of its $2^k - 1$ open intervals and takes the $2^k - 1$ values

$$\{1/2^k, 2/2^k, 3/2^k, \ldots, [2^k - 1]/2^k\}.$$

Thus, on the single interval removed at the first stage of the deletion process, the prescription for φ is

$$\varphi(x) = 1/2 \quad if \;\; x \in (1/3, 2/3).$$

On the three intervals that are removed in the first two stages, the prescription for φ is

$$\varphi(x) = \begin{cases} 1/4 & if\ x \in (1/9,\ 2/9) \\ 2/4 & if\ x \in (3/9, 6/9)\ = (1/3\,, 2/3) \\ 3/4 & if\ x \in (7/9,\ 8/9) \end{cases}$$

We extend φ to all of [0,1] by defining it on $\boldsymbol{C}$ as follows:

$\varphi(0) = 0$ and $\varphi(x) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, x)\}$if $x \in \boldsymbol{C} \sim \{0\}$.

4.7 THE CANTOR SET AND THE CANTOR LEBESGUE FUNCTION

Example 1: Show that for any set $A, m^*(A) = m^*(A + x)$ where $A + x = [y + x : y \in A]$, that is: outer measure is translation invariant.

Solution: For each $\epsilon > 0$ there exists a collection $[I_n]$ such that $A \subseteq \bigcup I_n$ and $m^*(A) \geq \sum l(I_n) - \epsilon$. But clearly $A + x \subseteq \bigcup(I_n + x)$. So, for each $\epsilon, m^*(A + x) \leq \sum l(I_n + x) = \sum l(I_n) \leq m^*(A) + \epsilon$. So $m^*(A + x) \leq m^*(A)$. But $A = (A + x) - x$ so we have $m^*(A) \leq m^*(A + x)$.

Example 2: For any sequence of sets $\{E_i\}$, $m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.

Solution: For each i, and for any $\epsilon > 0$, there exists a sequence of intervals $\{I_{i,j}, j = 1,2, \ldots\}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$ and $m^*(E_i) \geq \sum_{j=1}^{\infty} l(I_{i,j}) - \frac{\epsilon}{2^i}$. Then $\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$, that is : the sets $[I_{i,j}]$ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$. So

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i,j=1}^{\infty} l(I_{i,j}) \leq \sum_{i=1}^{\infty} m^*(E_i) + \epsilon.$$

But ϵ is arbitrary and the result follows.

Example 3: Show that, for any set A and any $\epsilon > 0$, there is an open set 0 containing A and such that $m^*(0) \leq m^*(A) + \epsilon$.

Solution: Choose a sequence of intervals I_n such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) - \frac{\epsilon}{2} \leq m^*(A)$. If $I_n = [a_n, b_n)$, let $I'_n = (a_n - \frac{\epsilon}{2^{n+1}}, b_n)$ so that $A \subseteq \bigcup_{n=1}^{\infty} I'_n$.

Hence if $0 = \bigcup_{n=1}^{\infty} I'_n$, 0 is an open set and

$$m^*(0) \leq \sum_{n=1}^{\infty} l(I'_n) = \sum_{n=1}^{\infty} l(I_n) + \frac{\epsilon}{2} \leq m^*(A) + \epsilon.$$

Example 4: Suppose that in the definition of outer measure , $m^*(E) = inf \sum l(I_n)$ for sets $E \subseteq R$, we stipulate

- **i.** I_n open,
- **ii.** $I_n = [a_n, b_n)$,
- **iii.** $I_n = (a_n, b_n]$,
- **iv.** I_n closed ,
- **v.** or mixtures are allowed, for different n, of the various types of intervals. Show that the same m^* is obtained.

Solution:

In case **(ii)** we obtain the Definition of m^* write the corresponding m^* in case **(i),** m^*_{oc} in **case (iii),** m^*_c in case **(iv),** m^*_m in case **(v).**

We show that each equals m^*_m.

Consider m^*_0, the proof in other cases being similar.

From the definition,

$m^*_m(E) \leq m^*_0(E)$

To prove the converse :

for each $\epsilon > 0$ and each interval I_n let I'_n be an open interval containing I_n with $l(I'_n) = (1+\epsilon)l(I_n)$.

Suppose that the sequence $\{I_n\}$ is such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$

and

$m_m^*(E) \leq \sum_{n=1}^{\infty} l(I_n) - \epsilon$.

Then

$m_m^*(E) + \epsilon \geq (1+\epsilon)^{-1}$

$\sum_{n=1}^{\infty} l(I'_n)$ but $E \subseteq \bigcup_{n=1}^{\infty} I'_n$ .A union of open intervals, so

$m_0^*(E) \leq (1+\epsilon)m_m^*(E) + \epsilon(1+\epsilon)$,

for any $\epsilon > 0$, so $m_0^*(E) \leq m_m^*(E)$, as required.

Example 5: Every interval is measurable.

Solution: We may suppose the interval to be of the form $[a,\infty)$ then by theorem . the result for the other types of interval. For any set A . We wish to show that

$m^*(A) \geq m^*\big(A \cap (-\infty, a)\big) + m^*\big(A \cap [a,\infty)\big)$.

Write $A_1 = A \cap (-\infty, a)$ and

$$A_2 = A \cap [a,\infty).$$

Then for any $\epsilon > 0$ thereexist interval I_n such that write $I'_n = I_n \cap (-\infty, a)$ and $I''_n = I_n \cap [a,\infty)$,

so that $l(I_n) = l(I'_n) + l\big(I''_n\big)$.

Then

$A_1 \subseteq \bigcup_{n=1} I'_n, \quad A_2 \subseteq \bigcup_{n=1}^{\infty} I''_n$

So, $m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} l(I'_n) + \sum_{n=1}^{\infty} l\big(I''_n\big)$

$$\leq \sum_{n=1}^{\infty} l(I_n) \leq m^*(A) + \epsilon$$

Example 6: The Constant functions are measurable.

Solution: Depending on the choice of α, the set $[x: f(x) > \alpha]$, where $'f'$ is constant is the whole real line or the empty set.

Example7: The characteristic function χ_A of the set A , is measurable iff A is measurable.

Solution: Depending on α, the set $[x : \chi_A(x) > \alpha] = A$.
R or ϕ and the result follows.

Example 8: Continuous functions are measurable.

Solution: If $'f'$ is continuous , $[x: f(x) > \alpha]$ is open and therefore, measurable

4.8 SUMMARY

This unit is an explanation of

- **i.** Definition of measure of a set.
- **ii.** Lebesgue measure defined with examples.
- **iii.** The Cantor set and The Cantor- Lebesgue Function.
- **iv.** Existence of Lebesgue Measure.

4.9 GLOSSARY

- **i.** The Real Numbers Sets
- **ii.** The Natural and Rational Numbers

iii. Sequences

iv. Functions

v. Countable and Uncountable Sets

vi. Lebesgue measure defined with examples.

vii. The Cantor set and The Cantor- Lebesgue Function.

CHECK YOUR PROGRESS

1. There not exist a non-measurable subset of R whose complement in R has outer measure zero? **True or False.**
2. There exist two non-measurable sets whose union is measurable**? True or False.**
3. If f : [0, 1] → R is continuous a.e., then f is not measurable. **True or False.**
4. If $f: [0, \infty) \to \mathbb{R}$ is differentiable, then f' is measurable. **True or False.**
5. The characteristic function of the Cantor set is Lebesgue integrable in [0, 1] but not Riemann integrable? **True or False.**

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4. Walter Rudin, Principle of Mathematical Analysis (3rd edition) McGraw-Hill Kogakusha, International Student Edition, 1976.

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4.12 TERMINAL QUESTIONS

Q-1 Which of the following(s) is /are correct ?

a) Singleton set is measurable.

b) Measure of Singleton set is zero.

c) Countable Set is measurable.

d) Measure of countable set is zero.

Q-2 Which of the following (s) is are correct?

a) Closed Set is lebesgue measure

b) Closed Set is not lebesgue measure

c) Open Set is lebesgue measure

d) Open Set is not lebesgue measure

Q-3 Let $\{f_n\}$ be a sequence of measurable function with the same domain, then

a) Inf $\{f_1, f_2, \dots, f_n\}$ is measurable.

b) Inf$\{f_1, f_2, \dots, f_n\}$ is not measurable.

c) Inf f_n is measurable.

d) Inf f_n is not measurable.

Q-4 Which of the following (s) is are true?

a) The set $[0,1]$ is not countable.

b) The set $[0,1]$ is countable.

c) $M^*[0,1] = 1$

d) $M^*[0,1] = 0$

Q-5 Which of the following (s) is are true?

a) The set of rational number is lebesgue measurable.

b) The set of rational numbers have lebesgue outer measure equal to zero.

c) The set of rational number is not lebesgue measurable.

d) The set of rational numbers have lebesgue outer measure equal to one.

Q-6 If c is constant and $'f'$ is measurable real-valued function, then

a) $f + c$ is measurable.

b) $f + c$ is not measurable.

c) cf is measurable.

d) cf is not measurable.

Q-7 Let $\{f_n\}$ be a sequence of measurable function with the same domain, then

a) $\sup\{f_1, f_2, \dots, f_n\}$ is measurable

b) $\sup\{f_1, f_2, \dots, f_n\}$ is not measurable

c) $sup\ f_n$ is measurable

d) $\sup f_n$ is not measurable

Q-8 If S is lebesgue measurable set, then

a) Each translate $S + k$ is also measurable

b) Each translate $S + k$ are not measurable

c) $S + k$ is measurable, if for some set A

$$m^*(A) = m^*\big(A \cap (S + k)\big) + m^*(A \cap (S + k)^c)$$

d) $S + k$ is measurable , if for some set A, $m^*(A) = 0$

Q-9 Which of the following (s) is/are true?

a) The interval $[a, \infty)$ is measurable

b) Every interval is measurable

c) Every open set in R is measurable

d) Every closed set in R is measurable

Q-10 Which of the following(s) is/are correct?

a) If f is measurable, then $|f|$ is measurable.

b) If f is a measurable function on$[a, b]$ and if $k \in R$, then $f + k$ and kf are measurable.

c) Constant function are measurable.

d) None of the above

Q-11 Given following statements

I The set [0,1] is not countable

II If E_1 and E_2 are lebesgue measurable, so $E_1 \cup E_2$ is also a lebesgue measurable.

a) Only **I** is true

b) Only **II** is true

c) Both **I** and **II** are true

d) Neither **I** nor **II** are true

Q-12 Given following statements

I The length of an interval I is the difference of end points of the interval.

II The lebesgue outer measure of an interval is its length.

a) Only I is true

b) Only II is true

c) Both I and II are true

d) Neither I and II is true

Q-13 Which of the following subsets of R has positive lebesgue measure?

a) $A = \{X \in Q^c | 0 \leq x \leq 1\}$

b) $B = [0, \infty)$

c) Set of natural numbers

d) $[0,1) \cup (2,3)$

Q-14 Which of the following subsets of R^2 has positive lebesgue measure?

a) $A = \{(x, y)|\ x^2 + y^2 = 1\}$

b) $A = \{(x, y)|\ x^2 + y^2 < 1\}$

c) $A = \{(x, y)|\ x^2 + y^2 > 1\}$

d) None of the above has positive lebesgue measure

Q-15 Which one of the following is true?

a) Intersection of two lebesgue measurable sets is lebesgue measurable set.

b) Intersection of two lebesgue measurable sets is not lebesgue measurable set

c) Both a) and b)

d) Neither a) and b)

Q-16. What is Lebesgue measure with example?...................................

……………………………………………………………………

……………………………………………………………………

Q-17 What is the difference between Lebesgue measure and Lebesgue

outer measure?..

………………………………………………………………………………

………………………………………………………………………………

.**Q-18** What is zero Lebesgue measure?...

………………………………………………………………………………

4.13 ANSWERS

CHECK YOUR PROGRESS

1. True.
2. True.
3. False
4. True
5. False

TERMINAL QUESTIONS:

1. a,b,c,d
2. a, c
3. a, c
4. a , c
5. a, b
6. a, c
7. a, c
8. a, c
9. a, b, c, d
10. a, b, c
11. d
12. c
13. a, b, d
14. b, c
15. a

BLOCK II:

MEASURABLE FUNCTIONS AND CONVERGENCE THEOREM

UNIT 5: MEASURABLE FUNCTIONS

5.1 INTRODUCTION

Before this unit we have explained about Sets and Lebesgue Measure. In this unit we are mainly describing about measurable function and it's properties. In measure theory, a measurable function is a function between the underlying sets of two measurable spaces that preserves the structure of the spaces: the preimage of any measurable set is measurable. In real analysis, measurable functions are used in the definition of the Lebesgue integral. In probability theory, a measurable function on a probability space is known as a random variable.

5.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of Measurable functions.
ii. Describe the notion of step function.
iii. Explain the concept of simple function.

5.3 MEASURABLE FUNCTIONS

Measurable functions are in some sense comparable to continuous functions in topology. They play an important role in the study of measure and integration. We will have instances where not all sets are measurable. We will see that many sets which arise in a natural way in certain constructions are measurable.

There are many sets which have infinite measure. For instance, the entire real line R has infinite measure, any unbounded interval in R has infinite measure, and so on. Also, we have functions taking values in the extended real number system, i.e. they take values ∞ or $-\infty$. To avoid any kind of restriction on the function, we shall consider the extended real number system, i.e. we include $\infty\, and\, -\infty$ in the real number system with the following conventions:

$b + \infty = \infty$ where b is any real number or $b = \infty$,

$b \times \infty = \infty$where $b > 0$,

$b \times \infty = -\infty$ where $b < 0, \infty \times \infty = \infty, 0 \times \infty = 0$.

Similar kind of algebra can be done using $-\infty$ in place of ∞. It is to be noted that $\infty + (-\infty)$ is not defined. Using these conventions we can easily handle functions or measures taking infinite values.

We now define the notion of Lebesgue measurable function.

Definition 1: Suppose f is an extended real valued function which is defined on a measurable set $\mathrm{E.f}$ is called Lebesgue measurable function or simply a measurable function if it satisfies the following:

for each real number α, the set $\{x: f(x) > \alpha\}$ is measurable.

We will now see equivalent definitions of a measurable functions. The following result gives equivalent definitions of a measurable function.

Proposition 1:

For an extended real valued functionf whose domain is measurable, the following statements are equivalent:

- **i.** f is a measurable function, i.e. for each real number α the set $\{x: f(x) > \alpha\}$ is measurable;
- **ii.** For each real number α the set $\{x: f(x) \geq \alpha\}$ is measurable;
- **iii.** For each real number α the set $\{x: f(x) < \alpha\}$ is measurable;
- **iv.** For each real number α the set $\{x: f(x) \leq \alpha\}$ is measurable.

As a consequence,

- **v.** For each extended real number α the set $\{x: f(x) = \alpha\}$ is measurable.

Proof:

Let us suppose the domain f is D.

We first show that (i) ⇒ (iv).

One easily sees that

$\{x: f(x) \leq \alpha\} = D \sim \{x: f(x) > \alpha\}$,

i.e. the set $\{x: f(x) \leq \alpha\}$

is the complement of the set $\{x: f(x) > \alpha\}$ in D.

As the difference of two measurable sets is measurable, we obtain the desired implication.

On similar lines,

we obtain the implications (iv) ⇒ (i) and (ii) ⇒ (iii).

We now show that (i) ⇒ (ii).

This follows from the observation that

$$\{x: f(x) \geq \alpha\} = \cap\{x: f(x) > \alpha - 1/k\}.$$

As the intersection of a sequence of measurable sets is measurable, we obtain the desired implication.

On similar lines, we obtain the reverse implication (ii) ⇒ (i) by observing that

$$\{x: f(x) > \alpha\} = \cup_k \left\{x: f(x) \geq \alpha + \frac{1}{k}\right\},$$

and the fact that union of a sequence of measurable sets is measurable.

So far we have shown that the first four statements are equivalent.

We now establish that if f is measurable, then the set $\{x: f(x) = \alpha\}$ is measurablefor each extended real number α.

If α is a real number,

then

$\{x: f(x) = \alpha\} = \{x: f(x) \geq \alpha\} \cap \{x: f(x) \leq \alpha\}$.

Now using (ii) and (iv) and the fact that intersection of two measurable sets is measurable, we obtain the desired result for α real.

We now consider the case when $\alpha = \infty$.

Observe that $\{x: f(x) = \infty\} = \cap_k \{x: f(x) > k\}$.

Using (i) and the fact that the intersection of a sequence of measurable sets is measurable we get the desired result.

On similar lines we can deal with the case $\alpha = -\infty$.

Remark 1:

Restricting ourselves to the class of measurable functions, the most important set associated with them are measurable.

We now give some examples of measurable functions.

Example 1:

Constant functions are measurable.

Solution:

Let f be a constant function, say $f(x) = c$ and $\alpha \in R$. We need to show that $\{x: f(x) > \alpha\}$ is measurable.If$\alpha < c$, the set$\{x: f(x) > \alpha\}$equals the whole real line R and if $\alpha \geq c$, thenthe set $\{x: f(x) > \alpha\}$

equals empty set. Hence, in both the situations, the set $\{x: f(x) > \alpha\}$ is measurable and so is f.

We now show that a continuous function defined on a measurable set is measurable.

Example 2:

A continuous function on a measurable set D is measurable.

Solution:

Let f be a continuous function having measurable domain D.

We need to show that for each $\alpha \in R, \{x \in D: f(x) > \alpha\}$ is measurable.

Observe that, $\{x \in D: f(x) > \alpha\} = f^{-1}(\alpha, \infty)$.

As f is continuous, $f^{-1}(\alpha, \infty)$ is an open set in the relative topology on D.

By definition of relative topology, $f^{-1}(\alpha, \infty) = D \cap O$, where O is an open set in R.

Hence, $\{x \in D: f(x) > \alpha\} = f^{-1}(\alpha, \infty) = D \cap O$.

This implies that f is measurable.

Proposition 2:

Suppose f and g are two measurable functions defined on the same measurable domain D. Then the following sets are measurable:

- **i.** $E_1 = \{x \in D: f(x) < g(x)\}$
- **ii.** $E_2 = \{x \in D: f(x) \geq g(x)\}$
- **iii.** $E_3 = \{x \in D: f(x) > g(x)\}$
- **iv.** $E_4 = \{x \in D: f(x) \leq g(x)\}$
- **v.** $E_5 = \{x \in D: f(x) = g(x)\}$

Proof: Suppose $x \in E_1$.

Then $f(x) < g(x)$.

By density of rationals Q in R, there exist some $r \in Q$ such that $f(x) < r < g(x)$.

As rationals are countable, we can express the set E_1 as $E_1 = \cup_{r\in Q} \{\{x \in D: f(x) < r\} \cap \{ x \in D: g(x) > r\}\}$.

(i) and (iii) from Proposition 1 above implies the set$\{x \in D: f(x) < r\} \cap \{ x \in D: g(x) > r\}$ is measurable.

As countable union of measurable sets is measurable, therefore the set E_1 is measurable.

Parts (ii)-(v) are left as simple exercises.

The following properties tells us that certain operations performed on measurable functions lead again to measurable function. In the next result, we will see how we can enlarge the class of measurable functions by doing some kind of algebra.

Proposition3:

Let c be any real number and f and gbe two measurable realvalued functions defined on the same measurable domainD. Then $\mathrm{f} + \mathrm{c}$, cf, $\mathrm{f} + \mathrm{g}$, $\mathrm{g} - \mathrm{f}$, and fg are also measurable.

Proof:

In order to prove this result, we shall use the condition (iii) of Proposition 1. Observe that,

$$\{x: f(x) + c < \alpha\} = \{x: f(x) < \alpha - c\},$$

As f is given to be measurable, so $f + c$ is measurable by equivalent definition of a measurable function.

We now show that cf is measurable.

If the constant $c = 0$, then from Example 1, cf is measurable.

If $c > 0$, then the set $\{x: cf(x) > \alpha\} = \{x: f(x) > c^{-1}\alpha\}$ which is a measurable set as f is given to be a measurable function.

A similar argument shows that $cf$is measurable when $c < 0$.

We now show the sum function $f + g$ to be measurable.

If $f(x) + g(x) > \alpha$, then $f(x) > \alpha - g(x)$and by the density of rationals$Q$ in R, there is a rational number $r$such that

$$\alpha - g(x) < r < f(x).$$

Observe that,

$$\{x: f(x) + g(x) > \alpha\} = \cup_{r \in Q} (\{x: f(x) > r\} \cap \{x: \alpha - r < g(x)\})$$

By the countability of rationals, we obtain the set on the right hand side to be measurable and so $f + g$ is measurable.

As $-g = (-1)g$ and g is measurable, so $-g$ is also measurable.

As a result, we have $f - g = f + (-g)$ is also measurable.

Observe that, $fg = \frac{1}{2}[(f + g)^2 - (f - g)^2]$.

It is sufficient to show that f^2 is measurable whenever f is so.Consider the set $\{x: f^2(x) > \alpha\}$.

If $\alpha < 0$, then this set equals entire real number R as square of any real number is always bigger than a negative real number and so the given set is measurable.

However, if $\alpha \geq 0$,

then observe that

$$\{x: f^2(x) > \alpha\} = \{x: f(x) > \sqrt{\alpha}\} \cup \{x: f(x) < -\sqrt{\alpha}\}$$

which is a measurable set as f is measurable.

Hence, we conclude that fg is a measurable function.

We now show that the pointwise supremum, infimum, limit superior and limit inferior of a sequence of measurable real valued functions is again measurable.

Proposition 4:

Suppose $\{f_n\}$ is a sequence of measurable functionsdefined on the same measurable domain E. Then the following holds:

- **i.** $\sup_{1 \leq j \leq n} f_j$ is measurable for each n;
- **ii.** $\inf_{1 \leq j \leq n} f_j$ is measurable for each n;
- **iii.** $\sup_n f_n$ is measurable;
- **iv.** $\inf_n f_n$ is measurable;
- **v.** $\lim \sup_n f_n$ is measurable;

vi. $\lim\inf_n f_n$ is measurable.

Proof:

i. We need to show that for each $\alpha \in R, \{x \in E: \sup_{1\le j\le n} f_j(x) > \alpha\}$ is a measurable set.

It is a simple consequence that $\left\{x \in E: \sup_{1\le j\le n} f_j(x) > \alpha\right\} = \cup_j f_j(x) > \alpha$.

As finite union of measurable sets is measurable, therefore, $\sup_{1\le j\le n} f_j$ is measurable for each n.

ii. Recall that, for any real valued function h defined on $D \subset R$, $\inf_{x\in D} h(x) = -\sup_{x\in D} -h(x)$.

Hence, $\inf_{1\le j\le n} f_j = -\sup_{1\le j\le n}(-f_j)$ and using (i), $\inf_{1\le j\le n} f_j$ is measurable.

iii. On similar lines to (i), $\left\{x: \sup_n f_n(x) > \alpha\right\} = \cup_n \{f_n(x) > \alpha\}$. As countable union of measurable set is measurable, therefore, $\sup_n f_n$ is measurable.

iv. On similar lines to (ii), $\inf_n f_n = -\sup_n -f_n$. Using (iii), $\inf_n f_n$ is measurable.

v. Recall that, $\lim\sup_n f_n = \inf_n\left(\sup_{j\ge n} f_j\right)$. Applying (iii) and (iv), $\lim\sup_n f_n$ is measurable.

vi. Recall that, $\liminf_n f_n = -\limsup_n(-f_n)$ and using (v), $\liminf_n f_n$ is measurable.

The pointwise limit of a sequence of measurable real valued functions is again measurable.

However, the pointwise limit of a sequence of continuous real valued functions may not be continuous, i.e., the pointwise convergence is not enough to guarantee the continuity of the limit function.

For continuity, of the limit function, we need to have a strong form of convergence, viz. uniform convergence.

We now define a property to hold almost everywhere with the help of Lebesgue measure.

A property holds ***almost everywhere*** **(a.e.)** if the set of points where it fails to hold has measure zero.

In particular, if f and g are two functions having the same domain, then saying that $f = g\ a.e.$ means $mE = 0$, where E is the set $E = \{x: f(x) \neq g(x)\}$.

We now see some consequence of this a.e. property. In some sense, this property is contagious.

Proposition 5:

Suppose f is a measurable function and $f = g\ a.e.$, then g is also a measurable function.

Proof: Suppose E is the set $E = \{x: f(x) \neq g(x)\}$.

By given hypothesis, $mE = 0$.

We need to show that for each α, the set $\{x: g(x) > \alpha\}$ is measurable. Observe that, the set $\{x: g(x) > \alpha\}$ may be expressed as the disjoint union of the following sets:

$\{x: g(x) > \alpha\} = \{x: f(x) > \alpha\} \cup \{\{x \in E : g(x) > \alpha\} \sim \{x \in E: g(x) \leq \alpha\}\}$.

As f is measurable, so the set $\{x: f(x) > \alpha\}$ is measurable.

Also, we know that subsets of measure zero set are measurable, so the other two sets on the right hand side are also measurable since $mE = 0$.

Therefore, the set $\{x: g(x) > \alpha\}$ is measurable for each α.

Hence, g is a measurable function.

5.4 STEP FUNCTION

Definition 1: (Step function)

A real valued function ϕ defined on an interval $[a, b]$ is called a $\boldsymbol{step\ function}$ if there is a partition of $[a, b] a = t_0 < t_1 < \cdots < t_n = b$ such that for each $1 \leq i \leq n, \phi$ assumes only one value in the open interval (t_i, t_{i+1}).

The $\boldsymbol{greatest\ integer\ function}$ on a bounded interval in R is an example of step function.

We will see in the exercise section that every step function on a measurable set is measurable.

If A is any set, we define the characteristic function, χ_A of the set A to be the function given by

$$\chi_A(x) = \begin{cases} 1, if x \in A \\ 0, if x \notin A \end{cases}$$

Example 1:

The characteristic function χ_A is measurable if and only if A is measurable.

Solution:

Depending on the choice of α, we shall see that $\{x: \chi_A(x) > \alpha\}$ equals R, A or empty set $\emptyset$.

If $\alpha < 0$, then it is clear that $\{x: \chi_A(x) > \alpha\} = R$.

If $0 \leq \alpha < 1$, then $\{x: \chi_A(x) > \alpha\} = A$.

Finally, if $\alpha \geq 1$, then $\{x: \chi_A(x) > \alpha\} = \emptyset$.

It follows that χ_A is measurable if and only if A is measurable.

An immediate consequence of above example is that,the existence of a non-measurable set implies the existence of a non-measurable function. We now list some simple properties of the characteristic function.

Properties of Characteristic function:

- **i.** $\chi_\emptyset = 0,\ \chi_E = 1$, i.e characteristic function of the empty set is 0 and that of the whole domain is 1.
- **ii.** If A and B are two sets such that $A \subset B$, then $\chi_A \leq \chi_B$.
- **iii.** If $\{E_n\}$ are disjoint subsets of the set E such that $E = \cup_n E_n$, then $\chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$
- **iv.** $\chi_{A\cap B} = \chi_A \cdot \chi_B$
- **v.** $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$
- **vi.** $\chi_{\bar{A}} = 1 - \chi_A$

Proof of the properties:

- **i.** The proof of properties (i) is obvious.
- **ii.** If $x \in A$, then $\chi_A(x) = 1$. Also, $x \in B$ as $A \subset B$, so $\chi_B(x) = 1$. Therefore, $\chi_A \leq \chi_B$.
- **iii.** We will see the hints to the proof of properties (iv)-(vi) in the exercises.

5.5 SIMPLE FUNCTIONS

Simple functions are going to play an important role in the study of integration of real valued functions.

Definition 1:

A real-valued function φ is called $simple$ if it assumes only a finite number of different values.

If $\varphi$is simple and takes values $a_1, a_2, \cdots, a_n$ then φ may be expressed as

$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$,

whereA_i is the set

$A_i = \{x: \varphi(x) = a_i\}, A_i \cap A_j = \emptyset$, for $i \neq j$ and $\cup_i A_i$ equals domain of φ.

Using the result (v) in Proposition 1, the sets A_i are measurable if the simple function φ is measurable.

In the next example, we show that the simple functions are closed undertaking sum and product.

Example 1: Suppose f and g are two simple functions defined on $A \subset R$. Then $f + g$ and fg are also simple function.

Solution:

As f, g are simple functions,

so let $f(x) = \sum_{i=1}^{m} a_i \chi_{A_i}(x)$

and $g(x) = \sum_{j=1}^{n} b_i \chi_{B_i}(x)$, where $\cup_i A_i = \cup_j B_j = A$.

Let $C_{ij} = A_i \cap B_j$.Now $A_i \subset A = \cup_j B_j$ and therefore, $A_i = A_i \cap (\cup_j B_j) = \cup_j C_{ij}$.

On similar lines, $B_j = \cup_i C_{ij}$. As the C_{ij} are disjoint, therefore, $\chi_{A_i}(x) = \sum_{j=1}^{n} \chi_{C_{ij}}(x)$ and , $\chi_{B_j}(x) = \sum_{i=1}^{m} \chi_{C_{ij}}(x)$.

This implies that, $f(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_i \chi_{C_{ij}}(x)$ and $g(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} b_j \chi_{C_{ij}}(x)$.

Hence, $(f+g)(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} (a_i + b_j) \chi_{C_{ij}}(x)$ and $(fg)(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_i b_j \chi_{C_{ij}}(x)$ are simple functions.

CHECK YOUR PROGRESS

Write **true or false** for the given statements:

i. Each subset of R is measurable.

ii. Every real valued function is a measurable function.

iii. Every step function is a simple function.

iv. Every simple function is a step function.

v. If f is a continuous function a.e. on an interval $[a, b] \subset R$, then f is measurable.

vi. Almost everywhere limit of a sequence of measurable functions is not a measurable function.

vii. The set of points on which a sequence of measurable functions $\{g_n\}$ converges is measurable.

viii. If f and g are measurable, then $\max\{f, g\}$ is also measurable.

ix. If f and g are measurable, then $\min\{f, g\}$ is also measurable.

x. If f is measurable and g is continuous on R, then $g \circ f$ is measurable.

xi. The characteristic function χ_A of a set A is measurable if and only if A is measurable.

xii. If A and B are any two sets, then $\chi_{A\cup B} = \chi_A + \chi_B$ holds in general.

xiii. If $A = \cup_n A_n$, then $\chi_A = \sum_n A_n$ holds in general.

xiv. χ_A is a monotone function for every subset $A \subset R$.

5.6 SUMMARY

Measurable functions play an important role in the study of measure and integration. They are more general than continuous functions. Many sets which arise in a natural way in some constructions happens to be measurable.The almost everywhere (a.e.) property allows certain abnormal behavior to hold on a set of measure zero. Simple functions assume only finitely many different values and may be expressed as finite linear combinations of characteristic functions on disjoint sets. Simple functions are closed undertaking sum, difference and products. The characteristic function plays an important role in the study of measurable functions as simple functions are based on them. It satisfies additive properties for disjoint union of sets.

5.7 GLOSSARY

i. The Real Numbers Sets

ii. Countable and Uncountable Sets

iii. Lebesgue measure defined with examples.

iv. Measurable functions

5.8 REFERENCES

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5.9 SUGGESTED READINGS

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5.10 TERMINAL QUESTIONS

1. Prove parts (ii)-(v) of Proposition 2.

2. Show that if f is a measurable function and E is a measurable subset of the domain of f, then the function obtained by restricting f to E is also measurable.
3. Show that every continuous function on the real line R is measurable.
4. Show that monotone functions are measurable.
5. Show that every step function on a measurable set is measurable.
6. Show that the difference of two simple functions is again a simple function.
7. Show that

    i. $\chi_{A\cap B} = \chi_A \cdot \chi_B$

    ii. $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

    iii. $\chi_{\bar{A}} = 1 - \chi_A$
8. Prove that every step function is simple, however, its converse may not be true.

5.11 ANSWERS

CHECK YOUR PROGRESS

i. False
ii. False
iii. True
iv. False
v. True
vi. False
vii. True
viii. True
ix. True
x. True
xi. True
xii. False
xiii. False
xiv. True

Hint to Terminal Questions

1. (ii) $E_2 = D \sim E_1$ and use the fact that difference of two measurable set is measurable.

(iii) Similar argument as in the proof of Proposition 2(i).

(iv) $E_4 = D \sim E_3$ and use the fact that difference of two measurable set is measurable.

(v) $E_5 = E_2 \cap E_3$ and use the fact that intersection of two measurable sets is measurable.

2. This is immediate.

3. For each $\alpha \in R, \{x: f(x) > \alpha\}$ is an open setand so measurable.

4. Suppose f is monotone and D be set of discontinuities of f. Then D is at most countable and hence of measure zero.$f|_D$ is measurable on D and $f|_{R\sim D}$ is measurable as f is continuous over there. Combining we get measurability of f.

5. Suppose f is a step function with a measurable domain $D = [a, b]$ and f assumes finitely many values $b_1, b_2, \cdots, b_n$ on D. For each $\alpha \in R, \{x: f(x) > \alpha\}$ turns out to be measurable in each of the three situations:

i. $\alpha < \min\{b_1, b_2, \cdots, b_n\}$

ii. $\alpha \geq max\{b_1, b_2, \cdots, b_n\}$

iii. $\min\{b_1, b_2, \cdots, b_n\} \leq \alpha < \max\{b_1, b_2, \cdots, b_n\}$

6. Analogous to Example 1 in Section 5.3

7. (i) $\chi_{A\cap B} = \chi_A \cdot \chi_B$ if and only if $\chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x)$. In both the situations when $x \in A \cap B$ and when $x \notin A \cap B$ equality holds.

(ii)$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$ if and only if $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x)$. In both the situations when $x \in A \cup B$ and when $x \notin A \cup B$ equality holds.

(iii) It can be done on similar lines.

8. A step function takes only finitely many different values on an interval $[a, b]$ where $[a, b]$ is partitioned into finitely many disjoint sub-intervals. Characteristic function over rationals is simple but not a step function.

UNIT 6:

LEBESGUE INTEGRAL OF A FUNCTION

CONTENTS:

6.1 INTRODUCTION

"I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral." -Henry Lebesgue wrote to Paul Montel.

...amongst the many definitions that have been successively proposed for the integral of real-valued functions of a real variable, I have retained only those which, in my opinion, are indispensable to understand the transformations undergone by the problem of integration, and to capture the relationship between the notion of area, so simple in appearance, and certain more complicated analytical definitions of the integral.

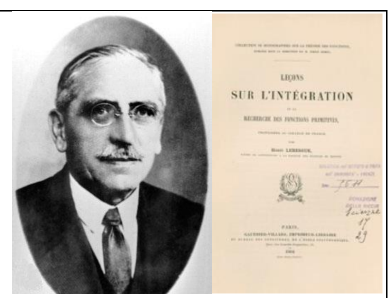
One might ask if there is sufficient interest to occupy oneself with such complications, and if it is not better to restrict oneself to the study of functions that necessitate only simple definitions.... As we shall see in this course, we would then have to renounce the possibility of resolving many problems posed long ago, and which have simple statements. It is to solve these problems, and not for love of complications, that I have introduced in this book a definition of the integral more generalthan that of Riemann.

--H. Lebesgue, 1903

Henry Lebesgue with cover page of his masterpiece.

https://en.wikipedia.org/wiki/Henri_Lebesgue

Fig 6.1.



Henri Lebesgue was born in Beauvais, France, in **1875.** The French mathematician Henri Leon Lebesgue developed the Lebesgue integral to overcome the shortcomings of the Riemann integral. The Lebesgue introduced his integration theory in his **1902** dissertation, "Integral, Length, Area", which is a generalization of the Riemann integral usually studied in elementary calculus. Lebesgue partitioned the rather than the of a function.

Let us understand a particular but important difficulty arising with Riemann integral: Limits of continuous functions.

Suppose $\{f_n\}$ is a sequence of real continuous functions defined on $[0,1]$. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every x, then what is the nature of the limiting function f?

If we suppose that the convergence is uniform, f is then everywhere continuous and things goes simple. However, dropping the assumption of uniform convergence, things may change drastically and the issues that arise can be quite subtle. For example, one can construct a sequence of continuous functions $\{f_n\}$ converging everywhere to f so that

- **i.** $0 \leq f_n(x) \leq 1$ for all x.
- **ii.** The sequence $f_n(x)$ is monotonically decreasing as $n \to \infty$.
- **iii.** The limiting function f is not Riemann integrable.

However, in view of (i) and (ii), the sequence $\int_0^1 f_n(x)dx$ converges to a limit. So it is natural to ask: what method of integration must be adopted to integrate f and obtain the desired one

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx?$$

Lebesgue integration solves this problem.

6.2 OBJECTIVE

After Completion of this unit learners will be able to

i. Define Lebesgue Integral for nonnegative function.

ii. Define the concept of general Lebesgue Integral

iii. Understand the Fatou's Lemma.

iv. Evaluate the different type of Lebesgue Integral with example.

v. Solve Problems on Lebesgue Integration .

6.3 RIEMANN INTEGRAL

Let us recall some definitions related to the Riemann integral. Let f be a bounded real-valued function defined on the interval $[a, b]$ and let

$$a = \xi_0 < \xi_1 < \cdots < \xi_n = b$$

be a subdivision of $[a, b]$.

Then for each subdivision we can define the sums$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1})M_i$ and$s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1})m_i$

where $M_i = \sup_{1<x\leq\xi_i} f(x), m_i = \inf_{1<x\leq\xi_i} f(x)$.

Now, we define the upper Riemann integral of f by $R\overline{\int_a^b} f(x)dx = \inf S$ with the infimum taken over all possible subdivisions of $[a, b]$.

Similarly, we define the lower integral

$$R\underline{\int_a^b} f(x)dx = \sup s.$$

The upper integral is always bigger than or equal to the lower integral, and if the two are equal we say that f is Riemann integrable and, we call this common value the Riemann integral of f. We shall denote it by

$$R\int_a^b f(x)dx.$$

By a step function we mean a function ψ which has the form

$$\psi(x) = c_i, \xi_{i-1} < x < \xi_i$$

for some subdivision of $[a, b]$ and some set of constant c_i. Practically anybody's definition of an integral we have

$$\int_a^b \psi(x)dx = \sum_{i=1}^{n} c_i(\xi_i - \xi_{i-1}).$$

With this in mind we see that

$$R\overline{\int_a^b} f(x)dx = \inf \int_a^b \psi(x)dx$$

for all step functions $\psi(x) \geq f(x)$. Similarly,

$$R\underline{\int_a^b} f(x)dx = \sup \int_a^b \varphi(x)dx$$

for all step functions $\varphi(x) \leq f(x)$.

6.3.1 LEBESGUE INTEGRAL

The Lebesgue Integral of a nonnegative measurable simple function

Definition 1:

The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 & x \notin E \\ 0 & x \notin E \end{cases}$$

is called the characteristic function of E. Note that characteristic function $\chi_E$of E is measurable if and only if the set $E$is measurable.

Definition 2:

A linear combination $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$

is called a simple function if the sets E_i are measurable.

This representation for φ is not unique. However, we note that a function φ is simple if and only if it is measurable and assumes only a finite number of real values.

If φ is a simple function and $\{a_1, \dots, a_n\}$ the set of nonzero values of φ, then $\varphi = \sum a_i \chi_{A_i}$

where $A_i = \{x : \varphi(x) = a_i\}$.

This representation for φ is called the canonical representation, and it is characterized by the fact that the A_i are disjoint and the a_i are distinct and nonzero.

Definition 3:

Let φ be a nonnegative measurable simple function on $\mathbb{R}$, we define the integral of φ by

$$\int \varphi(x)dx = \sum_{i=1}^{n} a_i m(A_i)$$

when φ has the canonical representation $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$. We sometimes abbreviate the expression for this integral to $\int \varphi$. If E is any measurable set, we define the Lebesgue integral of φ over E, denoted as $\int_E \varphi$, by

$$\int_E \varphi = \int \varphi \cdot \chi_E$$

It is often convenient to use representations which are not canonical, and the following lemma is useful:

Lemma 1:

Let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$, be nonnegative measurable simple function, with $E_i \cap E_j = \emptyset$ for $i \neq j$. Then

$$\int \varphi = \sum_{i=1}^{n} a_i m(E_i)$$

Proof:

The set $A_a = \{x: \varphi(x) = a\} = \bigcup_{a_i = a} E_i$.

Therefore, $am(A_a) = \sum_{a_i = a} a_i m(E_i)$ by the additivity of m, and so

$$\begin{aligned} \int \varphi(x)dx &= \sum am(A_a) \\ &= \sum a_i m(E_i). \end{aligned}$$

Proposition 2:

Let φ and ψ be nonnegative measurable simple functions and $a \geq 0, b \geq 0$. Then

$$\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$$

and if $\varphi \geq \psi$ a.e., then

$$\int \varphi \geq \int \psi$$

Proof:

Let $\{A_i\}$ and $\{B_i\}$ be the sets in canonical representations of φ and ψ. Let A_0 and B_0 be the sets where φ and ψ are zero. Then the E_k obtained by taking the intersections $A_i \cap B_j$ form a finite disjoint collection of measurable sets, and we may write

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k},$$

and so $$a\varphi + b\psi = \sum (aa_k + bb_k)\chi_{E_k},$$

hence $\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$ follows from Lemma 1. To prove the second statement, we note that

$$\int \varphi - \int \psi = \int (\varphi - \psi) \geq 0,$$

since the integral of a simple function which is greater than or equal to zero a.e. is nonnegative by the definition of the integral.

Remark:

It follows from this proposition that, if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$, then $\int \varphi = \sum a_i m(E_i)$, and so the restriction of Lemma 1 that the sets E_i be disjoint is unnecessary.

Let f be a bounded nonnegative real-valued measurable function and E a measurable set of finite measure. By analogy with the Riemann integral, if exists, we consider for nonnegative measurable simple functions ψ the number $\sup_{\psi \leq f} \int_E \psi$ to be Lebesgue integral. How can we extend this for nonnegative extended real valued measurable functions over unbounded measurable sets? The answer is given by the following proposition.

Proposition3:

Let f be nonnegative extended real valued measurable function then there is an increasing sequence $\{\psi_n\}$of nonnegative measurable simple functions such that $\{\psi_n(x)\}$ converges to $f(x)$a.e.

Proof:

Let f be nonnegative measurable. For n in N, let

$$E_n = \{x \in \mathbb{R}: f(x) \geqslant n\},$$

and for $1 \leqslant k \leqslant n \cdot 2^n$,

$$E_{n,k} = \left\{x \in \mathbb{R} : \frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n}\right\}$$

Then $E_n$and each $E_{n,k}$are measurable, disjoint, and have union $\mathbb{R}$. Thus,

The simple functions defined by $\psi_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n}$

are non-negative measurable function on $\mathbb{R}$.

We show that $\psi_n(x) \to f(x)$ for each$x$, and $\psi_n(x) \leqslant \psi_{n+1}(x)$ for each$x$, $\forall n \geqslant 1$.

Let $F = \{x \in X: f(x) = \infty\}$.

Case I-

If $x \in F$, then $f(x) = \infty$. Therefore, for each n

$$f(x) \geqslant n,$$

that is $x \in F_n$ for each n, therefore

$$\psi_n(x) = n \text{ foreach } n \geqslant 1,$$
$$\psi_n(x) = n \leqslant (n+1) = \psi_{n+1}(x)$$

and $\psi_n(x) \to \infty = f(x)$.

Case II-

If$x \notin F$, then $0 \leqslant f(x) < \infty$.

Let m be a natural number such that $m - 1 \leqslant f(x) < m$

Then,$\psi_1(x) = 1, \psi_2(x) = 2 \cdots, \psi_{m-1}(x) = (m-1)$

We have $f(x) \geqslant (m-1) = \frac{(m-1) \cdot 2^m}{2^m}$,

$$\psi_m(x) \geq \frac{(m-1)\cdot 2^m}{2^m} = (m-1) = \psi_{m-1}(x).$$

Thus, $\psi_1(x) \leq \psi_2(x) \leq \cdots \leq \psi_{m-1}(x) \leq \psi_m(x)$.

Then, $m-1 \leq f(x) < m \leqslant n$.

Since, $f(x) < n$.

We have, $\psi_n(x) = \frac{k-1}{2^n}$

Where, $\frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n}$

Hence, $\frac{2(k-1)}{2^{n+1}} \leqslant f(x) \leqslant \frac{2k}{2^{n+1}}$.

It follows that either $\frac{2k-2}{2n+1} \leqslant f(x) < \frac{2k-1}{2^{n+1}}$

or $\frac{2k-1}{2^{n+1}} \leqslant f(x) < \frac{2k}{2^{n+1}}$.

Therefore, either $\psi_{n+1}(x) = \frac{2\mathrm{k}-2}{2^{n+1}} = \frac{k-1}{2^n} = \psi_n(x)$

or $\psi_{n+1}(x) = \frac{2k-1}{2^{n+1}} > \frac{2k-3}{2^{n+1}} = \frac{k-1}{2^n} = \psi_n(x)$

Thus,

$\psi_{n+1}(x) \geq \psi_n(x) \forall n \geqslant m$

Hence, $\{\psi_n(x)\}_{n=1}^{\infty}$ is an increasing sequence. It remains to show that $\psi_n(x) \to f(x)$. For $n \geqslant m$, we have $\psi_n(x) = \frac{k-1}{2^n}$where $\frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n}$

Therefore $0 \leqslant f(x) - \psi_n(x) \leqslant \frac{1}{2^n}$

That is $|f(x) - \psi_n(x)| \leqslant \frac{1}{2^n} \forall n \geqslant m$

it follows that $\psi_n(x) \to f(x)$as $n \to \infty$.

This completes the proof of proposition.

6.3.2 LEBESGUE INTEGRAL OF A NONNEGATIVE MEASURABLE SIMPLE FUNCTION

Definition4:

If f is a nonnegative extended real valued measurable function defined on $\mathbb{R}$, we define the (Lebesgue) integral of f over $\mathbb{R}$ by

$$\int f(x)dx = \sup \int \psi(x)dx$$

for all nonnegative measurable simple functions $\psi \leq f$.

Definition5:

A nonnegative measurable function f is called integrable or summable if $\int f(x)dx < \infty$.

6.2.3 The General Lebesgue Integral

The positive part f^+ of a function f is defined as the function $f^+ = f \vee 0$; that is, $f^+(x) = \max\{f(x), 0\}$.

Similarly, we define the negative part f^- by $f^- = (-f) \vee 0$. If f is measurable, so are f^+ and f^-. We have
$f = f^+ - f^-$ and $|f| = f^+ + f^-$.
With these notions in mind we make the following definition.

Definition6: A measurable function f is said to be integrable over $\mathbb{R}$ if f^+ and f^- both are integrable over $\mathbb{R}$. In this case we define

$$\int f = \int f^+ - \int f^-.$$

6.3.4 Existence of Lebesgue Integral

i. If $\int f^+ = \infty$ and $\int f^- < \infty$, we take $\int f = \infty$

ii. If $\int f^+ < \infty$ and $\int f^- = \infty$, we take $\int f = -\infty$

iii. In case if $\int f^+ = \infty = \int f^-$, we say that Lebesgue integral does not exist.

Definition 6:

A measurable function f is said to be integrable over a measurable subset E of $\mathbb{R}$ if $f \cdot \chi_E$ is integrable over $\mathbb{R}$. The Lebesgue integral, in this case, is denoted by $\int_E f$ and is defined as

$$\int_E f = \int f \cdot \chi_E$$

Proposition5:

Let f and g be integrable over E. Then

i. The function cf is integrable over E, and $\int_E cf = c\int_E f$.

ii. The function $f + g$ is integrable over E, and

$$\int_E f + g = \int_E f + \int_E g.$$

iii. If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.

iv. If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof: Part (i) follows directly from the definition of the integral. To prove part (ii), we first note that if f_1 and f_2 are nonnegative integrable functions with $f = f_1 - f_2$, then $f^+ + f_2 = f^- + f_1$. We have

$$\int f^+ + \int f_2 = \int f^- + \int f_1,$$

and so $\int f = \int f^+ - \int f^- = \int f_1 - \int f_2$.

But, if f and g are integrable, so are $f^+ + g^+$ and $f^- + g^-$, and $(f + g) = (f^+ + g^+) - (f^- + g^-)$. Hence

$$\begin{aligned}\int (f+g) &= \int (f^+ + g^+) - \int (f^- + g^-) \\ &= \int f^+ + \int g^+ - \int f^- - \int g^- \\ &= \int f + \int g.\end{aligned}$$

Part (iii) follows from part (ii) and the fact that the integral of a nonnegative integrable function is nonnegative. For (iv) we have

$$\begin{aligned}\int_{A\cup B} f &= \int f\chi_{A\cup B} \\ &= \int f\chi_A + \int f\chi_B \\ &= \int_A f + \int_B f.\end{aligned}$$

Remark:

It should be noted that $f + g$ is not defined at points where $f = \infty$ and $g = -\infty$ and where $f = -\infty$ and $g = \infty$. However, the set of such points must have measure zero, since f and g are integrable. Hence the integrability and the value of $\int (f + g)$ are independent of the choice of values in these ambiguous cases.

If , we write $\int_a^b f$ instead of $\int_{[a,b]} f$. The following proposition shows that the Lebesgue integral is in fact a generalization of the Riemann integral.

Proposition6:

Let f be a bounded function defined on $[a, b]$.

If f is Riemann integrable on $[a, b]$, then it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx.$$

Proof: Since every step function is also a simple function, therefore conclusion is straightforward.

Lebesgue gave the following necessary and sufficient condition for the Riemann integrability of a bounded function.

Proposition7:

A bounded function f defined on $[a, b]$is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.

Monotone Convergence Theorem:

Let $\langle f_n \rangle$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim f_n$ a.e. Then

$$\int f = \lim \int f_n .$$

Proposition8:

If f and g are nonnegative measurable functions, then

i. $\int_E cf = c \int_E f, c > 0.$

ii. $\int_E \boldsymbol{f} + \boldsymbol{g} = \int_E \boldsymbol{f} + \int_E \boldsymbol{g}$.

iii. If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.

Proof: The parts (i) and (iii) follow directly, and part (ii) follows by proposition 3: there are increasing sequence $\{\psi_n\}$and $\{\varphi_n\}$of nonnegative measurable simple functions such that $\{\psi_n\}$and $\{\varphi_n\}$ converges to$f$ and ga.e. respectively, and then applyingMonotone Convergence Theorem.

6.3.5 Fatou's Lemma

Proposition (Fatou's Lemma):

If $\langle f_n \rangle$is a sequence of nonnegative extended real valued measurable functions.Then

$$\int \underline{\lim} f_n \leq \underline{\lim} \int f_n.$$

Proof: We have$\underline{\lim} f_n = \sup_{n\geqslant 1}\left(\inf_{k\geqslant n} f_k\right)$

Let $g_n = \text{Inf}_{k\geqslant n} f_k = \inf\{f_n, f_{n+1}, f_{n+1}, \cdots\}$, then

$$0 \leqslant g_n \leqslant g_{n+1} \forall n \geqslant 1.$$

Therefore,

$$\underline{\lim} f_n = \sup_{n\geqslant 1} g_n = \lim_{n\to\infty} g_n (\because g_n \leqslant g_{n+1} \forall n).$$

ApplyingM.C.T. to $\langle g_n \rangle$, we have,

$$\begin{aligned} \int \underline{\lim} f_n &= \int \lim_{n\to\infty} g_n \\ &= \lim_{n\to\infty} \int g_n \\ &= \underline{\lim} \int g_n \leq \underline{\lim} \int f_n. \end{aligned}$$

This completes the proof of Fatou's Lemma.

Remark: Fatou's Lemma has the weakest hypothesis: We need only that $f_n$be bounded below by zero (or more generally by an integrable function). Consequently, the conclusion of Fatou's Lemma is weaker than that of the others: We can only assert $\int f \leq \lim \int f_n$. The Monotone Convergence Theorem is something of a hybrid: It requires that the f_n be bounded from below by zero (or an integrable function) and above by the limit function f itself. Of course, if f is integrable, this isa special case of the Lebesgue Convergence Theorem, but the advantage ofFatou's Lemma and the Monotone Convergence Theorem is that they are applicable even if f is not integrable and are often a good way ofshowing that f is integrable. Fatou's Lemma and the MonotoneConvergence Theorem are very close in the sense that each can be derived from the other using only the fact that integration is positive and linear.

Problems:

1. **1.** Show that if f is integrable over E, then so is $|f|$ and

$$\left|\int_E f\right| \leq \int_E |f|.$$

2. **2.** Does the integrability of $|f|$ imply that of f ?

6. 4 SOLVED EXAMPLES

Example1:

If$f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$

then $R\overline{\int_a^b} f(x)dx = b - a$ and $R\underline{\int_a^b} f(x)dx = 0,$

This shows that f is not Riemann integrable.

Example 2:

(i)If$\psi(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{ifx is rational} \end{cases}$.

Then,$\psi = 1 \cdot \chi_{\mathbb{R}\sim\mathbb{Q}} + 0 \cdot \chi_{\mathbb{Q}}$

Therefore,$\int_{\mathbb{R}} \psi \cdot dm \quad = 1 \cdot m(\mathbb{R}\sim\mathbb{Q}) + 0 \cdot 0$

$= \infty$

(ii) Let $\psi(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$

Then ,$\psi = 0 \cdot \chi_{\mathbb{R}\sim\mathbb{Q}} + 1 \cdot \chi_{\mathbb{Q}}$

Therefore,$\int_{\mathbb{R}} \psi dm \quad = 0 \cdot m(\mathbb{R}\sim\mathbb{Q}) + 1 \cdot 0$

$= 0 \cdot 0 + 0 = 0.$

Example 3: Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions.

Solution:

Let$f_n(x) = 0$ if $x \leq n, f_n(x) = 1$ for $x > n$. Then

$$f_n = 0 \cdot \chi_{(-\infty,n]} + 1 \cdot \chi_{(n,\infty)}$$

Then f_n is nonnegative decreasing sequences of functions, and we have

$\lim f_n = 0$ but $\int f_n = \infty$. Therefore

$\infty = \lim\int f_n \neq \int \lim f_n = 0.$

Thus, MCT does not hold for decreasing sequence of functions.

Example4: Show that the strict inequality may hold in Fatou's Lemma.

Solution:

Let the sequence $\langle f_n \rangle$be defined by $f_n(x) = 1$ if $n \leq x < n + 1$, and $f_n(x) = 0$ otherwise. Then

$$f_n = \chi_{[n,n+1)}$$

and

$\lim f_n = 0$, and $\int f_n = 1 \ \forall n \geqslant 1$

Therefore,$\int \lim f_n$ =0 < 1= $\lim\int f_n$, that is, $\int \underline{\lim} f_n < \underline{\lim} \int f_n$.

Thus, strict inequality holds in Fatou's Lemma.

Example5: Show that theintegrabilityof $|f|$does not imply integrability of$f$.

Solutions: Let E be a non-measurable subset of $[0,1]$. Define

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathrm{E} \\ 1 & \text{if } x \notin \mathrm{E} \end{cases}$$

Then

$|f| = 1\ \forall x \in [0,1]$.

Thus, $|f|$ being a constant function, is integrable but f is not even a measurable function.

Example 6: Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$, find $\int f$.

Solutions:

Let $\psi(x) = \begin{cases} 0 \text{ for } x < 1 \\ 1 \text{ for } x \geqslant 1 \end{cases}$

Then $0 \leqslant \psi(x) \leqslant f(x)$

$$\psi = 0 \cdot x_{(-\infty,1)} + 1 \cdot X_{[1,\infty)}$$
$$\int \psi = 0.\infty + 1 \cdot \infty$$
$$= 0 + \infty$$
$$= \infty$$

We have, $\int f = \sup\{\int \psi : 0 \leqslant \psi \leqslant$

$f, \psi is$ measurable simple function $\}$

Thus, $\int_{\mathbb{R}} f = \infty$.

6.5 SUMMARY

This unit is an explanation of

i. Definition of Lebesgue integral.

ii. Lebesgue integrals defined with examples.

iii. Existence of Lebesgue integral.

iv. The Fatou's Lemma.

6.6 GLOSSARY

i. The Riemann Integral

ii. The Natural and Real Numbers

iii. Sequences

iv. Simple Functions

v. Lebesgue integral

vi. Monotone convergence Theorem.

vii. The Fatou's Lemma.

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6.8 SUGGESTED READINGS

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6.9 TERMINAL QUESTIONS

1. Which of the following(s) is /are correct ?

- **a)** A real valued constant function issimple.
- **b)** Riemann integrable function is Lebesgue integrable.
- **c)** Characteristic function of countable set is integrable.
- **d)** An extended real valued constant function is integrable.

2. Which of the following (s) is are correct?

- **a)** Lebesgue integrable function is Riemann integrable.
- **b)** Fatou's Lemma hold for every sequence of measurable functions.
- **c)** MCT holds for increasing sequence of measurable functions.
- **d)** A continuous function on a bounded set is Lebesgue integrable.

Q-3 Which of the following (s) is are correct?

- **a)** Sum of two simple function is simple.
- **b)** Positive part of a simple function is simple
- **c)** Negative part of integrable function is integrable
- **d)** Every Riemann integrable function is measurable.

Q-4 Which of the following (s) is are true?

a) The set of discontinuity of Riemann integrable function has a non-measurable set.

b) The identity function is integrable.

c) Every measurable function which is bounded above is integrable.

d) Every real valued constant function is integrable

Q-5 Which of the following (s) is are true?

a) Every measurable bounded function is integrable.

b) The positive part of a bounded non-measurable function is integrable.

c) An extended real valued function which is zero almost everywhere is integrable.

d) The set of irrational numbers is not Lebesgue measurable.

Q-6 If $c$is constant and is$f$ integrable function, then

a) cf is measurable.

b) $cf$is not measurable.

c) $cf$is integrable .

d) $|f|$is integrable.

Q-7 Let $f$be a measurable function defined over a measurable set E, then

a) integrability of $|f|$ imply that of f

b) f is integrable over E, then so is $|f|$

c) $\left|\int_E f\right| \leq \int_E |f|$.

d) $|f|$ is always integrable.

Q-8 If f is a nonnegative measurable function, then $\int f =$

a) $\sup\int \varphi$ over all simple functions $\varphi \leq f$

b) $\sup\int \varphi$ over all simple functions $\varphi \geq f$

c) $\inf \int \varphi$ over all simple functions $\varphi \leq f$

d) $\inf \int \varphi$ over all simple functions $\varphi \geq f$

Q-9 Which of the following (s) is/are true?

a) Integrals over sets of measure zero are zero

b) Integrals of zero over any measurable set is zero

c) Complement of a non-measurable set is measurable.

d) Interior of a non-measurable set is measurable.

Q-10 A bounded function f on $[a, b]$ is Riemann integrable if and only if

a) the set of points at which f is discontinuous has measure zero

b) the set of points at which f is continuous has measure zero

c) the set of points at which f is discontinuous has measure positive

d) the set of points at which f is continuous has measure positive

Q-11 Let f be a nonnegative measurable function. Then consider the following statements

I. $\int f = 0$ implies $f = 0$ a.e.

II. $f = 0$a.e.implies $\int f = 0$

Then

a) Only I is true

b) Only II is true

c) Both I and II are true

d) Neither I nor II are true

Q-12 Consider the following statements

I. The product of two characteristics functions of finite measures is integrable.

II. The sum of two integrable function is integrable. Then

a) Only I is true

b) Only II is true

c) Both I and II are true

d) Neither I and II is true

Q-13 Consider the following statements

I. A function integrable over a set is integrable over its subsets.

II. A function integrable over a set is measurable over its subsets.

Then

a) Only I is true

b) Only II is true

c) Both I and II are true

d) Neither I and II is true

Q-14 Which one of the following is true?

a) Integral of a simple function does not depends on its representations as linear combination of characteristic function.

b) Lebesgue integral is defined for extended real valued constant function.

c) Both a) and b)

d) Neither a) and b)

6.10 ANSWERS

1- a, b, c

2- None

3- All

4- d

5- c

6- a, c, d

7- b, c

8- a

9- a, b, d

10- a, d

11- c

12- c

13- d

14- c

UNIT 7:

GENERAL CONVERGENCE THEOREM

CONTENTS:

7.1 INTRODUCTION

In measure theory, Lebesgue's dominated convergence theorem provides sufficient conditions under the almost everywhere convergence of a sequence of functions implies convergence in the L^1 norm. Its power and utility are two of the primary theoretical advantage of Lebesgue integration over Riemann integration.

The Lebesgue Dominated Convergence Theorem is an important result in measure theory and real analysis, which provides conditions under which the limit of a sequence of measurable functions can be exchanged with the integral.

In the mathematical field of analysis, **Dini's theorem** says that if a monotone sequence of continuous functions converges pointwise on a compact space and if the limit function is also continuous, then the convergence is uniform.This is one of the few situations in mathematics where pointwise convergence implies uniform convergence; the key is the greater control implied by the monotonicity. The limit function must be continuous, since a uniform limit of continuous functions is necessarily continuous.

The continuity of the limit function cannot be inferred from the other hypothesis.In mathematics, more specifically measure theory, there are various notions of the **convergence of measures**.

For an intuitive general sense of what is meant by convergence of measures, consider a sequence of measures μ_n on a space, sharing a common collection of measurable sets. Such a sequence might represent an attempt to construct 'better and better' approximations to a desired measure μ that is difficult to obtain directly. The meaning of 'better and better' is subject to all the usual caveats for taking limits; for any error tolerance $\varepsilon > 0$, we require there be N sufficiently large for $n \geq N$ to ensure the 'difference' between μ_n and μ is smaller than ε.

Various notions of convergence specify precisely what the word 'difference' should mean in that description; these notions are not equivalent to one another, and vary in strength.

7.2 OBJECTIVES

After completion of this unit learners will be able to

i. Learners will understand the general convergence in measure.

ii. Explain the concept of Dini's theorem.

iii. Explain the dominated convergence theorem

7.3DOMINATED CONVERGENCE THEOREM

Definition.

Let (X, $\mathcal{B}, \mu$) is called measure space and f be a non-negative measurable function on X. Then Lebesgue integral of f with respect to μ over X, denoted by $\int f$ dμ is defined by

$$\int f \, d\mu = \sup \{\int \psi \, d\mu : 0 \leq \psi \leq f,$$
$$\psi \text{ is a simple measurable function on X}\}.$$

Definition.

Let (X, $\mathcal{B}, \mu$) is called measure space and f be an extended real valued measurable function on X then Lebesgue integral of f with respect to μ over X, denotes by $\int f \, d\mu$ is defined in the following ways.

i. If $\int f^+ \, d\mu = \int f^- \, d\mu = \infty$, then $\int f \, d\mu$ is not defined.

ii. If $\int f^+ \, d\mu = \infty$, $\int f^- \, d\mu \geq 0$, then $\int f \, d\mu = \infty$.

Theorem.

Let (X, $\mathcal{B}, \mu$) be a measurable space and ψ be a non-empty simple measurable function on X define $\vartheta(\psi)$E=$\int \psi d\mu$ for all E $\in \mathcal{B}$. Then $\vartheta(\psi)$ is a measure on X.

Theorem.

Lebesgue Monotone Convergence Theorem. Let (X, $\mathcal{B}, \mu$) be a measure space and {f_n} be an increasing sequence of non-negative functions on X. Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$

Fatou's Lemma.

Let (X, $\mathcal{B}, \mu$) be a measure space and {f_n} be a sequence of non-negative measurable function on X. Then

$$\int \lim_{n\to\infty} f_n \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu.$$

Lebesgue Dominated Convergence Theorem:

The Lebesgue Dominated Convergence Theorem is an important result in measure theory and real analysis, which provides conditions under which the limit of a sequence of measurable functions can be exchanged with the integral.In simpler terms, if we have a sequence of functions that converge pointwise almost everywhere and are dominated by an integrable function, then the limit of the integrals is the integral of the limit function.

The Lebesgue Dominated Convergence Theorem is a powerful tool in measure theory and is commonly used in probability theory, integration theory, and other areas of mathematics where the convergence of sequences of functions is studied in the context of integration.

Theorem 7.3.1. Lebesgue Dominated Convergence Theorem.

Let (X, $\mathcal{B}, \mu$) be a measure space and {f_n} be a sequence of extended real valued measurable function on X such that {$f_n(x)$} converges to point wise for all $x \in X$. then g be a non-negative measurable summable function on X such that $|f_n(x)| \leq g(x)$ for all $x \in X$ and $\forall\, n \geq 1$.

Then $\int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

Proof. Since $\{f_n(x)\}$ converges to f(x) for all $x \in X$, therefore

$$f(x) = \lim_{n\to} f_n(x) \text{ for all } x \in X$$

Therefore, f is measurable function on X and

$|f_n(x)| \leq g(x)$ for all $x \in X$

and $\forall\, n \geq 1$,

$-g(x) \leq f_n(x) \leq g(x)$, for all $x \in X$, $n \geq 1$.

$F_n(x) + g(x) \geq 0$ and $g(x) - f_n(x) \geq 0$,

for all $x \in X$.

Hence $f_n + g \geq 0$

$$\int (f + g)\, d\mu = \int \lim_{n\to\infty} (f_n + g)\, d\mu$$

$$= \int \lim_{n\to\infty} (f_n + g)\, d\mu$$

$$\leq \lim_{n\to\infty} \int (f_n + g)\, d\mu$$

$$\int f\, d\mu + \int g\, d\mu \leq \lim_{n\to\infty} \int f_n\, d\mu + \int g\, d\mu.$$

Since g is non-negative summable on X, therefore $0 \leq \int g\, d\mu < \infty$ then

$\int f\, d\mu \leq \lim_{n\to\infty} \int f_n\, d\mu$**….(1)**

Similarly, $g - f_n \geq 0$, $\int (g - f) d\mu = \int \lim_{n\to\infty} (g - f_n)\, d\mu$

$\int (g - f) d\mu = \int \lim_{n\to\infty} (g - f_n)\, d\mu$

$\int g\, d\mu - \int f\, d\mu \leq \lim_{n\to\infty} (g - f_n)\, d\mu$ (By Fatus lemma)

$\int g\, d\mu - \int f\, d\mu = \lim_{n\to\infty} \int (g + (-f_n))\, d\mu$

$$= \int g\, d\mu + \lim_{n\to\infty} \left(\int -f_n\, d\mu\right)$$

$$= \int g\, d\mu - \lim_{n\to\infty} \left(\int f_n\, d\mu\right)$$

$\int g \, d\mu - \int f \, d\mu \leq \int g \, d\mu - \lim_{n\to\infty} (\int f_n \, d\mu)$

Hence $\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$ **…. (2)**

Now by equation (1) and (2) we get

$\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$**…(3)**

Hence $\lim_{n\to\infty} \int f_n \, d\mu$ exists

and $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$.

7.4 DINI'S THEOREM

Dini's Theorem is a result in real analysis that provides conditions for the uniform convergence of a sequence of functions to its limit function. The theorem is named after the Italian mathematician Ulisse Dini, who first proved it in the late 19th century. Dini's Theorem is particularly useful when dealing with sequences of continuous functions converging to a continuous limit function.

Suppose (f_n) is a sequence of continuous functions defined on a closed interval [a, b] and converging pointwise to a function fon [a, b]. If the pointwise convergence is monotonic, meaning that

$f_{n+1}(x) \leq f_n(x)$ for all $x \in [a, b]$ and n is the natural numbers,

and the limit function f is also continuous, then the convergence is uniform on [a, b].

In other words, if you have a sequence of continuous functions on a closed interval, and they converge pointwise to a continuous function in a monotonic fashion, then the convergence is not only pointwise but also uniform.

Dini's Theorem is particularly valuable because it provides a situation where you can guarantee the uniform convergence of a sequence of functions. This is important in various areas of mathematics, including analysis and the study of convergence properties of functions. The theorem has applications in real analysis, functional analysis, and other fields were understanding the behaviour of sequences of functions is crucial.

Egorofl's Theorem

Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable.functions on E that converges pointwise on E to the real-valued function f.Then for each$\varepsilon > 0$, there is a closed set F contained in E for which

$\{f_n\} \rightarrow f$ uniformly on F and $m(E \sim F) < \varepsilon$.

Losin's Theorem

Let f be a real-valued measurable function on E. Then for each $\varepsilon > 0$, thereis a continuous function g on R and a closed set F contained in E for which

I = g on F and m (E ~ F) < E.

Dini’s Theorem

Let $\{f_n\}$ be an increasing sequence of continuous functions on [a, b] which converges pointwise on [a, b] to the continuous function f on [a, b]. Then theconvergence is uniform on [a, b].

7.5CONVERGENCE IN MEASURE

We have considered sequences of functions that converge uniformly, that converge pointwise,and that converge pointwise almost everywhere. To this list we add one more mode ofconvergence that has useful relationships both to pointwise convergence almost everywhereand to forthcoming criteria for justifying the passage of the limit under the integral sign.

Converge in measure:

Let $\{f_n\}$be a sequence of measurable functions on E and f a measurable functionon E for which f and,each in is finite a.e. on E.

The sequence $\{f_n\}$ is said to converge inmeasure on E to I provided for each $\varepsilon > 0$,

$$\lim_{n \to \infty} m \{x \in E: | f_n(x) - f(x) | > \varepsilon\} = 0.$$

When we write $\{f_n\} \to f$ in measure on E we are implicitly assuming that f and each f_n is measurable, and finite a.e. on E.

Observe that if $\{f_n\} \to f$ uniformly on E, and f is a real-valued measurable function on E, then $\{f_n\} \to f$ in measure on E since for$\varepsilon > 0$,

the set $\{x \in E: | f_n(x) - f(x) | > \varepsilon\}$ is empty for n sufficiently large. However, we also have the following much stronger result.

Theorem7.5.1.

Assume E has finite measure. Let$\{f_n\}$be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Then $\{f_n\} \to f$ in measure on E.

Proof. First observe that f is measurable since it is the pointwise limit almost everywhereof a sequence of measurable functions.

Let γ> 0.

To prove convergence in measure we let$\in$>0 and seek an index N such that m {x ∈ E: | $f_n(x) - f(x)$ | >γ} <ε for all n ≥ N.

Egoroff's Theorem tells us that there is a measurable subset F of E with m(E ~ F) <ε such that {f_n}→ f uniformly on F. Thus, there is an index N such that | f_n – f | <γ on F for all n ≥ N.

Thus, all n ≥ N, {x ∈ E: | $f_n(x) - f(x)$ | >γ} ⊆E ~ F, So the above expression holds for this choice of N.

The above Theorem is false if E has infinite measure. The following example showsthat the converse of this theorem also is false.

Example7.5.2. Consider the sequence of subintervals of $[0, 1]$, $\{I_n\}$ which has initial termslisted as

$[0, 1], [0, 1/2], [1/2, 1], [0, 1/3], [1/3, 2/3], [2/3, 1], [0, 1/4],$
$[1/4, 1/2], [1/2, 3/4], [3/4, 1] \ldots$

For each index n, define f_n to be the restriction to [0, 1] of the characteristic function of I_n.

Let f be the function that is identically zero on [0, 1].

We claim that $(f_n) \dashrightarrow f$ is measure Indeed, observe that $\lim_{n\to\infty} \ell\ (I_n) = 0$ since for each natural number m,

$$\text{If } n > 1 + \cdots + m = \frac{m(m+1)}{2}, \text{ then } \ell\ (I_n) < \frac{1}{m}.$$

Thus, for $0 < \varepsilon < 1$, since $\{x \in E : |\, f_n(x) - f(x)\, | > \varepsilon\} \subseteq I_n$.

$$0 \le \lim_{n\to\infty} m\{x \in E : |\, f_n(x) - f(x)\, | > \varepsilon\} \le \lim_{n\to\infty} \ell\ (I_n) = 0.$$

However, it is clear that there is no point x in [0, 1] at which $\{f_n(x)\}$ converges to $\{f(x)\}$ sincefor each point x in $[0,1]$, $f_n(x) = 1$for infinitely many indices n, while $f(x) = 0$.

Theorem7.5.3. (**Riesz**) If$\{f_n\}\rightarrow$ f in measure on E, then there is a subsequence $\{f_{nk}\}$ thatconverges pointwise a.e. on E to f.

Proof. By the definition of convergence in measure, there is a strictly increasing sequence ofnatural numbers $\{n_k\}$ for which

$$m \{x \in E: | f_j(x) - f(x) | > \frac{1}{k}\} < \frac{1}{2k} \text{ for all } j \geq n_k.$$

For each index k, define

$$E_k = \{x \in E: | f_{nk} - f(x) | > \frac{1}{k}\}$$

Then $m\{E_k\} < \frac{1}{2k}$and therefore$\sum_{k=1}^{\infty} m\{Ek\}<\infty$.

The Borel-Cantelli Lemma tellsus that for almost all x ∈ E, there is an index K(x) such that $x \notin E_k$ if $k \geq K(x)$, that is,

$$| f_{nk} - f(x) | \leq \frac{1}{k} \text{ for all } k \geq K(x).$$

Therefore,

$$\lim_{k\to\infty} f_{nk}(x) = f(x).$$

Corollary7.5.4. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then $\lim_{n\to\infty} \int f_n = 0$. If and only if $\{f_n\}\rightarrow 0$ in measure on E and $\{f_n\}$ is uniformly integrable and tight over E.

7.6 SOLVED EXAMPLE

Example

Define $f(x) = \begin{cases} 2n, & if\ x \in [1/2n,\ 1/n] \\ 0, & if\ x \in [0,\ 1] - [1/2n,\ 1/n] \end{cases}$

$$F_n(0) = 0$$

$$\lim_{n \to \infty} f_n(0) = 0 = f(0)$$

Let $x \in [0, 1]$, $x > 0$.

By Archimedean property, there is a natural number where n such that $\frac{1}{n} < x$, therefore $x \notin [\frac{1}{2n}, \frac{1}{m}]$ for all $m \geq n$.

Hence $f_n(x) = 0$, and $\lim_{n\to\infty} f_n = 0$,

and $\int \lim_{n\to\infty} f_n \, d\mu = 0$

$\int f_n \, dx = 2n \, [\frac{1}{n} - \frac{1}{2n}] = 1$.

Then $\lim_{n\to\infty} \int f_n \, d\mu = 1$.

This implies that

$0 = \int \lim_{n\to\infty} f_n \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu = 1$.

Hence Fatus lemma satisfied

$0 = \int \lim_{n\to\infty} f_n \, d\mu \neq \lim_{n\to\infty} \int f_n \, d\mu = 1$.

Lebesgue Dominated convergent theorem is not satisfied.

Suppose there is a non-negative summable function on [0, 1] such that $g(x) \geq f_n(x)$,

therefore

$g(x) \geq f_n(x) = 2n$ if $x \in [\frac{1}{2n}, \frac{1}{n}]$

$g(x) \geq 2$, if $x \in [1/2, 1]$,

and $g(x) \geq 4$, if $x \in [1/4, 1/2]$,

$g(x) \geq 6$, if $x \in [1/6, 1/3]$,

$g(x) \geq 8$, if $x \in [1/8, 1/4], \ldots.,$

$\int_0^1 g(x) = 1+1+1+1+, \ldots$

Therefore g is not summable, this is a contradiction, There is no g such that $|f_n(x)| \leq g(x)$ for all $x \in X = [0, 1]$ for all $n \geq 1$

CHECK YOUR PROGRESS

1. If f be a nonnegative measurable function on R.Then $\lim_{n\to\infty} \int f < \int f$. True/False
2. A measurable function f on E is said to be integrable over Eprovided | f |integrable over Ethen$\int f = \int f^{+} + \int f^{-}$.True/False
3. Is the Converse of Lebesgue dominating convergent theorem is true. True/False
4. Is the converse of Dini's theorem is true. True/False.

7.7 SUMMARY

This unit is complete combination of

- **i.** Concept of convergence in measure.
- **ii.** Concept of Lebesgue dominated convergence theorem .
- **iii.** Concept of Dini's theorem.

7.8 GLOSSARY

1. Measure space.
2. convergence in measure.
3. Dominated convergence theorem.
4. Dini's theorem.

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7.10 SUGGESTED READINGS

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7.11 *TERMINAL QUESTIONS*

1. Let $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Show that $\{f_n\} \to g$ in measure on E if and only iff = g a.e. on E.

2. Show that linear combinations of sequences that converge in measureon a set of finitemeasure also converge in measure.

3. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

4. Let E be a set of measure zero and define $f = \infty$ on E.

Show that $\int f = 0$.

5. Let f be a measurable function on E. Then $f^+$and f^- are integrable over if and only if if|f| is integrable over E.

6. Let I be integrable over E and C a measurable subset of E. Show that

$$\int f \;\; = \int f \, . \, \chi_C.$$

7. Let Let$\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int | f - f_n | \to 0$ if and only if $\lim_{n\to\infty} \int | f_n | = \int | f |$.

7.12 ANSWERS

CHECK YOUR PROGRESS

1. False
2. false
3. false
4. True

UNIT 8:

DIFFERENTIATIONOF AN INTEGRAL

CONTENTS:

8.1 INTRODUCTION

The fundamental theorems of integral and differential calculus, with respect to the Riemann integral, are the workhorses of calculus. In this chapter we formulate these two theorems for the Lebesgue integral. For a function f on the closed, bounded interval [a, b], when is

$$\int_a^b f' = f(b) - f(a) \text{ ………. } \mathbf{(1)}$$

Assume f is continuous. Extend f to take the value $f(b)$ on (b, b + 1], and for 0 < h≤ 1, define the divided difference function $Diff_h f$ and average value function $AV_h f$ on [a, b] by

$Diff_h\, f(x) = \frac{f(x+h)-f(x)}{h}$ and $Av_h\, f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt$ for all x ∈ [a, b].

A change of variables and cancellation provides the discrete formulation of (i) for the Riemann integral:

$$\int_a^b Diff_h\, f = Av_h\, f(b) - Av_h\, f(a).$$

The limit of the right - hand side as $h \to 0^+$ equals f(b) – f(a). We prove a striking theoremof Henri Lebesgue which tells us that a monotone function on (a, b) has a finite derivative almost everywhere. We then define what it means for a function to be absolutely continuous and prove that if f is absolutely continuous, then f is the difference of monotone functions and the collection of divided differences, $\{Diff_h f\}_{0<h<1}$, is uniformly integrable. Therefore, by the Vitali Convergence Theorem, (i) follows for $f$absolutely continuous by taking thelimit as $h \to 0^+$ in its discrete formulation. If f is monotone and (i) holds, we prove that f must be absolutely continuous. From the integral form of the fundamental theorem,

- we obtain the differential form, namely, if f is Lebesgue integrable over [a, b], then $\frac{d}{dx}\left[\int_a^b f\right] = f(x)$ for almost all x ∈ [a, b]……(2)

8.2 OBJECTIVES

After completion of this unit learners will be able to

i. Learners will understand the fundamentals of measure theory.

ii. Explain the concept of convergence in measure theory

iii. Define the Differentiation in measure space

iv. Explain the Absolute continuity in measure.

v. Define the convex and bounded variation function.

8.3CONTINUITY OF MONOTONE FUNCTIONS

Recall that a function is defined to be monotone if it is either increasing or decreasing. Monotone functions play a decisive role in resolving the question posed in the preamble. There are two reasons for this. First, a -theorem of Lebesgue asserts that a monotone function on an open interval is differentiable almost everywhere. Second, a theorem of Jordan tells us that a very general family of functions on a closed, bounded interval, those of bounded variation, which includes Lipschitz functions, may be expressed as the difference of monotone functions and therefore they also are differentiable almost everywhere on the interior of their domain. In this brief preliminary section, we consider continuity properties of monotone functions.

Theorem 8.3.1. Let f be a monotone function on the open interval (a, b). Then f is continuous except possibly at a countable number of points in (a, b).

Proof. Assume f is increasing. Furthermore, assume (a, b) is bounded and f is increasing on the closed interval $[a, b]$ otherwise, express (a, b) as the union of an ascending sequence of open, bounded intervals, the closures of which are contained in (a, b), and take the union of the discontinuities in each of this countable collection of intervals. For each $X_0 E\ (a, b)$, f has a limit from the left and from the right at x_0.

Define $x \to x_0^-$

$$f(x_0^-) = \lim_{x \to x\,0-} f(x) = \sup\{ f(x) \mid a < x < x_0 \}.$$

$$f(x_0^+) = \lim_{x \to x\,0+} f(x) = \inf\{ f(x) \mid x_0 < x < b \}.$$

Since f is increasing, $f(x_0^-) \le f(x_0^+)$.

Then the function f fails to be continuous at X_0

if and only if $f(x_0^-) < f(x_0^+)$,

in which case we define the open "jump" interval $J(x_0)$ by

$$J(x_0) = \{ y \mid f(x_0^-) < y < f(x_0^+)\}.$$

Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is disjoint.

Therefore, for each natural number n, there are only a finite number of jump intervals of length greater than $\frac{1}{n}$.

Thus, the set of points of discontinuity off is the union of a countable collection of finite sets and therefore is countable.

Theorem.1.2. Let C be a countable subset of the open interval (a, b). Then there is an increasing function on (a, b) that is continuous only at points in (a, b) ~C.

8.4 DIFFERENTIABILITYOF MONOTONE FUNCTIONS

A closed, bounded interval [c, d] is said to be non degenerate provided c < d.

A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set E in the sense of Vitali provided for each point x in E and $\epsilon > 0$, there is an interval I in $\mathcal{F}$ that contains x and has $\ell(I) < \epsilon$.

Lemma8.4.1. **The Vitali Covering Lemma**

Let E be a set of finite outer measure and $\mathcal{F}$a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\epsilon > 0$, there is afinite disjoint subcollection $\{I_k\}_{k=1}$ of $\mathcal{F}$ for which

$$m^*[E \sim \bigcup_{k=1}^{n} I_k] < \epsilon.$$

Lemme 8.4.2.

Let f be an increasing function on the closed, bounded interval [a, b]. Then, for each $\alpha > 0$,

$$m^*\{x \in (a, b) \mid \overline{D}f(x) \geq \alpha\} - \frac{1}{\alpha}[f(b) - f(a)]$$

and

$$m^*\{x \in (a, b) \mid \overline{D}f(x) = \infty\} = 0.$$

Theorem8.4.3. Lebesgue's Theorem

If the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).

Let f be integrable over the closed, bounded interval [a, b].

Extend f to take the value f(b) on (b, b+ 1]. For $0 < h \leq 1$, define the divided difference function $\text{Diff}_h f$ and average value function $AV_h\, f$ of [a, b] by

$$\text{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h} \text{ and } Av_h\, f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt \text{ for all } x \in [a, b].$$

By a change of variables in the integral and cancellation, for all $a \leq u < v \leq b$, $\int_u^v \text{Diff}_h f = Av_h f(v) - Av_h f(u)$.

Corollary 8.4.4.

Let 1 be an increasing function on the closed, bounded interval [a, b]. Then f ' is integrable over [a, b] and

$\int_a^b f' \leq$ f(b) – f(a). $x^2 sin\frac{1}{x}$

Remark 8.4.5. For a continuous function f on a closed, bounded interval [a, b] that is differentiable on the open interval (a, b), in the absence of a monotonicity assumption on f we cannot infer that its derivative f ' is integrable over [a, b].

8.5FUNCTIONS OF BOUNDED VARIATION

Lebesgue's Theorem tells us that a monotone function on an open interval is differentiable almost everywhere. Therefore, the difference of two increasing functions on an open interval also is differentiable almost everywhere. We now provide a characterization of the class of functions on a closed, bounded interval that may be expressed as the difference of increasing functions, which shows that this class is surprisingly large it includes, for instance, all lipschitz functions.

Let f be a real-valued function defined on the closed, bounded interval [a, b] and P = $(x_0, x_1, \ldots, x_k\}$ be a partition of [a, b]. Define the variation of f with respect to P by

$$V(f,p) = \sum_{i=0}^{k} |f(x_i) - f(x_i - 1)|$$

and the total variation of f on [a, b] by

TV(f) = sup {V(f, P) | P a partition of[a, b]}

For a subinterval [c, d] of [a, b], $TV(f_{[c, d]})$ denotes the total variation of the restriction off to [c, d].

Bounded Variation: A real-valued function f on the closed, bounded interval [a, b] is said to be bounded variation on [a, b]provided

$$TV(f) < \infty.$$

Example: Let f be an increasing function on [a, b]. Then f is of bounded variation on [a, b] andTV(f) = f(b) - f(a).

Indeed, for any partition P = (x_0, . . ,x_k} of [a, b],

$V(f, p) = \sum_{i=0}^{k} |f(xi) - f(xi - 1)| = f(b) – f(a).$

Lemma 8.5.1. Let the function f be of bounded variation on the closed, bounded interval [a, b].Then f has the following explicit expression as the difference of two increasing functions on[a, b]

$$f(x) = [\ f(x) + TV(f_{[a,\ x]})] – TV(f_{[a,\ x]})\ \forall\ x \in [a, b].$$

Theorem 8.5.2. **Jordan's Theorem** A function f is of bounded variation on the closed, bounded interval[a, b] if and only if it is the difference of two increasing functions on [a, b].

Proof. Let f be of bounded variation on [a, b]. The preceding lemma provides an explicit representation of f as the difference of increasing functions. To prove the converse, let f = g - h on [a, b] for any partition

P = (x_0, . . ,x_k} of [a, b],

$$\begin{aligned} V(f, p) &= \sum_{i=0}^{k} |f(xi) - f(xi - 1)| \\ &= \sum_{i=0}^{k} |[g(xi) - g(xi - 1)] + [h(xi) - h(xi - 1)]| \\ &\leq \sum_{i=0}^{k} |g(xi) - g(xi - 1)| + \sum_{i=0}^{k} |h(xi) - h(xi - 1)| \end{aligned}$$

V(f, p = [g(b) – g(a)] + [h(b) – h(a)].

Thus, the set of variations of f with respect to partitions of [a, b] is bounded above by [g(b) – g(a)] + [h(b) – h(a)] and therefore f is of bounded variation of [a, b].

We call the expression of a function of bounded variation f as the difference of increasing functions a Jordan decomposition of f.

Corollary 8.5.3. If the function f is of bounded variation on the closed, bounded interval [a, b] then it is differentiable almost everywhere on the open interval (a, b) and f′ is integrable over [a, b].

Proof: According to Jordan's Theorem, f is the difference of two increasing functions on [a, b]. Thus, Lebesgue's Theorem tells us that f is the difference of two functions which are differentiable almost everywhere on (a, b). Therefore, f is differentiable almost everywhere on (a, b). The integrability of f′ follows by above corollary.

8.6 DIFFERENTION OF AN INTEGRAL

Let f be a continuous function on the closed, bounded interval [a, b] and $\int_u^v Diff_h\, f = Av_h f(v) - Av_h f(u)$

take a = u andb = v to arrive at the following discrete formulation of the fundamental theorem of integral calculus.

$\int_u^v Diff_h\, f = Av_h f(b) - Av_h f(a)$.

Since f is continuous, the limit of the right-hand side as $h \to 0^+$ equal f(b) – f(a). We now show that if f is absolutely continuous, then the limit of the left -hand side as $h \to 0^+$ equals $\int_a^b f'$ and thereby establish the fundamental theorem of integral calculus for the Lebesgue integral.

Theorem 8.6.1. Let the function f be absolutely continuous on the closed, bounded interval [a, b]. Then f is differentiable almost everywhere on (a, b), its derivative f' is integrable over [a, b] and

$$\int_a^b f' = f(b) - f(a).$$

Theorem 8.6.2. A function f on a closed, bounded interval [a, b] is absolutely continuous on[a, b] if and only if it is an indefinite integral over [a, b].

Corollary 8.6.3. Let the function f be monotone on the closed, bounded interval [a, b] Then f is absolutely continuous on [a, b] if and only if

$$\int_a^b f' = f(b) - f(a).$$

Theorem8.6.4. Let f be integrable over the closed, bounded interval [a, b]. Then

$$\frac{d}{dx}\int_a^x f = f(x) \text{ for almost all } x \in (a, b).$$

8.7 ABSOLUTELY CONTINUOUS FUNCTION

A real-valued function f on a closed, bounded interval [a, b] is said to be absolutely continuous on [a, b] provided for each $\in > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{a_k, b_k\}$ intervals in (a, b),

$$\text{If } \sum_{k=1}^{n} [b_k - a_k] < \delta \text{ then } \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \in.$$

The criterion for absolute continuity in the case the finite collection of intervals consists of a single interval is the criterion for the uniform continuity of f on [a, b]. Thus, absolutely continuous functions are continuous. The converse is false, even for increasing functions.

Example8.7.1. The Cantor-Lebesgue function φ is increasing and continuous on [0, 1], but itis not absolutely continuousIndeed, to see that φ is notabsolutely continuous, let n be a natural number.

Theorem8.7.2. If the function f is Lipschitz on a closed, bounded interval [a, b], then it is absolutely continuous on [a, b].

Proof. Let c > 0 be a Lipschitz constant for f on [a, b], that is,

$$|f(u) - f(v)| \leq c\,|u - v| \text{ for all } u, v \in [a, b].$$

Then, regarding the criterion for the absolute continuity of f, it is clear that

$\delta = \frac{\varepsilon}{c}$

responds to any $\varepsilon > 0$ challenge.

There are absolutely continuous functions that fail to be Lipschitz the function f on [0, 1], defined by $f(x) = \sqrt{x}$ for $0 \leq x < 1$, is absolutely continuous but not Lipschitz.

Theorem 8.7.3. Let the function f be absolutely continuous on the closed, bounded interval [a,b]. Then f is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.

Theorem 8.7.4. Let the function f be continuous on the closed, bounded interval [a, b]. Then f is absolutely continuous on [a, b] if and only if the family of divided difference functions $\{Diff_h f\}_{0<h<\ 1}$ is uniformly integrable over [a, b].

Remark: For a non degenerate closed, bounded interval [a, b], let $Ƒ_{LIP}$, $Ƒ_{AC}$, $Ƒ_{BV}$ denote the family of functions on [a, b] that are Lipschitz, absolutely continuous, and of bounded variation, respectively. We have the following strict inclusions $Ƒ_{LIP} \subseteq Ƒ_{AC} \subseteq Ƒ_{BV}$.

8.8 CONVEX FUNCTONS

Throughout this section (a, b) is an open interval that may be bounded or unbounded.

A real-valued function φ on (a, b) is said to be convex provided for each pair ofpoints x_1, x_2 in (a, b) and eachλ with $0 \leq \lambda \leq 1$,

$\varphi(\lambda x_1 + (1-\lambda) x_2) \leq \lambda\varphi(x_1) + (1-\lambda) x_2$.

If we look at the graph of φ, the convexity inequality can be formulated geometrically by saying that each point on the chord between (x_1, φ(x_1)) and (x_2, φ(x_2)) is above the graph of φ.

Observe that for two points $x_1<x_2$ in (a, b), each point x in (x_1, x_2) may be expressed as

$x = \lambda x_1 + (1-\lambda) x_2$ where $\lambda = \frac{x2 - x}{x2 - x1}$.

Thus, the convexity inequality may be written as

$\frac{\varphi(x)-\varphi(x1)}{x - x1} \leq \frac{\varphi(x2)-\varphi(x)}{x2 - x}$ for $x_1< x < x_2$ in (a, b).

Therefore, convexity may also be formulated geometrically by saying that for $x_1< x < x_2$, the slope of the chord from (x_1, φ(x_1)) to (x, φ(x)) is no greater than the slope of the chord from (x, φ(x)) to (x_2, φ(x_2)).

Theorem 8.8.1. If φ is differentiable on (a, b) and its derivative φ ' is increasing, then φ isconvex. In particular, φ is convex if it has a nonnegative second derivative φ '' on (a, b).

Example 8.8.2. Each of the following three functions is convex since each has a non negative second derivative:

(i). $\varphi(x) = x^p$ on $(0, \infty)$ for $p \geq 1$.

(ii). $\varphi(x) = e^{ax}$ on $(-\infty, \infty)$.

(iii). $\varphi(x) = \ln\frac{1}{x}$ on $(0, \infty)$.

Corollary 8.6.3. Let φ be a convex function on (a, b). Then φ is Lipschitz, and thereforeabsolutely continuous, on each closed, bounded subinterval [c, d] of (a, b).

Theorem 8.6.4. Let φ be a convex function on (a, b). Then φ is differentiable except at acountable number of points and its derivative φ ' is an increasing function.

Theorem 8.6.5. **Jensen's Inequality** Let φ be a convex function on (-∞, ∞), f an integrable function over [0, 1] and $\varphi o f$ also integrable over [0, 1]. Then,

$$\varphi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 (\varphi\ o\ f)x\, dx.$$

8.9 MEASURE SPACE

Let X be a non-empty set and $\mathcal{B}$ be a σ – algebra of subsets of X.
A function μ on $\mathcal{B}$ into nonnegative extended real number i.e. $\mu: \mathcal{B} \to [0, \infty]$ is said to be measure on X if.

- **i.** $\mu(\varphi) = 0$.
- **ii.** μ is countable additive i.e. $A_n \in \mathcal{B}$ for all $n \in N$, $A_n \cap A_m = \varphi$, $m \neq n$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Let X be a non-empty set, $\mathcal{B}$ be a σ – algebra of subsets of X and μ is a measure on X then $(X, \mathcal{B}, \mu)$ is called measure space.

Example 8.9.1.

Let X any non-empty set and $\mathcal{B} = \{\varphi, X\}$, $\mathcal{B}$ is smallest σ – algebra of subsets of X. Define $\mu: \mathcal{B} \to [0, \infty]$, $\mu(\varphi) = 0$, and $\mu(X) = 1$

Let $A_n = \varphi = A_1$,

$A_n \cap A_m = \varphi = A_1 \cap A_2$

and $A_m = X = A_2$, $\cup A_n = A_1 \cup A_2 = \varphi \cup X = X$.

Then

$\mu(\varphi \cup X) = \mu(X) = 1$ and $\mu(\varphi) + \mu(X) = 0+1=1$.

Therefore $\mu(\varphi \cup X) = \mu(\varphi) + \mu(X)=1$.

Hence μ is measure on X.

Example 8.9.2. Let X be a non-empty set and $\mathcal{B}$ = P(X), Define μ on $\mathcal{B}$ by μ(A) = $\bar{\bar{A}}$, where $\bar{\bar{A}}$ denotes numbers of elements in A for all A ∈ $\mathcal{B}$ = P(X), i.e.

$$\mu(A) = \begin{cases} \text{Number of elements in A, where A is finite set} \\ \infty, \text{when A is infinite set} \end{cases}$$

Then μ is measure on X and this is known as Counting measure.

Properties of Measure 8.9.3

i. A measure μ is finitely additive, i.e. $A_1, A_2, A_3, \ldots, A_n \in \mathcal{B}$, $A_k \cap A_l = \varphi$, for $1 \le k \le n$, $1 \le l \le n$ and $k \ne l$. This implies that,

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

ii. A measure μ is additive, i.e. A, B ∈ $\mathcal{B}$ and $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$

iii. Measure μ is monotonic i.e. A, B ∈ $\mathcal{B}$, and $A \subset B$ then $\mu(A) \le \mu(B)$.

iv. A measure μ is countably sub additive i.e. $A_n \in \mathcal{B}$ for all n ∈ N then $\mu(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$.

v. A measure is finitely sub additive i.e. $A_1, A_2, A_3, \ldots, A_n \in \mathcal{B}$, then $\mu(\bigcup_{k=1}^{n} A_k) \le \sum_{k=1}^{n} \mu(A_k)$.

8.10 MEASURABLE FUNCTION

Let R be the set of all real numbers an extended real valued function f is said to be Lebesgue measurable function if each real number a the set { x ∈ R : f(x) > a} is a Lebesgue measurable subset of R,

$$\text{i.e. } f^{-1}(a, \infty] \in M_m,$$

where M_m is the σ – algebra of all Lebesgue measurable subset of R.

The function f is said to be Borel measurable function if for each real number a the set { x ∈ R : f(x) > a} is Borel set, i.e. $f^{-1}(a, \infty] \in \mathfrak{B}$, where $\mathfrak{B}$ is the smallest σ – algebra containing all open subset of R.

Theorem 8.10.1. Every continuous function on R into itself is Borel measurable hence Lebesgue measurable function.

Example 8.10.2. Define f: R → R by

$$F(x) = x \text{ for all } x \in R.$$

F is an identity function, let a be any real number {x ∈ R : f(x) > a}= (a, ∞) = $f^{-1}(a, \infty) \in \mathfrak{B}$, f is boral measurable hence Lebesgue measurable, identity function is Boral measurable Hence Lebesgue measurable function.

Theorem 8.10.3. Let (X, $\mathcal{B}$) be a measurable space and f be a real valued function on X. Then the following are equivalent.

1. {x ∈ X : f(x) >α }∈ $\mathcal{B}$ for each real number α.
2. {x ∈ X : f(x) ≥ α}∈ $\mathcal{B}$ for each real number α.
3. {x ∈ X : f(x) <α}∈ $\mathcal{B}$ for each real number α.
4. {x ∈ X : f(x) ≤ α}∈ $\mathcal{B}$ for each real number α.

Theorem 8.10.4. Let (X, $\mathcal{B}$) be a measurable space let f and g be measurable function on X. Then

1. Cf is measurable function on X for all c ∈ R.
2. F + g is measurable function on X.
3. F^2 is measurable function on X.
4. F.g is measurable function on X.
5. If f is measurable function on X then | f | is measurable function on X but not converse.

Theorem 8.10.5. Let (X, $\mathcal{B}$) be a measurable space let f and g be extended real valued measurable function on X then

1. $F \vee g$ is measurable function on X.
2. $F \wedge g$ is measurable function on X.
3. F^+ is measurable function on X.
4. F^- measurable function on X.

8.11 SOLVED EXAMPLES

Example 1. Let X any non-empty set and $\mathcal{B} = \{\varphi, X\}$, $\mathcal{B}$ is smallest σ – algebra of subsets of X, Define $\mu: \mathcal{B} \to [0, \infty]$

$\mu(\varphi) = 0$, and $\mu(X) = 1$

Let $A_n = \varphi = A_1$, $A_n \cap A_m = \varphi = A_1 \cap A_2$ and $A_m = X = A_2$, $\cup A_n = A_1 \cup A_2 = \varphi \cup X = X$.

Then $\mu(\varphi \cup X) = \mu(X) = 1$ and $\mu(\varphi) + \mu(X) = 0+1=1$. Therefore $\mu(\varphi \cup X) = \mu(\varphi) + \mu(X)=1$.

Hence μ is measure on X.

Example 2. Let X be a non-empty set and $\mathcal{B}$ be a σ – algebra of subsets of X. Let x ∈ X define $\mu_x : \mathcal{B} \to [0, \infty]$ by $\mu_x(A) = \begin{cases} 0, & if\ x \notin A \\ 1, & if\ x \in A \end{cases}$ for all A $\in \mathcal{B}$.

Then $\mu_x(A)$ is a measure on X and is called point mass at x.

Example 3. Let X be a non-empty set and $\mathcal{B}$ = P(X), Define μ on $\mathcal{B}$ by μ(A) = $\bar{\bar{A}}$, where $\bar{\bar{A}}$ denotes numbers of elements in A for all A $\in \mathcal{B}$ = P(X), i.e.

$$\mu(A) = \begin{cases} \text{Number of elements in A, where A is finite set} \\ \infty, \text{when A is infinite set} \end{cases}$$

Then μ is measure on X and this is known as Counting measure.

CHECK YOUR PROGRESS

1. Is the Vitali Covering Lemma does extend to the case in which the covering collection consists of non degenerate general intervals. True/False
2. Let f be continuous on R. Is there an open interval on which f is monotone. True/False
3. Is the f an increasing bounded function on the open, bounded interval (a, b).True/False
4. Is a continuous function f on [a, b] is lipschitz if its upper and lower derivatives are bounded. True/False
5. Let f and g be of bounded variation on [a, b].Then TV(f + g)= TV(f) + TV(g). True/False

8.12 SUMMARY

This unit is complete combination of

i. Definition of measure space and measurable function.

ii. Concept of differentiation and bounded variation.

iii. Definition of absolute continuity and convex functions.

8.13 GLOSSARY

1. Measure space.
2. Measurable function.
3. Convex function.
4. Differentiation of monotone functions.
5. Function of bounded variation.
6. Absolute continuity.

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8.16 TERMINAL QUESTIONS

1. Show that there is a strictly increasing function on [0, 1] that is continuous number in [0, 1].
2. Let f be continuous on R.Is there an open interval on which f is monotone?
3. Compute the upper and lower derivatives of the characteristic function of the rational.
4. Show that the linear combination of two functions of bounded variation is also of bounded variation. Is the product of two such functions also of bounded variation?

5. Show that both the sum and product of absolutely continuous functions are absolutely continuous.

6. State and prove a version of Jensen's Inequality on a general closed, bounded interval [a, b].

8.17ANSWERS

CHECK YOUR PROGRESS

1. True
2. True
3. True
4. True
5. False

BLOCK III:

L_p SPACE and

WEIERSTRASS APPROXIMATION THEOREM

UNIT 9: THE L^P – SPACES

CONTENTS:

9.1 INTRODUCTION

Recall that an σ-algebra (usually denoted by A) of subsets of set of real numbers R is a collection of subsets of real numbers that contains the empty set and is closed with respect to the formation of complements in R and with respect to the formation of countable unions and therefore by De Morgan's Identities to the formation of intersections. The collection of measurable sets of real numbers is a σ-algebra. The restriction of the set function outer measure to the class of measurable sets is called Lebesgue measure and it is denoted by m. Throughout this unit E denotes a measurable set of real numbers. Define Ƒ to be the collection of all measurable extended real-valued functions on E that are finite almost everywhere on E. Define two functions f and g in $\mathcal{F}$ to be equivalent, and write $f \sim g$, provided

$$f(x) = g(x) \text{ for almost all x in E.}$$

It is obvious that relation $\sim$ is an equivalence relation that is it is reflexive, symmetric and transitive. Hence it induces a partition of $\mathcal{F}$ into a disjoint collection of equivalence classes, which we denote by $\mathcal{F}/\approx$. It is easy to see that $\mathcal{F}/\approx$ has a natural linear structure.

Given two functions f and g in Ƒ, their equivalence classes $[f]$ and $[g]$ and real number c and d, we define the linear combination $c[f] + d[g]$ to be the equivalence classes of the functions in $\mathcal{F}$ that takes the value $cf(x) + dg(x)$ at the points x in E at which f and g take finite value at x.

These linear combinations are well defined such that they are independent of the choice of the representatives of the equivalence classes. The zero element of this linear structure is the equivalence class of the functions that vanish almost everywhere on E.

A subspace of a linear space is the subset of that space which is closed with respect to formation of linear combination, there is a natural family $(L^p(E))_{1 \le p \le \infty}$ of subspaces of $\mathcal{F}/\approx$.

For simplicity and convenience, we refer to the equivalence classes in $\mathcal{F}/\approx$ as functions and denote them by f rather than $[f]$. Therefore, to write $f \sim g$ means that $f - g$ vanishes almost everywhere on E.

This simplification imposes the obligation to check consistency when defining concepts for the L^p-Spaces. For instance, it is meaningful to assert that a sequence f_n in $L^p(E)$ converges pointwise almost everywhere on E to a function f in $L^p(E)$ since if $g_n \cong f_n$ for all n and $f \cong g$, then, since the union of a countable collection of sets of measure zero also is of measure zero, the sequence g_n also converges pointwise almost everywhere on E to g.

9.2 OBJECTIVES

After the completion of this unit learners will be able to

i. Define the concept of L^p-Spaces.

ii. Describe that L^p-Spaces are linear spaces.

iii. Explain and prove various inequalities like Young's inequality, Hölder's inequality, Cauchy-Schwarz inequality, Minkowski's inequality.

iv. After reading solved examples learners should be able to try the problems.

9.3 PRELIMINARIES, NORMED LINEAR SPACE AND L^p −SPACES

Definition 1:

For $1 \le p < \infty$, by $L^p(E)$, we mean a collection of equivalence classes $[f]$ for which $|f|^p$ is integrable. Thus

$$f \in L^p(E) \Leftrightarrow \int_E |f|^p < \infty.$$

Sometimes we denote the collection of such functions by the symbol L^p.

Definition 2:

A measurable function f on measurable set E is said to be an essentially bounded function if there exists $M_f > 0$ such that

$|f(x)| \le M_f$ for all most all $x \in E$.

We define $L^\infty(E)$ to be the collection equivalence classes $[f]$ for which f is essentially bounded functions on E.

Therefore $f \in L^\infty(E) \Leftrightarrow$ there exists $M_f > 0$ such that $|f(x)| \le M_f$ for almost all $x \in E$.

Definition 3:

For E a measurable set, $1 \le p < \infty$, and a function f in $L^p(E)$, we denote

$\|f\|_p := \left(\int_E |f|^p\right)^{1/p}$, and for $p = \infty$, $\|f\|_\infty = inf\{M_f > 0 :$

$|f(x)| \le M_f$ for almost all $x \in E\}$.

We prove that $L^p(E)$ is a vector space over R.

Theorem 1:

For $1 \le p \le \infty$, $L^p(E)$ is a vector space over R.

Proof: Let $1 \le p \le \infty$, for $f, g \in L^p(E)$, and $c \in R$ we note that

$|f+g| \leq |f| + |g| \leq 2.max\{|f|, |g|\}$.

Therefore for $p = \infty$, we have

$\|f+g\|_\infty \leq 2.max\{\|f\|_\infty, \|g\|_\infty\} < \infty$

$\|cf\|_\infty = c\|f\|_\infty < \infty$

And for the case $1 \leq p < \infty$, we have

$|f+g|^p \leq 2.max\{|f|^p, |g|^p\}$

Then

$$\int_E |f+g|^p \leq 2.max\{\int_E |f|^p, \int_E |g|^p\} < \infty$$

Hence $f + g \in L^p(E)$. Further

$$\int_E |cf|^p = c\int_E |f|^p < \infty.$$

Using these, all axioms of a vector space can be proved. Therefore $L^p(E)$ is a vector space over R.

When the vector spaces R^2 and R^3 are considered in the usual way, we have the concept of the length of a vector in R^2 and R^3 associated with each vector.

These are clearly elementary examples of vector spaces which gives us a deeper understanding of these vector spaces equipped with length. When we turn to other (possibly infinite-dimensional) vector spaces, we might hope to get more insight into these spaces if there is some way of assigning something similar to the length of a vector for each vector in the space. Accordingly, we look for a set of axioms which is satisfied by the length of a vector in R^2 and R^3•

This set of axioms will define the "norm" of a vector and throughout this unit we will mainly learn about some important example of normed vector spaces. Now we give the definition of normed vector spaces.

Definition 4:

Let X be a linear space. A real valued functional $\|.\|$ on X is called a norm provided for each f and g in X and each real number c,

(The triangle inequality)

$$\|f + g\| \leq \|f\| + \|g\|$$

(Positive homogeneity)

$$\|cf\| = |c|\|f\|$$

(Non negativity)

$\|f\| \geq 0$ and $\|f\| = 0$ if and only if $\mathrm{f} = 0$.

Moreover, the ordered pair $(X, \|.\|)$ is called a normed vector space or normed linear space.

A useful variation of triangle inequality is

$$|\|f - g\|| \leq \|f - g\|$$

For any f, g in X.

This also shows that a norm is a continuous function.

Example1:

Let N be the set of natural numbers, A be the collection of all the subsets of N and μ be the counting measure on N, then every real valued function on N is measurable. Hence, in this case, for any $p \geq 1$,

$L^p(N) = l^p(N) = l^p$

Here l^∞ is the space of all bounded sequences in R and for $1 \leq p < \infty$, l^p is the space of all sequences (a_n) in R such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$.

Example2:

Let $[a, b]$ be a closed, bounded interval. Then the linear space of continuous real- valued functions on $[a, b]$ is denoted by C$[a, b]$.

Since each continuous function $[a, b]$ takes a maximum value for $f \in [a, b]$, we can define

$$\|f\|_{\max} = max_{x \in [a,b]} |f(x)|$$

We will prove in the section of solved problems that $\|f\|_{max}$ is a norm.

Definition 5:

Let $p > 1$ and q be any two positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then q is called conjugate to p. For example, if $p = 2$, then $q = 2$, thus 2 is self-conjugate number.

9.4 THE INEQUALITIES OF YOUNG, HÖLDER

AND MINKOWSKI

The following inequality is named after William Henry Young. William Henry Young was an English mathematician. Young's inequality for products is a mathematical inequality about the product of two numbers and can be used to prove Hölder's inequality. The standard form of the inequality is the following:

Young's inequality:

Let a, b be non-negative real numbers and $p, q \in (1, \infty)$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof:

It is worth to note that function ϕ, defined by

$\phi(x) = e^x, x \in R,$ is a convex function, that is, for every $x, y \in R$ and $0 < \lambda < 1,$

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y).$$

Putting

$\lambda = \frac{1}{p}$, we have

$1 - \lambda = \frac{1}{q}$,

and

$$e^{\frac{x}{p}+\frac{y}{q}} \leq \frac{e^x}{p} + \frac{e^y}{q}.$$

Now consider $x > 0, y > 0$

such that

$a = e^{\frac{x}{p}}$ and $b = e^{\frac{y}{q}}$,

that is,

$\mathrm{x} = ln(a^p)$ and, $y = ln(b^q)$

and therefore we have

$$e^{\frac{ln(a^p)}{p}+\frac{ln(b^p)}{q}} \leq \frac{e^{ln(a^p)}}{p} + \frac{e^{ln(b^q)}}{q}\frac{e^{ln(b^q)}}{q}.$$

.

Therefore

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The German mathematician Otto Ludwig Hölder(1859-1937) worked in analysis and group theory. He proved this inequality in 1884. Hölder inequality is one of the fundamental inequalities between integrals and an indispensable tool for the study of L^p -spaces

Hölder'sinequality:

Let E be a measurable set $1 \leq \mathrm{p} \leq \infty, q$ the conjugate of the p

If f belongs to $L^p(E)$

and g belongs to $L^q(E)$

then their product belongs to $L^1(E)$

and $\int_E |f.g| \leq \|f\|_p . \|g\|_q .$

Proof.

First consider the case $p = 1,$ and $p = \infty$, it is easy to see that inequality holds. Now consider $1 < p < \infty,$ then $1 < q < \infty$. First note that if one of $\|f\|_p$ or $\|g\|_q \|g\|_q$ is 0 or ∞, then inequality holds. Hence suppose that,

$0 < \|f\|_p < \infty$ and $0 < \|g\|_q < \infty . < \|g\|_q < \infty$

Now putting $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$

for $x \in E$ in Young's inequality, we get,

$$\frac{|f|}{\|f\|_p} . \frac{|g|}{\|g\|_q} \leq \frac{|f|^p}{p||f||_p^p} + \frac{|g|^q}{q||g||_q^q}$$

Now taking integrals over E, we get

$$\frac{1}{\|f\|_p} . \frac{1}{\|g\|_q} \int_E |fg| \leq \frac{1}{p||f||_p^p} \int_E |f|^p + \frac{1}{q||g||_q^q} \int_E |g|^q$$

$$= \frac{1}{p} + \frac{1}{q}$$

$= 1$

Therefore

$$\int_E |fg| \le \|f\|_p \|g\|_q.$$

The most important special case of above theorem is when $p = q = 2$, and that is called Cauchy-Schwarz inequality

The Cauchy–Schwarz inequality (also called Cauchy–Bunyakovsky–Schwarz inequality) is an upper bound between two functions in $L^2(E)$space in terms of the product of the function norms. It is considered one of the most important and widely used inequality in functional analysis and measure theory. Baron Augustin-Louis Cauchy was a French mathematician, engineer, and physicist who made pioneering contributions to several branches of mathematics, including mathematical analysis and continuum mechanics while Karl Hermann Amandus Schwarz was a German mathematician, known for his work in complex analysis.

Cauchy-Schwarz inequality:

If f belongs to $L^2(E)$ and g belongs to $L^2(E)$

then their product belongs to $L^1(E)$ and

$$\int_E |f.g| \le \|f\|_2 . \|g\|_2 .$$

The following inequality is named after the German mathematician Hermann Minkowski the Minkowski inequality establishes that the L^p-Spaces are normed linear space. In fact Minkowski inequality is the triangle inequality in L^p -Spaces.

Minkowski's inequality:

Let E be a measurable set $1 \leq p \leq \infty$

If f and g belongs to $L^p(E)$ then so does their sum $f+g,$ and, moreover

$\|f+g\|_p \leq \|f+g\|_p$

Proof: For the case $p=1,$ and $p=\infty$,it is easy to see that inequality holds from inequality$|f+g| \leq |\mathrm{f}| + |\mathrm{g}|$. Now consider

$1 < p < \infty$,and $q > 1$ such that $\frac{1}{p} + \frac{1}{q}$=1. If

$\|f+g\|_p = 0$, then we are done. Hence assume that

$\|f+g\|_p \neq 0$. Now

$$\int_E |f+g|^p = \int_E |f+g|^{p-1}|f+g|$$

$$\leq \int_E |f+g|^{p-1}|f| + \int_E \int_E |f+g|^{p-1}|g|$$

Now by Holder inequality, we have

$$\int_E |f+g|^{p-1}|f| \leq \|f\|_p \left(\int_E |f+g|^{(p-1)q}\right)^{\frac{1}{q}} = \|f\|_p \|f+g\|_p^{\frac{p}{q}}$$

and,

$$\int_E |f+g|^{p-1}|g| \leq \|g\|_p \left(\int_E |f+g|^{(p-1)q}\right)^{\frac{1}{q}} = \|f\|_p \|f+g\|_p^{\frac{p}{q}}$$

Thus

$$\int_E |f+g|^p \ \le \|f\|_p ||f+g||_p^{\frac{p}{q}} + \|f\|_p ||f+g||_p^{\frac{p}{q}}$$

$$= (\|f\|_p + \|g\|_p) ||f+g||_p^{\frac{p}{q}}$$

Now cancelling out $||f+g||_p^{\frac{p}{q}}$ as $||f+g||_p^{\frac{p}{q}} \neq 0$, we obtain

$$\|f+g\|_p \le \ \|f \ + g\|_p \ .$$

Theorem 2: $(L^p(E), \|\|_p)$ is a normed linear space .

Proof:

To prove that $(L^p(E), \|\|_p)$ is a normed linear space, we have to prove that

- **i.** $\|f\|_{\mathrm{p}} \ge 0.$

  In particular $\|f\|_{\mathrm{p}} = 0$ if and only if $\mathrm{f} = 0$ almost everywhere
- **ii.** $\|cf\|_{\mathrm{p}} = |c|\|f\|_{\mathrm{p}}$
- **iii.** $\|f+g\|_{\mathrm{p}} \le \|f\|_{\mathrm{p}} + \|f\|_{\mathrm{p}}$

Since $|\mathrm{f}| \ge 0$ and so $\|f\|_{\mathrm{p}} := \left(\int_E |f|^p\right)^{1/p} > 0.$

Further $f \ = 0 \ \Rightarrow \|f\|_{\mathrm{p}} = 0$. Again if $f \ge 0$ and $\|f\|_{\mathrm{p}} = 0$ then $f \ = 0$.

$$\|cf\|_{\mathrm{p}} = \int_E |cf|^p \ = \ |c| \int_E |f|^p = |c|\|f\|_p.$$

And $\|cf\|_\infty = |c|\|f\|_\infty.$

Further by Minkowski inequality we have

$\|f+g\|_{\mathrm{p}} \leq \|f\|_{\mathrm{p}} + \|f\|_{\mathrm{p}}$. Thus $(L^p(E), \|\|_p)$ is a normed linear space.

9.5 SOLVED PROBLEMS

Problem 1:

Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$.

Then $L^{p_2}(E) \subset L^{p_1}(E)$. Furthermore,

$$\|f\|_{p_1} \leq c\,\|f\|_{p_2}$$

for all f in $L^{p_2}(E)$,

where $c = [m(E)]^{\frac{p_2-p_1}{p_1p_2}}$ if $p_2 < \infty$

and $c = [m(E)]^{\frac{1}{p_1}}$ if $p_2 = \infty$.

Solution:

Suppose $p_2 < \infty$.

Define $p = \frac{p_2}{p_1} > 1$ and let q be the conjugate of p that is $\frac{1}{p} + \frac{1}{q} = 1$.

Let f belongs to $L^{p_2}(E)$ and χ_E be the characteristic function of E then by Hölder inequality we have

$$\int_E |f|^{p_1} = \int_E |f|^{p_1}\chi_E \leq \left(\int_E |f|^{p_2}\right)^{\frac{p_1}{p_2}} \left(\int_E |\chi_E|^q\right)^{\frac{1}{q}} = \|f\|_{p_2}^{p_1}[m(E)]^{\frac{1}{q}}$$

Therefore

$$\left(\int_E |f|^{p_1}\right)^{\frac{1}{p_1}} = \|f\|_{p_2}[m(E)]^{\frac{p_1}{q}} < \infty.$$

For the $p_2 = \infty$ since E be a measurable set of finite measure it means $m(E) < \infty$ therefore

$$\left(\int_E |f|^{p_1}\right) \leq \|f\|_{\infty}^{p_1}.m(E).$$

Hence $L^{p_2}(E) \subset L^{p_1}(E)$.

Remark:

Neither containment holds in general (if E is not of finite measure).

For example,

let $E = R$, the set of real number with Lebesgue measure. Now suppose

$$f(x) = \begin{cases} 1/x, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

and

$$g(x) = \begin{cases} 1/\sqrt{x}, & 0 < x \leq 1 \\ 0, & otherwise \end{cases}.$$

Since

$$\|f\|_2^2 = \int_R \left|\frac{1}{x}\right|^2 = \int_1^\infty \frac{1}{x^2} = \left[\frac{-1}{x}\right]_1^\infty = 1,$$

$$\|f\|_1^1 = \int_R \left|\frac{1}{x}\right| = \int_1^\infty \frac{1}{x} = [log(x)]_1^\infty = \infty,$$

and

$$\|g\|_2^2 = \int_R \left|\frac{1}{\sqrt{x}}\right|^2 = \int_0^1 \frac{1}{x} = [log(x)]_0^1 \, is not finite,$$

$$\|g\|_1^1 = \int_R \left|\frac{1}{\sqrt{x}}\right| = \int_0^1 \frac{1}{\sqrt{x}} = \left[2\sqrt{x}\right]_0^1 = 2 \ .$$

Therefore $f \in L^2(R)\backslash L^1(R)$ and $g \in L^1(R)\backslash L^2(R)$.

Problem 2:

If p, q $\in (0,1), p + q = 1$,then show that$|f|^p . |g|^q \in L^1(E)$.

Solution :

Let $\boldsymbol{F} = |f|^p \; G =, |g|^q$

and $\alpha = \frac{1}{p}, \beta = \frac{1}{q}$,

then $\boldsymbol{F} \in L^{\alpha}(E)$

and $\boldsymbol{G} \in L^{\beta}(E)$

and then $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

and

so by Hölder inequality

$|f|^p.|g|^q = |f|^p.|g|^q \in L^1(E)$.

Problem 3:

Show that in Young's inequality there is equality if and only if $a^p = b^q$.

Solution:

Suppose $f(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab,$

treating b as independent of a.

Then equality holds if and only if $f(a) = 0$.Since $f'(a) = a^{p-1} - b$

and

$f'(a) = 0$

implies $b = a^{p-1}$.

Further $f''(a) = (p-1)a^{p-2} > 0$.

Therefore the minimum value of function f is 0 at $b = a^{p-1}$.

Hence equality holds if and only if $a^p = b^q$.

Problem 4:

Show that in Hölder's inequality there is equality if and only if there are constants α and β, not both zero, for which $\alpha|f|^p = \beta|g|^q$ almost everywhere on E.

Solution:

Suppose there are constants α and β, not both zero, for which $\alpha|f|^p = \beta|g|^q$ almost everywhere on E. Without loss of generality, assume $\beta \neq 0$. Thus we have

$$\int_E |f.g| = \int_E |f| \left|\frac{\alpha}{\beta} f^p\right|^{\frac{1}{q}} = \int_E |f|^{1+\frac{p}{q}} \left|\frac{\alpha}{\beta}\right|^{\frac{1}{q}} = \int_E |f|^p \left|\frac{\alpha}{\beta}\right|^{\frac{1}{q}} \ldots..(1)$$

Since $\alpha|f|^p = \beta|g|^q$

So

$$\alpha \int_E |f|^p = \beta \int_E |g|^q \ldots..(2)$$

Hence by (1) and (2) we get

$$\int_E |f.g| = \int_E |f|^p \left(\frac{\|g\|_q^q}{\|f\|_p^p}\right)^{\frac{1}{q}} = \|f\|_p . \|g\|_q .$$

If f or g are zero then converse holds. So suppose that $f \neq 0$ and $g \neq 0$ and

$$\int_E |f.g| = \|f\|_p.\|g\|_q \dots\dots.(3)$$

Then (3) can be rearranged as

$$\int_E \left(\left(\frac{|f|}{\|f\|_p}\right)^p.\frac{1}{p} + \left(\frac{|g|}{\|g\|_q}\right)^q.\frac{1}{q} - \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}\right) = 0$$

Almost everywhere on E. Thus

$$\left(\frac{|f|}{\|f\|_p}\right)^p.\frac{1}{p} + \left(\frac{|g|}{\|g\|_q}\right)^q.\frac{1}{q} = \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}$$

By problem 3 (condition for equality in Young's inequality) we get

$$\left(\frac{|f|}{\|f\|_p}\right)^p = \left(\frac{|g|}{\|g\|_q}\right)^q$$

Where $\alpha = \frac{1}{\|f\|_p^p}$ and $\beta = \frac{1}{\|g\|_q^q}$.

Problem 5:

Prove that for f in $C[a,b]$,

$$\|f\|_{max} = max_{x\in[a,b]}|f(x)|$$

is a norm.

Solution:

If $\|f\|_{max} = max_{x\in[a,b]}|f(x)| = 0$ if and only if $f = 0$ and triangle inequality and positive homogeneity are easy to verified.

Problem 6:

If $f \in L^2[0,1]$, then show that

$$\left|\int_0^1 f(x)dx\right| \le \left[\int_0^1 |f(x)|^2 dx\right]^{\frac{1}{2}}.$$

Solution: If $f, g \in L^2[0,1]$, then by Cauchy-Schwarz inequality, we get

$$||\mathrm{f.\,g}|| \le ||\mathrm{f}||_2 ||\mathrm{g}||_2$$

Or

$$\int_0^1 |fg|dx \le \left[\int_0^1 |f|^2 dx\right]^{\frac{1}{2}} \left[\int_0^1 |g|^2 dx\right]^{\frac{1}{2}}$$

Taking $g(x) = 1$ for all $xin[0,1]$ we get

$$\int_0^1 |f|dx \le \left[\int_0^1 |f|^2 dx\right]^{\frac{1}{2}}$$

Since

$$\left[\int_0^1 |g|^2 dx\right]^{\frac{1}{2}} = \left[\int_0^1 |1|^2 dx\right]^{\frac{1}{2}} = 1$$

So we get

$$\left|\int_0^1 f(x)dx\right| \le \int_0^1 |f(x)|dx \le \left[\int_0^1 |f(x)|^2 dx\right]^{\frac{1}{2}}$$

Therefore

$$\left|\int_0^1 f(x)dx\right| \le \left[\int_0^1 |f(x)|^2 dx\right]^{\frac{1}{2}}.$$

Problem 7:

If $p \ge 1$ and let $\lim_{n\to\infty} \|f_n - f\|_p = 0$ then prove that $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Solution:

Since we know that $\left|\|f_n\|_p - \|f\|_p\right| \le \|f_n - f\|_p$ so $\lim_{n\to\infty}\left|\|f_n\|_p - \|f\|_p\right| \le \lim_{n\to\infty}\left|\|f_n\|_p - \|f\|_p\right| = 0$ and therefore $\lim_{n\to\infty}\left|\|f_n\|_p - \|f\|_p\right| = 0$ which implies that $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

9.6 SUMMARY

This unit provides an explanation of

i. Normed linear spaces like of L^p -Spaces

ii. Describe the examples of L^p -Spaces.

iii. Proof of various inequalities like Young's inequality, Hölder's inequality, Cauchy-Schwarz inequality, Minkowski's inequality**.**

iv. Relation of inclusion between L^p -Spaces.

v. After reading solved examples learners should be able to try the problems

9.7GLOSSARY

i. **Function:** A function *f* from a set *X* to a set *Y* is an assignment of an element of *Y* to each element of *X*. The set *X* is called the domain of the function and the set *Y* is called the co domain of the function. If the element *y* in *Y* is assigned to *x* in *X* by the function *f*, one says that *f* maps *x* to *y*, and this is commonly written $y = f(x)$.

In this notation, *x* is the argument or variable of the function. A specific element *x* of *X* is a value of the variable, and the corresponding element of *Y* is the value of the function at *x*, or the image of *x* under the function.

ii. **Equivalence classes:**

An equivalence relation is a relation that satisfies three properties: reflexivity, symmetry, and transitivity. Equivalence classes partition the set S into disjoint subsets. Each subset consists of elements that are related to each other under the given equivalence relation.

iii. **Measurable set:**

If (X, Σ) is a measurable space, where Σ is a σ-algebra of subsets of X then elements of Σ is called measurable set.

iv. Conjugate numbers

v. L^p -Spaces.

vi. **almost everywhere:**

A property of a measurable space X is said to hold almost everywhere if the set of points in X where this property fails is contained in a set that has measure zero.

vii. Continuous function:

A continuous function can be formally defined as a function $f: X \rightarrow Y$ where the pre-image of every open set in X is open in Y. More concretely, a function $f(x)$ in a single variable x is said to be continuous at point x_0 if f(x_0) is defined, so that x_0 is in the domain of f with $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$

viii. Measurable function:

A measurable function is a function between the underlying sets of two measurable space that preserves the structure of the spaces that is the preimage of any measurable set is measurable.

That is if (X, Σ) and (Y, Γ) are two measurable space then a function $f: (X, \Sigma) \rightarrow (Y, \Gamma)$ is called a measurable function if for each $E \in \Gamma$

$$f^{-1}(E) = \{x \in X: f(x) \in Y\} \in \Sigma$$

CHECK YOUR PROGRESS

Question 1: Prove that

$$\int_0^{\pi} x^{\frac{-1}{4}} \sin x \; dx \; \leq \pi^{\frac{3}{4}} .$$

Question 2: If $1 \leq p_1 < p_2 \leq \infty$.

Then prove that $l^{p_1} \subset l^{p_2}$. Moreover this inclusion is proper.

Question 3: Assume $m(E) < \infty$. For $f \in L^{\infty}(E)$, show that $\lim_{n\to\infty} \|f\|_p = \|f\|_{\infty}$.

Question 4: If f in $L^1(E)$ and g in $L^{\infty}(E)$ then prove that $f.g \in L^1(E)$ and $\|f.g\| \leq \|f\|_1 \|g\|_{\infty}$.

Question 5: Give an example of a function which belongs to $L^{\infty}(R)$but not in $L^2(R)$.

Question 6: Show that norm is a continuous function.

9.8 REFRENCES

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3. G. D. Barra, Measure Theory and Integration (2023), New Age International.

9.9SUGGESTED READINGS

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9.10 TERMINAL QUESTION

Question1: Is For 1$\leq p \leq \infty$, $L^p(E)$ a vector space over R.

Question 2: Is For $p < 1$, $L^p(E)$ a normed linear space over R.

Question 3: Is For 1$\leq p \leq \infty$, $L^p(E)$ a normed linear space over R.

Question 4: Is $C[a,b]$ a normed linear space, where normed is defined as $\|f\|_{min} = min_{x\in[a,b]}|f(x)|$ for $\boldsymbol{f} \in C[a,b]$

9.11 ANSWERS

CHECK YOUR PROGRESS

Answer 1: Use Cauchy-Schwarz inequality.

Answer 2: Suppose$\{a_n\} \in l^{p_1}$ then$\sum_{n=1}^{\infty}|a_n|^{p_1} < \infty$. It means there exists a natural number m such that$|a_n|^{p_1} < 1$ for all $n > m$.

Since $p_1 < p_2$ therefore $|a_n|^{p_2} < |a_n|^{p_1}$for all $n > m$ and so $\sum_{n=1}^{\infty}|a_n|^{p_2} < \infty . < \sum_{n=1}^{\infty}|a_n|^{p_1} < \infty$. Hence proved.

For proper inclusion let $\{a_n\} = \{\frac{1}{n^{\frac{1}{p_1}}}\}$ for each natural number n.

Answer 3: Use problem 1.

Answer 4: Since$|g| \leq ||g||_{\infty}$ almost everywhere on E and $|f.g| \leq |f|||g||_{\infty}$.

Answer 5: The characteristic function of $[0, \infty]$ that is $\chi_{[0,\infty]}$.

Answer 6: Use the triangle inequality$|||f - g||| \leq ||f - g||$

TERMINAL QUESTION

Answers 1: Yes

Answers 2: No as triangle inequality for norm is not satisfied.

Answers 3: Yes

Answer 4: No

Unit 10 :

THEOREM IN LEBESGUE INTEGRATION

CONTENTS:

10.1 INTRODUCTION

Completeness of reals may be formulated by asserting that if $\{a_n\}$ is a sequence of real numbers for which $\lim_{n,m\to\infty}|a_n - a_m| = 0,$

there is a real number a such that

$$\lim_{n\to\infty}|a_n - a| = 0.$$

There is a corresponding completeness property for the Lebesgue integral. For E measurable set and $1 \le p < \infty$, define $L^p(E)$ to be the collection of measurable functions f for which $|f|^p$ is integrable over E.

If $\{f_n\}$ is a sequence of functions in $L^p(E)$ for which

$$\lim_{n,m\to\infty}|f_n - f_m|^p = 0,$$

there is a function f in $L^p(E)$ such that

$$\lim_{n\to\infty}|f_n - f|^p = 0.$$

This is the Riesz-Fischer theorem, the centerpiece of this unit. The Riesz-Fischer theorem is named for mathematicians Frigyes Riesz and Ernst Fischer who independently published the result for special case $\mathrm{p} = 2$ in 1907.

The Riesz-Fischer theorem usually described a number of results for convergence of Cauchy sequence in $L^p(E)$.

Most often Riesz-Fischer theorem is considered to state that $L^p(E)$ are complete, where E is a measurable subset of reals that is every Cauchy sequence in a $L^p(E)$converges to a function in $L^p(E)$.In this unit we show that, for $p \geq 1$, $L^p(E)$are complete. Furthermore, there are some consequence of Riesz-Fischer theorem which in turn also give necessary and sufficient conditions for the convergence in $L^p(E)$ norm for a sequence that converges pointwise.

10.2 OBJECTIVES

After the completion of this unit learners will be able to

i. Define the Cauchy sequence, rapidly Cauchy sequence.

ii. Understand the concept of completeness for $L^p(E)$.

iii. Understand the concept of convergence in $L^p(E)$ and pointwise convergence.

iv. Define Uniform integrable function, tight function on a measurable set E.

v. State necessary and sufficient condition for the equivalence of convergence in $L^p(E)$ and pointwise convergence.

vi. Explain examples and counterexamples.

10.3 PRELIMINARIES

Recall some important results and definitions useful in this unit

Lebesgue monotone convergence theorem:

Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E.

If $\{f_n\} \to f$ pointwise almost everywhere on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Lebesgue dominated convergence theorem:

Let $\{f_n\}$ be a sequence of measurable functions on E.

Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that

$|f_n| \le g$ for all n.

If $\{f_n\} \to f$ pointwise almost everywhere on E, then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Fatou lemma:

Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E.

$$\text{If} \{f_n\} \to f \text{ pointwise, then} \int_E f \leq \liminf \int_E f_n.$$

Minkowski's inequality:

Let E be a measurable set, $1 \leq p \leq \infty$ If f and g belong to $L^p(E)$ then so does their sum f + g, and, moreover,

$$\|f + g\|_p \leq \|f + g\|_p .$$

Egoroff'sTheorem:

Suppose E has a finite measure. Let $\{f_n\}$ be a sequence of measurable function on E that converges pointwise on E to the real valued function f. Then for each $\in > 0$, there is a closed set F Contained in E for which

$$\{f_n\} \to f \text{ uniformaly on F and } \mathrm{m}(E{\sim}F) < \in,$$

where m is Lebesgue measure.

The Vitali Convergence theorem:

Let $\{f_n\}$ be a sequence of functions on E that is uniformly integrable and tight over EIf $\{f_n\} \to f$ pointwise on E. Then

f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

10.4 RIESZ – FISHER THEOREM

Definition 1: Convergent sequence in $L^p(E)$

Let $\{f_n\}$ be a sequence of functions in $L^p(E)$, then $\{f_n\}$ is said to be convergent to a function f in $L^p(E)$ provided

$$\lim_{n\to\infty} \|f_n - f\|_p = 0$$

Or if for given

$\epsilon \geq 0,\ \exists\ n_0 \in N$

such that

$n \geq n_0 \Rightarrow \|f_n - f\|_p \leq \epsilon.$

Definition 2: Cauchy sequence in $L^p(E)$

Let E be a measurable set and $\{f_n\}$ is a sequence of functions in $L^p(E)$, then $\{f_n\}$ is said to be Cauchy sequence, if for given

$\epsilon \geq 0,\ \exists\ n_0 \in N$ such that

$\mathrm{m}, \mathrm{n} \geq n_0 \Rightarrow \|f_m - f_n\|_p \leq \epsilon.$

A normed linear space X is said to be complete provided every Cauchy sequence X in converges to a function in X. A complete normed linear space is called a Banach space.

Proposition 3:

Let X be a normed linear space. Then every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Proof: Let $\{f_n\} \to f$ in X.

By triangle inequality in norm,

$$\begin{aligned}\|f_m - f_n\| &= \|f_m - \mathrm{f} + \mathrm{f} - f_n\| \\ &\leq \|f_m - f\| + \|f_n - f\|\end{aligned}$$

for all m, n.Therefore $\{f_n\}$ is Cauchy.

Now let $\{f_n\}$ be a Cauchy sequence in X that has a subsequence $\{f_{n_k}\}$ which converges in X to f.

Suppose$\epsilon > 0$. Since $\{f_n\}$ is Cauchy, we may choose N such that $\|f_m - f_n\| \leq \frac{\epsilon}{2}$ for all$\mathrm{m}, \mathrm{n} \geq N$. But $\{f_n\}$ converges to f, we may have k such that $n_k \geq N$ and

$$\left\|f_{n_k} - f\right\| \leq \frac{\epsilon}{2},$$

then by triangle inequality we have for $n \geq N$

$$\|f_n - f\| = \left\|f_n - f_{n_k} + f_{n_k} - f\right\|$$

$$\leq \left\|f_n - f_{n_k}\right\| + \left\|f_{n_k} - f\right\| < \epsilon.$$

Thus $\{f_n\} \to f$ in X.

Definition 4:

Let X be a normed linear space. A sequence $\{f_n\}$ in X is said to be rapidly Cauchy provided there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \epsilon_k$ for which

$\|f_{k+1} - f_k\| \leq \epsilon_k^2$ for all k.

It is useful to note that if $\{f_n\}$ is a sequence in a normed linear space and sequence of nonnegative numbers $\{a_k\}$ has the property that

$$\|f_{k+1} - f_k\| \leq a_k$$

Then, since

$f_{n+k} - f_n = \sum_{j=n}^{n+k-1}\left(f_{j+1} - f_j\right)$ or all $n, k,$ and therefore, we have

$$\|f_{n+k} - f_n\| \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \leq \sum_{j=n}^{\infty} a_j$$

for all n, k.

Proposition 5: Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.

Proof: Let $\{f_n\}$ be a rapidly Cauchy sequence in X and $\sum_{k=1}^{\infty} \epsilon_k$ a convergent series of nonnegative numbers for which

$$\|f_{k+1} - f_k\| \leq \epsilon_k^2$$

for all k.

Thus, we have

$$\|f_{n+k} - f_n\| \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \leq \sum_{j=n}^{\infty} \epsilon_j^2$$

for all n, k.

Since the series

$\sum_{k=1}^{\infty} \epsilon_k$ converges, therefore the series

$\sum_{k=1}^{\infty} \epsilon_k^2$ also converges and so $\{f_n\}$ is a Cauchy.

Now assume that $\{f_n\}$ be a Cauchy sequence in X. For $\epsilon = \frac{1}{2}$, there exists $n_1 > 0$ such that

$$\|f_n - f_m\| \leq \frac{1}{2}$$

for all $n, m \geq n_1$.

Now for $\epsilon = \frac{1}{2^2}$, there exists $n_2 > n_1$ such that

$$\|f_n - f_m\| \leq \frac{1}{2^2}$$

for all $n, m \geq n_2$.

Again for $\epsilon = \frac{1}{2^3}$, there exists $n_3 > n_2$ such that

$$\|f_n - f_m\| \leq \frac{1}{2^3}$$

for all $n, m \geq n_3$.

Therefore, we have a subsequence $\{f_{n_k}\}$ of sequence $\{f_n\}$ such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\| \leq \frac{1}{2^k} \text{ for all } k \geq 1.$$

Theorem 6 (The Riesz-Fischer):

Let E be a measurable set, and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space.

Proof: First, we consider the case $1 \leq p < \infty$.Let $\{f_n\}$ be a Cauchy sequence in $L^p(E)$. By Proposition 5 there is a subsequence $\{f_{n_k}\}$ of sequence $\{f_n\}$ such that

$\|f_{n_{k+1}} - f_{n_k}\|_p \leq \frac{1}{2^k}$ for all $k \geq 1$.T

Therefore $\{f_{n_k}\}$ is a rapidly Cauchy sequence.By Proposition 5 to prove $\{f_n\}$ converges, it suffices to that $\{f_{n_k}\}$ converges in $L^p(E)$. Now consider the following two series

$$f_{n_1}(x) + \sum_{k=1}^{\infty}\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

And

$$|f_{n_1}(x)| + \sum_{k=1}^{\infty}\left|\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)\right|$$

The corresponding partial sums are as follow

$$\begin{aligned}S_{1,m}(x) &= f_{n_1}(x) + \sum_{k=1}^{m}\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)\\ &= f_{n_{m+1}}(x)\end{aligned}$$

$$S_{2,m}(x) = |f_{n_1}(x)| + k = 1\infty\left|\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)\right|.$$

Since $\{S_{2,m}(x)\}$ is an increasing sequence of nonnegative real numbers so the limit

$$g(x) := \lim_{n\to\infty} S_{2,m}$$

$$= |f_{n_1}(x)| + \sum_{k=1}^{\infty} \left|\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)\right|$$

always exists and $g(x)$ could be $+\infty$ at some point of E.

Now we claim that g belongs to $L^p(E)$. By the triangle inequality in $L^p(E)$ we have

$$\|S_{2,m}\|_p = \|f_{n_1}\|_p + \sum_{k=1}^{m} \left\|\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)\right\|_p .$$

Therefore, we have

$$\|S_{2,m}\|_p \le \|f_{n_1}\|_p + \sum_{k=1}^{m} \frac{1}{2^k} < \|f_{n_1}\|_p + 1$$

Therefore

$$\int_E |S_{2,m}|^p = \left|\left|S_{2,m}\right|\right|_p^p < \left(\|f_{n_1}\|_p + 1\right)^p < \infty$$

Hence, we have

$$\lim_{m\to\infty} \int_E |S_{2,m}|^p < \left(\|f_{n_1}\|_p + 1\right) < \infty \dots\dots\dots\dots.(1)$$

Since $\{S_{2,m}(x)\}$ is an increasing sequence of nonnegative real numbers so by Lebesgue monotone convergence theorem, by(1) we have

$$\int_E |g|^p = \int_E \lim_{m\to\infty} |S_{2,m}|^p = \lim_{m\to\infty} \int_E |S_{2,m}|^p$$
$$< \left(\|f_{n_1}\|_p + 1\right) < \infty.$$

Which shows that g belongs to $L^p(E)$. It also implies that g is finite almost everywhere on E. It means that $S_{2,m}(x)$ converges almost everywhere on E and therefore $S_{1,m}(x)$ converges to a finite value f(x):

$$\mathrm{f}(x) := \lim_{m\to\infty} S_{1,m}(x) = \lim_{m\to\infty} f_{n_m}(x)$$

Now we claim that

$\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$ and f belongs to $L^p(E)$.

For we note that

$$|f_{n_m} - \mathrm{f}|^p \leq \left(2\max\{\mathrm{f}, S_{1,m-1}\}\right)^{\mathrm{p}} < (2g)^p.$$

Since $(2g)^p$ is measurable on E, applying Lebesgue dominated convergence theorem, we have

$$\lim_{m\to\infty} \int_E |f_{n_m} - f|^p = \int_E \lim_{m\to\infty} |f_{n_m} - f|^p = 0.$$

This implies that

$$\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0.$$

And hence for $\epsilon = 1$ there exists a natural number N such that

$\|f_{n_m} - f\|_p < 1$ for all $\mathrm{m} \geq N$ and therefore we have

$\| f \|_p < \|f_{n_N} - f\|_p + \|f_{n_N}\|_p < 1 + \|f_{n_N}\|_p$. This implies that f belongs to $L^p(E)$.

Now for the case $\mathrm{p} = \infty$ we use the fact that a function is greater than its essential supremum only on a set of measure of zero.

Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(E)$. Define

$$A_{n,m} = \{\mathrm{x} \in E: |f_n(x) - f_m(x)| > \|f_\mathrm{n} - f_m\|_\infty\}$$

and

$$B_k = \{\mathrm{x} \in E: |f_n(x)| > \|f_\mathrm{n}\|_\infty\}.$$

Then if $E_0 = A_{n,m} \cup B_k$, then E_0 is the set of measure zero and $\{f_n(x)\}$ is a Cauchy sequence for each $\mathrm{x} \in E \sim E_0$, with limit $\mathrm{f}(x)$, say. Define f arbitrarily on E.

Since $\{f_n\}$is a Cauchy sequence in $L^\infty(E)$, given $\epsilon > 0$ there exists N such that $\|f_\mathrm{n} - f_m\|_\infty < \epsilon$ for $\mathrm{n}, \mathrm{m} \geq N$. So for $\mathrm{x} \in \mathrm{E} \sim E_0 |f_n(x) - f_m(x)| < \|f_\mathrm{n} - f_m\|_\infty < \epsilon$,and letting $\mathrm{n} \to \infty$, we get $\|\mathrm{f} - f_m\|_\infty < \epsilon$ and also $\|\mathrm{f}\|_\infty < \|f_m\|_\infty + \epsilon$, and hence f belongs to $L^\infty(E)$.

In conclusion we proved that for $1 \leq p \leq \infty, L^p(E)$is a Banach space.

Remarks:

Let N be the set of natural numbers, A be the collection of all the subsets of N and μ be the counting measure on N, then every real valued function on N is measurable. Hence, in this case, for any $p \geq 1$,

$$L^p(N) = l^p(N) = l^p$$

Here l^∞ is the space of all bounded sequences in R and for $1 \leq p < \infty$, l^p is the space of all sequences (a_n) in R such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$.

Remarks:

Since the function f defined in the proof of The Riesz-Fischer theorem is the pointwise limit of rapidly Cauchy sequence $\{f_{n_k}\}$. Therefore it is worth to note that every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to $L^p(E)$ norm and pointwise almost everywhere on E to a function in $L^p(E)$.

Hence, we have following corollary.

Corollary 7:

Let E be a measurable set, and $1 \leq p \leq \infty$. If $\{f_n\} \rightarrow f$ in $L^p(E)$, then there is a subsequence of $\{f_n\}$ converges pointwise almost everywhere on E to f.

Proof:

Let $\{f_n\}$be a Cauchy sequence in$L^p(E)$. According to Proposition 5, there is a subsequence $\{f_{n_k}\}$of $\{f_n\}$ that is rapidly Cauchy. The Riesz-fisher theorem tells that $\{f_{n_k}\}$ converges to a function f in $L^p(E)$both with respect to the $L^p(E)$ norm and pointwise almost everywhere onE. According to Proposition 3 the whole Cauchy sequence converges to fwith respect to the$L^p(E)$ norm.

The following example shows that a sequence $\{f_n\}$ in $L^p(E)$ that converges pointwise almost everywhere on Eto f in $L^p(E)$ will not in general converge in $L^p(E)$

Example 8:

For $\mathrm{E} = [0,1], 1 \leq \mathrm{p} < \infty$ and each natural number n

Let $f_n = n^{1/p}\chi_{(o,1\}}$.

The sequence converges pointwise on E to the function that is identically zero but does not converge to zero function with respect to the$L^p(E)$norm as $\|f_n - 0\|_p = 1$.

The following theorem gives a necessary and sufficient condition for the convergence in $L^p(E)$ norm for a sequence that converges pointwise.

Theorem9:

Let E be a measurable set, and $1 \leq p < \infty$. If $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise almost everywhere onE to the function f in $L^p(E)$, then

$$\{f_n\} \to f \text{ in } L^p(E)$$

if and only if

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p.$$

Proof:

By possibly excising from *E* a set of measure zero, we may assume *f* and each f_n is real-valued and convergence is pointwise on E. We have from Minkowski inequality that for each n,

$$\left| \|f_n\|_p - \|f\|_p \right| \leq \|f_n - f\|_p.$$

Therefore if $\{f_n\} \to f$ in $L^p(E)$

Then we have

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p.$$

To prove the converse part suppose

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p.$$

Now define a function $\phi(t) = t^p$ for all t.

Then ϕ is a convex function as its second derivative is nonnegative and therefore, we have

$$\phi\left(\frac{a+b}{2}\right) \le \frac{\phi(a)+\phi(b)}{2} \text{ for all a, b.}$$

Thus

$$\frac{|a|^p+|b|^p}{2} - \left|\frac{a-b}{2}\right|^p \ge 0 \text{ for all a, b.}$$

Therefore, for each natural number n, a nonnegative measurable function h_n is defined on E by

$$h_n(x) = \frac{|f_n(x)|^p+|f(x)|^p}{2} - \left|\frac{f_n(x)-f(x)}{2}\right|^p \ge 0$$

for all x in E.

Since $h_n \to |f|^p$ Pointwise on Ewe get from Fatou's lemma

$$\begin{aligned}\int_E |f|^p &\le \lim\inf \int_E h_n \\ &= \lim\inf \int_E \frac{|f_n(x)|^p+|f(x)|^p}{2} \\ &\quad - \left|\frac{f_n(x)-f(x)}{2}\right|^p \\ &= \int_E |f|^p - \limsup \int_E \left|\frac{f_n(x)-f(x)}{2}\right|^p\end{aligned}$$

Therefore

$$\lim\sup \int_E \left|\frac{f_n(x)-f(x)}{2}\right|^p \le 0.$$

Hence, we have $\{f_n\} \to f$ in $L^p(E)$.

10.5 THEOREM ON LEBESGUE INTEGRATION

Definition 10: A nonnegative measurable function f on a measurable set E is said to be integrable over E provided

$$\int_E |f|^p < 0$$

Theorem 11 (Chebyshev's inequality):

Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$

$$\text{m}\{\text{x} \in E \mid \text{f(x)} \geq \lambda\} \leq \frac{1}{\lambda} \int_E \text{f}.$$

Proof:

Define $\{E_\lambda\} = \{\text{x} \in \text{E} \mid \text{f(x)} \geq \lambda$.

First suppose $\text{m}\{E_\lambda\} = \infty$.

Let n be a natural number. Define $E_{\lambda,n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda . \chi_{E_{\lambda,n}}$.

Then ψ_n is a bounded measurable function such that closure of set $\{\text{x} \in \text{E} \mid \text{f(x)} \neq 0\}$ is finite.

Further $0 \leq \psi_\text{n} \leq \text{f}$ on E for all n and

$$\lambda.\mathrm{m}\left(E_{\lambda,n}\right) = \int_{\mathrm{E}} \psi_n.$$

Now we infer from continuity of measure that

$$\infty = \lambda.\mathrm{m}(E_\lambda) = \lambda.\lim_{n\to\infty} m\left(E_{\lambda,n}\right) = \lim_{n\to\infty}\int_E \psi_n \le \int_E f.$$

Thus, Chebyshev's inequality holds as both sides equal to infinity.

Now consider the case $\mathrm{m}(E_\lambda) < \infty$.

Now define a function $\mathrm{h} = \lambda.\chi_{E_\lambda}$ that is abounded measurable function and $0 \le h \le f$ on E.

By the definition of the integral of f over E, we have

$$\lambda.\mathrm{m}(E_\lambda) = \int_E h \le \int_E f.$$

Now by dividing both side by λ, we get

$$\mathrm{m}\{\mathrm{x} \in E \mid \mathrm{f(x)} \ge \lambda\} \le \frac{1}{\lambda}\int_E f$$

Theorem 12: Let f be a nonnegative measurable function on E. Then

$$\int_E f = 0 \text{ if and only if } \mathrm{f} = 0 \text{ almost everywhere.}$$

Proof: First assume that

$$\int_E f = 0.$$

Then by Chebyshev's inequality, for each natural number n, we have

$$m\{x \in E \mid f(x) \geq \frac{1}{n}\} = 0.$$

Now by countable additivity of Lebesgue measure we have

$$m\{x \in E \mid f(x) > 0\} = 0.$$

For the converse part suppose that

$f = 0$ almost everywhere on E.

Let ϕ be a simple function and h a bounded measurable function for which $0 \leq \phi \leq h \leq f$ on E.

Thus $\phi = 0$ almost everywhere on E and therefore $\int_E \phi = 0$. Since this holds for all such h we infer that

$$\int_E f = 0$$

Proposition 13: Let f be a nonnegative measurable function on E. Then f is finite almost everywhere on E.

Proof:

Let n be a natural number. By Chebyshev's inequality and monotonicity of measure we have

$$m\{x \in E \mid f(x) = \infty\} \leq m\{x \in E \mid f(x) \geq n\} \leq \frac{1}{n}\int_E f.$$

But $\int_E f$ is finite and therefore $m\{x \in E \mid f(x) = \infty\} = 0.$

Beppo Levi' Lemma:

Let $\{f_n\}$be an increasing sequence of nonnegative measurable function on E. If the sequence of integrals $\{\int_E f_n\}$is bounded, then $\{f_n\}$ converges pointwise on Eto a measurable function f that is finite almost everywhere on E and also

$$\lim_{n\to\infty} \int_E f_n = \int_E f < \infty.$$

Proof: Every monotone sequence of extended real number converges to an extended real number. Since $\{f_n\}$is an increasing sequence of extended real valued function onE, we may define the extended real valued nonnegative pointwise on E by

$$\mathrm{f}(x) = \lim_{n\to\infty} \mathrm{f_n}(x)$$

for all $x \in E$.

According to Lebesgue monotone convergence theorem

$$\{\int_E f_n\} \to \int_E f$$

But the sequence of real numbers $\{\int_E f_n\}$ is bounded, its limit is finite and so $\int_E f < \infty$. Thus f is finite almost everywhere on E.

Proposition 14: Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a set of measure zero, then

$$\int_E f = \int_{E\sim E_0} f.$$

That is called excision formula. excision formula.

Proof: Additivity over domains of integration follows from linearity as it did for bounded functions on sets of finite measure.

The excision formula follows from additivity over domains and the observations that by Theorem 12, the integral of a nonnegative function over a set of measure zero is zero.

Theorem 15: Let f be integrable over E. Then f is finite almost everywhere on E and

$$\int_E f = \int_{E\sim E_0} f \; .$$

If $E_0 \subset E$ and $m(E) = 0$

Proof:

Theorem 13 tells that $|f|$ is finite almost everywhere on E.Thus f is finite almost everywhere on E. Further by excision formula, we get the result.

Definition 16:

A family Ƒ of measurable functions on E is said to be uniformly integrable over Eprovided for each $\in > 0$ there is a δ>0 such that for each$f \in$ Ƒ, ifA $\subseteq E$ is measurable and $m(A) < \delta$, then $\int_E |f| < \in$.

Definition 17:

A family Ƒ of measurable functions on E is said to be tight over Eprovided for each $\in > 0$ there is a subset of finite measure $E_0 \subset E$ such that

$$\int_{E \sim E_0} |f| < \in \qquad \text{for each } f \in Ƒ$$

Proposition 18:

Let E be set of finite measure and $\delta > 0$ then Eis the disjoint union of a finite collection of sets, each of which has measure less thanδ.

Proof: By the continuity of measure, we have

$$\lim_{n\to\infty} m(E\sim[-n,n]) = 0.$$

Choose a natural number n_0 for which $\mathrm{m}(E\sim[-n_0,n_0]) < \delta$.

By choosing a finite enough partition of $[-n_0,n_0]$ so we can express $\mathrm{E} \cap [-n_0,n_0]$ as the disjoint union of a finite collection of sets each of which has measure less than δ.

Proposition 19:

Let f be a measurable function on E. If f is integrable over E, then for each $\in > 0$ there is a δ>0 for which

$$\text{if } \mathrm{A} \subseteq E \text{ is measurable and } \mathrm{m}(A) < \delta, \text{then} \int_E |f| < \in.$$

Conversely,

in the case $\mathrm{m}(E) < \infty$ if for each $\in > 0$ there is a δ>0 $\int_E |f| < \in$ such that $\mathrm{A} \subseteq E$ is measurable and $\mathrm{m}(A) < \infty$, then f is integrable over E.

Proof:

It suffices to prove it for positive part of the function f. Hence assume that $\mathrm{f} > 0$ First assume that f is integrable over E. Let $\in > 0$. By the definition of nonnegative integrable function, there is a bounded function $f_\in$ such that closure of $\{x \in E \mid f_\in(x) \neq 0\}$ is finite and for which

$$0 \leq f_\in \leq f \text{ on E}$$

And

$$0 \leq \int_E f - \int_E f_\in < \frac{\in}{2}$$

Since $f - f_\in \geq 0$ on E.

If $A \subseteq E$ is measurable then by linearity and additivity of the integral

$$\int_A f - \int_A f_\in = \int_A [f - f_\in] \leq \int_E [f - f_\in]$$
$$= \int_E f - \int_E f_\in < \frac{\in}{2}.$$

But $f_\in$ is bounded. Choose $M > 0$ for which $M > f_\in \geq 0$ on E.

Therefore if $A \subseteq E$ is measurable, then

$$\int_A f < \int_A f_\in + \frac{\in}{2} \leq M.\mathrm{m}(A) + \frac{\in}{2}.$$

For $\delta = \frac{\in}{2M}$ we have done. Now for converse part suppose $\mathrm{m}(E) < \infty$ if for each $\in > 0$ there is a $\delta>0$ for which $\int_E |f| < \in$ such that $A \subseteq E$ is measurable and $\mathrm{m}(A) < \infty$.

Let $\delta_0 > 0$ respond to the $\in = 1$. Since $\mathrm{m}(E) < \infty$, then by Proposition 18 we can express E as the disjoint union of finite collection of measurable subsets $\{E_k\}_{k=1}^N$ each of which has measure less than δ, therefore we have,

$$\sum_{k=1}^{N} \int_{E_k} f < N$$

It follows that if h is a nonnegative measurable function for which closure of $\{x \in E \mid f(x) \neq 0\}$ and $0 \leq h \leq f$ on E, then $\int_E h < N$.

This implies that f is integrable.

Example 20: Let g be nonnegative integrable function. Define

$$\mathcal{F} = \{f \mid f is measurable on E and |f| < g on E\}.$$

Then the family $\mathcal{F}$ is uniformly integrable.

This follows from Proposition 19 with f replaced by g and by the observation that for any measurable subset A of E, by the monotonicity of integration. If f is in to $\mathcal{F}$, then

$$\int_A |f| \leq \int_A g$$

Proposition 21:

$\{f_k\}_{k=1}^{n}$ be a finite collection of functions, each of which is integrable over E.

Then $\{f_k\}_{k=1}^{n}$ is uniformly integrable.

Proof:

Let $\in > 0$. For $1 \leq k \leq n$ by Proposition 19, there is a $\delta_k > 0$ for which if $A \subseteq E$ is measurable and $m(A) < \delta_k$ then $\int_A |f_k| \leq \in$.

Now define $\delta = \min_{1 \leq k \leq n}\{\delta_k\}$ then this is the required δ for given $\in > 0$ regarding the criterion $\{f_k\}_{k=1}^{n}$ to be uniformly integrable.

Note that The Vitali Convergence theorem tells us that if sequence of measurable functions is uniformly integrable and tight over and converges pointwise almost everywhere on E to f then f is integrable over E.

The following Propositions are the consequences of The Vitali Convergence theorem that is useful to give another necessary and sufficient condition for the equivalence of pointwise convergence and $L^p(E)$ norm convergence for sequence of functions in $L^p(E)$.

Proposition 22: Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise almost everywhere on E to $h = 0$.

Then

$$\lim_{n\to\infty}\int_E h_n = 0$$

if and only if

$\{h_n\}$ is uniformly integrable over E.

Proof:

Assume that $\{h_n\}$ is uniformly integrable over E, then by

Vitali convergence theorem

suppose that

$$\lim_{n\to\infty}\int_E h_n = 0$$

and $\in > 0$.

We may have a natural number N for which if $n > N$ then

$\int_E h_n < \in$.

Therefore, since $A \subseteq E$ is measurable and $n > N$,

then $\int_A h_n < \in$.

According to Proposition 18 and 19, the finite collection $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over E.

Let for given $\in > 0$ there is $\delta > 0$ regarding uniform integrability of family $\{h_n\}_{n=1}^{N-1}$.

We infer that same $\delta > 0$ serves for the criterion for uniform integrability of $\{h_n\}$.

Proposition 23:

Let $\{h\_n\}$ be a sequence of nonnegative integrable functions on E. Suppose $\{h\_n(x)\} \rightarrow 0$ for almost all $x \in E$. Then

$$\lim_{n\to\infty} \int_E h_n = 0$$

if and only if $\{h_n\}$ is uniformly integrable and tight over E

Proof:

If $\{h_n\}$ is uniformly integrable and tight over E, then $\lim_{n\to\infty} \int_E h_n = 0$

By the Vitali Convergence Theorem. Conversely, suppose $\lim_{n\to\infty} \int_E h_n = 0$. Pick a natural number N such that

$$\int_E h_n < \epsilon$$

for all $n \geq N$. As we will see in Problem 1 that finite collection of functions $\{h_n\}_{n=1}^{N}$ is tight. We can therefore find a set of finite measure E_0 a subset of E such that $\int_{E\sim E_0} |h_n| < \epsilon$ for all $n < N$.

Since

$$\int_{E\sim E_0} |h_n| < \int_E |h_n| < \epsilon$$

for all $n \geq N$.

Therefore we conclude that $\{h_n\}$is tight over E. For the proof of uniform integrability of $\{h_n\}$is same as in proof of Proposition 22.

We note that Theorem 9 gives a necessary and sufficient condition for the convergence in $L^p(E)$norm for a sequence that converges pointwise. Now we are in position to give another necessary and sufficient condition for the convergence in $L^p(E)$ norm for a sequence that converges pointwise.

Theorem 24: Let E be a measurable set and $1 \leq p < \infty$ and suppose $\{f_n\}$ be a sequence in $L^p(E)$that converges pointwise almost everywhere on E to the function fin $L^p(E)$.

Then $\{f_n\} \to f$ in $L^p(E)$ if and only if $\{|f_n|^p\}$ uniformly integrable and tight over E.

Proof:

The sequence of nonnegative integrable functions$\{|f_n - f|^p\}$ converges pointwise almost everywhere on E to zero function. According to Proposition 23, we have

$$\lim_{n\to\infty} \int_E |f_n - f|^p = 0$$

if and only if $\{|f_n - f|^p\}$is uniformly integrable and tight over E.

However, we infer from the following that for all natural number n,

$|f_n - f|^p \leq 2^p\{|f_n|^p + |f|^p\}$

and $\{|f_n|^p \leq 2^p\{|f_n - f|^p + |f|^p\}$almost everywhere on E. By assumption $|f|^p$ is integrable over E and therefore$\{|f_n - f|^p\}$ is uniformly integrable and tight over E if and only if the sequence $\{|f_n|^p\}$ is uniformly integrable and tight over E.

10.6 SOLVED PROBLEMS

Problem 1:

Let $\{f_k\}_{k=1}^n$be a finite family of functions each of which is uniformly integrable and tight overE.

Show that $\{f_k\}_{k=1}^n$is uniformly integrable and tight over E.

Solutions:

For eachf_k, by Theorem 15 we find a set of finite measureE_k of E that satisfies $\int_{E\sim E_0}|f_k|^p < \infty$. Define $E_0 = \bigcup_{k=1}^n E_k$ Then

$$\{m(E\}_0) = \text{m}\left(\bigcup_{k=1}^{n}\{E_k) = \sum_{k=1}^{n} m(E_k) < \infty\right\}$$

And hence we have $\int_{E\sim E_0}|f_k|^p < \int_{E\sim E_k}|f_k|^p <$ for each k.

This implies that $\{f_k\}_{k=1}^{n}$ is tight.

Further $\{f_k\}_{k=1}^{n}$ is uniformly integrable over E by Proposition 21.

Problem 2:

Let E be a set of measure zero. Show that if f is bounded function on E and $\int_E f = 0$.

Solution:

Since any set contained in a set of measure zero is of zero measure, therefore for any real number $\mathrm{c} > 0$ the set $\{\mathrm{x} \in \mathrm{E} \mid \mathrm{f(x)} \leq c\} \subset \mathrm{E}$ is measurable. Therefore f is measurable.

Since f is bounded so there is a $\mathrm{M} > 0$ such that $|f(x)| \leq M$, so we have

$$\int_E |f| \leq \int_E M \leq \mathrm{M}\,.\,\mathrm{m}(E) < \mathrm{M}.\,0 = 0$$

Problem 3:

Let E be a set of measure zero and define $\mathrm{f} \equiv 0$ on E. Show that $\int_E f = 0$.

Solution:

Let h be a bounded, measurable, non-negative function on E Then $0 < \mathrm{h} < \mathrm{f}$ on E and by Problem 2.Therefore $\int_E f = 0$.

Problem 4:

Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.

Solution:

Let $f_n = \sum_{k=1}^{n} a_k . \chi_{[k,k+1)}$.

Then $\{f_n\}$ is a sequence of simple measurable functions on E that converges to f.

Therefore by Lebesgue monotone convergence theorem we have

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_E \sum_{k=1}^{n} a_k . \chi_{[k,k+1)} = \sum_{n=1}^{\infty} a_n.$$

Problem 5:

Let f be integrable over E and g be a bounded measurable function on E. Show that f. g is integrable over E.

Solution:

Given that g is a bounded measurable function on E, so there is a $M > 0$ such that $|f(x)| \leq M$, so we have

$$\int_E |f.g| \leq \int_E |f|M \leq M . \int_E |f| < \infty.$$

Therefore f. g is integrable over E.

Problem 6:

Provide an example of a Cauchy sequence of real numbers that is not rapidly Cauchy.

Solution: Define $a_n = \begin{cases} 0\,, & if\ n = 1 \\ \sum_{i=1}^{n-1} \frac{1}{i^2}\,, & n = 2,3,\ldots \end{cases}$

Then

$$\lim_{n\to\infty} \mathrm{a_n} = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

Therefore $\{a_n\}$ is convergent and hence is Cauchy.

We note that .

If possible, let $\{a_n\}$ be rapidly Cauchy, then there is $\in_k$ for $k > 1$ such that $|\ a_{n+1} - a_n| < \in_k^2$ and

$$\sum_{k=1}^{\infty} \in_k^2 < \infty.$$

But $|\ a_{n+1} - a_n| = \frac{1}{n^2} < \in_k^2.$

This implies that $\frac{1}{n} < \in_k$ which in turn implies that

$$\sum_{k=1}^{\infty} \frac{1}{n^2} < \sum_{k=1}^{\infty} \in_k^2 < \infty$$

which is a contradiction. Hence $\{a_n\}$ is not rapidly Cauchy.

CHECK YOUR PROGRESS

Question 1: Prove that for $1 \leq p \leq \infty, l^p$ is a Banach space.

Question 2: Let the sequence of functions $\{f_n\}$ and $\{g_n\}$be uniformly integrable over E. Show that for any α and β, the sequence of linear combination $\{\alpha f_n + \beta g_n\}$ also is be uniformly integrable over E.

Question 3: Give an example of a sequence of function in $L^p(E)$ that is convergent in $L^p(E)$but not pointwise convergent to a function in $L^p(E)$.

Question 4: Give examples of a Cauchy sequence and a rapidly Cauchy sequence of real numbers.

10.7 SUMMARY

This unit provides an explanation of

- **i.** Completeness of $L^p(E)$.
- **ii.** Cauchy sequence and Rapidly Cauchy sequence
- **iii.** Describe the results regarding uniform integrable function and tight function.
- **iv.** Proof of Riesz-Fischer Theorem**.**
- **v.** Necessary and sufficient condition for pointwise convergence and norm convergence in $L^p(E)$.
- **vi.** Relation between convergence in $L^p(E)$, and pointwise convergence of sequence of integrable functions.

vii. After reading solved examples learners should be able to try the problems

10.8 GLOSSARY

i. Complete space.

ii. The space $L^p(E)$,

iii. Cauchy sequence

iv. Rapidly Cauchy sequence

v. Uniform integrable function and tight function.

vi. Riesz-Fischer Theorem**.**

vii. Norm convergence.

viii. Pointwise convergence.

10.9 REFRENCES

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3. G. D. Barra, Measure Theory and Integration (2023), New Age International.

10.10 SUGGESTED READINGS

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10.11 TERMINAL QUESTIONS

Question 1: Is a Cauchy sequence a rapidly Cauchy sequence?

Question 2: Has a Cauchy sequence a rapidly Cauchy sequence?

Question 3: Is a rapidly Cauchy sequence a Cauchy sequence?

Question 4: Are pointwise convergence and norm convergence equivalent in $L^p(E)$, in general?

Question 5: Is $L^p(E)$, complete?

Question 6: Is it true that if $\{f_n\}$ be a Cauchy sequence in $L^p(E)$, then is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that is rapidly Cauchy and $\{f_{n_k}\}$ converges to a function f in $L^p(E)$, both with respect to the $L^p(E)$, norm and pointwise almost everywhere on E?

Question 7: If $\{f_n\}$ converges pointwise to a function f in $L^p(E)$,then $\{f_n\} \to f$ in $L^p(E)$,if and only if $\{|f_n|^p\}$ uniformly integrable and tight over E.(True/False)

10.12 ANSWERS

CHECK YOUR PROGRESS

Answers 1: Use properties of convergent series.

Answers 2: Use definitions

Answers 3: $f_i^k(x) = \begin{cases} 1, & if \ \frac{i-1}{k} \leq x \ \frac{i}{k} \\ 0, & otherwise \end{cases}$

Defined in the [0,1). The sequence $\{f_i^k\}$ converges to zero function in $L^p([0,1))$ for $1 \leq p < \infty$ but not convergent to zero function pointwise.

Answers 4: Refer to problem 6.

TERMINAL QUESTIONS

Answers 1: No

Answers 2: Yes

Answers 3: Yes

Answers 4: No

Answers 5: Yes

Answers 6: Yes

Answers 7: Yes

Unit 11:
THE WEIERSTRASS APPROXIMATION THEOREM

CONTENTS:

11.1 INTRODUCTION

In this unit we focus on the famous theorem of Weierstrass which tell us about approximation of continuous real-valued functions defined on closed intervals by polynomials. This theorem important as it has a important consequence in analysis as well as generalized form discovered by mathematician Stone. The latter theorem is far reaching extension of Weierstrass Approximation Theoremand known as Stone Weierstrass Approximation Theorem.

11.2 OBJECTIVES

After the completion of this unit learners will be able to

i. State and prove Weierstrass Approximation Theorem

ii. State and prove Stone Weierstrass Approximation Theorem.

iii. Understand the concept of separability, separation of points, Algebra and function spaces of continuous functions.

iv. State necessary and sufficient condition for the denseness of an algebra in a $C(X)$.

v. Explain examples and counterexamples.

11.3 PRELIMINARIES

In this section, we collect some results and definitionsthat will be used in this unit.

Definition 1: A Hausdorff space is a topological space X in which for any two distinct points x and y of X there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Definition 2: A topological space$X$ is said to be first countable provided there is a countable base at each point. The space X is said to be second countable provided there is a countable base for the topology.

Urysohn's Lemma: Let A and B be disjoint closed subsets of normal topological space X. Then for any closed bounded interval$[a, b]$, there is a continuous real-valued function $f$defined on X that takes values in$[a, b]$,while$f = a$ on A and $f = b$ on B.

Definition 3: A topological space$X$ is said to be compact provided every open cover of X has a finite subcover.

Definition 4: A normal space is a topological space X in which for any two disjoint closed subsets C and D of X there exist disjoint open sets U and V such that $C \subset U$ and $D \subset V$.

Definition 5:A topological space$X$ is said to be separable provided X has a countable dense subset.

A topological space is said to be metrizable provided the topology is induced by a metric. Not every topology is induced by a metric. Since we know that a metric space is normal, so certainly the trivial topology on a set with more than one point is not metrizable. It is natural to ask if it is possible to identify those topological spaces that are metrizable. In the case the topological space$X$ is second countable, there is the following pretty well known necessary and sufficient criterion for metrizability.

Urysohn's Metrization Theorem: Let$X$ be a second countable topological space. Then X is metrizable if and only if it is normal.

Definition6: If n is a positive integer and k an integer such that $0 \leq k \leq n$ then the binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

.

Definition7: The polynomial B_n – one for each n – defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

are called the Bernstein polynomial associated with f.

11.4 THE WEIERSTRASS APPROXIMATION THEOREM

The Weierstrass Approximation Theorem

The following theorem is one of the jewels of classical analysis.

Let us consider a closed interval $[a, b]$ on the real line and a polynomial

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

with coefficients in reals and defined on $[a, b]$. We will use the fact that such polynomial is a continuous real valued function and limit of any uniformly convergent sequence of such polynomial is also a continuous real function.

Further we know that being a closed and bounded subset of set of reals,$[a, b]$ is compact and since set of real is Hausdorff, therefore $[a, b]$ is compact and Hausdorff space.

In general, for a compact Hausdorff space$X$,consider the linear space$C(X)$of continuous real valued functions on$X$with the maximum norm that is defined as follows

$$\|f\|_{max} = max\{|f(x)| \mid x \in X\}$$

Note that f attains its maximum value on X as X is compact.

It is important to note that$C(X)$has a product structure not possessed by all linear spaces, namely, the product$f.g$of two functions$f$ and $g$in$C(X)$is again in$C(X)$. A linear subspace$\boldsymbol{A}$ of $C(X)$is called an algebra provided the product of any two functions in$\boldsymbol{A}$also belongs to$\boldsymbol{A}$.A collection $\boldsymbol{A}$ of real -valued functions on X is said to separate points in X provided for any two distinct points x and y in X, there is an$f$ in $\boldsymbol{A}$ for which $f(x) \neq g(y)$ we observe that since X is compact and Hausdorff, it is normal, so we infer from Urysohn's lemma that whole algebra $C(X)$ separates the points in X.

Now we are set to prove famous theorem due to Weierstrass which explores a very interesting topological structure of space of continuous functions defined on $[a, b]$.

Several proofs this classic theorem are known, and the one we present here is perhaps as concise and elementary as most.

Theorem (The Weierstrass Approximation Theorem):

Let f be a continuous real-valued function on a closed bounded interval $[a, b]$.Then for each$\epsilon > 0$ there is polynomial p with real coefficients for which$|f(x) - p(x)| < \epsilon$for all x in $[a, b]$.In other words, the collection of polynomials is dense in $C[a, b]$.

Proof: First, we prove that it suffices to prove the theorem for the particular case $a = 0$ and $b = 1$, the conclusion follows at onceon taking p to be constant polynomial defined by $p(x) = f(a)$. Without loss of generality we assume that $a < b$. Now we observe that $x = [b - a]x' + a$ gives a continuous mapping of [0,1]onto $[a, b]$ so that the function g defined as $g(x') = f([b - a]x' + a)$ is acontinuous real function defined on $[0,1]$. If we prove theorem for the case $= 0$ and $b = 1$, then there exists a polynomialp' defined [0,1] on such that $\left|g(x') - p'^{(x')}\right| < \epsilon$ for all x' in $[a, b]$.Therefore,we have $\left|f(x) - p'\left(\frac{x-a}{b-a}\right)\right| < \epsilon$ for all x in $[a, b]$; and define a polynomial$p$ by $p(x) = p'\left(\frac{x-a}{b-a}\right)$ in order to prove theorem in the general case. Accordingly, we may assume $a = 0$ and $b = 1$. Now we prove the theorem by finding a Bernstein polynomial with required property.

We know that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = [x + (1 - x)]^n = 1 \dots \dots . . (1).$$

Now if we differentiate (1) with respect to x, we get

$$\sum_{k=0}^{n} \binom{n}{k} [kx^{k-1}(1 - x)^{n-k} - (n - k)x^k(1 - x)^{n-k-1}]$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k-1}(1 - x)^{n-k-1}(k - nx) = 0$$

and multiplying through by $x(1-x)$we get

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx)=0\ldots\ldots..(2)$$

On differentiating (2) with respect to x and considering $x^k(1-x)^{n-k}$ as one of the two factors in applying the product rule, we get

$$\sum_{k=0}^{n}\binom{n}{k}[-nx^k(1-x)^{n-k}+x^{k-1}(1-x)^{n-k-1}(k-nx)^2]=0\ldots\ldots(3)$$

Applying (1) to (3) we get

$$\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx)^2=n$$

And on multiplying this through by $x(1-x)$ we get

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx)^2=nx(1-x)$$

And on dividing both side byn^2, we get

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left(x-\frac{k}{n}\right)^2=\frac{x(1-x)}{n}\ldots\ldots.(4)$$

Identities (1) and (4) will be our main tools to show that$B_n(x)$is uniformly close to $f(x)$ for all sufficiently large n. Now by using (1) we have

$$f(x)-B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left[f(x)-f\left(\frac{k}{n}\right)\right]$$

So that

$$|f(x) - B_n(x)| \leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left|f(x) - f\left(\frac{k}{n}\right)\right| \ldots (5)$$

Being continuous on [0,1], f is uniformly continuous on [0,1]. So we can find a $\delta > 0$such that$\left|x - \frac{k}{n}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2}$.

We now split the sum on the right of (5) into two parts, denoted by **Σ**and **Σ**', where **Σ** is the sum of those terms for which $\left|x - \frac{k}{n}\right| < \delta$and where **Σ**' is the sum of remaining terms. It is easy to see that $\Sigma < \frac{\epsilon}{2}$. Now we complete the proof by showing that if n is taken sufficiently large, then **Σ**' can be made less than $\frac{\epsilon}{2}$ independently of x. Since$f$ is bounded, there exists a positive real number Ksuch that $|f(x)| \leq K$ for all x in [0,1].

From this it follows that

$$\Sigma' \leq 2\mathrm{K}\,.\,\Sigma \binom{n}{k} x^k (1-x)^{n-k}$$

Where the sum on the right side -denote it by$\Sigma'' -$ is taken over all $k$such that $\left|x - \frac{k}{n}\right| \geq \delta$.It now suffices that if $n$is taken sufficiently large, thenΣ''can be made less than$\frac{\epsilon}{4K}$independently of$x$.Therefore identity (4) shows that

$$\delta^2 \Sigma'' \leq \frac{x(1-x)}{n}$$

and so

$$\Sigma'' \leq \frac{x(1-x)}{n\delta^2}$$

Sincethe maximal value of $x(1-x)$ on [0,1]is $\frac{1}{4}$ so we have

$$\Sigma'' \leq \frac{1}{4n\delta^2}.$$

If we take any integer n greater than $\frac{K}{\epsilon\delta^2}$.,then

$\Sigma'' < \frac{\epsilon}{4K}$ and $\Sigma' < \frac{\epsilon}{2}$, $|f(x) - B_n(x)| < \epsilon$ for all x in $[0,1]$ so theorem is proved.

11.5 STONE-WEIERSTRASS APPROXIMATION THEOREM

The Stone-Weierstrass Approximation Theorem

Before going to prove the theorem, a few words concerning the strategy are in order.

Suppose X is compact and Hausdorff, it is normal. We infer from Urysohn's lemma that for each pair of disjoint closed subsets A and B of X and for $0 < \epsilon < \frac{1}{2}$, there is a function f in $C(X)$ for which

$f = \frac{\epsilon}{2}$ on A, $f = 1 - \frac{\epsilon}{2}$ on B and $\frac{\epsilon}{2} \leq f \leq 1 - \frac{\epsilon}{2}$ on X.

Therefore, if $|h - f| < \frac{\epsilon}{2}$ on , we have

$h < \epsilon$ on A, $h > 1 - \epsilon$ on B and $0 < h < 1$ on X ... ... (1).

Lemma: Let X be compact Hausdorff space and $\boldsymbol{A}$ an algebra of continuous functions on X that separates the points and contains the constant functions. Then for each closed subset F of X and point x_0 belonging to $X \sim F$, there is a neighborhood U of x_0 that is disjoint from F and has the following property: for each $\epsilon > 0$, there is a function h in $\boldsymbol{A}$ for which

$h < \epsilon$ on U, $h > 1 - \epsilon$ on F and $0 \leq h \leq 1$ on X ... ... (2)

Proof: We claim that for each point y in F, there is function g_y in $\boldsymbol{A}$ for which

$g_y(x_0) = 0$, $g_y(y) > 0$ and $0 \leq g_y \leq 1$ on X ....(3)

Since $\boldsymbol{A}$ separates points, there is a function f in $\boldsymbol{A}$ for which $f(x_0) \neq f(y)$. The function

$$g_y = \left[\frac{f - f(x_0)}{||f - f(x_0)||_{max}}\right]^2$$

belongs to $\boldsymbol{A}$and satisfies (3). Since g_y is continuous, there is a neighborhood N_y of y on which g_y takes only positive values. However, F is a closed subset of compact space X and henceF is compact.Therefore, we may have a finite collection of these neighborhoods$\{N_{y_1}, \ldots . N_{y_n}\}$ that covers F.

Define the function g in $\boldsymbol{A}$by

$$g = \frac{1}{n}\sum_{i=1}^{n} g_{y_i}.$$

Then

$g(x_0) = 0$, $g > 0$ on $F$and $0 \leq g \leq 1$ on X……..(4)

Since g is continuous on X so it attains a minimum value$c > 0$ on X

such that $g \geq c$ on F. By possibly multiplying it by a positive number, we may suppose $c < 1$. On the other hand, g is continuous on x_0, so there is a neighborhoodof x_0 for which $g < \frac{c}{2}$ on U. Thus g belongs to the algebra $\boldsymbol{A}$and

$g < \frac{c}{2}$ on U and $g \geq c$ on F, and $0 \leq g \leq 1$ on X…….(5)

Now we claim that (2) holds for this choice of neighborhood$U$. Let $\epsilon > 0$. By Weierstrass Approximation Theorem, we can find a polynomial p such that

$p < \epsilon$ on $\left[0, \frac{c}{2}\right]$, $p > 1 - \epsilon$ on$[c, 1]$, and $0 \leq p \leq 1$ on$[0,1]$…...(6)

Since p is a polynomial and f belongs to the algebra $\boldsymbol{A}$, the composition $h = pog$ also belongs to $\boldsymbol{A}$. By (5) and (6) we conclude that

$h < \epsilon$ on U, $h > 1 - \epsilon$ on F and $0 \leq h \leq 1$ on X.

Lemma: Let X be compact Hausdorff space and $\boldsymbol{A}$ an algebra of continuous functions on X that separates the points and contains the constant functions. Then for each closed subsets A and B of X and $\epsilon > 0$, there is a function h in $\boldsymbol{A}$ for which

$h < \epsilon$ on A, $h > 1 - \epsilon$ on B and $0 \leq h \leq 1$ on X.

Proof: By the preceding lemma, in the case $F = B$, for each point x in A, there is a neighborhood N_x of x that is disjoint from B and has the property (2). Being a closed subset of the compact space X, A is compact, and hence there is a finite collection of neighborhoods $\{N_{x_1}, \dots. N_{x_n}\}$ that covers A. Now we choose ϵ_0 for which $0 < \epsilon_0 < \epsilon$ and $\left(\frac{1-\epsilon_0}{n}\right)^n > 1 - \epsilon$. For $1 \leq i \leq n$, since N_{x_i} has the property (2) with $B = F$, we choose $h_i \in \boldsymbol{A}$ such that

$h_i < \frac{\epsilon_0}{n}$ on N_{x_i}, $h_i > 1 - \epsilon_0$ on B, and $0 \leq h_i \leq 1$.

Now we define $h = h_1.h_2 \dots h_n$ on X. Then h belongs to the algebra $\boldsymbol{A}$ as $\boldsymbol{A}$ is closed underproduct. Now since for each , $0 \leq h \leq 1$ on X. Also for each $i, h_i > \frac{1-\epsilon_0}{n}$ on B, so $h > \left(\frac{1-\epsilon_0}{n}\right)^n > 1 - \epsilon$ on B. Finally, for each point in $\boldsymbol{A}$, there is an index i for which x belongs to N_{x_i}. Therefore $h_i(x) \leq \frac{\epsilon_0}{n} < \epsilon$ and since for the other indices $j, 0 \leq h_j(x) \leq 1$. Hence, we conclude that $h(x) < \epsilon$.

Theorem (The Stone Weierstrass Approximation Theorem): Let X be compact Hausdorff space and $\boldsymbol{A}$ an algebra of continuous functions on X that separates the points and contains the constant functions. Then $\boldsymbol{A}$ is dense in $C(X)$.

Proof: Let f belongs to $C(X)$. Define $c = \|f\|_{max}$. If we can arbitrarily closely uniformly approximate the function $\frac{f+c}{\|f+c\|_{max}}$ by the functions in $\boldsymbol{A}$, we can do the same for f. Therefore we may assume that, $0 \leq f \leq 1$ on X. Suppose $n > 1$ is a natural number. Now we consider the uniform partition $\{0, \frac{1}{n}, \frac{2}{n}, \dots. \frac{n-1}{n}, 1\}$ of $[0,1]$ into n intervals. Fix $j, 1 \leq j \leq n$. Define

$A_j = \{x \in X \mid f(x) \leq \frac{j-1}{n}\}$ and $\boldsymbol{B}_j = \{x \in X \mid f(x) \geq \frac{j}{n}\}$.

Since f is continuous, A_j and B_j are closed and disjoint subsets. By the preceding lemma, with A= A_j, B= B_j and $\epsilon > \frac{1}{n}$, there is a function g_j in the algebra $\boldsymbol{A}$ for which

$g_j(x) < \frac{1}{n}$ if $f(x) \leq \frac{j-1}{n}$, $g_j(x) > 1 - \frac{1}{n}$ if $f(x) \geq \frac{j}{n}$ and

$$0 \leq g_j \leq 1 \text{ on } X \ldots\ldots(1)$$

Define

$$g = \frac{1}{n}\sum_{j=1}^{n} g_j$$

Then g belongs to $\boldsymbol{A}$. We claim that

$$\|f - g\|_{max} < \frac{3}{n} \ldots \ldots (2)$$

Once this claim established, proof of the theorem is complete, since given $\epsilon > 0$, we can select a natural number n such that $\frac{3}{n} < \epsilon$ and therefore $\|f - g\|_{max} < \epsilon$. Now to verify (2) we first show that

if $0 \leq k \leq 1$ and $f(x) \leq \frac{k}{n}$, then $g(x) \leq \frac{k}{n} + \frac{1}{n}$……..(3)

Indeed for $j = k+1, \ldots n$, since $f(x) \leq \frac{k}{n}, f(x) \leq \frac{j-1}{n}$ and therefore $g_j(x) \leq \frac{1}{n}$. Thus

$$\frac{1}{n}\sum_{j=k+1}^{n} g_j \leq \frac{(n-k)}{n^2} \leq \frac{1}{n}.$$

Consequently, since each $g_j(x) \leq 1$ for all j,

$$g(x) = \frac{1}{n}\sum_{j=1}^{n} g_j = \frac{1}{n}\sum_{j=k+1}^{k} g_j + \frac{1}{n}\sum_{j=k+1}^{n} g_j \leq \frac{1}{n}\sum_{j=1}^{k} g_j + \frac{1}{n} \leq \frac{k}{n} + \frac{1}{n}$$

Thus (3) holds. A similar argument shows that

if $1 \leq k \leq n$ and $\frac{(k-1)}{n} \leq f(x)$, then $\frac{(k-1)}{n} - \frac{1}{n} \leq g(x) \ldots \ldots \ldots \ldots (4)$

For x in X, choose k, $1 \leq k \leq n$, such that $\frac{(k-1)}{n} \leq f(x) \leq \frac{k}{n}$. From (3) and (4) we have

$$|f(x) - g(x)| < \frac{3}{n}.$$

Now we conclude this unit with the following consequence of Stone-Weierstrass Approximation Theorem.

Theorem (Riesz's Theorem): Let X be compact Hausdorff space then $C(X)$is separable if and only if$X$is metrizable(that is there is a metric that induces the topology of$X$).

Proof: First suppose that $X$is metrizable and therefore μ be a metric that induces topology on X. Then X, being a compact metric space, is separable. Let a countable dense subset$\{x_n\}$ of X. For each natural number n define $f_n(x) = \mu(x, x_n)$ for all x in X. Since μ be a metric that induces topology on X, and f_n is continuous. We infer from denseness of $\{x_n\}$ that the family $\{f_n\}$ separates points in X. Define $f_0 \equiv 1$ on X. Now let P be the collection of polynomials, with real coefficients, in a finite number of f_k, where $0 \leq f_k < \infty$. Then P is an algebra that contains the constant function and separates points in X as it contains each f_k. Now we infer from Stone Weierstrass Theorem, $P$is dense in $C(X)$.But the collection of functions $f$in P that arepolynomials with rational coefficients is a countable set that is dense in P.Hence $C(X)$ is separable.

For the converse part, suppose $C(X)$ is separable. Let $\{g_n\}$ be a countable dense subset of $C(X)$. For each natural number n define

$$O_n = \{x \in X \mid g_n(x) > \frac{1}{2}\}.$$

Then$\{O_n\}_{1 \leq n < \infty}$is a countable collection off open sets. We claim that every open set is the union of subcollection of $\{O_n\}_{1 \leq n < \infty}$,and therefore X is second countable. But X, being compact and Hausdorff, is normal. The Urysohn's Metrization Theorem tells us that $X$is metrizable.

To verify second countability, let the point x belongs to the open set O. Since X is normal, there is an open for which $x \in U$and U is contained in closure of O. By the Urysohn's Lemma, there is a g in $C(X)$such that $g(x) = 1$ on U and $g \equiv 0$ on $X \sim O$. By the denseness of $\{g_n\}$ in $C(X)$, there is a natural number n for which $|g - g_n| < \frac{1}{2}$ on X. Therefore$x \in O_n \in O$. This completes the proof.

11.6 SOLVED PROBLEMS

Problem 1: Prove that collection of polynomials with rational coefficients is countable.

Solution: Let P be the collection of polynomials with rational coefficients and $P_n$be the collection of polynomials of degree n with rational coefficients. Then there is one-one and onto map from P_n to the cartesian product of n copies of set of rationals that isQ_n. Therefore each P_n is countable and so

$$P = \bigcup_{n=1}^{\infty} P_n$$

is also countable as countable union of countable set is countable.

Problem 2: Let X be compact subset of real numbers, then show that $C(X)$ is separable.

Solution: Let P be the collection of polynomials with rational coefficients, then P separates the points in X and contains constant functions (constant polynomials) then by Stone-Weierstrass Theorem P is dense in $C(X)$, further P is countable by Problem 1, therefore P is countable and dense in $C(X)$. Hence $C(X)$ is separable.

Problem 3: Let ***A***be the vector space generated by the functions

1, $\sin x$, $sin^2 x$,…,$sin^n x$,….

defined on $\left[0, \frac{\pi}{2}\right]$.Then show that***A*** is an algebra and dense in$C\left[0, \frac{\pi}{2}\right]$.

Solution: Since ***A*** is algebra as product of two functions in ***A***is in ***A*** and also $\sin x$ separates the points in $\left[0, \frac{\pi}{2}\right]$, likewise ***A*** separates the points, therefore by Stone-Weierstrass Theorem, ***A*** isdense in $C\left[0, \frac{\pi}{2}\right]$.

Problem 4: Show that the algebra generated by the set $\{1, x^2\}$ is dense in $C[0,1]$ but fails to be dense in $C[-1,1]$.

Solution: Since ***A*** is algebra as product of two functions in ***A***is in and also x^2separates the points in $[0,1]$, likewise separates the points, and contains the constant functions, therefore by Stone-Weierstrass Theorem, ***A*** dense in $C[0,1]$. But for any f in the uniform closure of***A***we have $f(-1) = f(1)$, therefore***A***is not dense in $C[-1,1]$.

Problem 5: Let X be compact Hausdorff space and ***A***is an algebra then uniform closure of ***A***is also an algebra.

Solution: By the inequality

$$\|f_n g_n - fg\|_{max} \leq \|f\|_{max}\|g_n - g\|_{max} + \|g\|_{max}\|f_n - f\|_{max}$$

We conclude that $f_n g_n \to fg$ uniformly, so uniform closure of ***A***is also an algebra.

Problem 6: Let $\boldsymbol{A}$ be the algebra generated by the functions $1, x, x^2$ defined on $[0,1]$. Then show that $\boldsymbol{A}$ is dense in $C[0,1]$.

Solution: $\boldsymbol{A}$ separates the points and contains constant functions, therefore by Stone-Weierstrass Theorem, $\boldsymbol{A}$ is dense in $C[0,1]$.

CHECK YOUR PROGRESS

Question 1: For a compact metric space X, the space $C(X)$ is separable metric space. True/false

Question 2: $C[-10,-5]$ is separable. True/False

Question 3: Suppose $\boldsymbol{A}$ is an algebra generated by the functions $\sin x, sin^2 x, \ldots, sin^n x, \ldots$ defined on $\left[0, \frac{\pi}{2}\right]$. Then $\boldsymbol{A}$ is dense in $C\left[0, \frac{\pi}{2}\right]$. True/False

Question 4: Weierstrass Approximation Theorem is a consequence of Stone Weierstrass Approximation Theorem. True/False

Question 5: $C(X)$ is separable if and only if X is ………..

Question 6: A space is said to be separable if it has a ……..

Question 7: Suppose $\boldsymbol{A}$ is an algebra generated by the functions $1, \cos x, cos^2 x, \ldots, cos^n x, \ldots$ defined on $[0,1]$. Then $\boldsymbol{A}$ is …….in $C[0,1]$.

Question 8: Let X be compact Hausdorff space and $\boldsymbol{A}$ an algebra of continuous functions on X. Then necessary condition for $\boldsymbol{A}$ to be dense in $C(X)$ is $\boldsymbol{A}$…………

Question 9: The collection of all polynomials is ……….

Question 10: Let X be compact Hausdorff space and f be a continuous function then f attains it …… and …. values.

Question 11: Let $\boldsymbol{A}$ be the algebra generated by the functions $2, x, x^2$ defined on $[0,1]$. Then show that $\boldsymbol{A}$ is ………in $C[0,1]$.

Question 12: Let $\boldsymbol{A}$ be the algebra generated by the functions x, x^2 defined on $[0,1]$. Then show that $\boldsymbol{A}$ is ………in $C[0,1]$.

11.7 SUMMARY

This unit provides an explanation of

i. Weierstrass Approximation Theorem

ii. Stone Weierstrass Approximation Theorem

iii. Interesting problems regarding algebra generated by different functions.

11.8GLOSSARY

i. Compact Hausdorff space.
ii. Normal space
iii. Separable space
iv. Dense subset
v. Metrizable space
vi. $C[a,b]$, $C(X)$
vii. Uniform convergence and uniform closure.
viii. Bernstein polynomials.

11.9 REFRENCES

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11.10 SUGGESTED READINGS

1. P. R. Halmos, Measure theory (1950), Van Nostrand, Princeton, New Jersey.
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11.11 TERMINAL QUESTIONS

Question 1: Show that $C[a, b]$ is separable.

Question 2: Show that $C[a, b]$ separates the points in $[a, b]$.

Question 3: Suppose***A*** is an algebra generated by the functions

1, e^x, ,e^{2x},e^{3x},. ..….e^{nx} defined on $[-1,1]$. Since e^xseparates the points in $[-1,1]$, likewise ***A*** separates the points, therefore by Stone-Weierstrass Theorem, ***A*** is dense in $C[-1,1]$.

Question 4: Prove that $C(X)$ is an algebra.

Question 5: If a topological space X is equipped with indiscrete topology, then show that $C(X)$ does not separate the point in X

11.12 ANSWERS

CHECK YOUR PROGRESS

Answers 1: True

Answers 2: True

Answers 3: False

Answers 4: True

Answers 5: Metrizable

Answers 6: Countable dense subset

Answers 7: Dense

Answers 8: Separates the points and contains the constant functions

Answers 9: Countable

Answers 10: Maximum, minimum

Answers 11:dense

Answers 12:not dense

TERMINAL QUESTIONS

Answers 1: By Weierstrass approximation Theorem$C[a,b]$ is separable.

Answers 2: Since$[a,b]$ is normal topological space so by Urysohn's Lemma, $C[a,b]$ separates the points in $[a,b]$.

Answers 3: Suppose$\boldsymbol{A}$ is an algebra generated by the functions

1, e^x, ,e^{2x}, e^{3x}, ……. e^{nx} defined on $[-1,1]$. Then show that $\boldsymbol{A}$is dense in $C[-1,1]$.

Answers 4: By the definition $C(X)$ and the fact that product of two continuous functions is again a function so $C(X)$is an algebra.

Answers 5: If a topological space X is equipped with indiscrete topology, then show that $C(X)$contains only constant functions so $C(X)$ does not separate the point in X.

BLOCK IV:
SIGNED MEASURE
AND
PRODUCT MEASURE

UNIT 12:
SIGNED MEASURES

CONTENTS:

12.1 INTRODUCTION

A signed measure is defined as an extended real-valued countably additive set function on a σ-algebra and then shown to be the difference of two positive measures. Further, the Radon-Nikodym derivative of a signed measure with respect to a positive measure is defined as a function which we integrate with respect to the latter to obtain the former. The existence of the Radon-Nikodym derivative is then proved under the assumption that the former is absolutely continuous with respect to the latter and that both are σ -finite.

 

Johann Radon (16 December 1887 – 25 May 1956)

Otton Marcin Nikodym (13 August 1887 – 4 May 1974)

Ref: https://mathshistory.standrews.ac.uk

Fig 1.1

12.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of Signed measure.

ii. Obtain Hahn Decomposition.

iii. Obtain Jordan Decomposition

iv. Obtain Radon-Nikodym Theorem

v. Obtain Lebesgue Decomposition Theorem

12.3 Signed Measure

A function μ that associate an extended real number to certain sets is referred to as a set function.

Definition Let X be a set and $\mathcal{M}$ be σ- algebra of subsets of X. The pair $(X,\mathcal{M})$ is called measurable space. A set $E \subset X$ is a member of $\mathcal{M}$ then it is called measurable.

Definition An extended real-valued nonnegative set function $\mu: \mathcal{M} \to [0, \infty]$ for which $\mu(\phi) = 0$ is called measure if it is countably additive in the sense that for any countable disjoint collection $\{E_k\}$ of measurable sets,

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

A measurable space $(X,\mathscr{M})$ with a measure μ defined on $\mathcal{M}$ is referred to as a measure space $(X,\mathcal{M}, \mu)$.

Now we look at certain examples of measure spaces.

Example-1.

$(\mathbb{R}, \mathscr{L}, m)$ is a measure space, where $\mathbb{R}$ is real numbers set, $\mathscr{L}$ is the collection of real number sets that are Lebesgue measurable, and m is the Lebesgue measure.

Example-2.

$(\mathbb{R}, \mathscr{B}, m)$ is a second example of a measure space, in which m is Lebesgue measure and $\mathscr{B}$ is the set of Borel sets of real numbers.

Example-3.

For any given set one may construct a counting measure. Let X be a set and $\mathcal{M}$ be collection of subsets of X that is $\mathscr{M} = 2^X$. A set function $\mu: \mathscr{M} \to [0, \infty]$ such that

$$\mu(E) = \begin{cases} \#(E), & E \text{ is finite} \\ \infty, & otherwise \end{cases}$$

is called counting measure and $(X, \mathscr{M}, \mu)$ is called counting measure space.

Example-4.

For any given σ- algebra one may construct a Dirac measure. Let $\mathcal{M}$ be a σ- algebra of subsets of X. A set function $\delta_a : \mathscr{M} \to [0, \infty]$ such that

$$\delta_a(E) = \begin{cases} 1, & a \in E \\ 0, & otherwise \end{cases}$$

is called Dirac measure and $(X, \mathscr{M}, \delta_a)$ is called Dirac measure space.

Example-5.

Let X be an uncountable set and $\mathcal{C} = \{\text{E} \subset \text{X: either E is countable or its complement is countable}\}$. Verify that it is a a σ- algebra. A set function $\mu : \mathscr{M} \to [0, \infty]$ such that

$$\mu(E) = \begin{cases} 0, & E \text{ is countable} \\ 1, & E^c \text{ is countable} \end{cases}$$

is a measure and $(X, \mathcal{C}, \mu)$ is called measure space.

SIGNED MEASURE: THE JORDAN AND HAHN DECOMPOSITIONS

Let $(X, \mathscr{M}, \mu_1)$ and $(X, \mathscr{M}, \mu_2)$ be two measure spaces. Let $\mu = c_1\mu_1 + c_2\mu_2$ be a linear combination of μ_1 and μ_2 with $c_1, c_2 \in \mathbb{R}$. It is interesting to observe that μ is qualified as a measure if c_1 and c_2 are positive. In order to deal with other possibilities of coefficients, we consider a particular case $\mu = \mu_1 - \mu_2$ with $c_1 = 1$ and $c_2 = -1$. Easy to see that in this case μ can not able to maintain non negativity. Secondly $\mu(E) = \mu_1(E) - \mu_2(E)$ is not even defined when both $\mu_1(E) = \infty = \mu_2(E)$. We now provide the following definition keeping above points in mind.

Definition:

Let $(X, \mathcal{M})$ be a measurable space. An extended real valued set function $\nu: \mathcal{M} \to [-\infty, \infty]$ with the following characteristics:

i. At most, one of the values out of $-\infty$ and ∞ are assumed by ν.

ii. $\nu(\phi)$ is zero.

iii. For any countable disjoint collection $\{E_k\}$ of measurable sets,

$$\nu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \nu(E_k)$$

Where the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely if $\nu(\bigcup_{k=1}^{\infty} E_k) < \infty$.

A signed measure is generalization of a measure.

It is easy to understand that a signed measure is the difference of two measures, one of which is finite. The Jordan Decomposition Theorem will reveal in due course that each signed measure is actually the difference between two of such measures.

A measurable set A is called positive with respect to ν if $\nu(E) \geq 0$ for every measurable subset E of A.

Similarly, A measurable set B is called negative with respect to ν if $\nu(E) \leq 0$ for every measurable subset E of B.

A measurable set C is called null with respect to ν if $\nu(E) = 0$ for every measurable subset E of C.

The difference between a set of measure zero and a null set should be carefully noted.

A set of measure zero can be the union of two sets whose measures are not zero but are the opposite of each other, even though every null set must have a measure zero.

A set is null with regard to a measure if and only if it has measure zero. For a signed measure we do not have monotonicity results. However we do have following monotonicity-like result.

Exercise-1.
Let $A \subset B$. If $|\nu(B)| < \infty$ then $|\nu(A)| < \infty$.

Proposition-1.

Let $(X, \mathcal{M}, \nu)$ a signed measure space. Each measurable subset of a positive set is positive. The union of a countable collection of positive sets is also positive.

Proof.

By definition of positive set, each measurable subset of a positive set is positive. Let $A = \bigcup_{k=1}^{\infty} A_k$ with $\nu(A_k) \geq 0$. Let E be a measurable subset of A. Define

$$E_1 = E \cap A_1$$

$$E_k = (E \cap A_k) - (A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{k-1}) \qquad \forall\, k \geq 2\,.$$

By construction each E_k is subset of positive set A_k so $\nu(E_k) \geq 0$. Moreover, collection $\{E_k\}$ is disjoint collection. Then

$$\nu(E) = \nu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0.$$

Hence, A is a positive set.

12.4 HAHN'S LEMMA

Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there exists a positive set set $A \subset E$ with positive measure.

Proof.

If E is a positive set, then we are done. If E is not positive, then E is not positive, then it has a subset of negative measure. Let m_1 be the smallest natural number for which there is a measurable set out of which there is a measurable set of measure less than $-\frac{1}{m_1}$. Let $E_1 \subset E$ with $\nu(E_1) < \frac{-1}{m_1}$. Inductively define natural numbers $m_1, m_2, m_3, \dots, m_n$ and $E_1, E_2, E_3, \dots, E_n$ such that for $1 \leq k \leq n$, m_k is the smallest natural number for which there is a measurable subset of $E - \overset{k-1}{\underset{r=1}{\uplus}} E_r$ of measure less than $\frac{-1}{m_k}$ and E_k is a subset of $E - \overset{k-1}{\underset{r=1}{\uplus}} E_r$ for which $\nu(E_k) < \frac{-1}{m_k}$. If the process terminates at some point $n \in \mathbb{N}$, then set $A = E - \overset{n}{\underset{r=1}{\uplus}} E_r$ is a positive subset of E. If the process does not terminates, define $A = E - \overset{\infty}{\underset{r=1}{\uplus}} E_r$. Then $E = A \cup (\overset{n}{\underset{r=1}{\uplus}} E_r)$. Since $\overset{\infty}{\underset{r=1}{\uplus}} E_r \in \mathcal{M}$ and $\overset{\infty}{\underset{r=1}{\uplus}} E_r \subset E$, then

$$-\infty < \nu\left(\biguplus_{r=1}^{\infty} E_r\right) = \sum_{r=1}^{\infty} \nu(E_r) \leq \sum_{r=1}^{\infty} \frac{-1}{m_k}.$$

So $m_k \to \infty$. Now we show A is positive. Let $B \subset A$ be measurable. Then $B \subset A \subset E - \left(\overset{k-1}{\underset{r=1}{\uplus}} E_r\right)$ for each $k \in \mathbb{N}$. Since m_k is the smallest natural number such that there is a measurable subset of $E - \left(\overset{k-1}{\underset{r=1}{\uplus}} E_r\right)$ of measure less than $-\frac{1}{m_k}$ so $-\frac{1}{m_k - 1} < \nu(E_k) \leq -\frac{1}{m_k}$, then it must be that $\nu(B) > \frac{-1}{m_k - 1}$. Since this holds for all $k \in \mathbb{N}$ and $m_k \to \infty$, then $\nu(B) \geq 0$. So A is a positive set. Moreover, $E = A \uplus (\overset{\infty}{\underset{r=1}{\cup}} E_r)$ so $\nu(E) = \nu(A) + \nu\left(\overset{\infty}{\underset{r=1}{\cup}} E_r\right) > 0$ and since $\nu\left(\overset{\infty}{\underset{r=1}{\cup}} E_r\right) < 0$ so $\nu(A) > 0$.

12.5 HAHN DECOMPOSITION THEOREM

The Hahn Decomposition Theorem. Let ν be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of X.

Proof. Without loss of generality, suppose $+\infty$ is the infinite value omitted by ν (otherwise, replace ν with $-\nu$ and follow this proof). Let $\mathcal{P}$ be the collection of positive subsets of X and define
$\lambda = \sup\{\nu(E) \mid E \in \mathcal{P}\}$. Then $\lambda \geq 0$ since $\varnothing \in \mathcal{P}$. Let $\{A_k\}_{k=1}^{\infty}$ be a

sequence of positive sets such that $\lambda = \lim_{k\to\infty} \nu(A_k)$ (which exists by the definition of supremum). Define $A = \cup_{k=1}^{\infty} A_k$. By Proposition 1, set A is a positive set, and so $\lambda \geq \nu(A)$ (by the definition of supremum). Also, for each $k \in \mathbb{N}$, $A \setminus A_n \subset A$ and so $\nu(A \setminus A_k) \geq 0$ since A is positive. Thus $\nu(A) = \nu(A_k) + \nu(A \setminus A_k) \geq \nu(A_k)$. Hence $\nu(A) \geq \lambda$. Therefore $\nu(A) = \lambda$, and $\lambda < \infty$ since λ does not take on the value $+\infty$.

Let $B = X \setminus A$. ASSUME B is not negative.

Then there is a subset E of B with positive measure. So by Hahn's Lemma there is $E_0 \subset B$ such that E_0 is positive and $\nu(E_0) > 0$. But then $A \cup E_0$ is a positive set by Proposition 1 and by additivity,

$\nu(A \cup E_0) = \nu(A) + \nu(E_0) > \lambda$, a CONTRADICTION to the definition of λ (notice that $\lambda < \infty$ is needed here). So the assumption that B is not negative is false and hence B is a negative set. Therefore $\{A, B\}$ is a Hahn decomposition of X.

12.6 JORDAN DECOMPOSITION THEOREM

Theorem *If μ is a signed measure on a measurable space $(X, \mathcal{A})$, there exist positive measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$ and μ^+ and μ^- are mutually singular. This decomposition is unique.*

Proof. Let E and F be negative and positive sets, resp., for μ so that $X = E \cup F$ and $E \cap F = \emptyset$. Let $\mu^+(A) = \mu(A \cap F)$, $\mu^-(A) = -\mu(A \cap E)$. This gives the desired decomposition.

If $\mu = \nu^+ - \nu^-$ is another such decomposition with ν^+, ν^- mutually singular, let E' be a set such that $\nu^+(E') = 0$ and

$\nu^-((E')^c) = 0$. Set $F' = (E')^c$. Hence $X = E' \cup F'$ and $E' \cap F' = \emptyset$. If $A \subset F'$, then $\nu^-(A) \leq \nu^-(F') = 0$, and so

$$\mu(A) = \nu^+(A) - \nu^-(A) = \nu^+(A) \geq 0,$$

and consequently F' is a positive set for μ. Similarly, E' is a negative set for μ. Thus E', F' gives another Hahn decomposition of X. By the uniqueness part of the Hahn decomposition theorem, $F \Delta F'$ is a null set with respect to μ. Since $\nu^+(E') = 0$ and $\nu^-(F') = 0$, if $A \in \mathcal{A}$, then

$$\begin{aligned}\nu^+(A) &= \nu^+(A \cap F') = \nu^+(A \cap F') - \nu^-(A \cap F') \\ &= \mu(A \cap F') = \mu(A \cap F) = \mu^+(A),\end{aligned}$$

and similarly $\nu^- = \mu^-$. □

The measure

$$|\mu| = \mu^+ + \mu^- \tag{1}$$

is called the *total variation measure* of μ.

12.7 THE RADON – NIKODYM THEOREM

This theorem is a result in measure theory that expresses the relationship between two measures defined on the same measurable space.

Suppose f is non-negative and integrable with respect to μ. If we define ν by

$$\nu(A) = \int_A f \, d\mu,$$

then ν is a measure. The only part that needs thought is the countable additivity. If A_n are disjoint measurable sets, we have

$$\nu(\cup_n A_n) = \int_{\cup_n A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu = \sum_{n=1}^{\infty} \nu(A_n)$$

Moreover, $\nu(A)$ is zero whenever $\mu(A)$ is zero.

In this **unit** we consider the converse. If we are given two measures μ and ν, when does there exist f such that

$$\nu(A) = \int_A f \, d\mu,$$

holds? The Radon-Nikodym theorem answers this question.

Definition A measure ν is said to be *absolutely continuous* with respect to a measure μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. We write $\nu \ll \mu$.

Proposition *Let ν be a finite measure. Then ν is absolutely continuous with respect to μ if and only if for all ε there exists δ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$.*

Proof. Suppose for each ε, there exists δ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$. If $\mu(A) = 0$, then $\nu(A) < \varepsilon$ for all ε, hence $\nu(A) = 0$, and thus $\nu \ll \mu$.

Suppose now that $\nu \ll \mu$. If there exists an ε for which no corresponding δ exists, then there exists E_k such that $\mu(E_k) < 2^{-k}$ but $\nu(E_k) \geq \varepsilon$. Let $F = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$. Then

$$\mu(F) = \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} E_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} 2^{-k} = 0,$$

but

$$\nu(F) = \lim_{n\to\infty} \nu(\cup_{k=n}^{\infty} E_k) \geq \varepsilon;$$

This contradicts the absolute continuity. □

Lemma *Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{A})$. Either $\mu \perp \nu$ or else there exists $\varepsilon > 0$ and $G \in \mathcal{A}$ such that $\mu(G) > 0$ and G is a positive set for $\nu - \varepsilon\mu$.*

Proof. Consider the Hahn decomposition for $\nu - \frac{1}{n}\mu$. Thus there exists a negative set E_n and a positive set F_n for this measure,

$$\nu(E) \leq \nu(E_n) \leq \tfrac{1}{n}\mu(E_n) \leq \tfrac{1}{n}\mu(X).$$

Since ν is a positive measure, this implies $\nu(E) = 0$.

E_n and F_n are disjoint, and their union is X. Let $F = \cup_n F_n$ and $E = \cap_n E_n$. Note $E^c = \cup_n E_n^c = \cup_n F_n = F$.

For each n, $E \subset E_n$, so

$$\nu(E) \leq \nu(E_n) \leq \tfrac{1}{n}\mu(E_n) \leq \tfrac{1}{n}\mu(X).$$

Since ν is a positive measure, this implies $\nu(E) = 0$.

One possibility is that $\mu(E^c) = 0$, in which case $\mu \perp \nu$. The other possibility is that $\mu(E^c) > 0$. In this case, $\mu(F_n) > 0$ for some n. Let $\varepsilon = 1/n$ and $G = F_n$. Then from the definition of F_n, G is a positive set for $\nu - \varepsilon\mu$. □

We now are ready for the *Radon-Nikodym theorem.*

Theorem *Suppose μ is a σ-finite positive measure on a measurable space $(X, \mathcal{A})$ and ν is a finite positive measure on $(X, \mathcal{A})$ such that ν is absolutely continuous with respect to μ. Then there exists a μ-integrable non-negative function f which is measurable with respect to $\mathcal{A}$ such that*

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$. *Moreover, if* g *is another such function, then* $f = g$ *almost everywhere with respect to* μ.

The function f is called the *Radon-Nikodym derivative* of ν with respect to μ or sometimes the *density* of ν with respect to μ, and is written $f = d\nu/d\mu$. Sometimes one writes

$$d\nu = f \, d\mu.$$

The idea of the proof is to look at the set of f such that

$\int_A f \, d\mu \le \nu(A)$ for each $A \in \mathcal{A}$, and then to choose the one such that $\int_X f \, d\mu$ is largest.

Proof. *Step 1.* Let us first prove the uniqueness assertion. For every set A we have

$$\int_A (f - g) \, d\mu = \nu(A) - \nu(A) = 0$$

then we have $f - g = 0$ a.e. with respect to μ.

Step 2. Let us assume μ is a finite measure for now. In this step we define the function f. Define

$$\mathcal{F} = \left\{ g \text{ measurable} : g \ge 0, \int_A g \, d\mu \le \nu(A) \text{ for all } A \in \mathcal{A} \right\}.$$

$\mathcal{F}$ is not empty because $0 \in \mathcal{F}$. Let $L = \sup\{\int g \, d\mu : g \in \mathcal{F}\}$, and let g_n be a sequence in $\mathcal{F}$ such that $\int g_n \, d\mu \to L$. Let $h_n = \max(g_1, \ldots, g_n)$.

We claim that if g_1 and g_2 are in $\mathcal{F}$, then $h_2 = \max(g_1, g_2)$ is also in $\mathcal{F}$. To see this, let $B = \{x : g_1(x) \ge g_2(x)\}$, and write

$$\begin{aligned}
\int_A h_2 \, d\mu &= \int_{A\cap B} h_2 \, d\mu + \int_{A\cap B^c} h_2 \, d\mu \\
&= \int_{A\cap B} g_1 \, d\mu + \int_{A\cap B^c} g_2 \, d\mu \\
&\le \nu(A \cap B) + \nu(A \cap B^c) \\
&= \nu(A).
\end{aligned}$$

Therefore $h_2 \in \mathcal{F}$. By an induction argument, h_n is in $\mathcal{F}$.
The h_n increase, say to f. By monotone convergence, $\int f \, d\mu = L$
and

$$\int_A f \, d\mu \leq \nu(A)$$

for all A.

Step 3. Next we prove that f is the desired function. Define a measure λ by

$$\lambda(A) = \nu(A) - \int_A f \, d\mu.$$

λ is a positive measure since $f \in \mathcal{F}$.

Suppose λ is not mutually singular to μ. By Lemma , there exists $\varepsilon > 0$ and G such that G is measurable, $\mu(G) > 0$, and G is a positive set for $\lambda - \varepsilon\mu$. For any $A \in \mathcal{A}$,

$$\nu(A) - \int_A f \, d\mu = \lambda(A) \geq \lambda(A \cap G) \geq \varepsilon\mu(A \cap G) = \int_A \varepsilon\chi_G \, d\mu,$$

$$\nu(A) \geq \int_A (f + \varepsilon\chi_G) \, d\mu.$$

Hence $f + \varepsilon\chi_G \in \mathcal{F}$. But

$$\int_X (f + \varepsilon\chi_G) \, d\mu = L + \varepsilon\mu(G) > L,$$

a contradiction to the definition of L.

Therefore $\lambda \perp \mu$. Then there must exist $H \in \mathcal{A}$ such that $\mu(H) = 0$ and $\lambda(H^c) = 0$. Since $\nu \ll \mu$, then $\nu(H) = 0$, and hence

$$\lambda(H) = \nu(H) - \int_H f \, d\mu = 0.$$

This implies $\lambda = 0$, or $\nu(A) = \int_A f \, d\mu$ for all A.

Step 4. We now suppose μ is σ-finite. There exist $F_i \uparrow X$ such that $\mu(F_i) < \infty$ for each i. Let μ_i be the restriction of μ to F_i, that is, $\mu_i(A) = \mu(A \cap F_i)$. Define ν_i, the restriction of ν to F_i, similarly. If $\mu_i(A) = 0$, then $\mu(A \cap F_i) = 0$, hence $\nu(A \cap F_i) = 0$, and thus $\nu_i(A) = 0$. Therefore $\nu_i \ll \mu_i$. If f_i is the function such that $d\nu_i = f_i \, d\mu_i$, the argument of Step 1 shows that $f_i = f_j$ on F_i

if $i \leq j$. Define f by $f(x) = f_i(x)$ if $x \in F_i$. Then for each $A \in \mathcal{A}$,

$$\nu(A \cap F_i) = \nu_i(A) = \int_A f_i \, d\mu_i = \int_{A \cap F_i} f \, d\mu.$$

Letting $i \to \infty$ shows that f is the desired function. □

We now are ready for the *Lebesgue decomposition theorem.*

12.8 LEBESGUE DECOMPOSITION THEOREM

Theorem *Suppose μ and ν are two finite positive measures. There exist positive measures λ, ρ such that $\nu = \lambda + \rho$, ρ is absolutely continuous with respect to μ, and λ and μ are mutually singular.*

Proof. Define $\mathcal{F}$ and L and construct f as in the proof of the Radon-Nikodym theorem. Let $\rho(A) = \int_A f \, d\mu$ and let $\lambda = \nu - \rho$. Our construction shows that

$$\int_A f \, d\mu \leq \nu(A),$$

so $\lambda(A) \geq 0$ for all A. We have $\rho + \lambda = \nu$. We need to show μ and λ are mutually singular.

If not, by Lemma , there exists $\varepsilon > 0$ and $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \varepsilon\mu$. We get a contradiction exactly as in the proof of the Radon-Nikodym theorem. We conclude that $\lambda \perp \mu$. □

12.9 SUMMARY

This unit provides an explanation of

i. Concept of Signed measure.
ii. Hahn Decomposition Theorem.
iii. Jordan Decomposition Theorem.
iv. Radon-Nikodym Theorem
v. Lebesgue Decomposition Theorem

12.10 GLOSSARY

i. Complete space.
ii. The space $L^p(E)$,
iii. Cauchy sequence
iv. Norm convergence.
v. Pointwise convergence.
vi. σ- algebra.
vii. Measurable space
viii. Measure μ
ix. Measure space $(X,\mathcal{M},\mu)$.

CHECK YOUR PROGRESS

Fill in the Blanks:

1. Let X be a set and $\mathcal{M}$ be **…………**of subsets of X. The pair $(X,\mathcal{M})$ is called measurable space.
2. The Hahn decomposition theorem is based on the notions of **………**and **………..** measurable sets.
3. The Jordan decomposition states the existence of two nonnegative measures and which are mutually **………………………………….**

4. The Radon – Nikodym Theorem expresses the relationship between two **……..**defined on the **…………………..**
5. Lebesgue's decomposition theorem states that for every two **……..** on a measurable space there exist two σ-finite signed measures .

The following statements is true or false

6. Hahn decomposition is unique. True \False
7. Signed measure is a generalization of the concept of (positive) measure by allowing the set function to take negative values, i.e., to acquire sign. True\False
8. The difference of two measures, one of which only assumes finite values, is a signed measure. True\False.
9. The Radon–Nikodym theorem essentially states that, under certain conditions, any measure can be expressed in this way with respect to another measure on the same space. True\False
10. Lebesgue decomposition not gives a very explicit description of measures. True\False

12.11 REFRENCES

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3. G. D. Barra, Measure Theory and Integration (2023), New Age International.

12.12SUGGESTED READINGS

1. P. R. Halmos, Measure theory (1950), Van Nostrand, Princeton, New Jersey.
2. W. Rudin, (1966), Real and Complex analysis, McGraw Hill.

12.13 TERMINAL QUESTIONS

1. What is the signed measure of the Jordan Decomposition Theorem?...
....
2. What is the proof of Jordan decomposition?..........................
…………………………………………………………………………
3. State and proof of Radon – Nikodym Theorem. …………………...
…………………………………………………………………………
……………………………………………………………………
4. Defined the Signed measure. ……………………………
…………………………………………………………………………
5. State and proof Lebesgue Decomposition Theorem ………………
…………………………………………………………………………
6. ➨ Show that if $\phi(E) = \int_E f\, d\mu$ where $\int f\, d\mu$ is defined, then ϕ is a signed measure.
7. ➨ Give an example showing that a Hahn decomposition is not unique.
8. ➨ Show that if $\nu(E) = \int_E f\, d\mu$ for each $E \in \mathcal{S}$, where f is non-negative and measurable, and $f = \infty$ on a set of positive μ-measure, then ν is not σ-finite.
9. ➨ Show that the condition: μ σ-finite, is necessary in the Radon-Nikodym Theorem.

➡ Show that if μ and ν are measures such that $\nu \ll \mu$ and $\nu \perp \mu$, then ν is identically zero.

10.

12.14ANSWERS

CHECK YOUR PROGRESS

1 σ- algebra.

2 positive, negative.

3 singular.

4 Measures, same measurable space.

5 σ-finite signed measures.

6 False

7 True

8 True

9 True

10 False

UNIT 13:
PRODUCT MEASURE

CONTENTS:

13.1 INTRODUCTION

Lebesgue measure has been defined on the line. We now present a technique for building measures in many product spaces, including n-dimensional Euclidean spaces and the plane. One of the most significant theorems in measure theory is the Fubini’s theorem, which permits one to vary the order of integration.

Guido Fubini **(19 January 1879 – 6 June 1943)** **Ref**: https://mathshistory.st-andrews.ac.uk/Biographies/Fubini/pictdisplay/ ***Fig 1.1***	

13.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of Product Measure.

ii. Evaluate the iterated integrals.

iii. Understand role of Fubini's theoreom.

iv. Follow Tonelli 's theoreom.

13.3 PRODUCT MEASURE

Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be two complete measure spaces, and consider the direct product $X \times Y$ of X and Y. If $A \subset X$ and $B \subset Y$, we call $A \times B$ a rectangle. If $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, we call $A \times B$ a measurable rectangle. The collection $\mathfrak{R}$ of measurable rectangles is a semialgebra, since

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

and

$$\sim(A \times B) = (\tilde{A} \times B) \cup (A \times \tilde{B}) \cup (\tilde{A} \times \tilde{B}).$$

If $A \times B$ is a measurable rectangle, we set

$$\lambda(A \times B) = \mu A \cdot \nu B.$$

Lemma: *Let $\{(A_i \times B_i)\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$. Then*

$$\lambda(A \times B) = \sum \lambda(A_i \times B_i).$$

Proof: Fix a point $x \in A$. Then for each $y \in B$, the point $\langle x, y\rangle$ belongs to exactly one rectangle $A_i \times B_i$. Thus B is the disjoint union of those B_i such that x is in the corresponding A_i. Hence

$$\sum \nu B_i \cdot \chi_{A_i}(x) = \nu B \cdot \chi_A(x),$$

since ν is countably additive. Thus by the corollary of the Monotone Convergence Theorem, we have

$$\sum \int \nu B_i \cdot \chi_{A_i} \, d\mu = \int \nu(B) \cdot \chi_A \, d\mu$$

or

$$\sum \nu B_i \cdot \mu A_i = \nu B \cdot \mu A. \quad \blacksquare$$

The lemma implies that λ has a unique extension to a measure on the algebra $\mathcal{R}'$ consisting of all finite disjoint unions of sets in $\mathcal{R}$. Carathéodory Theorem allows us to extend λ to be a complete measure on a σ-algebra $\mathcal{S}$ containing $\mathcal{R}$.

This extended measure is called the product measure of μ and ν and is denoted by $\mu \times \nu$. If μ and ν are finite (or σ-finite), so is $\mu \times \nu$. If X and Y are the real line and μ and ν are both Lebesgue measure, then $\mu \times \nu$ is called two-dimensional Lebesgue measure for the plane.

The purpose of the next few lemmas is to describe the structure of the sets which are measurable with respect to the product measure $\mu \times \nu$. If E is any subset of $X \times Y$ and x a point of X, we define the *x cross section* E_x by

$$E_x = \{ y : \langle x, y \rangle \in E \},$$

and similarly for the y cross section for y in Y. The characteristic function of E_x is related to that of E by

$$\chi_{E_x}(y) = \chi_E(x, y).$$

We also have $(\tilde{E})_x = \sim(E_x)$ and $(\bigcup E_\alpha)_x = \bigcup (E_\alpha)_x$ for any collection $\{E_\alpha\}$.

Lemma: *Let x be a point of X and E a set in $\mathcal{R}_{\sigma\delta}$. Then E_x is a measurable subset of Y.*

Proof: The lemma is trivially true if E is in the class $\mathcal{R}$ of measurable rectangles. We next show it to be true for E in $\mathcal{R}_\sigma$. Let $E = \bigcup_{i=1}^{\infty} E_i$, where each E_i is a measurable rectangle. Then

$$\begin{aligned}\chi_{E_x}(y) &= \chi_E(x, y)\\ &= \sup_i \chi_{E_i}(x, y)\\ &= \sup_i \chi_{(E_i)_x}(y).\end{aligned}$$

Since each E_i is a measurable rectangle, $\chi_{(E_i)_x}(y)$ is a measurable function of y, and so χ_{E_x} must also be measurable, whence E_x is measurable.

Suppose now that $E = \bigcap_{i=1}^{\infty} E_i$ with $E_i \ \varepsilon \ \mathcal{R}_\sigma$. Then

$$\begin{aligned}\chi_{E_x} &= \chi_E(x, y)\\ &= \inf_i \chi_{E_i}(x, y)\end{aligned}$$

$$= \inf_i \chi_{(E_i)_x}(y),$$

and we see that χ_{E_x} is measurable. Thus E_x is measurable for any $E \in \mathcal{R}_{\sigma\delta}$. ∎

Lemma: *Let E be a set in $\mathcal{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$. Then the function g defined by*

$$g(x) = \nu E_x$$

is a measurable function of x and

$$\int g \, d\mu = \mu \times \nu(E).$$

Proof: The lemma is trivially true if E is a measurable rectangle. We first note that any set in $\mathcal{R}_\sigma$ is a disjoint union of measurable rectangles. Let $\langle E_i \rangle$ be a disjoint sequence of measurable rectangles, and let $E = \bigcup E_i$. Set

$$g_i(x) = \nu[(E_i)_x].$$

Then each g_i is a nonnegative measurable function, and

$$g = \sum g_i .$$

Thus g is measurable, and by the corollary of the Monotone Convergence Theorem , we have

$$\int g \, d\mu = \sum \int g_i \, d\mu$$

$$= \sum \mu \times \nu(E_i)$$

$$= \mu \times \nu(E).$$

Consequently, the lemma holds for $E \in \mathcal{R}_\sigma$.

Let E be a set of finite measure in $\mathcal{R}_{\sigma\delta}$. Then there is a sequence $\langle E_i \rangle$ of sets in $\mathcal{R}_\sigma$ such that $E_{i+1} \subset E_i$ and $E = \bigcap E_i$. It follows that $\mu \times \nu(E_1) < \infty$. Let $g_i(x) = \nu[(E_i)_x]$.

Since

$$\int g_1 \, d\mu = \mu \times \nu(E_1) < \infty,$$

we have $g_1(x) < \infty$ for almost all x. For an x with $g_1(x) < \infty$, we have $\langle (E_i)_x \rangle$ a decreasing sequence of measurable sets of finite measure whose intersection is E_x.

Thus we have

$$g(x) = \nu(E_x) = \lim \nu[(E_i)_x]$$

$$= \lim g_i(x).$$

Hence

$$g_i \to g \text{ a.e.,}$$

and so g is measurable. Since $0 \le g_i \le g_1$, the Lebesgue Convergence Theorem implies that

$$\int g\, d\mu = \lim \int g_i\, d\mu$$

$$= \lim \mu \times \nu(E_i)$$

$$= \mu \times \nu(E). \qquad \blacksquare$$

Lemma: *Let E be a set for which $\mu \times \nu(E) = 0$. Then for almost all x we have $\nu(E_x) = 0$.*

Proof: There is a set F in $\mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $\mu \times \nu(F) = 0$. It follows that for almost all x we have $\nu(F_x) = 0$.

But $E_x \subset F_x$, and so $\nu E_x = 0$ for almost all x since ν is complete. $\blacksquare$

Proposition: *Let E be a measurable subset of $X \times Y$ such that $\mu \times \nu(E)$ is finite. Then for almost all x the set E_x is a measurable subset of Y. The function g defined by*

$$g(x) = \nu(E_x)$$

is a measurable function defined for almost all x and

$$\int g\, d\mu = \mu \times \nu(E).$$

Proof: There is a set F in $\mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and

$$\mu \times \nu(F) = \mu \times \nu(E).$$

Let $G = F \sim E$. Since E and F are measurable, so is G, and

$$\mu \times \nu(F) = \mu \times \nu(E) + \mu \times \nu(G).$$

Since $\mu \times \nu(E)$ is finite and equal to $\mu \times \nu(F)$, we have $\mu \times \nu(G) = 0$.

Thus we have $\nu G_x = 0$ for almost all x. Hence

$$g(x) = \nu E_x = \nu F_x \text{ a.e.};$$

so g is a measurable function, and

$$\int g\, d\mu = \mu \times \nu(F)$$

$$= \mu \times \nu(E). \quad \blacksquare$$

13.4 FUBINI'S THEOREM

Theorem (Fubini): *Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be two complete measure spaces and f an integrable function on $X \times Y$. Then*

i. *For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is an integrable function on Y.*

i′. *For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is an integrable function on X.*

ii. $\int_Y f(x, y)\, d\nu(y)$ *is an integrable function on X.*

ii′. $\int_X f(x, y)\, d\mu(x)$ *is an integrable function on Y.*

iii. $\int_X \left[\int_Y f\,d\nu\right] d\mu = \int_{X\times Y} f\,d(\mu\times\nu) = \int_Y \left[\int_X f\,d\mu\right] d\nu.$

Proof: Because of the symmetry between x and y it suffices to prove (i), (ii), and the first half of (iii). If the conclusion of the theorem holds for each of two functions, it also holds for their difference, and hence it is sufficient to consider the case when f is nonnegative. Theorem is true if f is the characteristic function of a measurable set of finite measure, and hence the theorem must be true if f is a simple function vanishes outside a set of finite measure. Each nonnegative integrable function f is the limit of an increasing sequence $\langle\varphi_n\rangle$ of nonnegative simple functions, and, since each φ_n is integrable and simple, it must vanish outside a set of finite measure. Thus f_x is the limit of the increasing sequence $\langle(\varphi_n)_x\rangle$ and is measurable. By the Monotone Convergence Theorem

$$\int_Y f(x, y)\,d\nu(y) = \lim \int_Y \varphi_n(x, y)\,d\nu(y),$$

and so this integral is a measurable function of x. Again by the Monotone Convergence Theorem

$$\int_X \left[\int_Y f\,d\nu\right] d\mu = \lim \int_X \left[\int_Y \varphi_n\,d\nu\right] d\mu$$

$$= \lim \int_{X\times Y} \varphi_n\,d(\mu\times\nu)$$

$$= \int_{X\times Y} f\,d(\mu\times\nu). \quad \blacksquare$$

In order to apply the Fubini Theorem, one must first verify that f is integrable with respect to $\mu \times \nu$; that is, one must show that f is a measurable function on $X \times Y$ and that $\int |f| \, d(\mu \times \nu) < \infty$. The measurability of f on $X \times Y$ is sometimes difficult to establish, but in many cases we can establish it by topological considerations. In the case when μ and ν are σ-finite, the integrability of f can be determined by iterated integration using the following theorem:

13.5 TONELLI'S THEOREM

Theorem (Tonelli): *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces, and let f be a nonnegative measurable function on $X \times Y$. Then*

i. *For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y.*

i′. *For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is a measurable function on X.*

ii. $\int_Y f(x, y) \, d\nu(y)$ *is a measurable function on X.*

ii′. $\int_X f(x, y) \, d\mu(x)$ *is a measurable function on Y.*

iii. $$\int_X \left[\int_Y f \, d\nu \right] d\mu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \left[\int_X f \, d\mu \right] d\nu.$$

Proof: For a nonnegative measurable function f the only point in the proof of last Theorem where the integrability of f was used was to infer the existence of an increasing sequence $\langle \varphi_n \rangle$ of simple functions each vanishing outside a set of finite measure such that $f = \lim \varphi_n$. But if μ and ν are σ-finite, then so is $\mu \times \nu$, and any nonnegative measurable function on $X \times Y$ can be so approximated. ∎

13.6 SOLVED PROBLEMS

Example Show that if μ and ν are σ-finite measures, then $\mu \times \nu$ as

$$(\mu \times \nu)(V) = \int_X \nu(V_x)\, d\mu = \int_Y \mu(V^y)\, d\nu,$$

for each $V \in \mathcal{S} \times \mathcal{T}$, is the only measure on $\mathcal{S} \times \mathcal{T}$ giving to each measurable rectangle $A \times B$ the measure $\mu(A)\,\nu(B)$.

Solution: The required measure must have value $\sum_{i=1}^{n} \mu(A_i)\,\nu(B_i)$ on the elementary set which decomposes into disjoint measurable rectangles as $\bigcup_{i=1}^{n} (A_i \times B_i)$. Now $\mu \times \nu$ clearly takes the correct value on measurable rectangles and it is a measure on $\mathcal{E}$ so it takes the correct value on the sets of $\mathcal{E}$ and indeed is clearly a σ-finite measure on the σ-algebra $\mathcal{E}$. Hence, $\mathcal{S}(\mathcal{E}) = \mathcal{S} \times \mathcal{T}$ is unique.

Example : Let $X = Y = [0,1]$, $\mathcal{S} = \mathcal{T} = \mathcal{B}$. Take $\mu = m$ on the Borel subsets of $[0,1]$, and for ν take the counting measure on $[0,1]$, that is, $\nu(E) = \text{Card}\, E$. Take $V = [(x, y): x = y, (x, y) \in X \times Y]$. Then V is $\mathcal{S} \times \mathcal{T}$-measurable, for if n is any positive integer put $I_j = [(j-1)/n, j/n], j = 1, \ldots, n$ and $V_n = (I_1 \times I_1) \cup \ldots \cup (I_n \times I_n)$. So V_n is measurable, and so therefore is $V = \bigcap_{n=1}^{\infty} V_n$. However

$$\int_Y d\nu \int_X \chi_V\, d\mu = 0 \quad \text{but} \quad \int_X d\mu \int_Y \chi_V\, d\nu = 1.$$

Example ⁝: The condition $f \in L^1(\mu \times \nu)$ in Fubini's theorem is necessary if the order of integration is to be interchangeable.

Solution: Take X, Y, $\mathcal{S}$, $\mathcal{T}$ as in the last example and let $\mu = \nu = m$, restricted to $[0,1]$. Let $0 < \alpha_1 < \ldots < \alpha_n < \ldots < 1$, $\lim \alpha_n = 1$. For each n choose a continuous function g_n such that $[t: g_n(t) \neq 0] \subseteq (\alpha_n, \alpha_{n+1})$ and also $\int_0^1 g_n \, dt = 1$.

Let $f(x,y) = \sum_{n=1}^{\infty} g_n(y)(g_n(x) - g_{n+1}(x))$. For each (x, y) only one term in this series can be non-zero, so f is well defined. Also f is measurable, indeed f is continuous except at $(1,1)$. But

$$\int_0^1 f(x,y)\, dx = \int_0^1 \sum_{n=1}^{\infty} g_n(y)(g_n(x) - g_{n+1}(x))\, dx =$$

$$= g_n(y)\left(\int_{\alpha_n}^{\alpha_{n+1}} g_n \, dx - \int_{\alpha_{n+1}}^{\alpha_{n+2}} g_{n+1} \, dx\right) = 0$$

for each y. However

$$\int_0^1 f(x,y)\, dy = \sum_{n=1}^{\infty} (g_n(x) - g_{n+1}(x)) \int_0^1 g_n \, dy = g_1(x),$$

so $\int_0^1 dx \int_0^1 f(x,y)\, dy = 1$ and the iterated integrals are therefore unequal.

However, Fubini's theorem is not contradicted since f is not integrable. For, writing $I_i = (\alpha_i, \alpha_{i+1})$, we have

$$\int |f(x,y)|\, dx\, dy = \sum_{i,j=1} \int_{I_i \times I_j} \left|\sum_{n=1}^{\infty} g_n(y)(g_n(x) - g_{n+1}(x))\right| dx\, dy$$

$$= \sum_{i,j=1}^{\infty} \int_{I_i \times I_j} |g_j(y)(g_j(x) - g_{j+1}(x)| \, dx \, dy$$

$$= \sum_{j=1}^{\infty} \int_{I_j \times I_j} + \int_{I_{j+1} \times I_j} |g_j(y)(g_j(x) - g_{j+1}(x))| \, dx \, dy$$

$$= \infty.$$

CHECK YOUR PROGRESS

Fill in the Blanks:

1. If $A \times B$ is a measurable rectangle then $\lambda(A \times B) =$………………

2. If $\mu \times \nu(E) = 0$ then $\nu(E_x) = 0$ for …………

3. …….. theorem allows to interchange the order of integration.

4. Fubini's theorem enables to calculate integrals with respect to …………measure.

5. To apply Fubini's theorem one must verify that f is …………w.r.t. $\mu \times \nu$.

The following statements is true or false

6. If $X = Y = [0, 1]$ then each Borel set is measurable in $X \times Y$. True/False

7. In Fubini's theorem $f_x(y)$ is an integrable function on Y for almost all x. True/False

8. A function f is measurable w.r.t. $\mu \times \nu$ if f is measurable on $X \times Y$ and $\int |f| d(\mu \times \nu) = \infty$. True/False

9. The collection of measurable rectangles is a semi algebra. True/False

10. If $\mu \times \nu$ is a two dimensional Lebesgue measure on $\mathbb{R}^2$ then $d(\mu \times \nu) = dxdy$. True/False

13.7 SUMMARY

This unit is an explanation of

- **i.** Definition of Product measure.
- **ii.** Different type of iterated integrals.
- **iii.** The idea of interchanging order on integration.

13.8 GLOSSARY

- **i.** Measure
- **ii.** Product Measure
- **iii.** Iterated Integrals
- **iv.** Integrability of a function
- **v.** Fubini's theorem
- **vi.** Tonelli's theorem

13.9 REFERENCES

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3. G. D. Barra, Measure Theory and Integration (2023), New Age International.
4. https://faculty.etsu.edu/gardnerr/5210/Riemann-Lebesgue-Theorem.pdf

13.10 SUGGESTED READINGS

1. P. R. Halmos, Measure theory (1950), Van Nostrand, Princeton, New Jersey.
2. W. Rudin, (1966), Real and Complex analysis, McGraw Hill.
3. **https://archive.nptel.ac.in/courses/111/108/111108135/**

13.11 TERMINAL QUESTIONS

1. What is product measure...............................
2. Define two dimensional product measure for the plane…………….
3. If E is a subset of $X \times Y$ and $x \in X$ then define x −cross section………..
4. Let x be a point of X and E a set in $\mathcal{R}_{\sigma\delta}$. Show that E_x is a measurable subset of Y……
5. If E is a set in $\mathcal{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$. Show that $g(x) = \nu(E_x)$ is a measurable function………………………………..
6. If $\mu \times \nu(G) = 0$ for a set G. Show that $\nu(G_x) = 0$ for almost all x………………
7. State Lebesgue Convergence theorem…………………………..
8. State and prove Monotone convergence theorem………………
9. Give an example that σ −fiiniteness of μ is not necessary in Tonelli's theorem……
10. Give an example that we cannot remove hypothesis that f is integrable in Fubini's theorem……………….

13.11 ANSWERS

CHECK YOUR PROGRESS

1 $\mu(A)\nu(B)$

2 for almost all x

3 Fubini's

4 Product

5 integrable

6 True

7 True

8 False

9 True

10 True

UNIT 14:
RELATION BETWEEN RIEMANN AND LEBESGUE

CONTENTS:

14.1 INTRODUCTION

There are many kinds of integrals. The Riemann integral is historically the first one invented and is perhaps the simplest. In this unit we characterize Riemann integrable functions.

Georg Friedrich Bernhard Riemann (17 September 1826 – 20 July 1866) **Ref**: https://mathshistory.st-andrews.ac.uk/Biographies/Riemann/pictdisplay/ ***Fig 1.1***	
Henri Léon Lebesgue (28 June 1875 – 26 July 1941) **Ref**: https://mathshistory.standrews.ac.uk/Biographies/Nikodym/pictdisplay/ ***Fig 1.2***	

14.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of Riemann Integral.

ii. Define the concept of Lebesgue Integral.

iii. Evaluate the different type of Integralst.

iv. Describe the notion of measure zero set

v. See that Lebesgue Integral is more powerful than Riemann integral.

14.3 RIEMANN INTEGRAL

Definition: A *partition* $P = \{x_0, x_1, ..., x_n\}$ of the interval $[a, b]$ is a finite set of numbers $x_0, x_1, ..., x_n$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Definition: Let P and Q be partitions of $[a, b]$. We say that Q is a refinement of P if $P \subset Q$.

Definition. Let $x_{i-1}, x_i \in P$, where P is a partition of $[a, b]$. For f a bounded function on $[a, b]$, define

$$m_i(f) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and $\Delta x_i = x_i - x_{i-1}$. Let

$$\overline{S}(f; P) = \sum_{i=1}^{n} M_i(f)\Delta x_i \text{ and } \underline{S}(f; P) = \sum_{i=1}^{n} m_i(f)\Delta x_i.$$

$\overline{S}(f; P)$ and $\underline{S}(f; P)$ are the *upper Riemann sum* and *lower Riemann sum*, respectively, of f on $[a, b]$ with respect to partition P.

Definition. With the notation above, suppose $\overline{x}_i \in [x_{i-1}, x_i]$. Then

$$S(f; P) = \sum_{i=1}^{m} f(\overline{x}_i)\Delta x_i$$

is a *Riemann sum* of f on $[a, b]$ with respect to partition P.

Definition. With the notation above, define

$$\overline{S}(f) = \inf\{\overline{S}(f; P) \mid P \text{ is a partition of } [a, b]\} \text{ and}$$

$$\underline{S}(f) = \sup\{\underline{S}(f; P) \mid P \text{ is a partition of } [a, b]\}.$$

$\overline{S}(f)$ and $\underline{S}(f)$ are the *upper Riemann integral* and *lower Riemann integral*, respectively, of f on $[a, b]$.

Definition. Let f be bounded on $[a, b]$. Then f is said to be *Riemann integrable* on $[a, b]$ if $\overline{S}(f) = \underline{S}(f)$. In this case, $\overline{S}(f)$ is called the *Riemann integral of f on $[a, b]$*, denoted

$$\overline{S}(f) = \int_a^b f(x)\, dx = \int_a^b f.$$

Note. First, let's explore some conditions related to the integrability of f on $[a, b]$. Notice that these conditions are merely restatements of the definition and that the proofs follow from this definition, along with properties of suprema and infima.

14.4 CRITERION FOR RIEMANN INTEGRABILITY

Theorem Riemann Condition for Integrability.

A bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists a partition $P(\varepsilon)$ of $[a, b]$ such that

$$\overline{S}(f; P(\varepsilon)) - \underline{S}(f; P(\varepsilon)) < \varepsilon.$$

Definition. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. The *norm* (or *mesh*) of P, denoted $\|P\|$, is

$$\|P\| = \max\{x_i - x_{i-1} \mid i = 1, 2, 3, \ldots, n\}.$$

Theorem *A bounded function f is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if P is a partition with $\|P\| < \delta(\varepsilon)$ then*

$$\overline{S}(f; P) - \underline{S}(f; P) < \varepsilon.$$

Theorem.

If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Proof. Since f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$ Let $\varepsilon > 0$. Then by the uniform continuity of f, there exists $\delta(\varepsilon) > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta(\varepsilon)$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ with $\|P\| < \delta(\varepsilon)$. On $[x_{i-1}, x_i]$, f assumes a maximum and a minimum (by the Extreme Value Theorem), say at x_i' and x_i'' respectively. Thus

$$\overline{S}(f; P) - \underline{S}(f; P) = \sum_{i=1}^{n}(f(x_i') - f(x_i''))\Delta x_i < \frac{\varepsilon}{b - a}\sum_{i=1}^{n}\Delta x_i = \frac{\varepsilon}{b - a}(b - a) = \varepsilon.$$

So f is Riemann integrable on $[a, b]$. ∎

Note. We now introduce a new idea about the "weight" of a set. We will ultimately see that the previous result gives us, in some new sense, a classification of Riemann integrable functions.

14.5 LEBESGUE INTEGRAL

Definition. The (Lebesgue) *measure* of an open interval (a, b) is $b - a$. The measure of an unbounded open interval is infinite. The measure of an open interval I is denoted $m(I)$.

Definition. A set $E \subset \mathbb{R}$ has *measure zero* if for all $\varepsilon > 0$, there is a countable collection of open intervals $\{I_1, I_2, I_3, \ldots\}$ such that

$$E \subset \cup_{i=1}^{\infty} I_i \text{ and } \sum_{i=1}^{\infty} m(I_i) < \varepsilon.$$

Theorem *The union of a countable collection of sets of measure zero is a set of measure zero.*

Proof. Let $E_n \subset \mathbb{R}$ have measure zero and put $E = \cup_n E_n$. Let $\epsilon > 0$. Then for each n there exist a countable collection $\{I_{n,k}\}_{k=1}^{\infty}$ of open intervals such that $E_n \subset \cup_{k=1}^{\infty} I_{n,k}$ and $\sum_k l(I_{n,k}) < \frac{\epsilon}{2^n}$. Now $\{I_{n,k}\}_{k,n}$ is again a countable union of open intervals and $E \subset \cup_{k,n} I_{n,k}$ such that $\sum_{k,n} l(I_{n,k}) < \epsilon$. Hence E has measure zero. ☐

Lemma *Let $0 \leq f : [a, b] \to \mathbb{R}$ be a Riemann integrable function with $\int_a^b f = 0$. Then for all $c > 0$ the set $\{x \in [a, b] : f(x) \geq c\}$ has content zero.*

Proof. Let $c > 0$ and denote by E the set $\{x \in [a, b] : f(x) \geq c\}$. Let $\epsilon > 0$. Then there exists a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that $\mathcal{U}(\mathcal{P}, f) < \epsilon \cdot c$, where $\mathcal{U}(\mathcal{P}, f)$ denotes the Riemann upper sum corresponding to $\mathcal{P}$, i.e.,

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Denote $I = \{i : E \cap [x_{i-1}, x_i] \neq \emptyset\}$. If $i \in I$, then $M_i \geq c$. Hence we have

$$\epsilon \cdot c > \mathcal{U}(\mathcal{P}, f) \geq \sum_{i \in I} M_i \Delta x_i \geq c \sum_{i \in I} \Delta x_i.$$

From this it follows that $\sum_{i \in I} l([x_{i-1}, x_i]) = \sum_{i \in I} \Delta x_i < \epsilon$. Since E is covered by $\{[x_{i-1}, x_i] : i \in I\}$, it follows that E has content zero. □

We say that a property P holds almost everywhere (abbreviated by a.e.) on $[a, b]$, if the set $\{x \in [a, b] : P \text{ fails for x}\}$ has measure zero.

Corollary *Let $0 \leq f : [a, b] \to \mathbb{R}$ be a Riemann integrable function with $\int_a^b f = 0$. Then f is zero a.e. on $[a, b]$.*

Proof. The set $\{x \in [a, b] : f(x) \neq 0\} = \cup_{n=1}^{\infty} \{x \in [a, b] : f(x) \geq \frac{1}{n}\}$, which is by the above lemma a countable union of sets of content zero and has thus measure zero. □

14.6 LEBESGUE THEOREM

Theorem (Lebesgue). *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous a.e. on $[a, b]$.*

Proof. Assume first that f is Riemann integrable on $[a, b]$. Let $\{\mathcal{P}_k\}$ be a sequence of partitions of $[a, b]$ with $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ and such that the mesh $|\mathcal{P}_k| \to 0$ as $k \to \infty$. Let ϕ_k the upper function for f corresponding to $\mathcal{P}_k$ and by ψ_k the corresponding lower function. Then $\psi_k(x) \uparrow \leq f(x) \leq \phi_k(x) \downarrow$ for all $x \in [a, b)$ and $\int_a^b \psi_k \uparrow \underline{\int_a^b} f$ and $\int_a^b \phi_k \downarrow \overline{\int_a^b} f$. Let $g(x) = \lim_{k \to \infty} \psi_k(x)$ and $h(x) = \lim_{k \to \infty} \phi_k(x)$ for $x \in [a, b]$. It follows now that $\psi_k(x) \leq g(x) \leq f(x) \leq h(x) \leq \phi_k(x)$ for $x \in [a, b)$. Hence we

have

$$\int_a^b \psi_k \leq \underline{\int_a^b} g \leq \overline{\int_a^b} g \leq \int_a^b f \leq \underline{\int_a^b} h \leq \overline{\int_a^b} h \leq \int_a^b \phi_k.$$

Letting $k \to \infty$ we conclude that g and h are Riemann integrable and that $\int_a^b g = \int_a^b h = \int_a^b f$. As $h \geq g$ it follows from Corollary that $g = h$ a.e. Hence the set $E = \{x \in [a, b] : g(x) \neq h(x)\} \cup \bigcup_k \mathcal{P}_k$ has measure zero. We claim that f is continuous on $[a,b] \setminus E$. Let $x_0 \in [a,b] \setminus E$ and let $\epsilon > 0$. Then $g(x_0) = h(x_0)$ implies that there exists $k \in \mathbb{N}$ such that $\phi_k(x_0) - \psi_k(x_0) < \epsilon$. Now $\phi_k - \psi_k$ is constant in a neighborhood of x_0, since $x_0 \notin \mathcal{P}_k$. Hence there exists $\delta > 0$ such

that $\phi_k(x) - \psi_k(x) = \phi_k(x_0) - \psi_k(x_0)$ for all $|x - x_0| < \delta$. For $|x - x_0| < \delta$ we now have

$$-\epsilon < \psi_k(x_0) - \phi_k(x_0) \le f(x) - f(x_0) \le \phi_k(x_0) - \psi_k(x_0) < \epsilon,$$

which shows that f is continuous at x_0. This completes the proof that f is continuous except for a set of measure zero. Assume now that f is continuous on $[a,b] \setminus E$, where E has measure zero. Let $\epsilon > 0$ and M such that $|f(x)| \le M$ on $[a,b]$. Then $|f(x) - f(y)| \le 2M$ for all $x, y \in [a,b]$. Since E has measure zero, there exists open intervals $I_1, I_2, \ldots$ such that $E \subset \cup_n I_n$ and $\sum_n l(I_n) < \frac{\epsilon}{4M}$. For all $x \in [a,b] \setminus E$ there exists an open interval J_x with $x \in J(x)$ such that $|f(z) - f(y)| \le \frac{\epsilon}{2(b-a)}$ for all $y, z \in J_x \cap [a,b]$, since f is continuous at such x. Now $\{I_k\} \cup \{J_x : x \in [a,b] \setminus E\}$ is an open cover of $[a,b]$, so by compactness of $[a,b]$ there exists a finite cover $\{I_k\}_{k=1}^n \cup \bigcup_{i=1}^m \{J_{x_i} : x_i \in [a,b] \setminus E\}$ of $[a,b]$. Let $\mathcal{P} = \{a = t_0, \ldots, t_N = b\}$ be the partition of $[a,b]$ determined by those endpoints of $\{I_k\}_{k=1}^n$ and $\{J_{x_i} : x_i \in [a,b] \setminus E\}$, which are inside $[a,b]$. For each $1 \le j \le N$ the interval (t_{j-1}, t_j) is contained in some I_k or some J_{x_i}. Let $J = \{j : (t_{j-1}, t_j) \subset I_k \text{ for some } k\}$. Then we have that

$$\begin{aligned}
\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) &= \sum_{j=1}^{N} \Delta(t_j) \cdot \sup\{f(x) - f(y) : x, y \in [t_{j-1}, t_j]\} \\
&\le \sum_{j \in J} \Delta(t_j) \cdot 2M + \sum_{j \notin J} \Delta(t_j) \cdot \frac{\epsilon}{2(b-a)} \\
&< \frac{\epsilon}{4M} \cdot 2M + (b-a) \cdot \frac{\epsilon}{2(b-a)} = \epsilon.
\end{aligned}$$

Hence f is Riemann integrable. $\square$

14.7 SOLVED PROBLEMS

Example

Let $[a, b] = [0, 2]$. Let $P = \{0, 1, 2\}$ and $Q = \{0, 1/2, 1, 2\}$. Then Q is a refinement of P. The subintervals determined by P are $[0, 1]$ and $[1, 2]$. The subintervals determined by Q are $[0, 1/2], [1/2, 1]$, and $[1, 2]$. Note that $[0, 1] = [0, 1/2] \cup [1/2, 1]$.

Example

Let f be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We show $\underline{S}(f) = 0$ and $\overline{S}(f) = 1$.

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0, 1]$. Any interval $[x_{i-1}, x_i]$ contains both rational and irrational points, so

$$m_i(f) = 0 \text{ and } M_i(f) = 1, \quad i = 1, ..., n.$$

Thus

$$\underline{S}(f; P) = \sum_{i=1}^{n} m_i(f)\Delta x_i = \sum_{i=1}^{n} 0 \ \Delta x_i = 0$$

and

$$\overline{S}(f; P) = \sum_{i=1}^{n} M_i(f)\Delta x_i = \sum_{i=1}^{n} 1 \ \Delta x_i = 1$$

since the sum of the subintervals, $\sum \Delta x_i$, is equal to the length of the interval $[0, 1]$. Thus for any partition P, $\underline{S}(f; P) = 0$ and $\overline{S}(f; P) = 1$ so that $\underline{S}(f) = 0$ and $\overline{S}(f) = 1$.

Example

We show that a finite set has measure zero. Let $\{x_1, \ldots, x_N\}$ be a finite set and let $\epsilon > 0$ be given. Then

$$\left\{(x_1 - \frac{\epsilon}{4N}, x_1 + \frac{\epsilon}{4N}), ..., (x_N - \frac{\epsilon}{4N}, x_N + \frac{\epsilon}{4N})\right\}$$

is an open cover of $\{x_1, .., x_N\}$. There are N intervals each of measure $\frac{\epsilon}{2N}$ so that the open cover has measure $N\frac{\epsilon}{2N} = \epsilon/2$.

CHECK YOUR PROGRESS

Fill in the Blanks:

1. A partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ is a set such that………………

2. The norm of $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is defined as

3. $\bar{S}(f, P) =$………….$\underline{S}(f, P) =$…………

4. A function f is said to be Riemann integrable on $[a, b]$ if ……………………

5. If f is continuous on $[a, b]$, then f is …………………. on $[a, b]$.

The following statements is true or false

6. The Lebesgue measure of an open interval (a, b) is $b - a$. True \False

7. A subset of a set of measure zero can have non zero measure. True \False

8. The union of a countable collection of sets of measure zero is a set

9. of measure zero. True\False

10. A countable set has measure zero. True\False.

14.8 SUMMARY

This unit is an explanation of

i. Riemann Integral.

ii. Criterion for Riemann Integrability.

iii. Lebesgue Integral.

iv. Lebesgue characterization for Riemann Integrability.

14.9 GLOSSARY

i. Partition

ii. Bounded Functions

iii. Riemann Integral

iv. Measure

v. Measure zero set

14.9 REFERENCES

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!4.11 SUGGESTED READINGS

i. E. M. Stein, R. Shakarchi (2005), *Measure Theory Integration and Hilbert Spaces*, Princeton University Press.

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!4.12 TERMINAL QUESTIONS

1. What is Riemann integral...............................

2. What is Lebesgue integral...............................

3. What is Riemann's condition for integrability...............................

4. What do you mean by a set of measure of zero…………..

5. Construct a countable set with measure zero…………

6. Construct an uncountable set with measure zero…………

7. Construct a function on $[a, b]$ such that measure of its discontinuities is zero……………

8. Construct a function on $[a, b]$ such that measure of its discontinuities is non zero…………..

9. Which integral is suitable to integrate a broader class of function? Justify……

10. Discuss role of uniform convergence in convergence theorem for Riemann integration…………….

!4.13 ANSWERS

CHECK YOUR PROGRESS

1 $a = x_0 < x_1 < x_2 < \cdots < x_n = b.$

2 $||P|| = max\{x_i - x_{i-1} \mid i = 1, 2, 3, \ldots, n\}.$

3 $\bar{S}(f; P) = \sum M_i(f)\Delta x_i$ and $\underline{S}(f; P) = \sum m_i(f)\Delta x_i.$

4 If lower integral = upper integral.

5 Riemann integrable.

6 True

7 False

8 True

9 True

10 True.

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