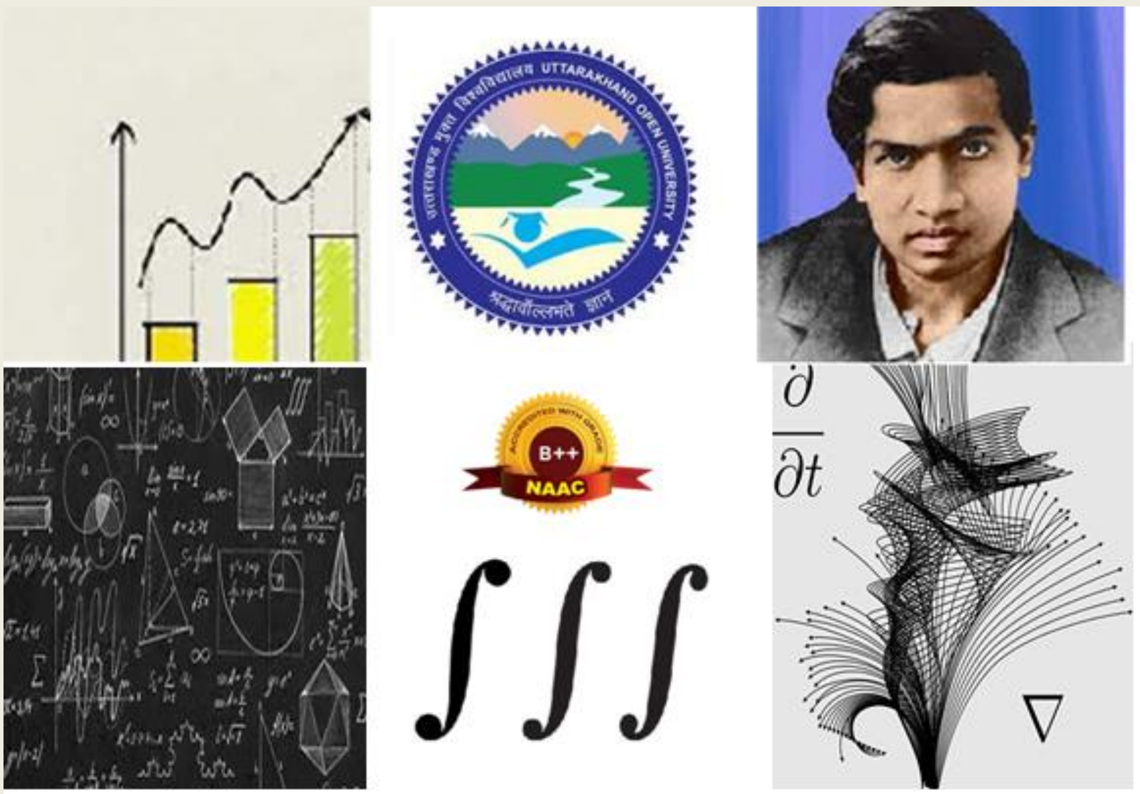
Master of Science

MATHEMATICS

Second Semester

MAT 506

TOPOLOGY



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: TOPOLOGY

COURSE CODE: MAT 506

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Uttarakhand Open University
Haldwani, Uttarakhand, India,
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Course Title and Code : TOPOLOGY (MAT-506)

Edition : 2024

Published By : Uttarakhand Open University, Haldwani, Nainital- 263139

ISBN :

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COURSE INFORMATION

The present self learning material **"Topology"** has been designed for M.Sc. (Third Semester) learners of Uttarkhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

First block is **Topological Spaces and Continuous Functions.** In this block Basics, Topological Spaces, Basis for a Topology, Order Topology, Product Topology and Subspace Topology, Closed Sets and Limit Points, Continuous Functions, Product Topology, Metric Topology and Quotient Topology defined.

Second block is **Connectedness and Compactness.** In this block Connected Spaces, Connected Sets in the Real Line, Components and Path Components, Local Connectedness, Compact Spaces, Compact set in the Real line, Limit Point Compactness and Local Compactness defined clearly.

Third block is **Countability and Separation Axioms** third block is composition of Countablity Axioms and Sepration Axioms.

Fourth block is **Tychonoff Theorem** which is a collection of The Urysohn Lemma and Tietze Extension Theorem, Baire Category Theorem, The Urysohn Metrization Theorem, Partitions of Unity, Tychonoff Theorem for Product Spaces. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the leaner's to grasp the subject easily.

Course Name: TOPOLOGY **Course Code: MAT 506**

Credit: 4

SYLLABUS

Topological Spaces and Continuous Functions: Basics, Topological Spaces, Basis for a Topology, The Order Topology, The Product Topology on $X \times Y$, The Subspace Topology, Closed Sets and Limit Points, Continuous Functions, The Product Topology, The Metric Topology, The Quotient Topology. Connectedness and Compactness: Connected Spaces, Connected Sets in the Real Line. Components and Path Components, Local Connectedness, Compact Spaces, Compact set in the Real line, Limit Point Compactness, Local Compactness. Countablity Axioms. Sepration Axioms. The Urysohn Lemma and Tietze Extension Theore, Baire Category Theorem, The Urysohn Metrization Theorem, Partitions of Unity, Tychonoff Theorem for Product Spaces

Reference Books:

1. K.D. Joshi (2017), *Introduction to General Topology*, New age International (P) Limited.
2. J. L. Kelly (2017), *General Topology*, Dover Publications Inc., 2017.
3. J. R. Munkres (1976), *Topology – A First Course*, Prentice Hall of India.
4. G.F. Simmons (2017), *Introduction to Topology and Modern Analysis*, Mc. Graw Hill Education.
5. https://archive.nptel.ac.in/courses/111/106/111106054/

Suggested Readings:

1. K. Ahmad (2020), *Introduction to Topology*, Alpha Science International Ltd.
2. W. J. Pervin (1964) *Foundations of General Topology*, Academic Press.
3. **https://archive.nptel.ac.in/noc/courses/noc22/SEM1/noc22-ma36/**
4. **https://archive.nptel.ac.in/courses/111/101/111101158/**

BLOCK- I:

TOPOLOGICAL SPACES AND CONTINUOUS FUNCTIONS

UNIT 1:
TOPOLOGICAL SPACES AND BASIS FOR A TOPOLOGY

CONTENTS:

1.1 INTRODUCTION

Topology is a branch of geometry. It is concerned with the structure of geometrical objects but not their exact shape or their size. Leonhard Euler first started the work in this field. Further contributions were made by Augustin-Louis Cauchy, Ludwig Schläfli, Johann Benedict Listing, Bernhard Riemann and Enrico Betti. Listing introduced the term "Topologie" in Vorstudien zur Topologie, written in his native German, in ***1847,*** having used the word for ten years in correspondence before its first appearance in print. The English form "topology" was used in ***1883*** in Listing's obituary in the journal Nature to distinguish "qualitative geometry from the ordinary geometry in which quantitative relations chiefly are treated". Their work was corrected, consolidated and greatly extended by Henri Poincaré.

Leonhard Euler

(15 April 1707 – 18 September 1783)

Ref: https://en.wikipedia.org/wiki/Leonhard_Euler#/media/File:Leonhard_Euler.jpg

Fig 1.1

In this unit we explain about topological space and we study a number of ways evaluating a topology on a set so as to make it into a topological space. We also consider some of the elementary concepts associated with topological spaces, Open and closed sets.

1.2 OBJECTIVES

After completion of this unit learners will be able to

- **i.** Define the concept of Topology.
- **ii.** Evaluate the different type of Topology on a set.
- **iii.** Describe the notion of Basis for a topology on a set
- **iv.** Explain the concept of subbasis for a topology on a set.

1.3 TOPOLOGY ON A SET

Definition 1: A topology on a set X is collection $\mathcal{T}$ of subsets of X having the following Properties:

- **i.** $\emptyset$ and X are in $\mathcal{T}$.
- **ii.** The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
- **iii.** The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

Note: In simple words, a topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.

Let X be a three – element set, $X = \{a, b, c\}$.

The open sets are: $\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$.$\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$

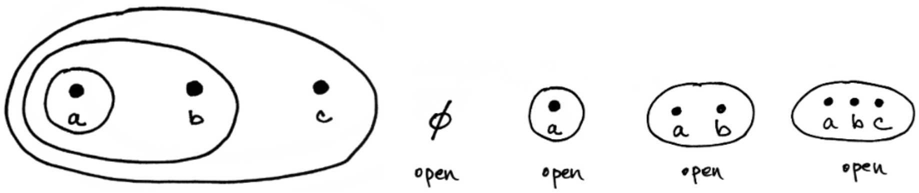

Fig 1.2

Let X be a three – element set, $X = \{a, b, c\}$.

The open sets are: $\emptyset$, $\{b, c\}$, $\{a, b\}$, $\{a, b, c\}$.

Here we cannot construct topological space. Because the property third of definition of topology on a set is not satisfied.

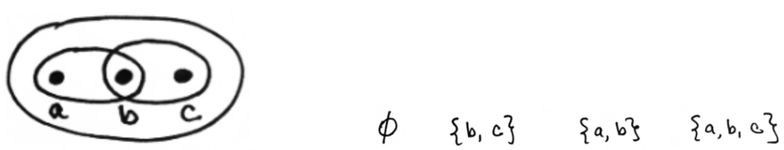

Fig 1.3

- If $X = \{1,2,3,4\}$, then $\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{1,2\}, X\}$ is clearly a topology on X.
- Consider the following classes of subsets of $X = \{a, b, c, d, e\}$,

$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

$\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

$\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$

We observe that

a) $\mathcal{T}_1$ is topology since it satisfies the all necessary three axioms.

b) $\mathcal{T}_2$ is not a topology since it not satisfies axiom (ii).

c) $\mathcal{T}_3$ is not a topology since it not satisfies axiom (iii).

With the convention that $\emptyset$ is the union of the empty collection of subsets of X, and X is the intersection of the empty collection of subsets of X, one may agree that **i** follows from **ii** and **iii**, but condition **i** is usually included for clarity. We express **ii** by saying that $\mathcal{T}$ is closed under (arbitrary) unions, and express **iii** by saying that $\mathcal{T}$ is closed under finite intersections.

To check that $\mathcal{T}$ is closed under finite intersections, it suffices to prove that if $U_1, U_2 \epsilon \mathcal{T}$ then $U_1 \cap U_2 \epsilon \mathcal{T}$. This follows by induction on n from the formula $U_1 \cap U_2 \ldots . \cap U_n = (U_1 \cap U_2 \ldots . \cap U_{n-1}) \cap U_n$.

Definition 2: If X is a topological space with topology $\mathcal{T}$, then a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$. A topological space is a set X together with a collection of subsets of X, called **open sets,** ϕ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Definition 3: A subset of X is said to be **closed** if its complement is in $\mathcal{T}$ (i.e., its complement is open).

Definition 4: If X is any set, the collection of all subsets of X is a topology on X; it is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is also a topology on X these two topology are trivial topology.

Definition 5: Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the **finite complement topology also called co-finite topology.** Both X and $\emptyset$ in$\mathcal{T}_f$, since $X - X$ is finite and $X - \emptyset$ is all of X.

Definition 6: Suppose that $\mathcal{T}$ and $\mathcal{T}^{'}$ are two topologies on a given set X. If $\mathcal{T}^{'} \supset \mathcal{T}$, it implies that $\mathcal{T}^{'}$ is **finer (stronger)** than $\mathcal{T}$; if $\mathcal{T}^{'}$ properly contains $\mathcal{T}$, it means that $\mathcal{T}^{'}$ is strictly finer (stronger) than $\mathcal{T}$. And in simple words $\mathcal{T}$ is **coarser (weaker)** than $\mathcal{T}^{'}$, or strictly coarser (weaker), in these two respective situations.

Definition 7: Given two topologies $\mathcal{T}_{\mathbf{1}}$ and $\mathcal{T}_{\mathbf{2}}$ if $\mathcal{T}_{\mathbf{1}} \not\subset \mathcal{T}_{\mathbf{2}}$ and $\mathcal{T}_{\mathbf{2}} \not\subset \mathcal{T}_{\mathbf{1}}$ then $\mathcal{T}_{\mathbf{1}}$ and $\mathcal{T}_{\mathbf{2}}$ are said to be **not comparable**.

Definition 8: Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be **equal** or the **same** iff $\mathcal{T}_1$ and $\mathcal{T}_2$ both contain the same open sets.

Definition 9: Given a non-empty set X if $\mathcal{T}$ is the class of all those subsets of X whose complements are countable together with the empty set, then $\mathcal{T}$ is said to be the **co-countable** topology for X.

Definition 10: If $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X, then their intersection is $\mathcal{T}_1 \cap \mathcal{T}_2$ is topology for X and their union $\mathcal{T}_1 \cup \mathcal{T}_2$ is not necessarily a topology for X.

Definition 11: If $(X, \mathcal{T})$ be a topological space and $Y(\neq \emptyset) \subset X$, then the topology on Y defined as $\mathcal{T}_y = Y \cap \mathcal{T}$, is **relative topology** if $(X, \mathcal{T}_y)$ is a topological space known as subspace of $(X, \mathcal{T})$.

Definition 12: Let $X = \{0,1\}$ then $\mathcal{T} = \{\emptyset, X, \{0\}\}$ is topology on X. The topological space $(X, \mathcal{T})$ is called **Sierpinski space**.

Definition 12: If every subspace of a topological space $(X, \mathcal{T})$ has a property of the topological space then this property as a **hereditary property.**

Definition 13: It is a topological space X if there exist a metric d on X s.t. (X, d) is a metric space.

Theorem 1: Let $\{\mathcal{T}_\lambda : \lambda \epsilon X\}$where X is an arbitrary set, be a collection of topologies for X. Then the intersection $\cap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$ is also a topology for X.

Proof. Here $\{\mathcal{T}_\lambda : \lambda \epsilon X\}$ is a collection of topologies on X.

We have to show that $\cap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$ is also a topology on X.

If $X = \phi$, then $\cap\{\mathcal{T}_\lambda : \lambda \epsilon \phi\} = P(X)$.

Thus in this case the intersection of topologies is the discrete topology.

Now, let $X \neq \phi$.

$\boldsymbol{\mathcal{T}_1}$: Since $\mathcal{T}_\lambda$ is a topology$\forall \lambda \epsilon X$,

It follows that $\phi, X \epsilon \mathcal{T}_\lambda \forall \lambda \epsilon X$.

But $\phi \epsilon \mathcal{T}_\lambda \forall \lambda \epsilon X \Longrightarrow \phi \epsilon \cap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$,

and $X \epsilon \mathcal{T}_\lambda \forall \lambda \epsilon X \Longrightarrow X \epsilon \cap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$.

$\boldsymbol{\mathcal{T}_2}$**:** Let $G_1, G_2 \epsilon \cap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$.

Then $G_1, G_2 \epsilon \mathcal{T}_\lambda \forall \lambda \epsilon X$.

Since $\mathcal{T}_\lambda$ is a topology for $X \epsilon \forall \lambda \epsilon X$

it follows that $G_1 \cap G_2 \epsilon \mathcal{T}_\lambda$.

Hence $G_1 \cap G_2 \epsilon \cap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$.

$\mathcal{T}_3$**:** Let $G_\alpha \in \bigcap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$ for $\alpha \epsilon Y$ where Y is an arbitrary set.

Then $G_\alpha \, \epsilon \mathcal{T}_\lambda \forall \lambda \epsilon X$ and $\forall \lambda \epsilon X$ and $\forall \, \alpha \epsilon Y$.

Since each $\mathcal{T}_\lambda$ is a topology for X,

it follows that $\bigcup\{G_\alpha : \alpha \epsilon Y\} \epsilon \mathcal{T}_\lambda \forall \lambda \epsilon X$. Hence $\bigcup\{G_\alpha : \alpha \epsilon Y\} \epsilon \bigcap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$.

Thus $\bigcap\{\mathcal{T}_\lambda : \lambda \epsilon X\}$ is a topology for X. ■

Some examples are following:

Example 1: Let $X = \{1,2,3\}$ and the set $\mathcal{T} = \{\emptyset, \{1\}, \{1,2,3\}\}$.

$\mathcal{T}$ is a topology on X. If we were given any other topology

$\mathcal{T}_1 = \{\emptyset, \ \{1,2,3\}\}$. $\mathcal{T}_1$ is a topology on X.

$\{1\}$ is open set under topology $\mathcal{T}$. $\{1\}$ is not open set under topology $\mathcal{T}_1$.

This example shows a set is only open under a particular topology. It does not strictly make sense to merely say that a set is open.

Example 2: Let $X = \{a, b\}$ be a two element set. There are four different possible topologies on X:

i. The minimal possibility is the trivial topology $\mathcal{T}_{trivial} = \{X, \emptyset\}$.

ii. An intermediate possibility is $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$.

iii. Another intermediate possibility is $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$.

iv. The maximal possibility is the discrete topology $\mathcal{T}_{discrete} = \{X, \{a\}, \{b\}, \{a, b\} \ \emptyset\}$.

Example 3: Let $X = \{a, b, c\}$. Here are some collections of subsets of X that are not topologies:

i. $\mathcal{T}_1 = \{ \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ does not contain $\emptyset$ and X.
ii. $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, X\}$ is not closed under unions.
iii. $\mathcal{T}_3 = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ is not closed under intersections.

CHECK YOUR PROGRESS

1. Let $X = \{a, b, c\}$. How many different topologies on X. Explain them……………………………………………………………………
…………………………………………………………………………
…………...

1.4 BASIS FOR A TOPOLOGY

In mathematics, a basis for the topology $\mathcal{T}$ of a topological space $(X, \mathcal{T})$ is a family U of open subsets of X such that every open set of the topology is equal to the union of some sub-family of U .

Basis is ubiquitous throughout topology. The sets in a basis for a topology, which are called *basic open sets,* are often easier to describe and use than arbitrary open sets. Many important topological definitions such as continuity and convergence can be checked using only basic open sets instead of arbitrary open sets. Some topologies have a basis of open sets with specific useful properties that may make checking such topological definitions easier.

1.4.1 DEFINITION

BASIS FOR A TOPOLOGY

If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X (called basis elements) such that

i. For each $x \epsilon X$, there is at least one basis element B containing x.

ii. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such $B_3 \subset B_1 \cap B_2$.

TOPOLOGY $\mathcal{T}$ GENERATED BY $\mathfrak{B}$

If $\mathfrak{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathfrak{B}$ is described as follows:

A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \epsilon U$, there is a basis element $B \epsilon \mathfrak{B}$ such that $x \epsilon B$ and $B \subset U$.

- Each element of $\mathfrak{B}$ is open in X under this definition. So that $\mathfrak{B} \subset \mathcal{T}$.
- Collection of subsets of X is a topology on X.

Some examples are following:

Example 4: Consider $X =$ Set of real numbers and collection $\mathfrak{B} = \{(a, b) \subseteq R | a < b\}$. Show that $\mathfrak{B}$ is a basis for any topology of set of real number?

The set of real number can be written $(a, b) = \{x \in R | a < x < b\}$.

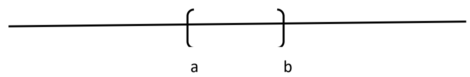

Fig 1.4.1

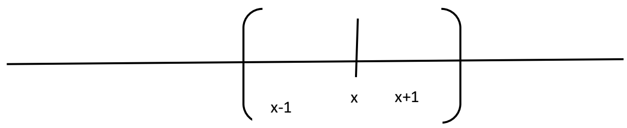

Fig 1.4.2

For showing the basis the following two conditions will be satisfied

i. Let $x \in R$ then $(x-1, x+1) \in \mathfrak{B}$ such that $x \in (x-1, x+1)$. First condition is satisfied.

ii. Let $\mathfrak{B}_1 = (a_1, b_1), \mathfrak{B}_2 = (a_2, b_2) \in \mathfrak{B}$ satisfies the condition $[a_1 < b_1, a_2 < b_2]$. If $x \in \mathfrak{B}_1 \cap \mathfrak{B}_2$. it implies $x \in (a_2, b_1) \subseteq \mathfrak{B}_1 \cap \mathfrak{B}_2 = (a_1, b_1) \cap (a_2, b_2)$. So $(a_2, b_1) \in \mathfrak{B}$. Second condition is satisfied.

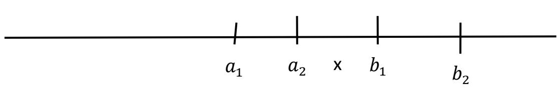

Fig 1.4.3

So the collection $\mathfrak{B}$ is a basis for any topology on X.

Example 5: Consider $X \neq \emptyset$. Show the collection $\mathfrak{B} = \{\{x\} \mid x \in X\}$ is a basis?.

For showing the basis the following two conditions will be satisfied

- **i.** For $x \in X$, there exists $\{x\} \in \mathfrak{B}$ such that $x \in \{x\}$. First condition is satisfied.
- **ii.** Here choosing two set in $\mathfrak{B}$ both are singleton and different. Since there is no element in common element in any two member of $\mathfrak{B}$. Second condition satisfied trivially.

So the collection $\mathfrak{B}$ is a basis for any topology on X.

Lemma 1: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be basis for the topology $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

- **i.** $\mathcal{T}'$ is finer than $\mathcal{T}$.
- **ii.** For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.

Lemma 2: Let $B_1, B_2, \dots\dots\dots.. B_n \in \mathfrak{B}$ (Basis) and let $x \in B_1 \cap B_2 \cap \dots .. \cap B_n$. Then there exists some member $B' \in \mathfrak{B}$ such that $x \in B' \subseteq B_1 \cap B_2 \cap \dots .. \cap B_n$.

Proof: We proof this by induction on n.

For $n = 2$.

Suppose $x \in B_1 \cap B_2$(which is an element of basis).

By definition of basis, there exists some

$B' \in \mathfrak{B}$

such that $x \in B' \subseteq B_1 \cap B_2$.

Suppose the result is true for $n - 1$

i.e. If $x \in B_1 \cap B_2 \cap \dots .. \cap B_{n-1}$.

Then there exist $B^* \in \mathfrak{B}$

such that

$x \in B^* \subseteq B_1 \cap B_2 \cap \ldots \ldots \cap B_{n-1}$ ……………………………………………… $(\mathbf{1})$.

Now suppose $x \in B_1 \cap B_2 \cap \ldots \ldots \cap B_n$

it implies that $x \in B_n$ and also $x \in B^*$(from (1)) it implies that $x \in B^* \cap B_n$.

By definition of basis,

there exists some $B^* \in \mathfrak{B}$ such that

$x \in B' \subseteq B^* \cap B_n \subseteq B_1 \cap B_2 \cap \ldots \ldots \cap B_{n-1} \ldots \cap B_n$

it implies that $x \in B' \subseteq B^* \cap B_n \subseteq B_1 \cap B_2 \ldots \cap B_n$. ■

Theorem 2: Let $\mathfrak{B}$ be basis on a set X. Define a collection $\mathcal{T}$ containing $\emptyset$ and every set that is equal to a union of Basis elements.

Then $\mathcal{T}$ is a topology on X (Called topology generated by $\mathfrak{B}$).

Proof.

i. $\emptyset \in \mathcal{T}$ (by definition). Since for each $x \in X$, there exists some $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq X \Longrightarrow X = \bigcup_{x \epsilon X}\{x\} \subseteq \bigcup_{x \in X} B_x \subseteq X$.

It implies that $X = \bigcup_{x \epsilon X} B_x$. So $X \in \mathcal{T}$.

ii. Let $V = \bigcup O_\alpha$(arbitrary union of open sets), where each $O_\alpha \in \mathcal{T}$ i.e. $O_\alpha = \phi$ or O_α is a union of member of $\mathfrak{B}$.

If each $O_\alpha = \phi$, then $V = \phi \epsilon \mathcal{T}$. If at least one $O_\alpha \neq \phi$, then V is the union of elements of $\mathfrak{B}$.

iii. Let $V = O_1 \cap O_2 \cap \ldots \ldots \ldots \cap O_n$ (finite intersection of open sets) where each $O_i \in \mathcal{T}$. If any one of the $O_i = \emptyset$, then $V = \phi \;\epsilon\; \mathcal{T}$.

If each $O_i \neq \emptyset$. Then for $x \in V$, $x \in O_1, x \in O_2 \ldots \ldots \ldots x \in O_n$.Since each O_i is the union of members of $\mathfrak{B}$, so there exists B_1,

$B_2 \ldots\ldots\ldots . B_n \in \mathfrak{B}$ such that $x \in B_1 \subseteq O_1, x \in B_2 \subseteq O_2 \ldots x \in B_n \subseteq O_n$.It implies that $x \in B_1 \cap B_2 \cap B_3 \ldots\ldots\ldots .. \cap B_n$. Then there exist $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq B_1 \cap B_2 \cap B_3 \cap \ldots \cap B_n \subseteq V$. It implies that $x \in B_x \subseteq V \Longrightarrow V = \bigcup_{x\epsilon V}\{x\} \subseteq \bigcup_{x\epsilon V} B_x \subseteq V$

$\Longrightarrow V = \bigcup_{x\epsilon V} B_x$

$\Longrightarrow V \in \mathcal{T}$.

So $\mathcal{T}$ is a topology on X. ■

Theorem 3: If $\{F_\lambda : \lambda \epsilon X\}$ is any collection of closed subsets of a topological space X, then $\cap \{F_\lambda : \lambda \epsilon X\}$ is a closed set.

Proof: F_λ is closed $\forall \lambda \epsilon X$

$\Longrightarrow F_\lambda'$ is open $\forall \lambda \epsilon X$

$\Longrightarrow \bigcup\{F_\lambda' : \lambda \epsilon X\}$ is open [arbitrary union of open sets]

$\Longrightarrow [\cap \{F_\lambda : \lambda \epsilon X\}]'$ is open [De-Morgan Law]

$\Longrightarrow \cap \{F_\lambda : \lambda \epsilon X\}$ is closed. [by definition of closed sets]. ■

Theorem 4: If F_1 and F_2 be two closed subsets of a topological space X. Then $F_1 \cup F_2$ is a closed set.

Proof: If F_1, F_2 are closed$\Longrightarrow F_1', F_2'$ are open.

$\Longrightarrow F_1' \cap F_2'$ are open [arbitrary union of open sets]

$\Longrightarrow [F_1 \cup F_2]'$ is open [De Morgan Law}

$\Rightarrow F_1 \cup F_2$ is closed. ■

- If F_1, F_2, ……… F_n be a finite number of closed subsets of X, then their union will also be a closed subset of X. This follows at once by the repeated application of the above theorem.

Theorem 5: The lower limit topology $\mathcal{T}'$ on $\mathbb{R}$ is strictly finer than the standard topology $\mathcal{T}$.

Proof: Given a basis element (a, b) for $\mathcal{T}$, and a point x of (a, b), the basis element $[a, b)$ for $\mathcal{T}'$ contains x and lies in (a, b).
Therefore, $\mathcal{T}'$ is finer than $\mathcal{T}$.
On the other hand, given a basis element $[x, d)$ for $\mathcal{T}'$, there is no open interval (a, b) satisfying the condition $x \epsilon (a, b) \subset [x, d)$;
Therefore, $\mathcal{T}$ is not finer than $\mathcal{T}'$. ■

Example 6: Consider $X = \{1,2,3,4\}$**,**
$\mathfrak{B} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ is a Basis for X
$\mathcal{T} = \{\emptyset$, union of elements of $\mathfrak{B}\}$

$$= \{\emptyset, \{1\}, \}\{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\},$$
$$\{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, X\}$$

- $\mathcal{T} =$ discrete topology on X.

  Let $X \neq \phi$. The collection $\mathfrak{B} = \{\{x\}: x \in X\}$ is a Basis for a discrete topology on X. It means the topology generated by $\mathfrak{B}$ is discrete topology on X.

1.4.2 STANDARD TOPOLOGY

If $\mathfrak{B}$ is the collection of all open intervals in the real line

$$(a, b) = \{x \mid a < x < b\}$$

The topology generated by $\mathfrak{B}$ is called the **standard topology (Usual topology or Euclidean topology for** $\mathbb{R}$ **)** on the real line. The usual topology generated by $\mathfrak{B}$. The usual topology is a non –trivial topology.

Example 7: If X be a metric space, then the topology defined as the class of all subsets of X which are open in the sense that a subset G of X is an open set, if given any point $x \in G$,$\exists$ a positive real number r such that the open sphere $S_r(x) \subset G$ $i.e.$ if each point $x \in G$ is the centre of some open sphere contained in G, then such a topology is the usual topology on a metric space and these sets are open sets generated by the metric on the space.

1.4.3 LOWER LIMIT TOPOLOGY

If $\mathfrak{B}'$is the collection of all half-open (left closed right open) intervals of the form

$$[a, b) = \{x \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathfrak{B}'$ is called the **lower limit topology** on the real line.

When R is given the lower limit topology, we denote it by R_l.

Note: It is easy to see that both $\mathfrak{B}$ and $\mathfrak{B}'$are bases; the intersection of two basis elements is either another basis element or is empty.

1.4.4 UPPER LIMIT TOPOLOGY

If $\mathfrak{B}'$is the collection of all half-closed (left open right closed) intervals of the form

$$(a, b] = \{x \mid a < x \leq b\},$$

where $a < b$, the topology generated by $\mathfrak{B}'$ is called the **upper limit topology** on the real line.

When R is given the upper limit topology, we denote it by R_u.

1.4.5 SUBBASIS FOR A TOPOLOGY

A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose unions equals X the topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

1.5 SOLVED PROBLEMS

Example 8: Let $\mathcal{T}$ be the collection of subsets of $\mathbb{N}$ consisting of empty set $\emptyset$ and all subsets of the form $G_m = \{m, m+1, m+2, \dots \dots .\}, m \in \mathbb{N}$. Show that $\mathcal{T}$ is a topology for $\mathbb{N}$. What are open sets containing 5?

Solution. $[\boldsymbol{\mathcal{T}_1}]$: $\emptyset \epsilon\, \mathcal{T}$ and $G_1 = \{1, 2, 3, \dots \dots .\} = \mathbb{N} \epsilon\, \mathcal{T}$.

$[\boldsymbol{\mathcal{T}_2}]$: Let $G_m\ \epsilon\ \mathcal{T}$ and $G_n \epsilon\, \mathcal{T}$, m, n ϵ $\mathbb{N}$.

Then$G_m \cap G_n = G_m$ or G_n according as $n > m$ or $n < m$.

Hence $G_m \cap G_n \in \mathcal{T}$.

$[\mathcal{T}_3]$: Let $G_\lambda \in \mathcal{T}$ for every $\lambda \in$ X where X is some subset of $\mathbb{N}$.

Since $\mathbb{N}$ is a well ordered set, X contains a smallest positive integer m_0 so that

$\bigcup\{G_\lambda : \lambda \in \mathrm{X}\} = \{m_0, m_0+1, m_0+2, \dots \dots .\} = G_{m_0}$ which belongs to $\mathcal{T}$. Hence $\mathcal{T}$ is a topology for $\mathbb{N}$.

The open sets containing $\mathcal{T}$ are the following:

$G_1 = \mathbb{N} = \{1, 2, 3, \dots \dots .\}, G_2 = \{2, 3, 4, \dots \dots .\}$,

$G_3 = \{3,4,5 \dots \dots .\}$,

$G_4 = \{4,5,6 \dots \dots .\}$,

$G_4 = \{5,6,7 \dots \dots .\}$,

Example 9: Show that for any family of topologies for X there exists a unique largest topology which is smaller than each member of the family.

Solution. Let $\{\mathcal{T}_\lambda : \lambda \in X\}$ be a non-empty family of topologies for X.

Then by the proceeding theorem $\bigcap\{\mathcal{T}_\lambda : \lambda \in X\}$ is also a topology for X. Further it is smaller than each member of the family.

We claim that it unique largest such topology.

If not, let Δ be the topology with this property. Then

$$\Delta \supset \cap \{\mathcal{T}_\lambda : \lambda \in X\} \dots \dots \dots \dots \dots \dots \dots \dots \dots . (1)$$

Now by our assumption, Δ is smaller than each member of the family $\{\mathcal{T}_\lambda : \lambda \in X\}$. Hence $\Delta \subset \cap \{\mathcal{T}_\lambda : \lambda \in X\} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (2)$

From (1) and (2), we have $\Delta = \cap \{\mathcal{T}_\lambda : \lambda \in X\}$.

Thus $\cap \{\mathcal{T}_\lambda : \lambda \in X\}$ is the greatest lower bound of the topologies $\mathcal{T}_\lambda$ ordered by inclusion relation.

Example 10: Give an example of a basis for the Euclidean topological space $(\mathbb{R}^n, \mathcal{T})$.

Solution. According to definition of a basis.

Let $(X,\mathcal{T})$ denote a topological space.

A family $\mathfrak{B} \subset \mathcal{T}$ is called a basis for $\mathcal{T}$ if each open set (i.e., an element of $\mathcal{T}$) is the union of members of $\mathfrak{B}$.

Note that if $\mathcal{T}$ is a topology on X, then $\mathfrak{B}$ is a basis for $\mathcal{T}$. In $(\mathbb{R}^n, \mathcal{T})$. The topology is the family of sets open in the sense of the Euclidean metric.

We shall show that in $\mathbb{R}^n$

$$\mathfrak{B} = \{\mathfrak{B}(x,r) : x \in \mathbb{R}^n, r > 0\} \quad \ldots\ldots\ldots\ldots\ldots. \qquad (1)$$

is a basis for the Euclidean topology.

Let A $\in \mathcal{T}$ denote an open subset of R^n. Then

$$A = \cup_{x \in A} \mathfrak{B}(X, r). \qquad \ldots. \quad \ldots\ldots\ldots \qquad (2)$$

Each open set can be represented as a union of balls.

Note that (1) is not the only possible basis for the Euclidean topology in R^n.

Example 11: Let $\mathbb{R}$ denote the set of the all real numbers and let Xdenote all the sets of the form

$\{x : x > a\}$……………………………(1)

$\{x : x < b\}$……………………………..(2)

$\mathcal{T}(\mathrm{A})$ consists of

$$\mathcal{T}(\mathrm{A}) = \left\{ \begin{matrix} \emptyset, \mathbb{R}, \text{all finite intersections of members of A,} \\ \text{all arbitrary unions of} \\ \text{finite intersections of members of A} \end{matrix} \right\}$$

Describe about the topology?

Solution. All finite intersections of members of A are open intervals (a,b). Hence, we conclude that the set of all open intervals $\subset \mathcal{T}(\mathrm{A})$.

The family of all open intervals forms a basis for $\mathcal{T}(\mathrm{A})$.

The family of all open intervals forms a basis for the Euclidean topology. Therefore, $\mathcal{T}(\mathrm{A})$ is the Euclidean topology.

Example 12: Let $\mathbb{R}$ denote the set of the all real numbers and let Xdenote all the sets of the form

$\{x: x > a\}$…………………………….(1)

$\{x: x \leq b\}$……………………………..(2)

Describe about the topology?

Solution. To find $\mathcal{T}(\mathrm{A})$, we must first obtain all finite intersections of elements of A.

The set $(a, b] = \{x \mid a < x \leq b\}$………………………(3)

form a basis for topology $\mathcal{T}(\mathrm{A})$,

which is not Euclidean because $(a, b]$ does not belong to the Euclidean topology.

Note that we were dealing with the situation, when a family of sets A was given and we had to find topology $\mathcal{T}(\mathrm{A})$ such that $A \subset \mathcal{T}(\mathrm{A})$.

Conversely, for a given topology $\mathcal{T}$, a family of sets $A \subset \mathcal{T}$ is called a subbasis for $\mathcal{T}$, If $\mathcal{T} = \mathcal{T}(\mathrm{A})$.

Example 13: Given $X \neq \emptyset$ and

$\mathcal{T} = \{\emptyset, all\ sets\ whose\ complements\ are\ countable\}$.

Show that $\mathcal{T}$ is a topology on X.

Solution. $\mathcal{T}_1$ is satisfied $\emptyset \epsilon\ \mathcal{T}$ and $X^{'} = \emptyset$ being countable $X \epsilon \mathcal{T}$

$i.e.$ $\quad \emptyset\ , X \epsilon \mathcal{T}$

$\mathcal{T}_2$ is satisfied since if $A_\alpha \in \mathcal{T} \forall \alpha \epsilon \Lambda$ (Λ being index set),

$A_\alpha^{'}$ is countable for all $\alpha \epsilon \Lambda$ and so by De Morgan's law

$[\cup \{ A_\alpha : \alpha \epsilon \Lambda\}^{'} = \cap \{A_\alpha : \alpha \epsilon \Lambda\} \in \mathcal{T}$.

Hence arbitrary $\cup \{ A_\alpha : \alpha \epsilon \Lambda \} \in \mathcal{T}$.

$\mathcal{T}_3$ is satisfied if $A, B \epsilon \mathcal{T}$ than their complements i.e. $A^{'}, B^{'}$ are countable and

so by De morgan's law

$(A \cap B)^{'} = A^{'} \cup B^{'}$ is countable, being union of two countable sets.

Hence $A \cap B \ \epsilon \ \mathcal{T}$ i.e. finite intersections belong to $\mathcal{T}$.

Conclusively $\mathcal{T}$ is a topology on X.

1.6. $\mathcal{T}_2$ −SPACES OR HAUSDORFF SPACES

Definition:

A topological space $(X, \mathcal{T})$ is said to be Hausdorff topological space (or Hausdorff space) if for $x, y \in X, x \neq y$ there exis $U, V \in \mathcal{T}$ such that

i. $x \in U, y \in V$,

ii. $U \cap V = \emptyset$

- Every discrete topological space is a $\mathcal{T}_2$-space.
- Every metric space is $\mathcal{T}_2$-space.

The complete description of Hausdorff space is in unit 10.

CHECK YOUR PROGRESS

2. Let A and B be subsets of topological space $(X,\mathcal{T})$. Then $(A\cup B)' \neq A' \cup B'$ .True/False

3. $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$ is a topology for $X = \{a, b, c\}$ True/False

4. $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$ is a topology for $X = \{a, b, c\}$
Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is topology True/False

5. $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$ is a topology for $X = \{a, b, c\}$
Then $\mathcal{T}_1 \cup \mathcal{T}_2$ is topology .True/False

6. An indiscrete topology has only elements

a) 1

b) 2

c) 3

7. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be usual topology and half-open topology on $\mathbb{R}$ respectively. Then

a) $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$.

b) $\mathcal{T}_2$ is weaker than $\mathcal{T}_1$.

c) They are not comparable.

8. The collection $\{(a, b) \subseteq \mathbb{R}: a, b \epsilon \mathbb{Q}\}$ is a.........for a topology on $\mathbb{R}$.

a) Basis

b) Not Basis

c) None of the above

1.7 SUMMARY

This unit is an explanation of

i. Definition of Topology on a set in a simple form.

ii. Different type of Topology defined with examples.

iii. Basis for a Topology.

iv. Standard topology and Lower limit topology in simple and clear manner.

v. the idea of Subbasis for s topology with example.

1.8 GLOSSARY

i.Functions

ii. Relations

iii. The integers and the Real Numbers

iv. Arbitrary Cartesian Products

v.Finite Sets

vi. Topology

vii. Basis for a Topology.

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1.10 *SUGGESTED READINGS*

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1.11 *TERMINAL QUESTIONS*

1. Show that for any collection of topologies on X there exists a unique smallest topology larger than each member of the collection?

2. Give an example to show that the union of an infinite collection of closed sets in a topological space is not necessarily closed.

3. Give an example of a topological space different from the discrete and indiscrete spaces in which open sets are exactly the same as closed sets.

4. Show that $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is a topology on $X = \{a, b, c\}$.

5. Let $(X, \mathcal{T})$ be a topological space. Then a subfamily $\mathfrak{B}$ of $\mathcal{T}$ is called base for $\mathcal{T}$.

6. Let $(X, \mathcal{T})$ be a topological space. Then a subfamily $\mathfrak{B}$ of $\mathcal{T}$ is called base for $\mathcal{T}$ if

   a) Every member of $\mathfrak{B}$ can be expressed as union of some members of $\mathfrak{B}$.
   b) Every member of $\mathfrak{B}$ can be expressed as finite 1 intersection of some members of $\mathfrak{B}$.
   c) Every member of $\mathfrak{B}$ can be expressed as union of some members of $\mathfrak{B}$.

7. Let X be a set and S is a family of subset of X and $\mathcal{T}$ be a topology on X generated by S. Then

   a) S is the basis for $\mathcal{T}$.
   b) S is the sub basis for $\mathcal{T}$.
   c) None of these.

8. For a set x, which among the following is the weakest topology that can be defined on X
   a) cofinite topology
   b) discrete topology
   c) indiscrete topology.

9. For a set x, which among the following is the strongest topology that can be defined on X

a) cofinite topology

b) discrete topology

c) indiscrete topology

1.13 *ANSWERS*

CHECK YOUR PROGRESS

1. 29 topologies. Construct according to definition of topology.

2. False

3. True

4. True

5. False

6. b

7. c

8. a

TERMINAL QUESTIONS

2. Let $(\mathbb{R}, \mathcal{T}_{usual})$ be the usual topological space and let $F_n = \{1/n, 1\}, n\epsilon\mathbb{N}$.

3. Let $X = \mathbb{R}$ and let $\mathcal{T}$ consist of the empty set and all those subsets G of $\mathbb{R}$ having the property that $x\epsilon G$ implies $-x\epsilon G$. Then $(\mathbb{R}, \mathcal{T})$ is the required space.

6. c
7. b
8. c
9. b

UNIT 2: ORDER TOPOLOGY, PRODUCT TOPOLOGY AND SUBSPACE TOPOLOGY

CONTENTS:

2.1 INTRODUCTION

In previous unit we have studied abut definition and example of topological space and basis for a topology. Now in this unit we are defining the order topology, product topology and subspace topology.

In mathematics, an ordinal topology is a particular topology that can be defined over any perfectly ordered set. This is a natural generalization of the topology of the real numbers to any totally ordered set.

In the field of topology product spaces are Cartesian products of the family of topological spaces with a natural topology called product topology.

A subset of a topological space has a naturally induced topology, called the subspace topology. In geometry, the subspace topology is the source of all funky topologies. In the area of topology a subspace of a topological space which is equipped with same topology called the subspace topology (or the relative topology, or the induced topology, or the trace topology).

2.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of order topology.

ii. Describe the notion of product topology.

iii. Explain the concept of subspace topology.

2.3 ORDER TOPOLGY

If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called "Order Topology".

Definition 1: Suppose X is a set having order relation $<$. Given $a < b \in X$, there are four subsets of X called intervals determined by a and b. They are as follows:

$(a, b) = \{x \mid a < x < b\}$
$(a, b] = \{x \mid a < x \leq b\}$
$[a, b) = \{x \mid a \leq x < b\}$
$[a, b] = \{x \mid a \leq x \leq b\}$

Definition 2: Let X be a simple order relation. Assume that X has more than one element. Let $\mathfrak{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathfrak{B}$ is a basis for a topology on X, which is called the **order topology.**

Remark: If X has no smallest element then there are no sets of type **2** and if X has no largest element then there are no sets of type **3.**

Note: One has to check that $\mathfrak{B}$ satisfies the requirements for a basis:

Let $x \epsilon X$.

If $x = a_0$ then $x \in [a_0, b)$.

If $x = a_0$ then $x \in (a, b_0]$.

These two cases are easy to verify.

Let us consider the case $x \neq a_0$ and $x \neq b_0$.

From definition, $(a, b) = \{z | a < z < b\}$.

As the order relation $<$ is defined on the set X, there will some element $a \in X$ and some element $b \epsilon X$ such that $a < x < b$.

In that case $x \in (a, b)$.

Hence we have verified that $\mathfrak{B}$ satisfies the first property for being a basis over the order topology on X.

Now we need to check for the second which $\mathfrak{B}$,

$\mathfrak{B}$ needs to satisfy for being a basis over the order topology on X.

Let us take $B_1 = [a_0, b)$, $B_2 = (a, b_0], B_3 = (a_1, b_1)$ and $B_3 = (a_2, b_2)$, where a_0 is the smallest element in X and b_0 is the largest element in X. The sets$B_1, B_{2,} B_3, B_4$ all belong to $\mathfrak{B}$.

If $x \in B_1 \cap B_2$, then $x \in (a, b)$.

The set (a, b) is thus subset of $B_1 \cap B_2$.

Hence $x \in (a, b) \subset B_1 \cap B_2 \in \mathfrak{B}$.

Thus if X has both largest and smallest element, then $\mathfrak{B}$ satisfies the property for being a basis.

We shall now observe the case that X has no largest element.

If $x \in B_1 \cap B_3$. Depending on the value of b, a_1 and b_1, we will be able to a set of the form (p, q), such that $x \in (p, q) \subset B_1 \cap B_3 \in \mathfrak{B}$.

Thus if X has no largest but a smallest element, then $\mathfrak{B}$ satisfies the property for being a basis.

In last we will consider the case when X has no largest element or smallest element.

So the basis of the order topology will consist of sets of the form (e, d), such that $x \in (e, d) \subset B_3 \cap B_4 \in \mathfrak{B}$.

Thus if X has no largest and smallest element, then $\mathfrak{B}$ satisfies the property for being a basis.
Thus we have considered all possible conditions which should be checked for $\mathfrak{B}$ for being a basis for the order topology on X.
Since $\mathfrak{B}$ has satisfied all of them, it is basis for the order topology on X.

Some examples are as following:

Example 1: The standard topology on $\mathbb{R}$ is just the order topology derived from the usual order on $\mathbb{R}$.

Example 2: Consider the set $\mathbb{R} \times \mathbb{R}$ in the dictionary order; we shall denote the general element of $\mathbb{R} \times \mathbb{R}$ by $x \times y$, to avoid difficulty with notation. The set $\mathbb{R} \times \mathbb{R}$ has neither a largest nor a smallest element, so the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection of all open intervals of the form $(a \times b, c \times d)$ for $a < c$ and collection of all open intervals of the form $(a \times b, a \times d)$ when $a = c, b < d$.

Example 3: The positive integers $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology on $\mathbb{Z}_+$, is the discrete topology, as the singletons are open sets. If $n > 1$, then the singleton $\{n\} = (n-1, n+1)$ is a basis element; and if $n = (n-1, n+1)$ is basis element; and if $n = 1$, the one-point set $\{1\} = [1,2)$ is basis element.

Example 4: The set $X = \{1,2\} \times \mathbb{Z}_+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting $1 \times n$ by an and $2 \times n$ by bn , we can represent X by a_1, a_2,.....; b_1, b_2,.....;

The order topology on X is not the discrete topology. Most one-point sets are open, but there is an exception the one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition), and any basis element containing b_1 contains points of the ai sequence.

RAYS

Definition 3: If X is an ordered set and a is an element of X, there are four subsets of X that are called the rays determined by a.

They are following:

$(-\infty, a) = \{x \mid x < a\}$
$(a, +\infty) = \{x \mid x > a\}$
$(-\infty, a] = \{x \mid x \leq a\}$
$[a, +\infty) = \{x \mid x \geq a\}$.

The sets of type $(-\infty, a)$ and $(a, +\infty)$ are called open rays. Similarly $(-\infty, a]$ and $[a, +\infty)$ are called closed rays.

We now need to verify that the open rays belong to the order topology on X. We shall first consider the open ray $(a, +\infty)$.
If X contains a largest element b_0 then $(a, +\infty)$ is of the form $(a, b_0]$ which is a basis element for the order topology on X.

Thus in this case $(a, +\infty)$ is open.
If X has no largest element, then $(a, +\infty) = \bigcap_{x>a}(a, x)$.
To prove this statement, let $z \in (a, +\infty)$.
From definition, $z > a$.
Hence if we consider $\bigcap_{x>a}(a, x), z$ will belong at least one of the subsets of the form (a, x).

Thus $z \in \bigcap_{x>a}(a, x)$.

Hence $(-\infty, a) \subset \bigcap_{x>a}(a, x)$.

Now let $p \in \bigcap_{x>a}(a, x)$.

So p will belong to at least one of the open intervals of the form (a, x).

If p belongs to some open interval of the form (a, x), then from definition $a < p < x$.

From definition of the open rays, $z \in (a, +\infty)$.

Thus $\bigcap_{x>a}(a, x) \subset (-\infty, a)$.

Also we have proven $(-\infty, a) \subset \bigcap_{x>a}(a, x)$.

So $(-\infty, a) = \bigcap_{x>a}(a, x)$.

We shall now consider the open ray $(-\infty, a)$.

If X contains a smallest element a_0 then $(-\infty, a)$ is of the form $[a_0, a)$, which is a basis element for the order topology on X.

Thus in this case $(-\infty, a)$ is open.

If X has no smallest element, then $(-\infty, a) = \bigcap_{x<a}(x, a)$.

To prove this statement, let $w \in (-\infty, a)$.

From definition $w < a$.

Hence if we consider the set $\bigcap_{x<a}(x, a), w$ will belong to at least one of subsets of the form (x, a).

Thus $w \in \bigcap_{x<a}(x, a)$.

Hence $(-\infty, a) \subset \bigcap_{x<a}(x, a)$.

Now let $q \in \bigcap_{x<a}(x, a)$.

So q will belong to at least one of open intervals of the form (x, a).

If p will belong to some open interval of the form (x, a), then from definition $x < q < a$.

From definition of the open rays $q \in (-\infty, a)$.

Thus $\bigcap_{x<a}(x, a) \subset (-\infty, a)$.

Also, we have proven that $(-\infty, a) \subset \bigcap_{x<a}(x, a)$. So $(-\infty, a) = \bigcap_{x<a}(x, a)$.

Hence both the open rays belong to the order topology on X.

Remark: The open rays form a sub-basis for the order topology on X.

Proof. The open rays $(-\infty, a)$ and $(a, +\infty)$ are open sets in the order topology defined on X.

Hence the topology generated by $(-\infty, a)$ and $(a, +\infty)$ are contained in the order topology on X.

If $\mathcal{T}_R$ be the topology generated by the open intervals and if $\mathcal{T}$ be the order topology on X, then we write $\mathcal{T}_R \subset \mathcal{T}$.

If we consider the intersection of the open rays of the form $(-\infty, b)$ and $(a, +\infty)$, then it is the open interval of the form (a, b).

The set (a, b) is a basis element of the order topology on X.

If X has a smallest element a_0, then $(-\infty, b)$ is of the form $[a_0, b)$.

Then the intersection of $[a_0, b)$ with $(a, +\infty)$ will produce an interval of the form (a, b), which is a basis element for the order topology on X.

Similarly, if X has a largest element b_0, then (a, ∞) is of the form $(a, b_0]$.

Then the intersection of $(a, b_0]$ with $(-\infty, b)$, will produce an interval of the form (a, b) which is a basis element for the order topology on X.

For both the largest element and smallest element cases, we have assumed that the intersection between the sets is non-empty. If it is empty, then the basis elements of the form $(a, b_0]$ or $[a_0, b)$ which both again are subsets of the order topology on X.

Thus, finite intersection of the open rays yield the basis elements for the order topology on X.

Thus, finite intersection of the open rays yield the basis elements for the order topology on X.

Also $X = (-\infty, a) \cap (a, +\infty)$. Hence the open rays satisfy the criteria for being a sub-basis for the order topology on X. ■

CHECK YOUR PROGRESS

1. The order topology of a simply ordered set X is the standard topology defined using the order relation. **True\False.**

2. The order topology on the real line is the standard topology. **True\False**

3. The open rays not form a sub-basis for the order topology on X. **True\False**

4. The order topology on the set of positive integers is the discrete topology. **True\False**

5. The order topology on the product $\{1,2\} \times \mathbb{Z}_+$ is the dictionary order has a basis consisting of all singletons except $(2,1)$ and all the intervals $((1,n),(2,1)]$ **True\False**

2.4 PRODUCT TOPOLGY ON $X \times Y$

A product topology is a type of topology in which the open sets of a product space are defined as the union of open sets from each of the individual spaces.

In other words, it's a way of defining the topology of a space made up of multiple other spaces, by specifying how the open sets of each of those spaces interact with each other.

This is useful in many areas of mathematics, including algebraic topology and differential geometry.

Definition 4: Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathfrak{B}$ of all sets of the form $U \times V$ where U is an open set in X and V is an open set in Y.

We need to check whether $\mathfrak{B}$ is a basis over $X \times Y$.

Let $(x, y) \epsilon X \times Y$.

The collection $\mathfrak{B}$ contains elements of the form $U \times V$, where U is an open set in X and V is an open set in Y.

So $U\ \epsilon X$ and $V\ \epsilon Y$.

The element (x, y) belongs to the product topology on $X \times Y$.

So there must be some $U\ \epsilon X$ and $V\ \epsilon Y$ such that $x\ \epsilon X$ and $y\ \epsilon V$.

Thus $(x, y) \epsilon U \times V \subset X \times Y$. Now $U \times V \epsilon\ \mathfrak{B}$.

So the elements of the set $\mathfrak{B}$ satisfy the first criteria for being a basis of the product topology on $X \times Y$.

Let us take $B_1 \in \mathfrak{B}$ and $B_2 \in \mathfrak{B}$ such that $B_1 = U \times V$ and $B_2 = T \times W$.

The sets U and T are open in X and the sets V and W are open in Y.

So we can write $B_1 \cap B_2 = (U \times V) \cap (T \times W)$.

Now $B_1 \cap B_2$ can also written as $(U \times V) \cap (T \times W)$.

If $(a, b) \in B_1 \cap B_2$, then $(a, b) \in (U \cap T) \times (V \cap W)$.

Since the sets U and T are open in X and the sets V and W are open in Y, so $(U \cap T)$ and $(V \cap W)$ are open in X and Y respectively.

Let $U_0 = (U \cap T)$and $V_0 = (V \cap W)$.

Thus we have $(a, b) \in (U_0 \cap V_0) \subset (U \cap T) \times (V \cap W)$. Also $(U_0 \cap V_0) \in \mathfrak{B}$.

Thus the elements of $\mathfrak{B}$ satisfy the two necessary conditions for a basis of the product topology on $X \times Y$.

Hence the elements of $\mathfrak{B}$ form a basis.

Theorem 1: If $\mathfrak{B}$ be the basis for a topology on X and $\mathcal{P}$ be the basis for a topology on Y, then, $\mathcal{D} = \{B \times C | B \in \mathfrak{B}\ and\ C \in \mathcal{P}\}$ is the basis for the topology on $X \times Y$.

Proof. Let us consider the element $W \times T$ which belongs to the product topology on $X \times Y$.

By definition of product topology, there exists an element $U \times V$ in $X \times Y$ such that $(x, y) \in\ U \times V, x\ \epsilon U$ and $y\ \epsilon V$.

The sets $\mathfrak{B}$ and $\mathcal{P}$ are bases for X and Y respectively.

Hence we can find $B\ \epsilon \mathfrak{B}$ and $C\ \epsilon \mathfrak{B}$ such that $x \in B \subset U$ and $y \in C \subset V$.

So we can write $(x, y) \in\ B \times C \subset\ W \times T$.

If $\mathcal{D} = \{B \times C | B \in \mathfrak{B}\ and\ C \in \mathcal{P}\}$, then from **lemma 1** of previous unit $\mathcal{D}$ satisfies the property for being a basis for the product topology on $X \times Y$. ■

Definition 5: Let $\pi_1 : X \times Y \to X$ be defined by the $\pi_1(x, y) = x$.

Also let $\pi_2 : X \times Y \to Y$ be defined by the $\pi_2(x, y) = y$.

The maps π_1 and π_2 are called the projections of $X \times Y$ onto its first and second factors respectively.

The word "onto" is used here because the mapping is subjective.

Let X and Y be topological spaces.

Let us consider the product topology on $X \times Y$. Assume, $U \subset X$ and $V \subset Y$.

Then we have the following,

a) $\pi_1(u, y) = x$ and ${\pi_1}^{-1}(u) = (u, y)$.

b) $\pi_2(x, v) = v$ and ${\pi_2}^{-1}(v) = (x, v)$.

Theorem 2: The collection, $\mathcal{S} = \{{\pi_1}^{-1}(U) | U \in X\} \cup \{{\pi_2}^{-1}(V) | V \in Y\}$, is a sub-basis for product topology on $X \times Y$.

Proof: Let $\mathcal{T}_{\mathcal{S}}$ be the topology generated by $\mathcal{S}$ and let $\mathcal{T}$ be the product topology on $X \times Y$.

Each and every element of $\mathcal{S}$ belongs to $\mathcal{T}$.

So the arbitrary unions of finite intersections of the elements of $\mathcal{S}$ also belong to $\mathcal{T}$.

Hence $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}$.

On the other hand, all the sets of the form $U \times V$, where U and V are open in X and Y respectively, form the basis for the product topology on $X \times Y$.

Since $\mathcal{S}$ is the sub-basis for the product topology, the union of the elements of $\mathcal{S}$ generates the entire set $X \times Y$.

Also, if we consider the finite intersection of the elements of $\mathcal{S}$, ${\pi_1}^{-1}(U) \cap {\pi_2}^{-1}(V)$, then it will be equal to $U \times V$.

Thus $U \times V \in \mathcal{T}_{\mathcal{S}}$. So we have $\mathcal{T} = \mathcal{T}_{\mathcal{S}}$.

From this, we can conclude that the topology generated by $\mathcal{S}$ is same as the product topology on $X \times Y$. ■

Some examples are as given below:

Example4: Let us consider the order topology on $\mathbb{R}$. The product of this topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R}$. Now we can write $\mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^2$. By definition, the collection of all sets of the form $U \times V$, where U of the form (p, q) and V of the form (r, s) form the basis for the product topology on $\mathbb{R}^2$. By **theorem 1** of this unit the basis for the product topology on $\mathbb{R}^2$ can also be represented by: $\mathcal{D} = \{B \times C | B \in$

$\mathbb{R}$ and $C \in \mathbb{R}\}$. In the above both B and C are basis elements of $\mathbb{R}$ and is of the form (a, b).

Example 5: Let $\mathcal{T}_1 = \{\emptyset, \{1\}, X_1\}$ be a topology on $X_1 = \{1,2,3\}$ and $\mathcal{T}_2 = \{\emptyset, X_2, \{\mathrm{a}\}, \{\mathrm{b}\}\ \{\mathrm{a, b}\}, \{\mathrm{c, d}\}, \{\mathrm{a, c, d}\}, \{\mathrm{b, c, d}\}\}$ be a topology for $X_2 = \{a, b, c, d\}$. Find a base for the product topology $\mathcal{T}$?

Solution: Let $\mathfrak{B}_1$ be a base for $\mathcal{T}_1$ and $\mathfrak{B}_2$ be a base for $\mathcal{T}_2$. Then $\mathfrak{B} = \{\mathfrak{B}_1 \times \mathfrak{B}_2 : \mathfrak{B}_1 \in \mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{B}_2\}$ is a base for the product topology $\mathcal{T}$.

We can take $\mathfrak{B}_1 = \{\{1\}, X_1\}, \mathfrak{B}_2 = \{\{a\}, \{b\}, \{c, d\}\}$, The elements of $\mathfrak{B}$ are $\{1\} \times \{a\}, \{1\} \times \{b\}, \{1\} \times \{c, d\}, \{1,2,3\} \times \{a\}, \{1,2,3\} \times \{b\}, \{1,2,3\} \times \{c, d\}$.

This is to say

$\mathfrak{B} = \{1, a\}, \{1, b\}, \{1, c\}, \{1, d\}, \{1, a\}, \{2, a\}, \{3, a\}, \{1, b\},$
$\{2, b\}, \{3, b\}, \{1, c\}, \{2, c\}, \{3, c\}, \{1, d\}, \{2, d\}, \{3, d\}$

is a base for $\mathcal{T}$.

CHECK YOUR PROGRESS

6. A product topology is a type of topology in which the open sets of a product space are defined as the union of ....

7. The product topology on $X \times Y$ is the topology having as .... the collection $\mathfrak{B}$ of all sets of the form $U \times V$ where U is an open set in X and V is an open set in Y.

2.5 SUBSPACE TOPOLOGY

Definition 6: Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection $\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$ is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace of** X; its open sets consist of all intersections of open sets of X with Y.

- It is easy to see that $\mathcal{T}_Y$ is a topology.

Lemma 1: If $\mathfrak{B}$ is basis for the topology of X, then the collection

$$\mathfrak{B}_Y = \{B \cap Y | B \in \mathfrak{B}\}$$

is a basis for the subspace topology on Y.

Proof. Given U open in X and given $y \in U \cap Y$,
we can choose an element B of $\mathfrak{B}$ such that $y \in B \subset U$.
Then $y \in B \cap Y \subset U \cap Y$.
It follows from Theorem 1 from unit 1 that $\mathfrak{B}_Y$ is a basis for the subspace topology on Y. ■

When dealing with a space X and a subspace Y, one needs to be careful when one uses the term "open set". Does one mean an element of the topology of Y or an element of the topology of X? We make the following definition: If Y is a subspace of X, then a set U is open in Y (or open relative to Y) if it belongs to the topology of Y: this implies in particular that it is a subset of Y. We say that U is open in X if it belongs to the topology of X. There is a special situation in which every set open in Y is also open in X.

Lemma 2: Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in $Y, U = Y \cap V$ for some set V open in X.
Since Y and V are both open in X, so is $Y \cap V$. ■

Theorem 2: If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. The set $U \times V$ is the general basis element for $X \times Y$,
where U is open in X and V is open in Y.
Therefore, $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$.
Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

The conclusion we draw is that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the topologies are the same. ■

Example 5: Consider the subset $Y = [0,1]$ of the real line $\mathbb{R}$, in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$, where (a, b) is an open interval in $\mathbb{R}$. Such a set is of one of the following types:

$$(a,b) \cap Y = \begin{cases} (a,b) & \text{if } a, b \in Y \\ [a,b) & \text{if } a \notin Y, b \in Y \\ (a,1] & \text{if } b \notin Y, a \in Y \\ Y \text{ or } \phi & \text{if } b \notin Y, a \notin Y \end{cases}$$

By definition, each of these sets is open in Y . But sets of the second and third types are not open in the larger space $\mathbb{R}$.

Note: These sets form a basis for the order topology on Y.

Thus, we see that in the case of the set $Y = [0,1]$ its subspace topology and its order topology are the same.

Example 6: Let Y be the subset $[0,1) \cup \{2\} \in \mathbb{R}$. In the subspace topology on Y the one-point set $\{2\}$ is open, because $\{2\} = \left(\frac{3}{2}, \frac{5}{2}\right) \cap Y$.

But in the order topology on Y , the set $\{2\}$ is not open.

Any basis element for the order topology on Y that contains 2 is of the form $\{x | x \in Y \; and \; a < x \leq 2\}$.

For some $a \in Y$, such a set necessarily contains points of Y less than 2.

Example 7: Let $I = [0,1]$ The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$.

However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

- For example, the set $\{l/2\} \times (1/2,1]$ is open in $I \times I$ in the subspace topology, but not in the order topology.

Definition 7: The set $I \times I$ in the dictionary order topology is called the ordered square, and is denoted by I_0^2.

Definition 8: Given an ordered set X , let us say that a subset Y of X is convex in X if for each pair of points $a < b$ of Y , the entire interval (a, b) of points of X lies in Y.

- Intervals and rays in X are convex in X.

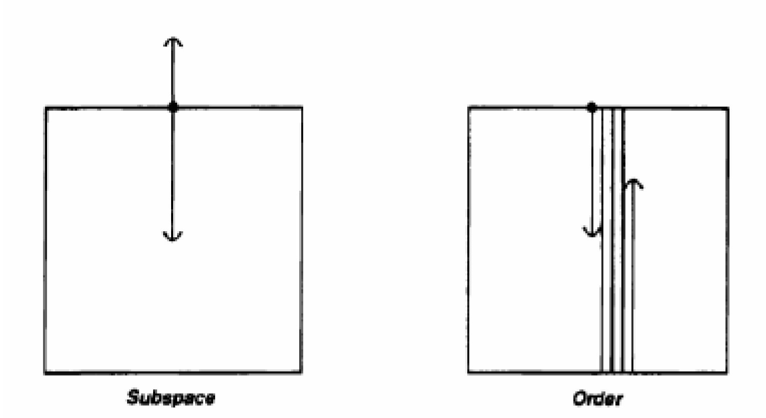

Fig 2.1

Theorem 3: Let X be an ordered set in the order topology, let Y be a subset of X that is convex in X.

Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof: Consider the ray $(a, +\infty)$ in X. If $a \in Y$, then

$$(a, +\infty) \cap Y = \begin{cases} \{x : x \in Y, x > a\}, a \in Y \\ Y, \text{a} = \text{lower bound on } Y, \text{a} \notin \text{Y} \\ \emptyset, \text{a} = \text{lower bound on } Y, \text{a} \notin \text{Y}. \end{cases}$$

A similar remark shows that the intersection of the ray (∞, a) with Y is either an open ray of Y, or Y itself, or empty.

Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a \cap Y)$ form a sub-basis for the subspace topology on Y.

And since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of Y equals the intersection of an open ray of X with Y.

So it is open in the subspace topology on Y.

Since the open rays of Y are a sub-basis for the order topology on Y, this topology is contained in the subspace topology. ■

CHECK YOUR PROGRESS

8. If Y is a subspace of X It is possible for a set U to be open in Y but not open in X. Explain by example..

2.6 SUMMARY

This unit is explanation of

1. Concept of order topology.

2. Definition and examples of product topology and subspace topology.

3. These concepts explained with the help of definitions, examples, different lemma and different theorem with proves.

2.7 GLOSSARY

1. Set.
2. Subset.
3. Topology on a set.
4. Function.
5. Relation.
6. Basis for a Topology.

7. Order topology.

8. Product topology.

9. Subspace topology.

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2.10 *TERMINAL QUESTIONS*

1. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

2. Proof that the subspace topology satisfies the axioms for a topology.

3. What is the product topology in topology**?..**

4. What is the order topology in topology**?..................................**

5. What is the subspace topology in topology**?..................................**

2.11 *ANSWERS*

CHECK YOUR PROGRESS

1. True.

2. True.

3. False.

4. True.

5. True.

6. Open sets.

7. Basis.

8. Let $X = R$ and $Y = [0, 2]$ where R has the standard topology and Y has the subspace topology. Then $U = [0, 1)$ is open in Y since $U = (-1, 1) \cap [0, 2]$ where $(-1, 1)$ if open in R, but U is not open in R. The following result gives a condition under which open sets in the subspace topology are also open in the "superspace" topology.

UNIT 3: CLOSED SETS AND LIMIT POINTS, CONTINUOUS FUNCTIONS.

CONTENTS:

3.1 INTRODUCTION

In previous unit we have defined the order topology, product topology and subspace topology. For continuation of the work related to topology we have need the different concepts Now in this unit define the notion of closed set, Limit point: Derived set, Continuity: Continuous Map and Continuity on a Set**,** Homeomorphism, Open and Closed Map which will be beneficial to understand the concept of topology theory.

3.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the closed set.
ii. Explain the concept of limit points and derived set.
iii. Describe the concept of continuity.

3.3 CLOSED SETS

A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Example 1: The subset $[a, b]$ of R is closed because it's complement
$R - [a, b] = (-\infty, a) \cup (b, +\infty)$, is open.
Similarly, $[a,+\infty)$ is closed, because its complement $(-\infty, a)$ is open. These facts justify our use of the terms "closed interval" and "closed ray". The subset $[a, b)$ of R is neither open nor closed.

Example 2: In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

Example 3: Consider the following subset of the real line:

$Y = [0,1] \cup (2,3)$,in the subspace topology.

In this space, the set $[0,1]$ is open, since it is the intersection of the open set $\left(-\frac{1}{2}, \frac{3}{2}\right)$ of R with Y.

Similarly, $(2,3)$ is open as a subset of Y; it is even open as a subset of R. Since $[0,1]$ and $(2,3)$ are complements in Y of each other, we conclude that both $[0,1]$ and $(2,3)$ are closed as subsets of Y.

Example 4: In the plane R^2, the set $\{x \times y | x \geq 0 \text{ and } y \geq 0\}$ is closed, because its complement is the

union of the two sets $(-\infty, 0) \times R$ and $R \times (-\infty, 0)$,

each of which is a product of open sets of R and is therefore open in R^2.

Theorem1: Let X be a topological space. Then the following conditions hold:

i. $\emptyset$ and X are closed.

ii. Arbitrary intersections of closed sets are closed.

iii. Finite unions of closed sets are closed.

Proof:

i. $\emptyset$ and X are closed, because they are the complements of the open sets $\emptyset$ and X, respectively.

ii. Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J,}$ we apply DeMorgan's law, $X - \bigcap_{\alpha \in J} A_n = \bigcup_{\alpha \in J}(X - A_\alpha)$.Since the sets $X - A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\bigcap A_\alpha$ is closed.

iii. Similarly, if A_i is closed for $i = 1,2, \dots \dots \dots n$, consider the equation $X - \cup_{i=1}^{n} (X - A_i)$.

The set on the right side of this equation is finite intersection of open sets and is therefore open. Hence $\cup A_i$ is closed. ■

Theorem 2: Let Y be a sub space X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof: Assume that $A = C \cap Y$, where C is closed in X.
Then $X - C$ is open in X , so that $(X - C) \cap Y$ is open in Y,
by definition of the subspace topology.
But $(X - C) \cap Y = Y - A$.
Hence $Y - A$ is open in Y, so that A is closed in Y.

Conversely, assume that A is closed in Y.

Then $Y - A$ is open in Y, so that by definition it equals the intersection of an open set U of X with Y.
The set $X - U$ is closed in X, and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y, as desired. ■

3.1.1 INTERIOR OF A SET

Given a subset A of a topological space X.
A point $p \in A$ is called an interior point of A if p belongs to an open set G contained in A:

$$p \in G \subset A \text{ where } G \text{ is open.}$$

The set of interior points of A, denoted by int (A), $\overset{\circ}{A}$ or A° is called the interior of A.

Note:

The interior of A is defined as the union of all open sets contained in A.

- A° is open.
- A° is the largest open subset of A i.e. if G is an open subset of A then $G \subset A^\circ \subset A$;
- A is open iff $A = A^\circ$.

The exterior of A, written ext (A), is the interior of the complement of A, i.e. $\text{int}(A^c)$. The boundary of A, written $b(A)$, is the set of points which do not belong to the interior or the exterior of A.

3.1.2 CLOSURE OF A SET

Given a subset A of a topological space X, the closure of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by Int A or by $\dot{A}$ and the closure of A is denoted by $Cl\ A$ or by $\bar{A}$. Int A is an open set and $\bar{A}$ is a closed set:

Furthermore, int $A \subset A \subset \bar{A}$.

If A is open, $A =$ Int A; while A is closed, $A = \bar{A}$.

Theorem 3: Let A be any subset of a topological space X. Then the closure of A is the union of the interior and boundary of A, i.e.

$$\bar{A} = A^\circ \cup \text{b}(\text{A}).$$

Example 5: Consider the four intervals $[a,b],(a,b),[a,b),[a,b)$ whose endpoints are a and b. The interior of each is the open interval (a,b) and the boundary of each is the set of endpoints, i.e. $\{a,b\}$.

Example 6: Consider the topology
$\mathcal{T}=\big\{X,\emptyset,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\big\}$ on $X=\{a,b,c,d,e\}$, and the subset $A=\{b,c,d\}$ of X.
Here int $(A)=\{c,d\}$. ext $(A)=\{a,e\}$. $\text{int}(A^c)=\{a\}$ and $b(A)=\{b,e\}$.

Example 7: Consider the set $\mathbb{Q}$ of rational numbers. Since every open subset of $\mathbb{R}$ contains both rational and irrational points, there are no interior or exterior points of $\mathbb{Q}$; so $\text{int}(\mathbb{Q})=\emptyset$ and $\text{int}(\mathbb{Q}^c)=\emptyset$.
Hence the boundary of $\mathbb{Q}$ is the entire set of real numbers, i.e. $\text{b}(\mathbb{Q})=\mathbb{R}$.

A subset A of a topological space X, is said to be nowhere dense in X if the interior of the clousure of A is empty, i.e. int $(\bar{A})=\emptyset$.

Example 8: Consider the subset $A=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\dots\right\}$ of $\mathbb{R}$. Observe that $\bar{A}$ has no interior points; so A is nowhere dense in $\mathbb{R}$.

Example 9: Let A consist of the rational points between 0 and $1, i.e. A=\{x : x\in\mathbb{Q}, 0<x<1\}$, Observe that the interior of A is empty, i.e. int $(A)=\emptyset$. But A is not nowhere dense in $\mathbb{R}$; for the closure of A is $[0,1]$, and so $\text{int}(\bar{A})=\text{int}\ ([0,1])=(0,1)$.

Theorem 4: Let Y be a subpace of a topological space X; Let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A}\cap Y$.

Proof. Let B denote the closure of A in Y.

The set $\bar{A}$ is closed in X. So $\bar{A} \cap Y$ is closed in Y by Theorem 2.

Since $\bar{A} \cap Y$ contains A, and since by definition B equals the intersection of all closed subsets of Y containing A,

We must have $B \subset (\bar{A} \cap Y)$.

On the other hand, we know that B is closed in Y.

Hence by Theorem 2 $B = C \cap Y$ for some set C closed in X.

Then C is a closed set of X containing A: because $\bar{A}$ is the intersection of all such closed sets.

We conclude that $\bar{A} \subset C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$. ■

Theorem 5: Let A be a subset of the topological space X.

- **i.** Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.
- **ii.** Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof. Consider the statement in (i.)

It is a statement of the form $P \Leftrightarrow Q$.

Let us transform each implication to its contra positive, thereby obtaining the logically equivalent statement (not P) $\Leftrightarrow$(not Q).

Written out, it is the following:$x \notin \bar{A} \Leftrightarrow$ there exists an open set U containing x that does not intersect A.

If x is not in $\bar{A}$, the set $U = X - \bar{A}$ is an open set containing x that does not intersect A, as desired.

Conversely, if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A.

By definition of the closure $\bar{A}$, the set $X - U$must contain $\bar{A}$;

therefore , x cannot be in $\bar{A}$.

Statement (ii) follows readily.

If every open set containing x intersects A, so does every basis element B containing x intersects A.

So does every open set U containing x, because U contains a basis element that contains x. ■

3.1.3 NEIGHBORHOODS AND NEIGHBORHOOD SYSTEMS

Let p be a point in the topological space X.

A subset N of X is a neighborhood of p iff N is a superset of an open set G containing p:

$$p \in G \subset N \text{ where } G \text{ is an open set.}$$

In other words, the relation "N is a neighborhood of a point p" is the inverse of the relation "p is an interior point of N".

The class of all neighborhoods of $p \in X$, denoted by $\mathcal{N}_p$, is called the neighborhoods system of p.

Example 10: Let a be any real number, i.e. $a\epsilon\mathbb{R}$. Then each closed interval $[a-\delta, a+\delta]$, with centre a, is a neighborhood of a since it contains the open interval $(a-\delta, a+\delta)$ containing a. Similarly, if p is a point in the plane $\mathbb{R}^2$, then every closed disc $\{q \in \mathbb{R}^2 : d(p,q) < \delta \neq 0\}$, with center p, is a neighborhood of p since it contains the open disc with center p.

Neighborhood axioms:

- **i.** $\mathcal{N}_p$ is not empty and p belongs to each member of $\mathcal{N}_p$.
- **ii.** The intersection of any two members of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.
- **iii.** Every superset of a member of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

iv. Each member $\mathcal{N} \in \mathcal{N}_p$ is a superset of a member $G \in \mathcal{N}_p$ where G is a neighbourhood of each of it's points, i.e. $G \in \mathcal{N}_a$ for every $a \in G$.

- If A is a subset of the topological space X, then $x \in \bar{A}$ if and only if every neighbourhood of x intersects A.

Example 11: Let X be the real line R. If $A = (0, 1]$, then $\bar{A} = [0,1]$, for every neighborhood of 0 intersects$A$, while every point outside $[0,1]$ has a neighborhood disjoint from A.

- If $B = \{1/n | n \in Z_+\}$, then $\bar{B} = \{0\} \cup B$. If $C = \{0\} \cup (1,2)$, then $\bar{C} = \{0\} \cup [1,2]$. If Q is the set of rational numbers, then $\bar{Q} = R$. If Z_+ is the set of positive integers, then $\overline{Z_+} = Z_+$. If R_+ is the set of positive reals, then the closure of R_+ is the set $R_+ \cup \{0\}$.

Example 12: Consider the subspace $Y = (0, 1]$, of the real line R. The set $A = \left(0, \frac{1}{2}\right)$ is a subset of Y; its closure in R is the set $\left[0, \frac{1}{2}\right]$, and its closure in Y is the set $\left[0, \frac{1}{2}\right] \cap Y = \left(0, \frac{1}{2}\right]$.

3.2 LIMIT POINT

If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point (or "cluster point," or "point of accumulation") of A,

if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not.

Example 13: Consider the real line R. If $A = (0, 1]$, then the point 0 is a limit point of A and so is the point $\frac{1}{2}$. In fact, every point of the interval $[0,1]$ is a limit point of A, but no other of R is a limit point of A.

- If Q is the set of rational numbers, every point of R is a limit point of Q.
- If Z_+ is the set of positive integers, no point of R is a limit point of Z_+.
- If R_+ is the set of positive reals, then every point of $\{0\} \cup R_+$ is a limit point of R_+.
- Let $X = \{a, b, c\}$ with topology

$\mathcal{T} = \{\phi, \{a, b\}, \{c\}, X\}$ and $A = \{a\}$, then b is the only limit point of A, because the open sets b namely $\{a, b\}$ and X also contains a point of A.

Whereas, 'a' and 'b' are not limit point of $C = \{c\}$,
because the open set $\{a, b\}$ containing these points do not contain any point of C. The point c is also not a limit point of C, since then open set $\{c\}$ containing 'c' does not contain any other point of C different from c. Thus, the set $C = \{c\}$ has no limit points.

- Prove that every real number is a limit point of R.

Solution. Let $x \epsilon R$ then every neighbourhood of x contains at least one point of R other than x. Therefore x is a limit point of R. But x
was arbitrary. Therefore every real number is a limit point of R. ■

- Prove that every real number is a limit point of $R - Q$.

Solution. Let x be any real number, then every neighbourhood of X contains at least one point of $R - Q$ other than x. Therefore x is a limit point of $R - Q$. But x was arbitrary. Therefore every real number is a limit point of $R - Q$. ■

3.2.1 DERIVED SET

The set of all limit points of A is called the derived set of A and is represented by $D(A)$.

Theorem 6: Every derived set in a topological space is a closed.

Proof: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

First we are showing that $D(A)$ is a closed set.

As we know that B is a closed set if $D(A) \subset B$.

Hence, $D(A)$ is a closed set iff $D[D(A)] \subset D(A)$.

Let $x \in D[D(A)]$ be arbitrary,

then x is a limit point of $D(A)$ so that

$(G - \{x\}) \cap D(A) \neq \emptyset \forall G \in \mathcal{T}$ with $x \in G$

it implies that $(G - \{x\}) \cap A \neq \emptyset$.

It implies that $x \epsilon D(A)$. ■

Theorem 7: In any topological space, prove that $A \cup D(A)$ is closed.

Proof. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

For proving $A \cup D(A)$ is closed set.

Let $x \in X - A \cup D(A)$ be arbitrary then $x \notin A \cup D(A)$

so that $x \notin A, x \notin D(A)$.

$x \notin D(A)$ it implies that there exists $G \in \mathcal{T}$ with $x \in G$ such that $(G - \{x\}) \cap D(A) = \emptyset$ it implies that

$G \cap A = \emptyset (\because x \notin A) \dots \dots \dots \dots \dots \dots \dots . . (1)$

For this G, we also claim $G \cap D(A) = \emptyset$.

Let $y \in G$ be arbitrary. Now G is an open set containing $'y'$ such that $G \cap A = \emptyset$, showing that $y \notin D(A)$,

any $y \in G \Rightarrow y \notin D(A) = \emptyset$.

It implies that $G \subset X - A \cup D(A)$. T

his proves that x is an interior point of $X - A \cup D(A)$.

Since x is arbitrary point of $X - A \cup D(A)$.

Hence, every point of $X - A \cup D(A)$ is an interior point of $X - A \cup D(A)$. Therefore $X - A \cup D(A)$ is open.

It implies that $A \cup D(A)$ is closed. ■

Theorem 8: In any topological space, prove that $\bar{A} = A \cup D(A)$.

Proof. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

To show $\bar{A} = A \cup D(A)$

Since $A \cup D(A)$ is closed and hence

$\overline{A \cup D(A)} = A \cup D(A)$……………(1)

Since $A \subset A \cup D(A)$

Therefore $\bar{A} \subset \overline{A \cup D(A)} = A \cup D(A)$ [Using (1)].

$\bar{A} \subset A \cup D(A)$…………………………..(2)

Now, we are to prove that

$A \cup D(A) \subset \bar{A}$…………………………..(3)

But, $A \subset \bar{A}$……………………………….(4)

To prove (3), we are to prove $D(A) \subset \bar{A}$………………………………(5)

i.e., to show that $D(A) \subset \cap \{F \subset X : F \text{ is closed } F \supset A\}$……………….(6)

Let $x \in D(A)$ be arbitrary,

$x \in D(A) \Rightarrow x$ is a limit point of A.

$\Rightarrow x$ is a limit point of all those sets which contain A.

$\Rightarrow x$ is a limit point of all those F appearing on R.H.S of (6).

$\Rightarrow x \in D(F_i) \subset F_i (\because F$ is closed).

$\Rightarrow x \in F_i$ for each i.

$\Rightarrow x \in \cap \{F_i \subset X : F_i$ is closed$\}$

$\Rightarrow \ x \in \bar{A}$.

Thus any $x \in D(A) \Rightarrow x \in \bar{A}$.

$D(A) \subset \ \bar{A}$.

Hence the result (5) proved.

From (4) and (5), we get

$A \cup D(A) \subset \bar{A} \cup \bar{A} = \bar{A}$.

i.e., $A \cup D(A) \subset \bar{A}$

Hence the result (3) proved.

Combining (2) and (3) we get the required result. ■

Theorem 9: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$, then A is closed iff $A' \subseteq A$, or $A \supseteq D(A)$. A subset A of X in a topological space $(X, \mathcal{T})$ is closed iff A contains each of it's limit points.

Proof. Let A is closed $\Rightarrow A^c$ is open.

Let $x \in A^c$

then A^c is open set containing x but containing no point of A other than x.

This shows that x is not a limit point of A.

Thus, no point of A^c is a limit point of A.

Consequently, every limit point of A is in A and therefore $A^c \subseteq A$.

Conversely, Let $A' \subseteq A$,

To show : A is closed.

Let x be an arbitrary point of A^c.

Then $x \in A^c \Rightarrow x \notin A \Rightarrow x \notin A$ and $x \notin A'$ (Since $A' \subseteq A$).

$\Rightarrow x \notin A$ and x is not a limit point of A.

$\Rightarrow \exists$ an open set G such that $x \in G$ and $G \cap A = \emptyset \Rightarrow \ G \subseteq A^c$.

$\Rightarrow x \in G \subseteq A^c$.

$\Rightarrow A^c$ is the neighbourhood of each of it's points and therefore $A^c$is open.

Hence A is closed. ■

Example 14: Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ then $D(\{a\}) = \{c\}, D(\{c\}) = \{\emptyset\}$.

Example 15: If $\mathbb{R}$ is endowed with it's usual Euclidean topology then the derived set of the half-open interval $[0,1)$ is the closed interval $[0,1]$.

Example 16: Consider $\mathbb{R}$ with topology (open sets) consisting of the empty set and any subset of $\mathbb{R}$ that contains 1. The derived set of $A \coloneqq \{1\}$ is $A^c = \mathbb{R}/\{1\}$.

CHECK YOUR PROGRESS

1) The closure of a subset Y of a topological space is defined as the

a) union of all the closed set containing the subset Y

b) maximal closed set containing the subset Y.

c) intersection of all the closed set containing the subset Y.

2) Let X be a topological space and $A \subset X$. Then interior of $A\big(int(A)\big)$ is:

a) union of all open sets contained in A

b) largest open subset of X contained in A

c) Both (a) and (b)

3) The interior of an empty set $\emptyset$ is

a) $\emptyset$

b) Not defined

c) The entire space

4)

a) A subset A of a topological space X is said to be closed

b) The interior of A is defined as

c) $D(A)$.

d) x is a limit point of A

i. the union of all open sets contained in A.

ii. The set of all limit points of A

iii. if every neighborhood of x intersects A in some point other than x itself.

iv. if the set $X - A$ is open.

A. a)iv , b) i, c) iii, d) ii

B. a)i , b) iv, c) ii, d) iii

C. a)iv , b) i, c) ii, d) iii

D. None of the above

5) Consider the R with usual topology there is no limit points of set of integers. **True/False**

6) For the subset $\mathrm{A} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.\right\}$ of R. 0 is the limit point **True/False**

3.5 CONTINUOUS FUNCTION

3.5.1 CONTINUITY OF A FUNCTION

Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological spaces. A function $f: X \longrightarrow Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

In other words if $V \epsilon \mathcal{T}_Y$, then its inverse image $f^{-1}(V)\ \epsilon \mathcal{T}_x$.

Note: $\mathrm{f}^{-1}(\mathrm{V})$ is the set of all points x of X for which f(x) ∈ V; it is empty if V does not intersect the image set f(X) of f.

Note: Continuity of a function depends not only upon the function f itself, but also on the topologies specified for it's domain and range.

Note: f is continuous relative to specific topologies on X and Y.

Proposition 1. A function $f: X \longrightarrow Y$ is continuous if for each $x \in X$ and each neighborhood N of $f(x)$ in Y, the set $f^{-1}(N)$ is a neighborhood of x in X.

Proof. Let x be an arbitrary element of X and N an arbitrary neighborhood of $f(x)$ in Y.
Then, $f^{-1}(N)$ and contains x and by definition, is open in X.
Hence, for each $x \in X$ and each neighborhood N of $f(x)$ in Y , the set $f^{-1}(N)$ is a neighborhood of x in X.

Conversely, let for each $x \in X$ and each neighborhood N of $f(x)$ in Y , the set $f^{-1}(N)$is a neighborhood of x in X.
Let V be an arbitrary open subset of Y.

i. If $V \cap f(X) = \emptyset$ where $f(X)$ is the range of f, then $f^{-1}(V) = \emptyset$ and hence is open in X.
ii. If $V \cap f(X) \neq \emptyset$ then V is a neighborhood of each of its points (let $f(X)$ be one such point for some $x \in X$).

By assumption, $f^{-1}(V)$ $(\subseteq X)$must be a neighborhood of each of its points (including the said x) in X and hence, $f^{-1}(V)$ is open in X. ■

Proposition 2. If the topology of the range space Y is, given by a basis $\mathfrak{B}$. Let $f: X \longrightarrow Y$ where X is topological space. Then f is continuous. For proving the continuity of f it suffices to show that the inverse image of every basis element is open.

Suppose for each member $B \in \mathfrak{B}, f^{-1}(B)$ is an open subset of X. Let H be an open subset of Y; then $H = \bigcup_i B_i$, a union of members of $\mathfrak{B}$. But $f^{-1}[H] = f^{-1}[\bigcup_i B_i] = \bigcup_i f^{-1}[B_i]$ and each $f^{-1}[B_i]$ is open by hypothesis; hence $f^{-1}[H]$ is the union of open sets and is therefore open. Accordingly, f is continuous.

Proposition 3. A function $f: X \longrightarrow Y$ is continuous iff the inverse of each member of a basis $\mathfrak{B}$ for Y is an open subset of X.

Proposition 4. Let $\mathcal{S}$ be a subbasis for a topological space Y. Then a function $f: X \longrightarrow Y$ is continuous iff the inverse of each number of $\mathcal{S}$ is an open subset of X.

Proposition 4. Let X be a topological space with discrete topology and Y be any topological space then every function $f: X \longrightarrow Y$ is continuous.
For if H is any open subset of Y, it's inverse $f^{-1}[H]$ is an open subset of X Since every subset of a discrete space is open.

Theorem 10. Let $X$and Y be topological spaces. Let function $f: X \longrightarrow Y$.Then the following are equivalent:

- **i.** f is continuous.
- **ii.** For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.
- **iii.** For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.

Proof. i ⇒ ii.

Consider that f is continuous.

Lat A be a subset of X.

We show that if $x \epsilon \bar{A}$, then $f(x) \epsilon \overline{f(A)}$.

Let V be a neighborhood of $f(x)$.

Then $f^{-1}(V)$ is an open set of X containing x;

it must intersect A in some point y.

Then V intersect $f(A)$ in the point $f(y)$.

Hence $f(x) \epsilon \overline{f(A)}$.

ii ⇒ iii.

Let B be closed in Y and let $A = f^{-1}(B)$.

We are proving that A is closed in X.

We show that if $\bar{A} \subset A$.

We have $f(A) \subset B$.

Therefore, if x is a point of $\bar{A}$,

$f(x) \epsilon f(\bar{A}) \subset \overline{f(A)} \subset B = B$.

So that $x \epsilon f^{-1}(B) = A$. Therefore $\bar{A} \subset A$.

iii ⇒ i.

Let V be an open set in Y.

Let $B = Y - V$.

Now, $f^{-1}(V) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$,

so that $f^{-1}(V)$ is open. ■

Example 17: Consider the following topologies $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$ respectively:

$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$,

$\mathcal{T}^* = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$, Also consider the function $f: X \longrightarrow Y$ and $g: X \longrightarrow Y$ defined by the diagrams below:

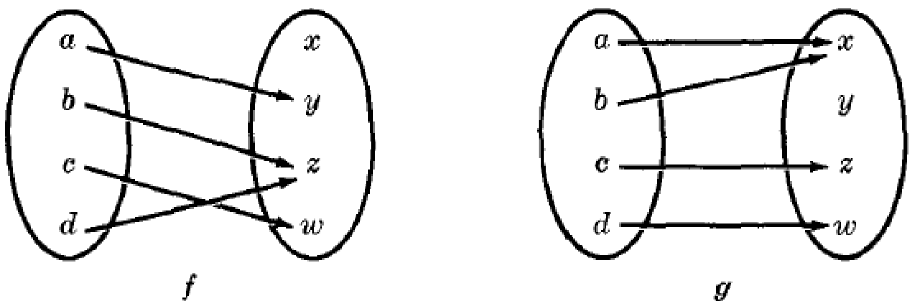
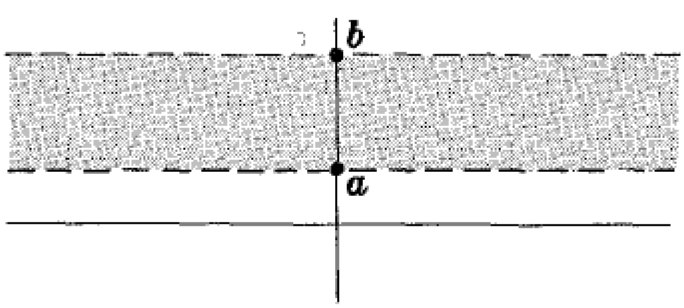


Fig 3.5.1

The function f is continuous since the inverse of each member of the topology $\mathcal{T}^*$ on Y is a member of the topology $\mathcal{T}$ on X. The function g is not continuous since $\{y, z, w\} \in \mathcal{T}^*, i.e.$ is an open subset of $g^{-1}[\{y, z, w\}] = \{c, d\}$ is not an open subset of $X, i.e.$ does not belong to $\mathcal{T}$.

Example 18: The projection mapping from the plane $\mathbb{R}^2$ into the line $\mathbb{R}$ are both continuous relative to the usual topologies. Consider, for example, the projection $\pi: \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi(x, y) = y$. Then the inverse of any open interval (a, b) is an infinite open strip as described below:



$\pi^{-1}[(a, b)]$ is shaded

Fig 3.5.2

From Proposition 3, the inverse of every open subset of $\mathbb{R}$ is open in $\mathbb{R}^2$, it means $\pi(x, y) = y$ is continuous.

- **How can you define a continuity in the context of topology and how does it differ from the concept of continuity in calculus?**

In topology, continuity is defined through open sets. This definition doesn't require any notion of distance or limit unlike in calculus. Continuity in calculus, on the other hand, is typically defined using limits. The key difference lies in their definitions' reliance on different structures. Topological continuity relies on the structure of open sets while calculus continuity depends on the structure of a metric space.

Rules for constrcting continuous functions:

Let X, Y, and Z be topological spaces:

(Constant function): If $f: X \longrightarrow Y$ maps all of X into the single point y_0 of Y, then f is continuous.

Proof. Let $f(x) = y_0$ for every x in X.

Let V be open in Y. The set $f^{-1}(V)$ equals X or $\emptyset$ depending on whether V contains y_0 or not. In either case, it is open. ■

(Inclusion): If A is a subspace of X, the inclusion function $j: A \longrightarrow X$ is continuous.

Proof. If U is open in X, then $j^{-1}(U) = U \cap A$, which is open in A by definition of the subspace topology. ■

(Composites): If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are continuous, then the map $gof: X \longrightarrow Z$ is continuous.

Proof. If U is open in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}\big(g^{-1}(U)\big)$ is open in X.

But $f^{-1}\big(g^{-1}(U)\big) = (gof)^{-1}(U)$. ■

(Restricting the domain): If $f: X \longrightarrow Y$ is continuous, and if A is a subspace of X, then the restricting function $f/A : A \longrightarrow Y$ is continuous.

Proof. Let $f: X \longrightarrow Y$ be continuous.

If $f(X) \subset Z \subset Y$, we show that the function $g: X \longrightarrow Z$ obtained from f is continuous.

Let B be open in Z.

Then $B = Z \cap U$ for some open set U of Y. Because Z contains the entire image set $f(X)$, $f^{-1}(U) = g^{-1}(B)$. Since $f^{-1}(U)$ is open, so is $g^{-1}(B)$.

To show $h: X \longrightarrow Z$ is continuous if Z has Y as a subspace, note that h is the composite of the map $f: X \longrightarrow Y$ and the inclusion map $j: Y \longrightarrow Z$. ■

(Local formulation of continuity): The map $f: X \longrightarrow Y$ is continuous if X can be written as union of open sets U_α such that f/U_α is continuous for each α.

Proof. By hypothesis, we can write X as a union of open sets U_α such that f/U_α is continuous for each α.

Let V be an open set in Y. Then

$$f^{-1}(V) \cap U_\alpha = (f/U_\alpha)^{-1}(V),$$

because both expression represent the set of those points x lying in U_α and hence open in X. But

$$f^{-1}(V) = \cup_\alpha \left(f^{-1}(V) \cap U_\alpha\right),$$

so that $f^{-1}(V)$ is also open in X. ■

(Continuity at each point): The map $f: X \longrightarrow Y$ is continuous if for each $x \in X$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset V$.

If the above condition holds for a particular point x of X, we say that f is continuous at the point x.

Proof. Let V be an open set of Y; let x be a point of $f^{-1}(V)$.

Then $f(x) \in V$, so that by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subset V$.

Then $U_x \subset f^{-1}(V)$.

It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open. ■

Theorem 11. (The pasting lemma) Let $X = A \cup B$, where A and B are closed in X.

Let $f: X \longrightarrow Y$ and $g: B \longrightarrow Y$ be continuous.

If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h: X \longrightarrow Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Proof. Let C be a closed subset of Y.

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$.

Since f is continuous, $f^{-1}(C)$ is closed in A and therefore closed in X. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. T
heir union $h^{-1}(C)$ is thus closed in X. ■

Note: This theorem also holds if A and B are open set in X.

Theorem 12.(Maps into products). Let $f: A \longrightarrow X \times Y$ be given by the equation $f(a) = \left(f_1(a), f_2(a)\right)$.
Then f is continuous if and only if the functions
$f_1: A \longrightarrow X$ and $f_2: A \longrightarrow Y$ are continuous.
The maps f_1 and f_2 are called the coordinate function of f.

Proof. Let $\pi_1: X \times Y \longrightarrow X$ and $\pi_2: X \times Y \longrightarrow Y$ be projections onto the first and second factors, respectively. These maps are continuous:
For ${\pi_1}^{-1}(U) = U \times Y$ and ${\pi_1}^{-1}(U) = X \times V$, and these sets are open if U and V are open. Note that for each $a \in A$,

$$f_1(a) = \pi_1(f(a)) \text{ and } f_2(a) = \pi_2(f(a))$$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions and therefore continuous. Conversely, suppose that f_1 and f_2 are continuous. We show that for each basis element $U \times V$ for the topology of $X \times Y$, it's inverse image $f^{-1}(U \times V)$ is open. A point a is in $f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, that is, if and only if
$f_1(a) \in U$ and $f_2(a) \in V$. Therefore,
$f^{-1}(U \times V) = {f_1}^{-1}(U) \cap {f_2}^{-1}(U)$.
Since both of the sets ${f_1}^{-1}(U)$ and ${f_2}^{-1}(U)$ are open, so is their intersection. ■

3.5.1 HOMEOMORPHISM

Let $X$and Y be topological spaces ; let function $f: X \longrightarrow Y$ be a bijection. If both the function f and the inverse function

$$f^{-1}: Y \to X$$

are continuous, then f is called a homeomorphism.

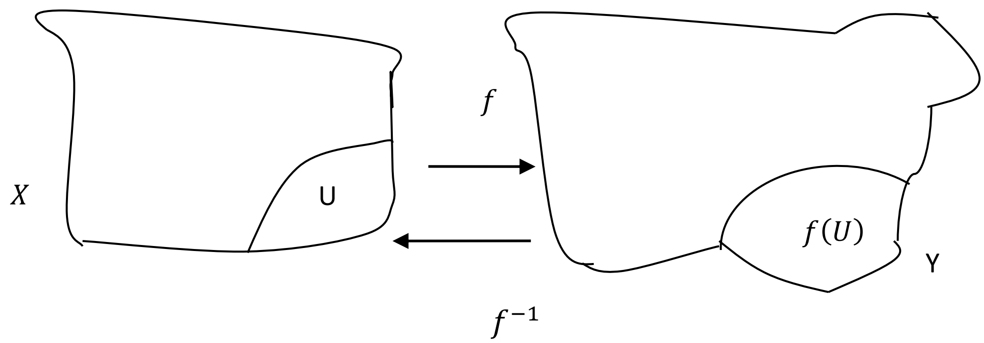


Fig 3.5.2

The condition that f^{-1} be continuous says that for each open set U of X, the inverse image of U under the map $f^{-1}: Y \to X$ is open in Y. But the inverse image of U under the map f^{-1} is the same as the image of U under the map f^{-1} is the same as the image of U under the map f.

Note: Homomorphism is a continuous stretching and bending of the object into a new shape.

Note: A homeomorphism $f: X \longrightarrow Y$ gives us a bijective correspondence not only between X and Y but between the collection of open sets of X and Y but between the collections of open sets of X and Y.

Note: Any property of X that is entirely expressed in terms of the topology of X yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called a topological property of X.

Note: An isomorphism is a bijective correspondence that preserves the algebraic structure involved. The analogous concept in topology is that of homemorphism; it is a bijective correspondence that preserves the topological structure involved.

Topological imbedding: Suppose that $f: X \longrightarrow Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f': X \longrightarrow Z$ obtained by restricting the range of f is bijective. If f' happens to be a homemorphism of X with Z. The map $f: X \longrightarrow Y$ will be called a topological imbedding or simply an imbedding of X in Y.

Example 14: Let $X = (-1,1)$. The function $f: X \longrightarrow \mathbb{R}$ defined by $f(x) = tan\frac{1}{2}\pi x$ is one-one, onto and continuous. Furthermore, the inverse function f^{-1} is also continuous. Hence the real line $\mathbb{R}$ and the open interval $(-1,1)$ are homeomorphic**.**

Example 15: Let X and Y be discrete spaces. Then as seen in Proposition – 4, all functions from one to the other are continuous. Hence X and Y are homemorphic iff there exists a one-one, onto function from one to the other,i.e. iff they are cardinally equivalent.

Note: The relation in any collection of topological spaces defined by "***X* is homeomorphic to *Y*"** is an equivalence relation.

CHECK YOUR PROGRESS

7. Let X and Y be topological spaces. Under which condition a function

$f : X \to Y$ is said to be continuous

a) If and only if preimages of open sets are open

b) If open sets in X are mapped to open sets in Y

c) If closed sets in X are mapped to closed sets in Y

8. Let a function $f : X \longrightarrow Y$ be a function and U, V be topologies on X, Y respectively. Then f is said to be continuous if

a) for all $V \epsilon V$, $f^{-1}(V) \epsilon U$

b) for all $V \epsilon U$, $f(V) \epsilon U$

c) none of the above

9. Which of the following statements about continuous functions are true?

a) inverse of continuous function is always continuous

b) continuous functions is always one-one

c) composition of continuous functions are continuous

10. Which of the following statements about continuous functions are false?

a) any function from a discrete space is continuous

b) any function into an discrete space is continuous

c) any function into an indiscrete space is continuous

11. The equivalent definition for a homeomorphic function f f is a bijection and f is a open map. **True\False.**

3.6 SOLVED EXAMPLES

Problem 1: The class

$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, c, d\}, \{b, c, d, e\}\}$ defines a topology on

$X = \{a, b, c, d, e\}$.

The closed subsets of X are $\emptyset$, $X, \{b, c, d, e\}$, $\{a, b, e\}$, $\{b, e\}$, $\{a\}$ that is, the complements of the open subsets of X.

Note that there are subsets of X, such as $\{b, c, d, e\}$, which are both open and closed, and there are subsets of X, such as $\{a, b\}$, which are neither open nor closed.

Problem 2: Let X be a discrete topological space,

i.e. every subset of X is open.

Then every subset of X is also closed since it's complement is always open. In other words, all subsets of X are both open and closed.

Problem 3: Let X be an indiscrete topological space, i.e. X and $\emptyset$ are the only open subsets of X.

Then X is the only open set containing any point $p \in X$.

Hence p is an accumulation point of every subset of X except the empty set $\emptyset$ and the set consisting of p alone,
i.e. the singleton set $\{p\}$. Accordingly, the derived set $D(A)$ of any subset A of X is as follows:

$$D(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{p\}^c = X/\{p\} & \text{if } A = \{p\} \\ X & \text{if } A \text{ contains} \\ & \text{two or more points.} \end{cases}$$

Problem 4: The class
$\mathcal{T} = \big\{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\big\}$ defines a topology on
$X = \{a,b,c,d,e\}$.
Consider the subset $A = \{a,b,c\}$ of X.
Observe that $b \in X$ is a limit point of A since the open sets containing b are $\{b,c,d,e\}$ and X, and each contains a point of A different from b,
i.e. c.
On the other hand, the point $a \in X$ is not a limit point of A since the open set $\{a\}$, which contains a, does not contain a point of A different from a. Similarly, the points d and e are limit points of A and the point c is not a limit point of A. So $A' = \{b,d,e\}$ is the derived set of A.

Problem 5: Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^*)$ be topological spaces defined as follows:
$X = \{A,B,C\}, \mathcal{T} = \{\emptyset, \{A\}, \{B\}, \{A,B\}, \{A,C\}, X\}$,
$Y = \{1,2,3\}, \mathcal{T}^* = \{\emptyset, \{1\}, \{1,2\}, Y\}$.
Let f and g be bijective mapping defined as
$f(A) = 1, f(B) = 2$ and $f(C) = 3$.
Then f is continuous since
$f^{-1}(\emptyset) = \emptyset, f^{-1}(\{1\}) = A, f^{-1}(\{1,2\}) = \{A,B\}, f^{-1}(Y) = X$

all of which are open in X.

However, its inverse maps g, with

$g(1) = A, g(2) = B, g(3) = C$ is not continuous since

$g^{-1}(\{C\}) = \{3\} \notin \mathcal{T}, g^{-1}(\{A, B\}) = \{1,3\} \notin \mathcal{T}^*$.

Problem 6: The unit step function $f: \mathbb{R} \longrightarrow \{0,1\}$ is given by

$$f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$$

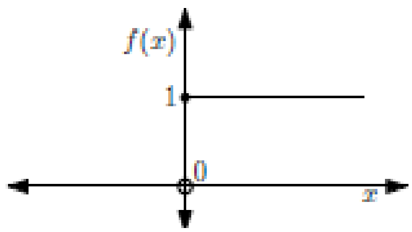


Fig 3.6.1

Let $\mathbb{R}$ be equipped with the standard topology, i.e., all open intervals are open, and the set $\{0,1\}$ be equipped with the discrete topology. Then, $u^{-1}(0) = (-\infty, 0)$ is open in the standard topology on $\mathbb{R}$, but $u^{-1}(1) = [0, \infty)$ is not. Hence, the unit step function is discontinuous.

Problem 7: Let R denote the set of real numbers in its usual topology, and R_l denote the same set is the lower limit topology.

Let $f: R \longrightarrow R_l$

Be the identity function;

$f(x) = x$ for every real number x.

Then f is not a continuous function; the inverse image of open set $[a, b)$ of R_l equals itself, which is not open R.

On the other hand, the identity function $g: R_l \longrightarrow R$ is continuous, because the inverse image of (a, b) is itself, which is open in R_l.

Problem 8: The function $f: R \longrightarrow R$ given by

$f(x) = 3x + 1$ is a homeomorphism.

If we define $g: R \longrightarrow R$ by the equation

$g(y) = \frac{1}{3}(y - 1)$

then $f\big(g(y)\big) = y$ and $g((f(x)\big) = x$ for all real numbers.

It follows that f is bijective and $g = f^{-1}$; and $f(x)$ and $g(y)$ is continuous.

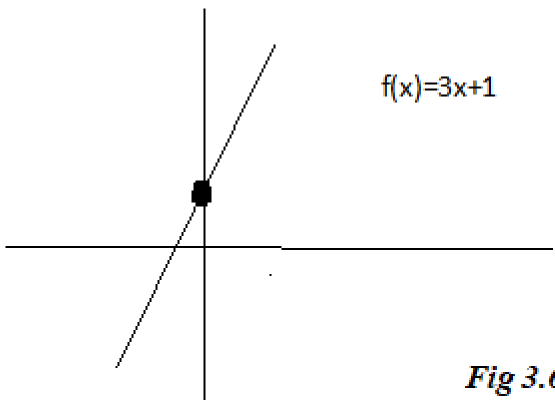


Fig 3.6.2

3.7 SUMMARY

This unit is complete combination of

- **i.** Definition closed set with examples.
- **ii.** Limit point and derived set with examples.
- **iii.** Continuity on a Set and point with different examples.
- **iv.** Homeomorphism with examples.

3.8 GLOSSARY

1. Set.
2. Subset.
3. Topology on a set.
4. Function.
5. Closed set.
6. Limit point.
7. Derived set.
8. Continuous maps.

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3.11 TERMINAL QUESTIONS

- In the plane R^2, the set $\{x \times y \mid x \geq 0 \;\&\; y \geq 0\}$is closed or open. Describe...
.
- Let A, B and A_α denote subsets of a space X. Prove the following

a) $\overline{A \cup B} = \bar{A} \cup \bar{B}$,

b) $\overline{\cup A_\alpha} = \cap \overline{A_\alpha}$.

- What is the concept of a neighborhood in topology? Can you list some of its properties?..
- What is the use of open and closed sets in general topology? Can you describe the relationship between them? ….................................
- Can you define a homeomorphism? What is its significance in topology? …………………………………………………………
- Consider $X = \{(x, y): x = 1/2^n, n = 1,2 \ldots \ldots .. y \in [0,1]\} \cup \{(0,0), (0,1)\}$ with the subspace topology inherited from R^2. Prove that any subset of X that is both open and closed and that contains $(0, 0)$ must also contain $(0, 1)$.
-

Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(a) $A = \{x \times y | y = 0\}$

(b) $B = \{x \times y | x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{x \times y | x \text{ is rational}\}$

(e) $E = \{x \times y | 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{x \times y | x \neq 0 \text{ and } y \leq 1/x\}$

12. Show that the subspace (a, b) of R is homeomorphic with $(0, 1)$, and the subspace $[a, b]$ of $R$is homeomorphic with $[0,1]$.

3.12 ANSWERS

CHECK YOUR PROGRESS

1. c.

2. c.

3. a.

4. C.

5. True.

6. True.

7. a.

8. a.

9. c.

10.b.

11. False**.**

TERMINAL QUESTIONS

1. Closed.

15.

Set	Bd	+	Int	=	Cl
A	A		$\emptyset$		A
B	$\{0\} \times \mathbb{R} \cup \mathbb{R}_+ \times \{0\}$		B		$\overline{\mathbb{R}}_+ \times \mathbb{R}$
C	$\mathbb{R}_- \times \{0\} \cup \{0\} \times \mathbb{R}$		$\mathbb{R}_+ \times \mathbb{R}$		$\mathbb{R}_- \times \{0\} \cup \overline{\mathbb{R}}_+ \times \mathbb{R}$
D	$\mathbb{R} \times \mathbb{R}$		$\emptyset$		$\mathbb{R} \times \mathbb{R}$
E	$\{(x,y) \| \|x\| = \|y\|$ or $x^2 - y^2 = 1\}$		$\{(x,y) \| 0 < x^2 - y^2 < 1\}$		$\{(x,y) \| 0 \leq x^2 - y^2 \leq 1\}$
F	$\{(x,y) \| (x \neq 0$ and $y = 1/x)$ or $x = 0\}$		$\{(x,y) \| x \neq 0$ and $y < 1/x\}$		$\{(x,y) \| (x \neq 0$ and $y \leq 1/x)$ or $x = 0\}$

UNIT 4: PRODUCT TOPOLOGY, METRIC TOPOLOGY AND QUOTIENT TOPOLOGY

CONTENTS:

4.1 INTRODUCTION:-

In previous unit we have defined the notion of closed set, Limit point: Derived set, Continuity: Continuous Map and Continuity on a Set**,** Homeomorphism. In this unit the concept of product topology, metric topology and Quotient topology are discussed in the simple manner. The product topology is also defined in unit-2 but in this unit we defined the generalization of product topology.

4.2 OBJECTIVES:-

After completion of this unit learners will be able to

i. Describe the generalisation of product topology.
ii. Explain the concept of metric topology.
iii. Define the concept of Quotient topology.

4.3 PRODUCT TOPOLOGY:-

In unit -2 we have defined a topology on the product $X \times Y$ of two topological spaces. In this unit the definition defined for arbitrary cartesian products. There are two ways of generalizing the definition; one that will later prove to be the more important we shall call the product topology.

- One way to impose a topology on a product space is the following; Is a direct generalization of the way we defined a basis for the product topology on $X \times Y$.

Definition. Let $\{X\}_{\alpha\in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha\in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha\in J} \boldsymbol{U_\alpha}$$

where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the box topology.

Note: This topology is not the most useful one for the product space $\prod X_\alpha$.

- A second method to generalize the previous definition is to generalize the subbasis formulation of the definition.

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ open in } X_\beta\},$$

and let $\mathcal{S}$ denote the union of these conditions,

$$\mathcal{S} = \cup_{\beta\in J} \mathcal{S}_\beta.$$

The topology generated by the subbasis $\mathcal{S}$ is called the product topology. In this topology $\prod_{\alpha\in J} X_\alpha$ is called a product space.

Theorem 4.3.1: (Comparison of the box and product topologies).

The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod \mathrm{U}_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod \mathrm{U}_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Note: Box topology is in general finer than the product topology.

Note: Whenever we consider the product $\prod X_\alpha$, we shall assume it is given the product topology unless we specifically state otherwise.

Theorem 4.3.2:

Suppose the topology on each space X_α is given by a basis $\mathfrak{B}_\alpha$. The collection of all sets of the form has as basis all sets of the form $\prod_{\alpha\epsilon J} \mathrm{B}_\alpha$, where $\mathrm{B}_\alpha \in \mathfrak{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha\epsilon J} \mathrm{X}_\alpha$.

The collection of all sets of the same form, where where $\mathrm{B}_\alpha \in \mathfrak{B}_\alpha$ for finitely many indices α and $\mathrm{B}_\alpha = \mathrm{X}_\alpha$ for all the remaining indices, will serve as a basis for the product topology on $\prod_{\alpha\epsilon J} \mathrm{X}_\alpha$. ■

Theorem 4.3.3:

Let $A_\alpha$be a subspace of X_α, for each $\alpha\epsilon J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both product are given the product topology. ■

Theorem 4.3.4:

Let $f: A \longrightarrow \prod_{\alpha\epsilon J} \mathrm{X}_\alpha$be given by the equation $f(a) = \big(X_\alpha\ (\mathrm{a})\big)_{\alpha\epsilon J}$, where $f: A \longrightarrow \mathrm{X}_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous. ■

Example 1: Consider Euclidean $n -$ space R^n. A basis for R consists of all open intervals in R; hence a basis for the topology of R^n consists of all products of the form

$$(a_1, b_1) \times (a_2, b_2) \dots \dots .(a_n, b_n).$$

Since R^n is a finite product, the box and product topologies agree. Whenever we consider R^n, we will assume that it is given this topology, unless we specifically state otherwise.

4.4 METRIC TOPOLOGY:-

Before describe the concept of metric topology the following definitions describe briefly.

Definition: A metric on a set X is a function

$$d: X \times X \rightarrow R$$

having the following properties:

$(\boldsymbol{M1})$ $d(x, y) \geq 0 \ \forall x, y \in X$ (self distance)

$(\boldsymbol{M2})$ $d(x, y) = 0$ if and only if $x = y \forall x, y \in X$ (Positivity)

$(\boldsymbol{M3})$ $d(x, y) = d(y, x); \forall x, y \in X$ (Symmetry property)

$(\boldsymbol{M4})$ $d(x, y) \leq d(x, z) + d(z, y); \forall x, y, z \in X$ (Triangle inequality)

A metric space is an ordered pair (X, d) where X is a nonempty set and d is a metric on X.

Example 2. Prove that with $d(x, y) = |x - y|$, the absolute value of the difference $x - y$, for each $x, y \in \mathbb{R}, (\mathbb{R}, d)$ is a metric space.

Proof. It is given that $d(x, y) = |x - y|$

Clearly we see that $d(x, y)$ satisfied $(M1)$,$(M2)$and $(M3)$conditions

Now for all $x, y, z \in \mathbb{R}$

$$\begin{aligned} d(x, y) &= |x - y| \\ &= |(x - z) + (z - y)| \\ &\leq |x - z| + |z - y| \\ &= d(x, z) + d(z, y). \end{aligned}$$

$d(x, y)$ satisfied $(M4)$ conditions. Hence $(\mathbb{R}, d)$ is a metric space.

Example 3. Let X be the set of all continuous functions $f : [a,b] \to \mathbb{R}$. Then$d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}$ is a metric on X.

Proof. It is given that $d(x,y) = |x-y|$

Clearly we can see that $d(x,y)$satisfied $(M1)$ and $(M3)$ conditions

$$(M2)\ d(f,g) = 0 \Leftrightarrow \sup\{|f(x) - g(x)|\} = 0 \Leftrightarrow |f(x) - g(x)| = 0$$

$$\Leftrightarrow f = g$$

$\Leftrightarrow d(x,y)$satisfied $(M2)$ conditions.

$$(M4) d(f,g) = \sup\{|f(x) - g(x)|\}$$

$$= \sup\{|f(x)| - h(x) + h(x) - g(x)|\}$$

$$\leq \sup\{|f(x) - h(x)|\} + \sup\{|h(x) - g(x)|\}$$

$$\leq d(f,h) + d(h,g)$$

Hence $d(f,g)$ is a metric on X.

Diameter: Let (X,d) be a metric space and let Y be a non empty subset of X. Then the diameter of Y, denoted by $\delta(Y)$ be defined as

$$\delta(Y) = \sup\{d(x,y) : x, y \in Y\}$$

i.e. diameter is the supremum of the set of all distance between point of Y.

Distance between point and set: let Y be a non empty subset of X and $p \in X$ then distance between point p and Y is defined as

$$d(p,Y) = \inf\{d(p,x) : x \text{ in } Y\}\ .$$

If $p \in Y$ then $d(p,Y) = 0$.

Distance between two set: Let $Y_1$and Y_2 be a non empty subset of X then distance between $Y_1$and Y_2 is defined as

$$d(Y_1,Y_2) = \inf\{d(x,y) : x \text{ in } Y_1 \text{ and } y \text{ in } Y_1\}.$$

Bounded Metric spaces: Let (X, d) be a metric space. Then X is said to be bounded if there exists $K \in \mathbb{R}^+$ such that

$$d(x, y) \leq K \text{ for all } x, y \in X.$$

Bounded Metric spaces: Let (X, d) be a metric space. Then X is said to be unbounded if it is not bounded.

Consider the set

$$B_d(x, \epsilon) = \{y | d(x, y) < \epsilon\}$$

of all points y whose distance, from x is less than ϵ. It is called the ϵ −ball centered at x. We can denote $B(x, \epsilon)$.

Now we describe metric topology as given below:

Metric topology : If d is a metric on the set X, then the collection of all ϵ −balls $B_d(x, \epsilon)$,

for $B_d(x, \epsilon), \forall\, x \in X$ and $\epsilon > 0$,

is a basis for a topology on X, called the metric topology induced by d.

A set U is open in the metric topology induced by d

if and only if for each $y \in U$, there is a

$\delta > 0$ such that $B_d(y, \delta) \subset U$.

Definition: If X is a topological space, X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.

A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

Theorem 4.4.1. Let X be a metric space with metric d. Defined

$\bar{d}: X \times X \rightarrow R$ by the equation

$\bar{d}(x, y) = \min\{d(x, y), 1\}$.

Then $\bar{d}$ is a metric that induces the topology of X.

The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof. Checking the first two conditions for a metric is trivial.

Let us check the triangle inequality:

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$

Now if either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, then the right side of this inequality is at least 1; since the left side is (by definition) at most 1, the inequality holds.

It remains to consider the case in which

$d(x, y) < 1$ and $d(y, z) < 1$.

In this case, we have

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z) = \bar{d}(x, y) + \bar{d}(y, z).$$

Since $\bar{d}(x, z) \leq \bar{d}(x, z)$ by definition, the triangle inequality holds for $\bar{d}$.

The fact that $\bar{d}$ and d induces the same topology follows from the inclusions:

$$B_d(x, \epsilon) \subset B_{\bar{d}}(x, \epsilon),$$
$$B_{\bar{d}}(x, \epsilon) \subset B_d(x, \epsilon),$$

where $\delta = \min\{\epsilon, 1\}$.

We merely apply the following lemma. ∎

Lemma 4.4.1. Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively.

Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each x in X and each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$. Given the basis element $B_d(x, \epsilon)$ for $\mathcal{T}$, there is a basis element (from lemma 1 of unit 1) B' for the topology $\mathcal{T}'$,

$$\text{such that } x \epsilon B' \subset B_d(x, \epsilon).$$

Within B' we can find a ball $B_{d'}(x, \delta)$ centered at x.

Conversely, suppose the $\delta - \epsilon$ condition holds.

Given a basis element B for $\mathcal{T}$ containing x, we can find within B a ball $B_d(x, \epsilon)$ centered at x.By the given condition, there is a δ

$$\text{such that } B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

Then by lemma 1 of unit 1 applies to show that $\mathcal{T}'$ is finer than $\mathcal{T}$. ■

Now let us show that R^n and R^ω are metrizable.

Definition. Given $x = (x_1, \dots \dots \dots, x_n)$ in R^n, we define the norm of x by the equation

$$\|x\| = (x_1^2 + \cdots x_n^2)^{1/2};$$

and we define the Euclidean metric d on R^n by the equation

$$d(x, y) = \|x - y\| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}.$$

If we define $\rho(x, y) = \max\{|x_1 - y_1|, \dots . |x_n - y_n|\}$.

We can show that ρ is a metric is easier**………..(Practice for learner)**

……………………………………………………………………………………

……………………………………………………………………………………

On the real line $R = R^1$, these two metrics coincide with the standard metric for R.

In the plane R^2, the basis elements under d, can be pictured as circular regions, while the basis elements under ρ can be pictured as square regions.

Each of these metrics induces the usual topology on R^n.

Theorem 4.4.2. The topologies on R^n induced by the Euclidean metric d and square metric ρ are the same as the product topology on R^n.

Example 4. Let X be any set and define the function $d : X \times X \to \mathbb{R}$ by $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

Then d is a metric on X and called the discrete metric.

The topology it induces is the discrete topology.

$\mathcal{T} =$ discrete topology on X.

Let $X \neq \phi$. The collection

$\mathfrak{B} = \{\{x\}: x \in X\}$ is a Basis for a discrete topology on X. It means the topology generated by $\mathfrak{B}$ is discrete topology on X.

Note: Metrizability of a space depends only on the topology of the space in question, but properties that involve a specific metric for X in general do not. Therefore they are not topological properties.

4.5 QUOTIENT TOPOLOGY

The quotient topology is a construction that makes the idea of ‘gluing’ in a topological space precise. We can construct certain interesting surfaces, at least intuitively, by gluing together appropriate edges of rectangle. For example, the cylinder or the Mobius band can be built by identifying the edges of the rectangle. The gluing process can be described as a map $f: X \longrightarrow Y$ where X is the rectangle and Y is either the cylinder or the Mobius band. The map f is surjective but not certainly injective because of the gluing. Two points on opposite edges of the rectangle that are glued together have the same image under f, but other than this, f is injective.

Is it possible to describe the topology on Y just in terms of the topology on X and the map f? If so, then we can construct the space Y characterizing the open set in Y in terms of the open sets in X.

Definition: Let X and Y be topological spaces; let $\rho: X \to Y$ be a surjective map. The map ρ is said to be a quotient map, provided a subset U of Y is open in Y if and only if $\rho^{-1}(U)$ is open in X.

Note: This condition is stronger than continuity.

An equivalent condition is to require that a subset A of Y be closed in Y if and only if $\rho^{-1}(A)$ is closed in X. Equivalence of the two conditions follows from the equation

$$f^{-1}(Y - B) = X - f^{-1}(B),$$

Saturated set: A subset C of X is saturated (with respect to the surjective map $\rho: X \to Y$) if C contains every set $\rho^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals $\rho^{-1}(p(C))$.

ρ is a quotient map is equivalent to saying that ρ is continuous and ρ maps saturated open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

Two special kinds of quotient maps are the open maps and the closed maps.

Open map: A map $f: X \to Y$ is said to be an open map if for each open set

U of X, the set $f(U)$ is open in Y.

Closed map: A map $f: X \to Y$ is said to be an closed map if for each closed set U of X, the set $f(U)$ is closed in Y.

- If $p: X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map. There are quotient maps that are neither open nor closed.

Definition. If X is a topological space and Y is a set and if $p: X \to Y$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on Y relative to which p is a quotient map; it is called the quotient topology induced by p. or in other meaning the quotient topology on Y is the collections of subsets of $V \subset Y$ such that $p^{-1}(V)$ is open in X.

Lemma 1:
The quotient topology is a topology on Y.

Proof.

i. $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(Y) = X$ are both open in X, so $\emptyset$ and Y are open in the quotient topology on Y.

ii. If $\{V_\alpha\}_{\alpha \in J}$ is a collection of open subsets of Y then each $p^{-1}(V_\alpha)$ is open in X, so

$$p^{-1}\left(\cup_{\alpha \in J} V_\alpha\right) = \cup_{\alpha \in J} p^{-1}(V_\alpha).$$

is a union of open sets in X hence is open in X, so $\cup_{\alpha \in J} V_\alpha$ is open in the quotient topology on Y .

iii. If $\{V_1, V_2 \ldots\ldots, V_n\}$ is a finite collection of open subsets of Y**,** then each $p^{-1}(V_i)$ is open in X, so $p^{-1}(V_1 \cap V_2 \cap \ldots\ldots \cap V_n) = p^{-1}(V_1) \cap p^{-1}(V_2) \ldots\ldots p^{-1}(V_n)$

is a finite intersection of open sets in X hence is open in X, so $V_1 \cap V_2 \cap \ldots\ldots \cap V_n$ is open in the quotient topology on Y. ■

Lemma 2:

A quotient map $\rho: X \to Y$ is continuous.

Proof. When Y has the quotient topology from X. $V \subset Y$ is open in (if and) only if $p^{-1}(V)$ is open in X. In particular ρ is continuous. ■

Lemma 3:

Let $\rho: X \to Y$ be a surjective function, where X is a topological space. The quotient topology is the finest topology on Y such that $\rho: X \to Y$ is continuous.

Proof. Let $\mathcal{T}$ be a topology on Y such that $\rho: X \to Y$ is continuous. Then for each $V \in \mathcal{T}$ we have $p^{-1}(V)$ is open in X, so V is in the quotient topology . Hence $\mathcal{T}$ is coarser than the quotient topology. ■

Lemma 4:

A bijective quotient map $\rho: X \to Y$ is a homeomorphism, and conversely. ■

Definition. Let X be a topological space, and let X^* be a partition of X Into disjoint subsets whose union is X. Let $p: X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space of X. X^* called a decomposition space or identification space of X.

Topology of X^*: A subset U of X^* is a collection of equivalence classes, and the set $\rho^{-1}(U)$ is just the union of the equivalence class belonging to U. Thus the typical open set of X^* is a collection of equivalence classes whose union is an open set of X.

Theorem 4.5.1. The composite of two quotient maps is a quotient map.

Proof. Let X, Y, Z be topological spaces and $p: X \to Y, q: Y \to Z$ be quotient maps. Let $U \subset Z$ be given.

- Since $q: Y \to Z$ is a quotient map, is open if and only if $q^{-1}(U)$ is open

Since $p: X \to Y$ is a quotient map, is open if and only if $p^{-1}(q^{-1}(U))$ is open. Therefore, U is open if and only if $p^{-1}(q^{-1}(U))$ is open. Since, $p^{-1}(q^{-1}(U)) = (qop)^{-1}(U), U$ is open if and only if $(qop)^{-1}(U)$ is open. Therefore qop is a quotient map. ■

- The product of two quotient maps is in general not a quotient map.

Theorem 4.5.2. Let $p: X \to Y$ be a quotient map. Let Z be a space and $g: X \to Z$ be a continuous map that is constant on each set $\rho^{-1}(\{y\}), for\ y \in Y$. Then g induces a continuous map $f: Y \to Z$ such that $fop = g$. ■

Example 5: Let X be the subspace $[0,1] \cup [2,3]$ of $\mathbb{R}$ and let Y be the subspace $[0,2]$ of $\mathbb{R}$. The map $p: X \to Y$ defined by:

$$p(x) = \begin{cases} x\ for\ x \in\ [0,1], \\ x - 1\ for\ x \epsilon\ [2,3] \end{cases}$$

is readily seen to be surjective, continuous and closed. It is not, however, an open map; the image of the open set $[0,1]$ of X is not open in Y.

Note: If A is the subspace $[0, 1]$ $[2, 3]$ of X, then the map $q: A \to Y$ obtained by restricting p is continuous with surjective but it is not a quotient map. For the set $[2, 3]$ is open in A and is saturated w.r.t q, but its image is not open in Y.

4.6 SOLVED EXAMPLES

Example 1. Let X be the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Then$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$ is a metric on X.

Proof. It is given that $d(x, y) = |x - y|$

Clearly we can see that $d(x, y)$satisfied $(M1)$ and $(M3)$ conditions

$$(M2)\ d(f, g) = 0 \Leftrightarrow \sup\{|f(x) - g(x)|\} = 0 \Leftrightarrow |f(x) - g(x)| = 0$$
$$\Leftrightarrow f = g$$

$\Leftrightarrow d(x, y)$satisfied $(M2)$ conditions.

$$(M4) d(f, g) = \sup\{|f(x) - g(x)|\}$$
$$= \sup\{|f(x)| - h(x) + h(x) - g(x)|\}$$
$$\leq \sup\{|f(x) - h(x)|\} + \sup\{|h(x) - g(x)|\}$$
$$\leq d(f, h) + d(h, g)$$

Hence $d(f, g)$ is a metric on X.

Example 2: The standard metric $d(x, y) = |x - y|$, the absolute value of the difference $x - y$, for each $x, y \in \mathbb{R}, (\mathbb{R}, d)$ is a metric space…**……………………..(Check by learners).**

The topology it induces is the same as the order topology: Each basis element (a, b) for the order topology is a basis element for the metric topology; indeed,

$$(a, b) = B(x, \epsilon),$$

where $x = a + b/2$ and $\epsilon = b - a/2$. And conversely, each ϵ −ball $B(x, \epsilon)$ equals an open interval: the interval $(x - \epsilon, x + \epsilon)$.

Example 3: Let p be the map of the real line R onto the three – point set $A = \{a, b, c\}$ defined by :

$$p(x) = \begin{cases} a \text{ } if \text{ } x > 0, \\ b \text{ } if \text{ } x < 0 \\ c \text{ } if \text{ } x = 0. \end{cases}$$

The quotient topology on A induced by p is one indicated in figure given below.

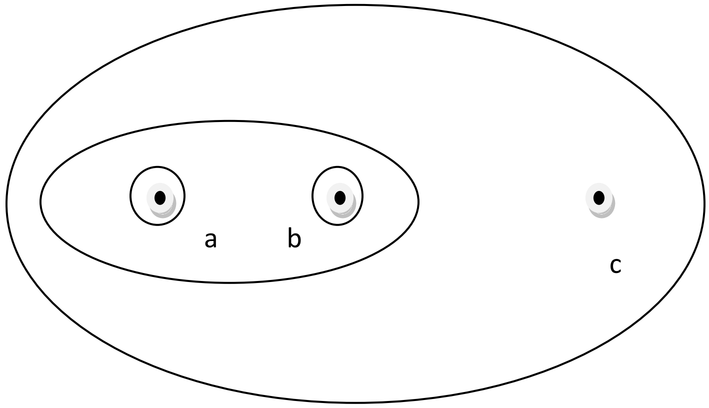

Fig. 4.5.1

Example 4: Prove that an injective quotient map is a homeomorphism.

Solution. Let X, Y be topological spaces and $q: X \to Y$, be an injective quotient map.

- q is injective.
- Since q is a quotient map, q is continuous.
- Since q is a quotient map, q is surjective.

Let $U \subset X$ be an open set. Then $U = q^{-1}(q(U))$ because $q: X \to Y,$ is injective. Since q is a quotient map, $q(U)$ must be an open. Therefore q is an open map. Therefore, q is bijective, continuous, and open, so it is indeed a homeomorphism.

Example 5: Let $\pi_1: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate, then π_1 is continuous and surjective. Furthermore, π_1 is an open map. For if $U \times V$ is a non-empty basis element for $\mathbb{R} \times \mathbb{R}$ then $\pi_1(U \times V) = U$ is open in ; it follows that π_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. However, π_1 is not a closed map. The subset

$$C = \{x \times y | xy = 1\}$$

of $\mathbb{R} \times \mathbb{R}$ is closed, but $\pi_1(C) = \mathbb{R} - \{0\}$, which is not closed in $\mathbb{R}$.

Note: If A is the subspace of $\mathbb{R} \times \mathbb{R}$ that is the union of C and the origin $\{0\}$, then the map $q: A \to \mathbb{R}$ obtained by restricting π_1 is continuous and surjective, but it is not a quotient map. For the one-point set $\{0\}$ is open in A and is saturated with respect to q. But its image is not open in $\mathbb{R}$.

CHECK YOUR PROGRESS

1. The set N of natural numbers equipped with the discrete topology is metrizable. True/False
2. The box topology is the topology not generated by the basis. True/False
3. A quotient map $\rho: X \to Y$ is continuous. True/False
4. A bijective quotient map $\rho: X \to Y$ is a homeomorphism. True/False
5. Let $\rho: X \to Y$ be a surjective function, where X is a topological space. The quotient topology is not the finest topology on Y. True/False

4.7 SUMMARY

This unit is complete combination of

- **i.** Definition of generalisation of product topology and it's properties.
- **ii.** Metric space and it's properties
- **iii.** Concept of metric topology and it's related results.
- **iv.** Definition of Quotient topology and it's properties.

4.8 GLOSSARY

1. Set.
2. Subset.
3. Topology on a set.

4. Function.
5. Continuous maps.
6. Metric space.

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4.11 *TERMINAL QUESTIONS*

1. Prove that the product of two quotient maps needs not be a quotient map.

2. Show that if f is a continuous, closed mapping of X onto Y, then the topology for Y must be the quotient topology.

3. Let $C[a,b]$ be the set of all continuous functions $f : [a,b] \to \mathbb{R}$. Then $d(f,g) = \left(\int_a^b (f(x) - gx)^2 dx\right)^{\frac{1}{2}}$ is a metric on $C[a,b]$.

4. Is box topology finer than product topology

....

....

....

4.12 ANSWERS

CHECK YOUR PROGRESS

1. True.

2. False.

3. True.

4. True.

5. False.

BLOCK- II:

CONNECTEDNESS AND COMPACTNESS

UNIT 5: CONNECTED SPACES AND CONNECTED SETS IN REAL LINE

CONTENTS:

5.1 INTRODUCTION

From the intuitive point of view, a connected space is a topological space which consists of a single piece. This property is perhaps the simplest which a topological space may have, and yet it is one of the most important for the applications of topology to analysis and geometry.

On the real line, for instance, intervals are connected subspaces, and we shall see that they are the only connected subspace. Continuous real functions are often defined on interval, and functions of this kind have many pleasant properties.

For example, such a function assumes as a value every number between any two of its values (The weierstrass intermediate value theorem) furthermore, its graph is connected subspace analysis, for the regions on which analytic functions are studied are generally taken to be connected open subspaces of the complex plane.

In the portion of topology which deals with continuous curves and their properties, connectedness is of great significance, for whatever else a continuous curve may be, it is certainly a connected topological space. Space which are not connected are also interesting.

One of the outstanding characteristics of the Cantor set is the extreme degree in which it fails to be connected. Much the same is true of the subspace.

5.2 OBJECTIVES

After completion of this unit learners will be able to

- **i.** Understand the concept of connectedness and disconnectedness in real line.
- **ii.** Apply the concept of connectedness to particle situation or real-word problem.
- **iii.** Develop critical thinking skills to assess relationships and connections between connectedness, disconnectedness, compactness, continuity.
- **iv.** Define the concept of continuity and connectedness in real line.

5.3 SEPARATED SETS

(5.3.1). Definition. Let (X, $\mathcal{T}$) be a topological space. Two non-empty subsets A and B of X are said to $\mathcal{T}$-separated if and only if

A ∩ $\overline{B}$=∅ and $\overline{A}$ ∩ B = ∅.

These two conditions are equivalent to the single conditions

$(A \cap \overline{B}) \cup (\overline{A} \cap B) = \emptyset$.

Thus, A and B are separated if and only if A and B are disjoint and neither of them contains limit point of other.

For we have,

$A \cap \overline{B} = \emptyset$

$\Leftrightarrow$ A ∩ [B ∪ derived set (B)] =∅

$\Leftrightarrow$ (A ∩ B) ∪ (A ∩ derived set (B))=∅

$\Leftrightarrow$ A ∩ derived set (B) = ∅

$\Leftrightarrow$ A contains no limit point of B.

Note that any two separated sets are disjoint.

But two disjoint sets are not necessarily separated.

For example, the subsets A= $(-\infty, 0)$ and B= $[0, \infty)$ of $\mathbb{R}$ are disjoint but not separated with respect to the usual topology since

A $\cap$ $\overline{B}$ = $(-\infty, 0] \cap [0, \infty) = \{0\} = \emptyset$.

(5.3.2). Example.

Consider the following subsets of $\mathbb{R}$ (with usual topology)

A = (2, 3), B = (3, 4] and C = [3, 4).

The sets A and B are separated since

$\overline{A}$= [2, 3] and $\overline{B}$= [3, 4], so that A $\cap$ $\overline{B}$ = $\emptyset$,

And $\overline{A}$ $\cap$ B = $\emptyset$. But A and C are not separated since

$\overline{A} \cap C = [2, 3) \cap [3, 4) = \{3\} \neq \emptyset$.

(5.3.3). Theorem.

Let (Y, $\mathcal{T}_y$)be a subspace of a topology space (X, $\mathcal{T}$) and A , B be two subsets of Y. Then A, B are $\mathcal{T}$-separated if and only if they are $\mathcal{T}_y$ -separated.

(5.3.4). Theorem.

If A and B are separated subsets of a space X and C $\subset$ A and D $\subset$ B, then C and D are also separated.

Proof. We are given that

A $\cap$ $\overline{B}$=$\emptyset$ and $\overline{A}$ $\cap$ B = $\emptyset$. … (1)

Also, C ⊂ A this implies that $\bar{C} \subset \bar{A}$and D⊂ B this implies that $\bar{D} \subset \bar{B}$… (2)

It follows from (1) and (2) that

$$C \cap \bar{D} = \emptyset \text{ and } \bar{C} \cap D = \emptyset.$$

Hence C and D are separated.

(5.3.5). Theorem.

Two closed (open) subset A, B of a topological space are separated if and only if they are disjoint.

Proof. Since any two separated sets are disjoint, we need only that two disjoint closed (open) sets are separated.

If A and B are both disjoint and closed, then

$$A \cap B = \emptyset, \bar{A} = A \text{ and } \bar{B} = B$$

So that $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$,

Showing that A and B are separated.

If A and B are both disjoint and open, then A' and B' are both closed so that

$\overline{(A')} = A'$ and $\overline{(B')} = B'$.

Also, $A \cap B = \emptyset \Rightarrow A \subset B'$ and $B \subset A'$

$\Rightarrow \bar{A} \cap B = \emptyset$ and $\bar{B} \cap A = \emptyset$

⇒ A and B are separated.

(5.3.6). Theorem.

Two disjoint sets A and B are separated in a topological space (X, $\mathcal{T}$) if and only if they are both open and closed in the subspace A ∪ B.

(5.3.7). Theorem.

Let X be a space and A, B subset of X. Then the following statements are equivalent.

(a). A ∪ B = X and $\bar{A} \cap \bar{B}$ = ∅.

(b).A ∪ B = X, A ∩ B = ∅ and A, B are both closed in X.

(c).B = X – A and A is open (i.e., closed as well as open) in X.

(d). B = X – A and ∂A (that is, the boundary of A) is empty.

(e). A ∪ B = X, A ∩ B = ∅ and a, B are both open in X.

CHECK YOUR PROGRESS – 1

6. Consider the space $(\mathbb{R}, U)$ $and\ let$

A= (0, 1), B = (1, 2) and C = [1,2]

Is the A and B are separated or A and B are not separated **(True/False).**

7. If A and B are subsets of a space X and both A and B are closed or both open, then show A-B is ………….. from B-A.

8. Is empty set and singleton sets are connected sets in any topology **(True/False).**

9. Is A = (1, 4) and B = (5, 8) are separated sets in usual topology **(True/False).**

5.4 CONNECTED AND DISCONNECTED SETS

Intuitively speaking a connected topological space is one which consists of a single piece.

(5.4.1). Definition.

Let (X, $\mathcal{T}$) be a topological space. A subset A of X is said to be $\mathcal{T}$ −disconnected if and if it is the union of two non-empty $\mathcal{T}$-separated sets, that is, if and only if there exists two non-empty sets C and D such that $C \cap \overline{D} = \emptyset$, $\bar{C} \cap D = \emptyset$ and $A = C \cup D$.

(5.4.2). Definition.

A is said to be connected if and only if it is not disconnected.

Note: The empty set is trivially connected. Also, every singleton set in a space is connected.

(5.4.3). Definition.

Two points a and b of a topological space X are said to be connected if and if they are contained in a connected subset of X.

(5.4.4). Theorem.

Let (Y, $\mathcal{T}_y$) be a subspace of a topological space (X, $\mathcal{T}$) and $A \subset Y$. Then A is $\mathcal{T}$ − disconnected if and only if it is $\mathcal{T}_y - disconnected$, or equivalently, A is $\mathcal{T}$- connected iff it is $\mathcal{T}_y - connected$.

Proof. By theorem (1.3), two non-empty subsets of Y are $\mathcal{T}$-separated if and only if they are $\mathcal{T}_y$ – separated. Therefore, A is the union of two $\mathcal{T}$-separated sets.

(5.4.5). Corollary.

Let (Y, $\mathcal{T}_y$) be a subspace of a topological space (X, $\mathcal{T}$). Then Y as a subset of X is $\mathcal{T}$- disconnected if and only if the subspace (Y, $\mathcal{T}_y$) is disconnected as a subspace in its own right.

(5.4.6). Theorem.

A topological space X is disconnected if and only if there exists a non-empty proper subset of X which is both open and closed in X.

Proof. Let X be disconnected.

Then there exist non-empty subsets A and B of X such that

$A \cap \overline{B}=\emptyset, \overline{A} \cap B = \emptyset$ and $A \cup B = X$.

Since $A \subset \bar{A}, \overline{A} \cap B = \emptyset \Rightarrow A \cap B = \emptyset$.

Hence $A = B'$.

Since B is non-empty, and $B \cup B' = X$, it follows that $B = A'$ is a proper subset of X.

Now $A \cup \bar{B} = X$.

Also, $A \cap \overline{B}=\emptyset \Rightarrow A =\overline{(B)}'$ and similarly $B = \overline{(A)}'$.

Since $\bar{B}$ and $\bar{A}$ are closed sets, it follows that A and B are open sets.

Since $A = B'$, A is also closed.

Thus, A is non-empty proper subsets of X which is both open and closed. Similarly, B is also a non-empty proper subset of X which is both open and closed.

Conversely, Let A be a non-empty proper subset of X which is both open and closed. We have to show that X is disconnected.

Let $B = A'$, then B is non-empty since A is proper subset of X. Moreover, $A \cup B = X$ and $A \cap B = \emptyset$. Since A is both closed and open, B is also both closed and open. Hence $\bar{A} = A$ and $\bar{B} = B$.

It follows that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

Thus, X has been expressed as a union of two separated sets and so X is disconnected.

(5.4.7). Theorem.

A topological space X is disconnected if and only if any one of the following statements holds.

(i). X is the union of two non-empty disjoint open sets.

(ii). X is the union of two non-empty disjoint closed sets.

(5.4.8). Corollary.

A subset Y of a topological space X is disconnected if and only if Y is the union of two non-empty disjoint sets both open (closed) in Y.

(5.4.9). Theorem.

Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. Then Y is disconnected if and only if there exists non-empty sets G and H both open (closed) in X such that

$$G \cap Y \neq \emptyset,\ H \cap Y \neq \emptyset,\ Y \subset G \cup H \text{ and } G \cap H \subset X\text{-}Y.$$

Proof. By the above corollary, Y is disconnected if and only if there exists non-empty sets G and H both open (closed) in X such that

$G \cap Y \neq \emptyset$, $H \cap Y \neq \emptyset$, $(G \cap Y) \cap (H \cap Y) = \emptyset$

And $\quad (G \cap Y) \cup (H \cap Y) = Y$.

$(G \cap Y) \cap (H \cap Y) = \emptyset$

$\Leftrightarrow (G \cap H) \cap Y = \emptyset$

$\Leftrightarrow G \cap H \subset X\text{-}Y$.

And $\quad (G \cap Y) \cup (H \cap Y) = Y$

$\Leftrightarrow (G \cup H) \cap Y = Y$

$\Leftrightarrow Y \subset G \cup H$.

Hence $G \cup H$ is called a disconnection of Y.

(5.4.10). Example.

Consider the topology $\mathcal{T} = \{\emptyset, \{b\}, \{a, c\}, X\}$ on $X = \{a, b, c\}$.

Then the space $(X, \mathcal{T})$ is disconnected since $\{b\}$ is non-empty proper subset of X which is both $\mathcal{T} -$ open and $\mathcal{T} -$closed.

(5.4.11). Example.

Let $X = \{a, b, c, d\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$.

Here $\mathcal{T} -$ closed subsets of X are

$$X, \{b, c, d\}, \{c, d\}, \{d\} \text{ and } \emptyset.$$

We see that there exists no non-empty proper subset of X which is both $\mathcal{T} -$ open and $\mathcal{T} -$ closed. Hence $(X, \mathcal{T})$ is connected.

(5.4.12). Question.

Show that if $(X, \mathcal{T})$ is connected and $\mathcal{T}'$ is coarser than $\mathcal{T}$, then $(X, \mathcal{T}')$ is connected.

(5.4.13). Theorem.

Let (X, $\mathcal{T}$) be a topological space and let E be a connected subset of X such that E ⊂ A ∪ B where A and B are separated sets. Then E ⊂ A or E ⊂ B.

(5.4.14). Theorem.

If E is a connected subsets of a space X such that E ⊂ A ∪ B where A, B are disjoint open (closed) subsets of X, then A and B are separated.

Proof. If A and B are open with A ∩ B = ∅, then

$$A \subset B' \Rightarrow \bar{A} \subset \overline{(B)'} = B' \qquad [\text{ since } B' \text{ is closed}]$$

$$\Rightarrow \bar{A} \cap B = \emptyset.$$

Similarly, $A \cap \bar{B} = \emptyset$. Hence A, B are separated.

(5.4.15). Theorem.

Let E be a connected subset of a space X. If F is a subset of X such that E ⊂ F ⊂ $\bar{E}$, then F is connected. In particular, $\bar{E}$ is connected.

Proof. Suppose F is disconnected. Then there exist non-empty sets A and B such that $A \cap \overline{B}=\emptyset$, $\overline{A} \cap B = \emptyset$ and A ∪ B = F. Since E ⊂ F =A ∪ B, it follows from theorem (2.14) that E ⊂ A or E ⊂ B. let E ⊂ A which implies that

$\bar{E} \subset \bar{A} \Rightarrow \bar{E} \cap B \subset \bar{A} \cap B = \emptyset.$

Since ∅ is a subset of every set, we have $\bar{E} \cap B = \emptyset$. Also, $A \cup B = F \subset \bar{E} \Rightarrow B \subset F \subset \bar{E} \Rightarrow \bar{E} \cap B = B$.

Hence $B = \emptyset$ which is a contradiction since B is non-empty. Hence F must be connected.

Again $E \subset F \subset \bar{E}$, we see that $\bar{E}$ is connected.

(5.4.16). Theorem.

If every two points of a subset E of a topological space X are contained in some connected subset of E, then E is a connected subset of X.

(5.4.17). Theorem.

Let $\{ C_\lambda : \lambda \in \Delta\}$ be a family of connected subsets of a space X such that $\cap \{ C_\lambda : \lambda \in \Delta\} \neq \emptyset$.

Then $\cup\{ C_\lambda : \lambda \in \Delta\}$ is a connected set.

(5.4.18). Theorem.

A subset E of $\mathbb{R}$is connected if and only if it is an interval. In particular $\mathbb{R}$ is connected.

Proof. Let E be connected.

If E is singleton set or $\emptyset$, then prove is done.

Now let E contain more than one point and suppose, if possible, that E is not an interval.

Then there exist real numbers a, p, b with $a < p < b$ such that $a, b \in E$but $p \notin E$.

Let $G = [p, \infty) \subset E$.

Then G is non-empty (since $b \in G$) and proper subset of E.

Also, G is open in E and since $[p, \infty)$ is closed in $\mathbb{R}$.

It follows that G is closed in E.

Thus G is non-empty proper subset of E which is both open and closed in E. Hence E is disconnected by (5.3.6).

But this is against our hypothesis.

Hence E must be an interval.

Conversely, let E be an interval and suppose, if possible, E is disconnected.

Then there exists non-empty disjoint sets a and B, both closed in E such that $E = A \cup B$.

Choose $a \in A$ and $b \in B$. Since $A \cap B = \emptyset$, $a \neq b$.

Thus either $a < b$ or $a > b$.

Without loss of generality, we may assume $a < b$. Since E is an interval and $a, b \in E$, we have $[a, b] \subset E = A \cup B$.

Let $u = \sup([a, b] \cap A)$.

Evidently $a \leq u \leq b$. To each $\in > 0$, there exists some $v \in [a, b] \cap A$ such that $u - \in < v \leq u$.

This shows that every neighborhood of u contains a point of $[a, b] \cap A$ and hence a point of A and therefore either $u \in A$ or u is a limit point of A.

Since A is closed, in either case, $u \in A$.

Then $u \notin B$ [since $A \cap B = \emptyset$].

Again, since $b \in B$, we have $b \neq u$.

This gives us the strict inequality $u < b$.

Moreover, the definition of u shows that $u + \in$ belong to B for every $\in > 0$ such that $u + \in \leq b$.

This means that every nhd of u contains a point of B distinct from u and so u is a limit point of B since B is closed, $u \in B$.

Hence $u \in A \cup B$ which is a contradiction since A and B are disjoint. Hence E must be connected.

(5.4.19). Theorem.

$\mathbb{R}$is Connected.

CHECK YOUR PROGRESS – 2

1. Is the indiscrete topological space (X, I) is connected **(True/False).**
2. Is the discrete space (X, Ɗ) where X contains more than one point is connected **(True/False).**
3. Is A = (1, 4) and B = (5, 8) are separated sets in usual topology. **(True/False).**

5.5 CONTINUITY AND CONNECTEDNESS

(5.5.1). Theorem.

Continuous image of connected space is connected.

Proof. Let f: X → Y be a continuous mapping of a connected space X into an arbitrary topological space Y.

We show that f(X) is connected as a subspace of Y.

Assume that f(X) is disconnected.

Then there exists G_1 and G_2 both open in Y such that

$G_1 \cap f(X) \neq \emptyset$, $G_2 \cap f(X) \neq \emptyset$,

$(G_1 \cap f(X)) \cup (G_2 \cap f(X)) = f(X)$.

It follows that

$\emptyset = f^{-1}(\emptyset) = f^{-1}[(G_1 \cap f(X)] \cap [G_1 \cap f(X)]$

$\emptyset = f^{-1}[(G_1 \cap G_2) \cap f(X)]$

$\emptyset = f^{-1}(G_1) \cap f^{-1}(G_2) \cap f^{-1}f(X)$

$\emptyset = f^{-1}(G_1) \cap f^{-1}(G_2) \cap X$

$\emptyset = f^{-1}(G_1) \cap f^{-1}(G_2)$.

And $\quad X = f^{-1}[f(X)] = f^{-1}[(G_1 \cap f(X)] \cup [G_2 \cap f(X)]$

$X = f^{-1}(G_1 \cup G_2) \cap f(X)$.

$X = f^{-1}(G_1 \cup G_2) \cap f^{-1}[f(x)]$

$X = f^{-1}(G_1) \cup f^{-1}(G_2)$.

Since f is continuous and G_1 and G_2 are open in Y both intersect f(X), it follows that $f^{-1}(G_1)$ and $f(G_2)$ are non-empty open subsets of X. Thus, X has been expressed as a union of two disjoint non-empty subsets of X and consequently X is disconnected, which is contradiction.

Hence f(X) must be connected.

(5.5.2).Corollary.

If f is a continuous mapping of a connected space onto an arbitrary topological space Y.

(5.5.3).Corollary.

If f is continuous mapping of a connected space X into $\mathbb{R}$**,** then f(X) is an interval.

(5.5.4). Theorem.

A topological space X is disconnected if and only if there exists a continuous mapping of X onto the discrete two-point space {0, 1}.

Proof. Let X be disconnected.

Then there exists disjoint open subsets G_1 and G_2 of X such that $X = G_1 \cup G_2$.

Define a mapping f of X onto {0, 1} by setting $f(x) = 0$ if $x \in G_1$ and $f(x) = 1$ if $x \in G_2$.

Since {0, 1} is discrete, its open sets are $\emptyset$, {0}, {1} and {0, 1}.

Since $G_1 \cap G_2 = \emptyset$, the definition of f show that

$f^{-1}[\{0\}] = G_1$ and $f^{-1}[\{1\}] = G_2$.

Also, $f^{-1}[\emptyset] = \emptyset$ and $f^{-1}[\{0, 1\}] = X$.

Thus, we have shown that the inverse image under f of every open subset of {0, 1} is open in X and therefore f is continuous.

Conversely, if there exists such a mapping then X is disconnected.

For if X were connected, then {0, 1} would be connected by the theorem 5.5.1.

But this impossible since every discrete space is disconnected. Hence proof is done.

(5.5.5). Theorem.

A space X is connected if and only if every continuous function *f* from X into the discrete two points space {0, 1} is constant.

CHECK YOUR PROGRESS - 3

1. Is the graph of continuous real function defined on an interval is a connected subspace. **(True/False)**
2. Is the connectedness is a topological invariant. **(True/False)**
3. Is a continuous bijection from $\mathbb{R}\, \boldsymbol{to}\, \mathbb{R}$must be a homeomorphism. **(True/False)**
4. Is A = (3, 7) and B = (5, 11) are separated sets in usual topology. **(True/False)**

5.6 SUMMARY

This unit is complete combination of

i. Definition of connectedness and disconnectedness of sets in real line.

ii. Separation of sets in real line.

iii. Concept of continuous in connectedness.

5.7 GLOSSARY

1. Separated sets.
2. Connected set
3. Disconnected set.
4. Continuity.
5. Continuous maps.

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5.10 TERMINAL QUESTIONS

1. Prove or disprove that the interior and boundary of a connected set are connected.

2. Show that a co finite topological space X is connected if X is infinite and disconnected if X is finite.

3. If A and B are connected subsets of a topological space and $A \cap B$ is non-empty, prove that $A \cup B$ is connected.

4. **4.** If two topological spaces X and Y are homeomorphic, prove that X is connected if and only if Y is.
5. **5.** Show that connectedness is a topological invariant.
6. **6.** Let $(X, \mathcal{T})$ be a topological space. Prove that X is connected iff X cannot be express as the union of two non-empty, disjoint, open set.
7. **7.** Show that the closed interval [0, 1] is a connected set in $\mathbb{R}$ ....
....
....

5.11 ANSWERS

CHECK YOUR PROGRESS - 1

1. True
2. Separated
3. True
4. True

CHECK YOUR PROGRESS–2

1. True
2. False
3.True

CHECK YOUR PROGRESS – 3

1.True
2. True
3. True
4. False

UNIT 6:

COMPONENTS ,

PATH COMPONENTS AND

LOCAL CONNECTEDNESS

CONTENTS:

6.1 INTRODUCTION

In previous units we have studied the connectedness of topological space. In this section we will analyze the idea of connectedness something more. A topological space can be partitioned into one or more than one parts. This specific partitioned parts are referred as connected components and this specific partition gives the information about connectedness of that particular topological space. After this we will extend the idea of connectedness into path connectedness.

Connectedness is a useful property for a space to possess. But for some purposes, it is more important that the space satisfy a connectedness condition locally. Roughly speaking, local connectedness means that each point has “arbitrary small” neighbourhoods that are connected. Therefore, in this unit, we shall explain two important topics components and local connectedness.

6.2 OBJECTIVES

After completion of this unit learners will be able to

i. Understand the concept of connected components.
ii. Explain the concept of path components.
iii. Understand the concept of local connectedness.

6.3 CONNECTED COMPONENTS

Recall that a relation on a set is said to be an equivalence relation, if it is “reflexive”, “symmetric” and “transitive”. Also we state here a important result regarding equivalence relation on a set and partition associated with the equivalence relation on that set: “Let$X$ be a non-empty set. Then any equivalence relation on X defines a partition on set X, and also any partition on set X defines an equivalence relation on set X.” This is done by as follows: let R be an equivalence relation on set X, then this relation gives equivalence class $[x] = \{y: xRy\}$, collection of all these equivalence class gives partition on set X. In a vice versa manner for a given partition on set X, we can define equivalence class.

Now, we come to our main topic: connected components.

Definition 6.3.1. Given a topological space X, define an equivalence relation on X by defining xRy, if there is a connected subset of X containing both x and y. The equivalence classes generated by this equivalence relation are called the connected components (or simply components) of X.

Example 6.3.1. Consider $\mathbb{R}$ with usual (Euclidean) topology. Then this space is connected. And hence there is only one component, $\mathbb{R}$.

Example 6.3.2. Consider $[1,2] \cup [3,4]$ with usual topology. Then in this space $[1,2]$ and $[3,4]$ are connected components.

The components of a topological space X can also be described as follows:

Theorem 6.3.1.

The components of X are connected disjoint subsets of X whose union is X, such that each connected subset of X intersects only one of them.

Proof. Since components are equivalence classes, therefore components of X are disjoint and their union is whole space X.

Consider a component U. We have to show that U is connected.

Choose a point x_0 of U. For each point x of U, we know that $x_0 \sim x$, so there is a connected set U_x containing x_0 and x. $U_x \subset U$.

Therefore, $U = \cup_{x \in U} U_x$.

Since the sets U_x are connected and have the point x_0 in common, their union is connected.

Now, for the last part of theorem, let U be a connected set in X.

We will show that U intersects only one of the connected components.

For if U intersects the more than one components, say C_1 and C_2 of X.

Then there are atleast two points x_1 and x_2, $U \cap C_1$ and $U \cap C_2$ in respectively.

Since U is a connected set containing x_1 and x_2,

therefore $x_1 \sim x_2$. This is possible only when $C_1 = C_2$.

Remark 6.3.1.

A component C of a topological space is a maximal connected subspace. By maximal connected means, if C is component and $C \subseteq U \subseteq X$, then either $C = U$ or $U = X$.

Remark 6.3.2.

Recall from previous unit that if Y is connected subspace of a topological space, then $\bar{Y}$ is also connected subspace.

Theorem 6.3.2.

Each component of a topological space is a closed set.

Proof: Let C be a component of a topological space X. Then by definition it is connected. Also by previous remark $\bar{C}$ is connected. And also since $C \subseteq \bar{C} \subseteq X$, then by maximality of C, $C = \bar{C}$. Hence C is closed.

Remark 6.3.3.

Recall from connectedness preserves under the continuous mapping.

6.4 PATH COMPONENTS

In this section we give some other notion of connectedness in a topological space. For this we first define path in a topological space.

Definition 6.4.1 Let X be a topological space and x, y be in X. Then a path from x to y in X is a continuous mapping $f: [a, b] \to X$, such that $f(a) = x$ and $f(b) = y$.

Note that, we can take $[a, b] = [0,1]$, by using a mapping $\sigma: [0,1] \to [a, b]$ by $\sigma(t) = a + t(b - a)$.

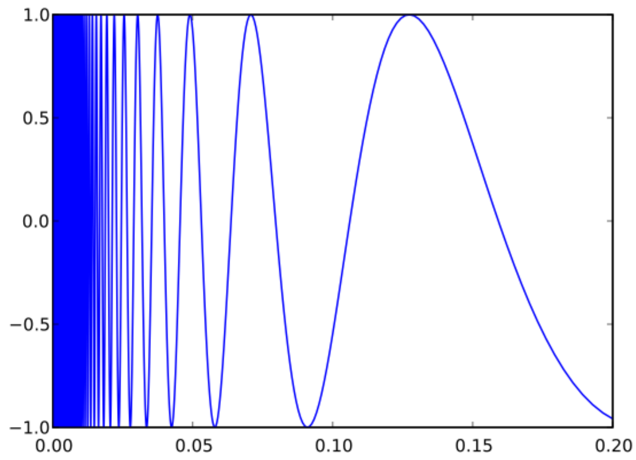
Definition 6.4.2. A topological space X is said to be path connected if every pair of points of X can be joined by a path in X.

Proposition 6.4.1: A path connected space is connected, but the converse is not true.

Proof: Consider a topological space X which is path connected. On the contrary suppose that X is not connected. Then $X = A \cup B$ is a separation of X. Let $f: [a, b] \to X$ be any path in X. Being the continuous image of a connected set, the set $f([a, b])$ is connected, so that it lies entirely in either A or B. Therefore, there is no path joining a point of A to a point of B. This is a contradiction.

To see the converse of the theorem does not hold, consider the following example:

Example 6.4.1: Let $S = \left\{x \times \sin\left(\frac{1}{x}\right): 0 < x \leq 1\right\}$. Since $(0,1]$ is a connected set and $\sin\left(\frac{1}{x}\right)$ is a continuous function. Therefore S is connected in $\mathbb{R}^2$. And hence it closure $\bar{S}$ is connected in $\mathbb{R}^2$. But this space is not path connected. The space $\bar{S}$ is a classic example in topology called the topologist's sine curve. The following graph is representation of topologist's sine curve.



Ref: https://en.wikipedia.org/wiki/Topologist%27s_sine_curve

Fig. 6.4.1

Now, we are going to define path components, in the same manner as we defined connected components.

Definition 6.4.3. Define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are calledthe path components of X.

Here we first show that the above defined relation is an equivalence relation. First we note that if there exists a path $f : [a, b] \to X$ from x to y whose domain is the interval $[a, b]$, Then (because any two closed intervals $[a, b]$, and $[c, d]$ in $\mathbb{R}$ are homeomorphic) there is also a path g from x to y having $[c, d]$ as its domain. Now the fact that $x \sim x$ for each x in X follows from the existence of the constant path $f: [a, b] \to X$ defined by the equation$f(t) = x$ for all t. Symmetry follows from the fact that if $f: [0,1] \to X$ is a path from x to y, then the "reverse path" $g: [0,1] \to X$ defined by $g(t) = f(1 - t)$ is a path from y to x. Finally, transitivity is proved as follows: Let $f: [0,1] \to X$ be a path from x to y, and let $g: [1,2] \to X$ be a path from y to z; the path h will be continuous by the pasting lemma.

By the definition of path connected and equivalence classes, we have the following result:

Theorem: The path components of X are path-connected disjoint subsets of X whose union is X, such that each path-connected subset of X intersects only one of them.

Proof: Left as an exercise.

Example 6.4.1. The components of the subspace

$$Y = [-1,0) \cup (0,1]$$

of the real line R are the two sets [-1,0) and (0,1]. These are also the path components of Y.

Example 6.4.2. The deleted comb space D is a space having a single connected component (because it is connected) and two path components. If we form a space Y by adjoining to the deleted comb space D all the irrational points of the interval $0\times [0,1]$, we obtain a space having only one component but uncountably many path components.

We saw earlier that a connected components is always closed. And if a space has finitely many components then the components are also open. But in case of path components, they need not be closed nor open. Consider the following example:

Example6.4.3: The topologist sine curve $\bar{S}$, where $S = \left\{x \times \sin\left(\frac{1}{x}\right) : 0 < x \leq 1\right\}$. It has two path components, one is S and the other is vertical interval $V = 0 \times [-1,1]$. Here S is not closed and V is not open in $\bar{S}$.

CHECK YOUR PROGRESS-1

1. Prove that the components of C corresponding to different points of E are either equal or disjoint..

6.5 LOCAL CONNECTEDNESS:

So far we have studied about connectedness and path connectedness on the whole space. Sometimes it is a useful to study connectivity condition in a space locally. In topology and other branches of mathematics, a topological space X is locally connected if every point admits a neighbourhood basis consisting of open connected sets. A connected space need not be locally connected; counterexamples include the comb space and broom space. Conversely, a locally connected space need not be connected; an easy counterexample is the union of two disjoint open intervals of the real line.The following definition gives the idea of locally connectedness:

Definition 6.5.1. A space X is said to be locally connected at a point$x \in X$ if for every neighborhood U of x, there is aconnected neighborhood $V$of x contained in U. If X is locally connected at each of its points, then the space X is said to be locally connected.

In another words, X is locally connected if there is a basis for X consisting of connected sets.

Note that, local connectedness and connectedness of a space are not related to one another; a space may posses one or both of these properties, or neither.

Example6.5.1. Each interval and each ray in the real line is both connected and locally connected. The subspace $[-1,0) \cup (0,1]$ of $\mathbb{R}$ is not connected, but it is locally connected.

Example 6.5.2. The deleted comb space discussed in previous units is connected but not locally connected.

Example 6.5.3. The set of all rationals in $\mathbb{R}$ is neither connected nor locally connected.

The most important facts about locally connected spaces are given in the following theorems:

Theorem 6.5.1. A space X is locally connected if and only if for every open set U of $X,$ each component of $U$is open in$X$.

Proof. Suppose that X is locally connected.Let U be an open set in X and C be a component of U. If x is a point of C , we can choose a connected neighborhood V of U. Therefore, C is open in X.

Conversely, suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of $x,$ let C be the component of Ucontaining$x$. Now $C$is connected; since it is open in $X$by hypothesis, $X$is locally connected at x.

There is also a notion of local path connectivity, just like as local connectedness defined above.

Definition 6.5.2. A space X is said to be locally path connected at x if for every neighborhood U of x, there is a path-connected neighborhood Vof x connected in U. If X is locally path connected at each of its points, then it is said to be locally path connected.

Example6.5.4. $\mathbb{R}^{\mathrm{n}}$ is locally path connected, since each of the basis elements $(\mathrm{a}_1, \mathrm{b}_2) \times \ldots \times (\mathrm{a}_{\mathrm{n}}, \mathrm{b}_{\mathrm{n}})$ is path connected. Similarly, $\mathbb{R}^{\infty}$is locally path connected; each of its standard basis elements is path connected.

Example 6.5.5. If we form a space Y by adjoining to the deleted comb space D all the points of the form $0\times\left(\frac{1}{n}\right)$for $n \in Z_{+}$, we obtain a space that is locally connected at the origin but not locally path connected at the origin.

The relation between path components and components is given in the following theorem:

Theorem 6.5.2. If X is a topological space, each path component of X lies in a component of X. if X is locally path connected, then the components and the path components of X are the same.

Proof. Let C be a component of X; let x be a point of C; let P be the path component of X containing x. Since P is connected, $P \subset C$. We wish to show that if X is locally path connected, $P = C$. Suppose that $P \neq C$. Let Q denote the union of all the path components of X that are different from P and intersect C; each of them necessarily lies in C, so that

$$C = P \cup Q.$$

Because X is locally path connected, each path component of X is open in X. Therefore, P(which is a path component) and Q(which is a union of path components) are open in X, so they constitute a separation of C. This contradicts the fact that C is connected.

6.6 SOLVED EXAMPLES:

Problem 6.1.1. On a topological space X, define a relation xRy, if there is a connected subset of X containing both x and y. Show that this relation is an equivalence relation.

Solution:

Reflexivity: Since every singleton set is connected set in a topological space, and $x \in \{x\}$, therefore xRx.

Symmetricity: Let xRy. Then there is connected set A in X such that $x, y \in A$. Hence in this case yRx also.

Transitivity: Let xRy and yRz. Then there is two connected sets A and B such that $x, y \in A$ and $y, z \in B$. Also since A, B are connected and y contained in both A and B, therefore $A \cup B$ is also connected containing x, z. Thus xRz.

Hence R is an equivalence relation.

Problem 6.1.3. Consider a topological space X with finite number of components, then show that each components are clopen subsets of X.

Solution: Let $C_1, C_2, \dots, C_n$ be components of X. Then $C_1 \cup C_2 \cup \dots \cup C_n = X$. We already known that each $C_i's$ are closed. Now, ${C_1}^c = C_2 \cup \dots \cup C_n$. Since finite union of closed sets is a closed set, therefore $C_2 \cup \dots \cup C_n = {C_1}^c$ is a closed set. Hence C_1 is a open set. Similarly each $C_i's$ are clopen set.

The above statement may not be true in case if X has infinitely many components. For example, consider the topological space $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with usual topology. Then $\{0\}$ is a connected component of X, but $\{0\}$ is not open set in X.

Problem: What are the components of $\mathbb{R}$ in discrete topology?

Solution: Every subset with more than one elements are disconnected subset. Hence in this space every singleton subsets are connected components.

Problem 6.1.1. Let $f: X \to Z$ be a continuous function and A be path component in X. Then $f(A)$ is path connected in Z.

Solution: Consider$f: X \to Z$ be a continuous function and A be path component in X, and $u, v \in f(A)$. Then there is $x, y \in A$ such that $f(x) = u$ and $f(y) = v$. Since A is path component therefore there is path $g: [0,1] \to X$, such that $g(0) = x$ and $g(1) = y$. Now, consider $h = fog: [0,1] \to Z$, then h is continuous, hence a path in Z. Also $h(0) = u$ and $h(1) = v$. Thus $f(A)$ is path connected in Z.

6.7 SUMMARY

This unit is complete combination of

i. An extend analysis of connectedness.
ii. Idea of connected components.
iii. Analysis of path components.
iv. Concept of locally connectedness.

6.8 GLOSSARY

1 Equivalence relation.
2 Equivalence classes and partition.
3 Connected space.
4 Path Connected space.
5 Connected components.
6 Path connected components.
7 Locally connected space.
8 Discrete space.

9 Open set.

10 Partition.

CHECK YOUR PROGRESS-2

1. Every discrete space is locally connected **True\False**

2. Every component of a locally connected space is not open. **True\False**

3. Local connectedness neither implies nor is implied by connectedness. **True\False**

4. Let $\rho: X \longrightarrow Y$ be a quotient map. Show that if X is locally connected, then Y is ....

5. Connected subspace of a locally connected space has a finite number of............

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6.11 TERMINAL QUESTIONS

TQ1. What are the components of $\mathbb{R}$ in finite complement topology?

TQ2. Show that if A is a both open and closed, non-empty, connected subset of a topological space X, then A is a connected component.

TQ3. Let X be locally path connected. Show that every connected open set in X is path connected.

TQ4. Show that homeomorphism maps components into components. And homeomorphic spaces have the same number of components.

TQ5. Show that a space X is locally path connected if and only if for every open set Uof X, each path component of U is open in X.

6.12 ANSWERS

CHECK YOUR PROGRESS-2

CHQ1 : True

CHQ2: False.

CHQ3: True

CHQ4: Local connectedness.

CHQ5: Components.

TERMINAL QUESTIONS

TQ1: Only one component, which is the whole space $\mathbb{R}$.

TQ2: Suppose that A is a both open and closed, non-empty, connected subset of a topological space X. Let $A \subseteq B \subseteq X$. Then A is also clopen set in B. Therefore A and $B \cap A^c$ forms a separation of B. Thus B is not connected. Hence A is a maximal connected set in X. Thus A is a connected components in $+X$.

TQ3: Hint: Use remark 6.3.3.

TQ4: Hint: Use the technique of proof of theorem 6.5.1.

UNIT 7: COMPACT SPACES, COMPACT SET IN THE REAL LINE

CONTENTS:

7.1 INTRODUCTION

In previous units we have studied the local connectedness of topological space. The notion of compactness is not nearly so natural as that of connectedness. From the beginning of topology, it was clear that the closed interval [a, b] of the real line had a certain property that was crucial for proving such theorems as the maximum value theorem and the uniform continuity theorem. But for a long time, it was not clear how this property should be formulated for an arbitrary topological space. It used to be thought that the crucial property of [a, b] was the fact that every infinite subset of [a, b] has a limit point, and this property was the one dignified with the name of compactness.

Later, mathematicians realized that this formulation does not lie at the heart of the matter, but rather that a stranger formulation, in terms of open coverings of the space, is more central. The latter formulation is what we now call compactness. It is not as natural of intuitive as the former; some familiarity with it is needed before its usefulness becomes apparent.

In Euclidean spaces the subsets which are both closed and Bounded and the study of these type of subsets may yields a characterization of compactness in terms of open sets.

7.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of open covering of a topological space;

ii. Understand the definition of a compact spaces;

iii. Solve the problems based on compact spaces and compact subspace on real line.

iv. Established several equivalent forms of compactness which are useful in applications.

7.3 COMPACT SPACES

Definition:

A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the element of A is equal to X. If its elements are open subset of X then it is called open covering of X.

In other words X is said to be an open cover if each $x\in A\subset X\exists$ at least one G_i, ie, $\cup G_i = A$ such that $x\in G_i$.

If each G_i is T-open then cover is known as T-Open cover.

SUBCOVER AND FINITE SUBCOVER OF A SET:-

A subclass of an open cover which is itself an open cover is said to be sub cover.

Let G be an open cover of a set S. A sub collection C^* of G is called a sub cover of S if C^* too is a cover of S. Further, if there are only a finite number of sets in C*, then we say that C* is a finite sub cover of the open cover G of S. Thus if G is an open cover of a set S, then a collection C* is a finite subcover of the open cover G of S provided the following three conditions hold.

(i) C* is contained in G.
(ii) C* is a finite collection.
(iii) C* is itself a cover of S.

Example1: if $\mathcal{T} = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}, \{4\}, \{1,4\}\{2,3,4\}, X\}$ is the topology on X= {1,2,3,4} the consider the following collection of subset of X as C = {{1},{4},{2,3}}, here each G_i namely {{1},{4},{2,3}} is $\mathcal{T}$-open and its $\cup_i G_i$ = {1}∪ {4}∪{2,3} = {1,2,3,4}= X hence C is an open cover of X.

Consider the two collection of subset of X
C_1= {{1,4},{2,3}}and C_2= {{1},{2,3,4}} both are open cover of X. Here C is not a sub cover of any C_1 and C_2 but if we consider other collection C_3= {{1},{2,3},{2,3,4}}, then it is clear that C_3 is an open cover of X and also C_2 is sub cover of C_3 as $C_2 \subset C_3$.

Example 2: Let X= {1,2,3,4} , $\mathcal{T}$= {∅,{1}, {2}, {1,2}}
Let A= {2}⊂ X, C= {{1}.{1,2}} then ∪C= {1}∪{1,2}= {1,2}
So, A⊂ {∪C}
⇒ C is open cover of A.

Result: C is open cover of all proper subsets of union of C.

Example 3: Let (ℝ,U) be usual topology.
If we take C= {(-n,n): n∈N} then ∪C= ∪{(-n,n): n∈N}= ℝ
So, C is open cover of ℝ.

Now we know that C^*= {(-2n,2n): n∈ N} is a subcollection of C and C^* is also open cover of ℝ.
So, C^* is subcover of C of ℝ.

7.3.2 COMPACT SPACES

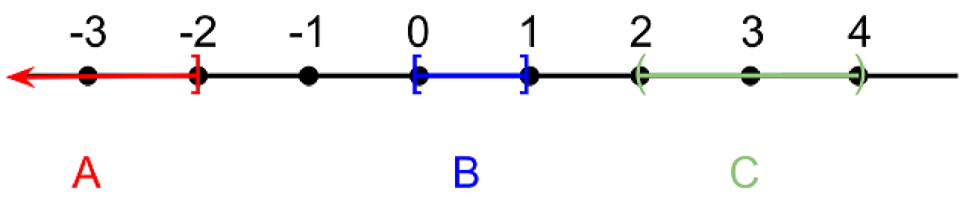
Compact Sets:

A set is said to be compact of every open cover of it is reducible to a finite cover or
in other words if every open cover of the set contain a finite subcover.

Example 4: If A= (0,1) on the real line R with the usual topology then A is not compact since if C = { $(\frac{1}{3},1),(\frac{1}{4},\frac{1}{2}),(\frac{1}{5},\frac{1}{3}),(\frac{1}{6},\frac{1}{4})$,…..} then consider A= $\bigcup_{n=1}^{\infty} G_n$, where $G_n =(\frac{1}{n+2},\frac{1}{n})$ which show that C is an open cover of A.
But C does not contain any finite subcover since if $(C)_0$= {(x_1,y_1), (x_2,y_2),(x_3,y_3)………(x_m,y_m) is a finite subclass of C i.e., $(C)_0 \subset C$ and
j= min ($x_1,x_2,x_3,$……… x_m), then j > 0 and
$(x_1,y_1)\cup(x_2,y_2)\cup(x_3,y_3)\cup\ldots\cup(x_m,y_m) \in (j,1)$
But $(0,j)\cap(j,1)=\emptyset \Rightarrow (C)_0$ is not a cover of A. Hence A is not compact.

The collection G = {]-n, n[: n ∈N} is an open cover of ℝ but does not admit of a finite subcover of ℝ. Therefore the set ℝ is not a compact set.

Thus you have seen that every finite set is always compact. But an infinite set may or may not be a compact set. The question, therefore, arises, "What is the criteria to determine when a given set is compact?" This question has been settled by a beautiful theorem known as Heine-Borel Theorem named in the honour of the German mathematician **H.E. Heine**[1821-18811 and the French mathematician **F. E.E**. Borel [1871-19561, both of whom were pioneers in the development of Mathematical Analysis.



Ref: **https://en.wikipedia.org/wiki/Compact_space#/media/File:Compact.svg**

Fig.7.3.2.1

Heine- Borel Theorem:

If A= [a,b] be a closed and bounded interval and $\{G_i\}$ is a class of open sets s.t.A $\subset \cup G_i$, then there can be a finite number of open sets say $G_{i_1}, G_{i_2}, G_{i_3} \dots \dots G_{i_m}$

s.t. $A \subset G_{i_1} \cup G_{i_2,} \cup\ G_{i_3} \dots \dots \cup G_{i_m}$

or in other words ,

every open cover of closed and bounded interval A = [a, b] is reducible to finite cover.

or as follows:-

Heine-Borel Theorem –

Every closed and bounded subset of R is compact. The immediate consequence of this theorem is that every bounded and closed interval is compact.

Compact Space:

A space X is said to be compact if every open covering A of X contains a finite sub-collection that also covers X.

Let A be a subset of a topological space $(X, \mathcal{T})$thenA is said to be compact for every set I and every family of open sets O_i, i∈ I such that A $\subseteq \cup_{i \in I} O_i$ there exist a finite sub collection Oi_1 ,Oi_2…… Oi_n such that A $\subseteq$ $Oi_1 \cup Oi_2 \cup \ldots\ldots \cup Oi_n$

Compact Subspace:

A subspace of a topological space, which is compact as a topological space in its own right, is said to be compact subspace.

Finite Intersection Property (FIP):

A class A* = $\{A_i\}$ is said to have the finite intersection property if the intersection of member of every finite subclass let say $(A^*)_0 = \{ Ai_1, Ai_2 \ldots\ldots Ai_m \}$ is non empty or we can say that $Ai_1 \cap Ai_2 \cap \ldots\ldots \cap Ai_m \neq \emptyset$.

Example 5:

The real line ℝ is not compact, for the covering ofℝ by open intervals A = {(n, n + 2)/ n ∈Z } contains no finite sub-collection that coversℝ.

Example 6:

The following subspace of$\mathbb{R}$ is compact

X = {0} $\cup$ {1/n $\in$ Z+ }. Given an open covering A of X, there is an element U of A containing O. The set U contains all but finitely many of the point 1/n; choose, for each point of X not in U, an element of A containing it. The collection consisting of these elements of A, along with the element U, is a finite sub-collection of A that covers X.

Example 7:

Let X= {a,b,c,d} $\mathcal{T} = \{\emptyset\ \{a\},\{c\},\{c,b,d\},\{a,c\},X\}$ be topology on X.

Let A= {c} and G= {{a},{c}, {a,c}}

Here A $\subset$$\{\cup G\}$$\Rightarrow$ G is open cover of A.

Now let G*= {{c},{a,c}}. Since G*$\subset$G and A$\subset$$\{\cup G^*\}$

$\Rightarrow$ G* is also open cover of A.

$\Rightarrow$ G* is finite subcover of G.

Therefore (X, $\mathcal{T}$) is compact space.

Lemma (i): Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite sub-collection covering Y.

Proof: Suppose that Y is compact and A = $\{A_x\}\}\alpha\in T$ is a covering of Y by sets open in X. Then the collection $\{A_x \cap Y \mid \alpha \in J\}$ is a covering of Y by sets open in Y; hence a finite sub-collection $\{A_{\alpha_1} \cap Y \ldots\ldots A_{\alpha_1} \cap Y\}$ covers Y. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a sub-collection of A that covers Y. Conversely, suppose the given condition holds; we wish to prove Y compact. Let A′ = $\{A'\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set $A\alpha$ open in X such that $A'\alpha = A\alpha \cap Y$

The collection A = {$A\alpha$ } is a covering of Y by sets open in X. By hypothesis, some finite sub-collection {A_{α_1},...., A_{α_n}} covers Y. Then {A'_{α_1}, ...,A'_{α_n} } is a sub-collection of A′ that covers Y.

Theorem 1: Every closed subspace of a compact space is compact.

Proof: Let Y be a closed subspace of the compact space X. Given a covering A of Y by sets open in X, let us form an open covering B of X by A joining to A the single open set X – Y that is B = A∪ {X – Y} Some finite sub-collection of B covers X. If this sub-collection contains the set X – Y, discard X – Y; otherwise, leave the sub-collection alone. The resulting collection is a finite sub-collection of A that cover Y.

Theorem 2: Every compact subspace of a Hausdorff space is closed.

Proof: Let Y be a compact subspace of the Hausdorff space X. We shall prove that X – Y is open. So that Y is closed. Let x_0 be a point of X – Y. We show there is a neighborhood of x0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods Uy and Vy of the points x_0 and y, respectively (using the Hausdorff condition). The collection {Vy / y ∈Y} is a covering of Y by sets in X; therefore, finitely many of them Vy_1, ...,Vy_n cover Y. The open set V = Vy_1 ....Vy_n contains Y, and it is disjoint from the open set U = $Uy_1 \cap \ldots \cap Uy_n$ formed by taking the intersection of the corresponding neighborhoods of x_0 . For if z is a point of V, then z $\in Vy_i$ for some i hence z $\notin U_{yi}$ also z$\notin$ U Then U is a neighborhood of x_0 , disjoint from Y, as desired.

Theorem 3: The image of a compact space under a continuous map is compact.

Proof: Let f : X $\rightarrow$ Y be continuous; let X be compact. Let A be a covering of the set f(X) by sets open in Y. The collection $\{f^{-1}(A) \mid A \in A\}$ is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them. Say $f^{-1}(A_1)$, ..., $f^{-1}(A_n)$, cover X, then the sets A_1 ...., A_n cover f(X).

Theorem 4: Let f : X $\rightarrow$ Y be a bijective function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof: We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact by theorem (1). Therefore by the theorem just proved f(A) is compact. Since Y is Hausdorff, f(A) is closed in Y by theorem (2)

Theorem 5: A closed subset of a countably compact space is countably compact.

Proof: Let Y be a closed subset of a countably compact space (X, $\mathcal{T}$).

Let $\{G_n : n \in N\}$ be a countable $\mathcal{T}$-open cover of Y, then $Y \subset \bigcup_n G_n$

But $X = Y' \cup Y$. Hence $X = Y' \cup \{G_n : n \in N\}$, This shows that the family consisting of open sets Y′, G1 , G2 , G3 ,…..forms an open countable cover of X which is known to be countably compact. Hence this cover must be reducible to a finite subcover, say Y′, G1 , G2 , ..., Gn so that $X = Y' \cup [\bigcup_{i=1}^{n} G_i] \Rightarrow Y \subset \bigcup_{i=1}^{n} G_i$. It means that $\{G_i : 1 \leq i \leq n\}$ is finite subcover of the countable cover $\{G_n : n \in N\}$ Hence Y is countably compact.

CHECK YOUR PROGRESS

1. Show that C*= { $(0,\frac{n}{n+1})$: n∈ N} is cover of (0,1).
2. Give an example of a compact space which is not Hausdorff.
3. Prove that a topological space is compact if every basic open cover has a finite sub-cover.
4. Show that the usual topological space (ℝ,U) is not compact.
5. Is every indiscrete space (X,I) Compact?
6. Show that (0,1) and (0,1] are not compact.

7.3.4 COMPACT SET IN THE REAL LINE

The theorems of the preceding section enable is to construct new compact spaces from existing ones, but in order to get very far we have to find some compact spaces to start with. The natural place to begin is the real line. Application include the extreme value theorem and the uniform continuity theorem of calculus, suitably generalised.

The theorems of the preceding section enable is to construct new compact spaces from existing ones, but in order to get very far we have to find some compact spaces to start with. The natural place to begin is the real line. Application include the extreme value theorem and the uniform continuity theorem of calculus, suitably generalised.

Theorem : Extreme Value Theorem

Let f: X → Y be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that f(c) ≤ f(x) ≤f(d) for every x ∈ X. The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof: Since f is continuous and X is compact, the set A = f(X) is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have m = f(c) and M = F(d) for some points c and d of X. If A has no largest element, then the collection $\{(-\infty, a) | a \in A\}$ forms an open covering of A. Since A is compact, for some finite subcollection $\{(-\infty, a_1), ..., (-\infty, a_n)\}$ covers A.
If a_i is the largest of the elements $a_1 ,..., a_n$, then a_i belongs to none of these sets, contrary to the fact that they cover A.
A similar argument shows that A has a smallest element.

Definition: Let (X, d) be a metric space; let A be a non-empty subset of X. For each $x \in X$, we define the distance from x to A by the equation
$d(x, A) = \inf \{d(x, a) \mid a \in A\}$. It is easy to show that for fixed A, the function d (x, A) is continuous function of x.
Given $x, y \in X$, one has the inequalities
$d(x, A) \le d(x, a) \le d(x, y) + d(y, a)$,
for each $a \in A$. It follows that $d(x, A) - d(x, y) \le \inf d(y, a) = d(y, A)$So that $d(x, A) - d(y, A) \le d(x, y)$. The same inequality holds with x and y interchanged, continuity of the function d(x, A) follows.

Theorem 8: Every closed and bounded interval on the real line is compact.

Proof: Let $I_1 = [a, b]$ be a closed and bounded interval on P. If possible, let I_1 be not compact. Then there exists an open covering C = {Gi} of I_i , having no finite sub covering.

Let us write $I_1 = [a, b] = \left[a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right] \ldots$ (1)

Since I_1 is not covered by a finite sub-class of C and therefore at least one of the intervals of the union in (1) cannot be covered by any finite sub-class of C. Let us denote such an interval by $I_2 = [a_1, b_1]$.

Now writing $I_2 = [a_1, b_1] = \left[a_1, \frac{a_1+b_1}{2}\right] \cup \left[\frac{a_1+b_1}{2}, b_1\right] \ldots$(2) As the above argument, at least one of the intervals in the union of (2) cannot be covered by a finite sub-class of C.

Let us denote such an interval by $I_3 = [a_2, b_2]$. On continuing this process we obtain a sequence $\langle I_n \rangle$ of closed intervals such that none of these intervals I_n can be covered by a finite sub-class of C.

Clearly the length of the interval $I_n = \frac{a-b}{2^n}$ Thus $\lim|I_n| = 0$

Hence, by the nested closed interval property, $\cap I_n \neq \phi$.

Let $p \in \cap I_n$, then $p \in I_n \forall n \in N$ .In particular $p \in I_1$. Now since C is an open covering of I_1, there exists some $A\alpha_0$ in C such that $p \in A\alpha_0$Since $A\alpha_0$ is open then there exists an open interval $(p - \varepsilon, p + \varepsilon)$ such that $p \in (p - \varepsilon, p + \varepsilon) \subseteq A\alpha_0$. Since $\lim(I_n) \to 0$ as $n \to \infty$, there exists some $In_0 \subseteq (p - \varepsilon, p + \varepsilon) \subseteq A\alpha_0$.

This contradicts occur as our assumption that no I_n is covered by a finite number of members of C. Hence [a, b] is compact.

Example 8: The real line is not compact.

Solution: Let X = {] –n, n [: $n \in N$}. Then each member of C is clearly an open interval and therefore, a U-open set. Also if p is any real number, then there exists a positive integer n_p such that $n_p > |p|$.

Then $p \in\ ] -n_p , n_p [\ \in C$. Thus each point of $\mathbb{R}$ is contained in some member of C and therefore C is an open covering of $\mathbb{R}$.

Now if C * is a family of finite number of sets in C, say

C * = {] $-n_1$, n_1 [,] $-n_2$, n_2 [, …,] $-n_k$, n_k [} and if n* = max {n_1, n_2 , …, n_k}, then n* $\notin \cup_{i=1}^{k}(]-n_k, n_k[)$

Thus it follows that no finite sub-family of C cover ℝ. Hence (ℝ, U) is not compact.

Theorem 9: A closed and bounded subset (subspace) of ℝ is compact.

Proof: Let $I_1 = [a_1 , b_1]$ be a closed and bounded subset of ℝ. Let G = {(c_i, d_i) :i∈Δ} be an open covering of I_1 .

To prove that there exist finite subcover of the original cover G. Suppose that we assume in contradiction way that ∃ no finite subcover of the cover G. Divide I_1 into two equal closed intervals.

$$\left[a_1, \frac{a_1+b_1}{2}\right] and \left[\frac{a_1+b_1}{2}, b_1\right]$$

Then, by assumption, at least one of these two intervals will not be covered by any finite subclass of the cover G. Name that interval by I_2. Write $I_2= [a_2 , b_2]$ Then $[a_2 , b_2] = \left[a_1, \frac{a_1+b_1}{2}\right] or \left[\frac{a_1+b_1}{2}, b_1\right]$.

Divide I_2 into two equal closed intervals

$\left[a_2, \frac{a_2+b_2}{2}\right] and \left[\frac{a_2+b_2}{2}, b_2\right]$. Again by assumption, at least one of these two intervals will not be covered by any finite sub-family of the cover G. Name that I_3 . Write $I_3 = [a_3 , b_3]$. Repeating this process an infinite number of times, we get a sequence of intervals I_1 , I_2 , I_3 , … with the properties.

iii. $I_n \supset I_{n+1} \forall$ n ∈N.

iv. I_n is closed ∀ n ∈ N.

v. I_n is not covered by any finite sub-family of G.

vi. $\lim_{n\to\infty} [I_n] = 0$, where $|I_n|$ denotes the length of the interval I_n and similar is the meaning of $|[a_m, b_m]|$.

Evidently the sequence of intervals $\langle I_n \rangle$ satisfies all the conditions of nested closed interval property.

This $\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$ So that $\exists$ a number $p_0 \in \bigcap_{n=1}^{\infty} I_n$.

7.4 SOLVED PROBLEMS

Example 9: Let $\mathcal{T}$ be the Cofinite topology on any set X. Show that $(X, \mathcal{T})$ is a compact space.

Solution. Let G= $\{G_i\}$ be an open cover of X. choose $G_0 \in$ G. since $\mathcal{T}$ is the cofinite topology, G_o^c is a finite set say $G_o^c = \{a_1, \ldots.. a_m\}$. Since G is cover of X. for each $a_k \in G_o^c \exists G i_k \in$ G such that $a_k \in G i_k$.
Hence $G_o^c \subset G i_1 \cup G i_2 \ldots\ldots \cup G i_m$ and
$X = G_0 \cup G_o^c = G i_1 \cup G i_2 \ldots\ldots \cup G i_m$. Thus X is compact.

Example 10: Show by means of an example that a compact subset of a topological space need not be closed.

Solution: Suppose (X, I) is an indiscrete topological space such that X contains more than one element. Let A be a proper subset of X and let (A, I_1) be a subspace of (X, I). Here, we have $I_1 = \{\phi, A\}$. For $I = \{\phi, X\}$. Hence, the only I_1 – open cover of A is {A} which is finite. Hence A is compact. But A is not I-closed. For the only I-closed sets are ϕ, X. Thus A is compact but not closed.

Example 11: Show that a finite union of compact subspaces of X is compact.

Solution. Suppose that $A_i \subset X$ is compact for $1 \le i \le n$, and suppose that U is a family of open subsets of X whose union contains $\cup_i A_i$. Then for each i there is a finite subfamily U_i whose union contains A_i . If we take U* to be the union of all these subfamilies then it is finite and its union contains $\cup_i A_i$. Therefore the latter is compact.

Example 12: Show that any infinite subset A of a discrete topological space X is not compact.

Solution. Since A is not compact when an open cover has not finite subcover. Consider the class A= $\{\{a\}: a \in \boldsymbol{A}\}$ of singleton subset of **A**. observe that (i) A is a cover of **A** (ii) A is an open cover of **A** since all subset of discrete space are open. (iii) No proper subclass of A is a cover of **A** (iv) A is infinite since **A** is infinite. Hence the open cover A of **A** contain no finite subcover, so **A** is not compact.

CHECK YOUR PROGRESS

1. When is a topological space X is said to be compact

- **a.** there exits a finite open cover for X
- **b.** every open cover of X has a finite subcover
- **c.** if and only if X has finite many elements.

2. A topological space (X, T) is said to be compact if every open cover of X has a finite subcover. T/F

3. Let I be the topology on R whose members are φ, R and all sets of the form (a, ∞) for $a \in R$. Then (R, I) is a

- **a.** T0-space
- **b.** T1-space

c. Hausdorff space

4. Every compact subset of Hausdorff space is –

(A) Closed set (B) Open set (C) Null set (D) None

5. If A is compact then $d(x, A) = d(x, a)$ for some $a \in A$. T/F

6. The real line is not compact. T/F

7. Is R with usual topology is a compact topological space. T/F

8. Let D is subset of a topological space R^2 with usual topology. If finitely many points from the set $D = \{ (x,y): x^2 + y^2 \leq 1$ is the resulting set compact. T/F

9. An indiscrete topology has only elements

a. 1 b. 2 c. 3

10. All Hausdorff spaces with countably many points are compact.

a. True b. False

7.5 SUMMARY

This unit is an explanation of

i. Definition of cover and subcover on a set in a simple form.

ii. Define Heine- Borel Theorem and Finite Intersection Property

iii. Compact set and compact space defined with examples.

iv. Compact subspace on a real line.

v. Give various example and theorems.

7.6 GLOSSARY

i. Closed Open Set

ii. Open cover and subcover

iii. Compact set

iv. Compact space

v. Homeomorphism

vi. Countably Compact

vii. Indiscrete Topology

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7.9 TERMINAL QUESTIONS

1. Define open cover of a topological space.

2. Write an example of T 1 space.

3. Which among the following is NOT a absolute property of a topological space a. compactness b. denseness c. Connectedness

4. Is The surjective map f : [0, 1] → S 1 given by f(t) = (cos(2πt),sin(2πt)) a quotient map.

5. Show by means of an example that a compact subset of a topological space need not be closed.

6. X and Y be compact spaces. Then X × Y is compact. T/F

7. Every continuous image of a compact space is ………………. .

(a) Compact (b) Dense (c) Separable (d) Lindelofff

8. Show that if A is compact then d(x, A) = d(x, a) for some a ∈ A

7.10 ANSWERS

CHECK YOUR PROGRESS

1. B

2. True

3. A

4. A

5. True

6. True

7. False

8. False

9. B

10. False

TERMINAL QUESTIONS

1. Let X be any topological space and $S \subseteq X$ any subset. An open cover of S is any collection of open sets in X whose union contains S..Now let X = R and S = (0, 1). The two collection of two sets $\{(-1/2, 1/2), (0, 3/2)\}$ is an open cover. Since it consists of only two sets, we say it is a finite cover.

2. set of real number with usual topology.

3 b

4 The surjective map $f : [0, 1] \rightarrow S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is quotient map, since [0, 1] is compact and $S^1 \subset R$ is Hausdorff. Similarly for $f \times f : [0, 1] \times [0, 1] \rightarrow S^1 \times S^1$.

5 Suppose (X, I) is an indiscrete topological space such that X contains more than one element. Let A be a proper subset of X and let (A, I_1) be a subspace of (X, I). Here, we have $I_1 = \{\phi, A\}$. For $I = \{\phi, X\}$. Hence, the only I_1 – open cover of A is {A} which is finite. Hence A is compact. But A is not I-closed. For the only I-closed sets are ϕ, X. Thus A is compact but not closed.

6 True

7 a

8 The function f(a) = d(x, a) is continuous and d(x, A) is the greatest lower bound for its set of values. Since A is compact, this greatest lower bound is a minimum value that is realized at some point of A.

UNIT 8: LIMIT POINT COMPACTNESS AND LOCAL COMPACTNESS

<u>CONTENTS:</u>

8.1 INTRODUCTION

In previous units we have studied the compact space. The property limit point compactness is more genuine and perceptive than that of compactness. In this unit we introduce the concept of limit point compactness. In this unit we also explained the concept Local compactness and we prove the theorems that every continuous image of a locally compact space is locally compact and many other theorems.

8.2 OBJECTIVES

After completion of this unit learners will be able to

- **i.** Explained the concept of limit point compactness
- **ii.** Defined the Sequentially Compact
- **iii.** Discussed the Local Compactness
- **iv.** Described the One – point compactification
- **v.** Understand the theorem, Lemma, Corollary and Examples based on limit point compactness and Local Compactness.

8.3 LIMIT POINT COMPACTNESS

Definition:

A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

In a topological space, subsets without limit point are exactly those that are closed and discrete in the subspace topology. So a space is limit point compact if and only if all its closed discrete subsets are finite. has a countably infinite closed discrete subspace.

8.3.1 THEOREM AND LEMMA

Theorem 1: Compactness implies limit point compactness, but not conversely.

Proof.

Let X be a compact space and A is infinite subset of X.

Suppose $A^{'} = \phi$.

That is A does not have any limit point.

Note that, $\bar{A} = A \cup A^{'} = A$ implies A is a closed set,

then $x \in A$ imply $x \notin A^{'}$ implies there exists an open set U_x such that $x \in U_x, U_x \cap A \backslash \{x\} = \emptyset$.

It means $U_x \cap A \cap \{x\}^c = \emptyset \Rightarrow U_x \cap A = \{x\}$.

Now, $\{U_x : x \in A\}$ is an open cover for the closed subset A,

of the given compact topological space.

Hence there exists a natural number n and $x_1, x_2, \ldots . x_n \in A$,

such that $A \subseteq U_{x_1} \cup \ldots \ldots . . \cup U_{x_n}$.

This gives that

$A = \left(U_{x_1} \cup U_{x_2} \ldots \ldots . U_{x_n}\right) \cap A = \left(U_{x_1} \cap A\right) \cup \left(U_{x_2} \cap A\right) \cup$

$\ldots \ldots . \left(U_{x_n} \cap A\right) = \{x_1, x_2, \ldots . x_n\}$.

Hence we have arrived at a contradiction by assuming $A^{'} = \phi$.

Therefore $A^{'} \neq \phi$.

Now we introduce the notion of Lebesgue number.

Recall that the diameter of a bounded subset A of a metric space (X, d) is the number

$$\text{lub } \{d(a_1, a_2) | a_1, a_2 \epsilon A\}.$$

Lemma (1) (The Lebesgue number Lemma):

Let A be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of A containing it.

The number δ is called a Lebesgue number for the covering A.

Proof:

Let A be an open covering of X.

If X itself is an element of A, then any positive number is a Lebesgue number of A.

So assume X is not an element of A.

Choose a finite subcollection $\{A_1, ..., A_n\}$ of A that covers X.

For each i, set $C_i = X - A_i$, and define f: $X \to R$ be letting f(x) be the average of the numbers $d(x, C_i)$.

That is, $f(x) = \frac{1}{n} \sum\sum_{i=1}^{n} (x, C_i)$.

We show that $f(x) > 0$ for all x.

Since given $x \in X$, choose i so that $x \in A_i$.

Then choose $\in$ so $\in$-neighbourhood of x lies in A_i.

Then $d(x,c_i) \geq \in$, so that $f(x) \geq \in/n$.

Since f is continuous, it has a minimum value δ_i we show that δ is our required Lebesgue number.

Let B be a subset of X of diameter less that δ.

Choosing a point x_0 of B; then B lies in the δ-neighbourhood of x_0.

Now $\delta \leq f(x_0) \leq d(x_0, C_m)$, where $d(x_0, Cm)$ is the largest of the number $d(x_0, C_i)$.

Then the δ-neighbourhood of x_0 is contained in the element $A_m - X - C_m$ of one covering A.

8.3.2 SEQUENTIALLY COMPACT

Definition: If every sequence in a space X has a convergent subsequence then X is **sequentially compact.**

Every metric space is naturally a topological space, and for metric spaces, the notions of compactness and sequential compactness are equivalent (if one assumes countable choice). However, there exist sequentially compact topological spaces that are not compact, and compact topological spaces that are not sequentially compact.

Definition: Uniformly Continuous

A function F from the metric space (X, d_x) to the metric (Y, d_y) is said to be uniformly continuous if given $\in > 0$, there is a $\delta > 0$, such that for every pair of points x_0 , x_1 of X,

$$d_x(x_0, x_1) < d \Rightarrow d_y\big(f(x_0), f(x_1)\big) < \epsilon$$

8.3.3 THEOREM

Theorem 2: Uniform Continuity Theorem

Let f: X → Y be a continuous map of the compact metric space (X, d_x) to be metric space (Y, d_y). Then f is uniformly continuous.

Proof: Given $\in > 0$, take the open covering of Y by balls B (y, $\in/2$) of radius $\in/2$.

Let A be the open covering of X by the inverse images of these balls under f.

Choose δ to be a Lebesgue number for the covering A.

Then if x_1 and x_2 are two points of X such that $d_x(x_1, x_2) < \delta$, the two point set $\{x_1, x_2\}$ has diameter less than δ.

So that its image $\{f(x_1), f(x_2)\}$ lies in some ball B (y, $\in/2$).

Then $dy(f(x_1), f(x_2)) < \in$, as desired.

Finally, we prove that the real numbers are uncountable.

The interesting thing about this proof is that it involves no algebra at all-no decimal or binary expansions of real numbers or the like-just the other properties of R .

Theorem 3: Let X be a metrizable space. Then the following are equivalent:

- **i.** X is compact.
- **ii.** X is limit point compact.
- **iii.** X is sequentially compact.

Proof.

Using the theorem 1, 2 and lemma 1 X is compact implies X is limit point compact implies X is sequentially compact.

We have also proved that sequential compactness implies that every open covering of X has a Lebesgue number.

We now proving that sequentially compactness of X imply compactness of$X$. So assume that X is sequentially compact.

In first step:

First we show that for every $\in > 0$, there exists a finite covering of X by $\in -$ balls.

And once again, we prove the contrapositive:

If for some $\in > 0, X$ cannot be covered by finitely $\in -$ balls, then X is not sequentially compact.

So suppose that X cannot be covered by finitely many $\in -$ balls.

Construct a sequence of points x_n of X as follows:

First choose x_1 to be any point of X.

Nothing that the ball $B(x_1, \in)$ is not all of X

(Otherwise X could be covered by a single $\in -$ ball),

choose x_2 to be a point of X not in $B(x_1, \in)$.

In general $x_1, \dots \dots \dots, x_n$, choose x_{n+1} to be a point not in the union

$$B(x_1, \in) \cup \dots \dots \dots . \cup B(x_n, \in),$$

using the fact that these balls do not cover X.

Note that by construction $d(x_{n+1}, x_i) \geq \in$ for, $i = 1, \dots \dots \dots n$.

Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\in/2$ any contain x_n for at most one value of n.

In Second step:

Now we prove that X is compact.

Let A be an open covering of X.

Since X is sequentially compact, the covering A has a Lebesgue δ.

Using first step choose a finite covering of X by balls of radius $\delta/3$.

Each of these balls has a diameter at most $2\delta/3$,

so we can choose for each of these balls an element of A containing it.

We thus obtain a finite subcollection of A covers X.

8.3.4 EXAMPLES

Example 1:

- **i.** Every countably compact space (and hence every compact space) is limit point compact.
- **ii.** Some examples of spaces that are not limit point compact: (1) The set $\mathbb{R}$ of all real numbers with its usual topology, since the integers are an infinite set but do not have a limit point in $\mathbb{R}$; (2) an infinite set with the discrete topology; (3) the countable complement topology on an uncountable set.

- **iii.** The space of all real numbers with the standard topology is not sequentially compact; the sequence $\langle x_n \rangle$ given by $x_n = n$, for all natural numbers n is a sequence that has no convergent subsequence.

Example 2:

Let $X = \{0,1\}, \mathcal{T} = \{\emptyset, X\}$ and $Y = \mathbb{N} = \{1,2, \dots.\}$, the set of all natural numbers and $\mathcal{T}'$ is

discrete topology on $\mathbb{N}$.

Let $X_0 = X \times Y$ be the product space.

In this example $\{X \times \{n\}\}$ is an open cover for $X \times Y$. But, for any fixed $k \in \mathbb{N}, X \times Y = X \times \mathbb{N} \nsubseteq (X \times \{1\} \cup \dots \dots .. \cup (X \times \{k\}))$.

It is noted that $(1, k+1) \notin \cup_{j=1}^{k} X \times \{j\}$.

This gives that $X \times Y$ is not a compact topological space.

Now let A be a nonempty subset of $X \times Y$.

Then there exists $k \in \mathbb{N}$ such that $(0, k) \epsilon A$ or $(1, k) \in A$.

In this case we claim that $(1, k) \in A'$.

Take a basic open set U containing $(1, k)$ then $U = X \times \{k\}$.

Now, $(0, k) \in U \cap A\{(1, k)\} \neq \emptyset$.

Hence, we have proved that $(1, k)$ is a limit point of A.

It is also noted that if $(1, k) \in A$ then we can prove that $(1, k)$ is a limit point of A.

So we have proved that every nonempty subset A of $X \times Y$ has a limit point.

In particular every infinite subset of $X \times Y$ has a limit point.

Therefore, $X \times Y$ is a limit point compact.

8.4 LOCAL COMPACTNESS

In topology a topological space is called locally compact if, each small portion of the space looks like a small portion of a compact space. More precisely, it is a topological space in which every point has a compact neighborhood. A topological space *X* is locally compact iff every point has a local base of compact neighborhoods. (Note that these neighborhoods do not have to be open themselves but need only contain an open set containing the given point.) Other definitions may be found in the literature, as discussed in the **Non-Hausdorff spaces** . The various definitions of local compactness all coincide for Hausdorff spaces. Almost all locally compact spaces studied in applications are Hausdorff, and this unit is thus primarily concerned with locally compact space.

A space X is said to be **locally compact** at x if there is some compact subset C of X that contains a neighborhood of x. ($\mathbb{R}$ is locally comapct since $x \epsilon \mathbb{R}$ lies in neighborhood $(x-1, x+1)$ which is in the compact space $[x-1, x+1]$.

If X is locally compact at each of its points, X is said simply to be locally compact.

.

Note: A compact space is automatically locally compact

8.4.1 ONE-POINT COMPACTIFICATION

In the field of topology, the Alexandroff extension or one-point compactification is a way to extend a noncompact topological space by adjoining a single point in such a way that the resulting space is compact. It is named after the Russian mathematician Pavel Alexandroff.

The advantages of the Alexandroff compactification or one-point compactification lie in its simple, often geometrically meaningful structure and the fact that it is in a precise sense minimal among all compactifications; the disadvantage lies in the fact that it only gives a Hausdorff compactification on the class of locally compact, noncompact Hausdorff spaces.

Let X be a locally compact Hausdorff space. Take some object outside X, denoted by the

Symbol ∞ for convenience, and adjoin it to X, forming the set $Y = X \cup \{\infty\}$.

Topologize Y by defining the collection of open sets in Y to be all sets of the following types:

- **i.** U, where U is an open subset of X.
- **ii.** $Y - C$, where C is a compact subset of X.

The space Y is called the one – point compactification of X.

We need to check that this collection is, in fact, a topology on Y. The empty set is a set of type (i), and the space Y is a set of type (ii).

Given a topological space X, we wish to construct a compact space Y by connecting one point:

$\mathrm{Y} = \mathrm{X} \cup \{\infty\}$. This is called a **one-point compactification** of X.

8.4.2 THEOREM

Theorem 4:

Let X be a locally compact Hausdorff space which is not compact: let Y be the one – point compactification of X. Then Y is a compact Hausdorff space; X is a subspace of Y; the set $Y - X$ consists of a single point; and $\bar{X} = Y$.

Proof.

First we show that X is a subspace of Y and $\bar{X} = Y$.

Given any open set of Y, it's intersection with X is open in X.

Since $U \cap X = U$ and $(Y - C) \cap X = X - C$, both of which are open in X.

Conversely, any set open in X is a set of type (i) and therefore open in Y.

The collection A must contain an open set of type (ii), say $Y - C$, since none of the open sets of type (i) contain the point ∞.

Take all the members of A different from $Y - C$ and intersect them with X; they from a collection of open sets in X covering C.

Because C is compact, finitely many of them cover C; the corresponding finite collection of elements of A will, along with the element $Y - C$, cover all of Y.

To show that Y is Hausdorff, let x and y be two points of Y.

If both of them lie in X, there are disjoint sets U and V open in X containing them, respectively. On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact set C in X containing a neighbourhood U of x.
Then U and $Y - C$ are disjoint neighborhoods of x and ∞, respectively, in Y.

8.4.3 LEMMA

Let X be a Hausdorff space. Then X is locally compact at x if and only if for every neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

8.4.4 COROLLARY

1. Let X be a locally compact Hausdorff space; let Y be a subspace of X. If Y is closed in X or open in X, then Y is locally compact.

2. A space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff.

8.4.4 EXAMPLES

Example3:

The Euclidean spaces $\mathbb{R}^n$ with the standard topology: their local compactness follows from the Heine-Borel theorem. The complex plane $\mathbb{C}$ carries the same topology as $\mathbb{R}^2$ and is therefore also locally compact.

Example 4:

Any discrete space is locally compact, since the singletons can serve as compact neighborhoods.

Example 5:

Consider the real line **R** with the usual topology. Observe that each point $p \in \mathbf{R}$ is interior to a closed interval, e.g. $[p - \delta, p + \delta]$, and that the closed interval is compact by the Heine-Borel Theorem. Hence **R** is a locally compact space. On the other hand, **R** is not a compact space; for example, the class

$$\boldsymbol{A} = \{\ldots\ldots.., (-\mathbf{3}, -\mathbf{1}), (-\mathbf{2}, \mathbf{0}), (-\mathbf{1}, \mathbf{1}), (\mathbf{0}, \mathbf{2}), (\mathbf{1}, \mathbf{3}) \ldots\ldots\ldots\}$$

is an open cover of ℝ but contains no finite subcover.

Example 6:

Consider the real line **R** with the usual topology $\mathcal{U}$. We adjoin two new points, denoted by ∞ and $-\infty$, to **R** and call the enlarged set $\mathbf{R}^* = \mathbf{R} \cup \{-\infty, \infty\}$ the *extended real line*. The order relation in **R** can be extended to $\mathbf{R}^*$ by defining $-\infty < a < \infty$ for any $a \in \mathbf{R}$. The class of subsets of $\mathbf{R}^*$ of the form

$$(a, b) = \{x : a < x < b\}, \quad (a, \infty] = \{x : a < x\} \quad \text{and} \quad [-\infty, a) = \{x : x < a\}$$

is a base for a topology $\mathcal{U}^*$ on $\mathbf{R}^*$. Furthermore, $(\mathbf{R}^*, \mathcal{U}^*)$ is a compact space and contains $(\mathbf{R}, \mathcal{U})$ as a subspace, and so it is a compactification of $(\mathbf{R}, \mathcal{U})$.

Example 7:

The one – point compactification of the real line ℝ is homeomorphic with the circle. The one-point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere S^2.

Example 8:

Show that the rationals Q is not locally compact.

Solution: $[a, b] \cap Q$ are not compact as we may take a sequence converging to an irrational number (in R) and no subsequence converges to a point in Q (sequential compactness is equivalent to compactness for metric spaces). Suppose some compact (and, therefore, closed) subset S of Q contains an open subset of Q. Then it contains an interval $[a, b] \cap Q$. The interval is closed in S and, therefore, compact contradiction. Therefore, there are no compact subsets of Q that contain any open subset. Hence, Q is not locally compact.

Example 9:

Show that the one – point compactification of Z_+ is homeomorphic with the subspace
$\{0\} \cup \{1/n \,|n \epsilon Z_+\}$ of $\mathbb{R}$.

Solution:

Z_+ is homeomorphic to the set $A = \{1/n \,|n \epsilon Z\}$ in the discrete topology, which is equivalent to the topology inherited from the standard topology of the real line. $A \cup \{0\}$ is a compact and Hausdorff space, therefore, it is a one-point compactification of the subspace A.

8.5 SUMMARY

This unit is an explanation of

i. Definition of limit point compactness

ii. Define the Sequentially Compact.

iii. Definition of Local Compactness

iv. One – point compactification defined with examples.

v. Give various example and theorems.

8.6 GLOSSARY

i. Topological space.

ii. Set.

iii. Subset.

iv. Limit point.

v. Continuity.

vi. Uniform Continuity.

vii. Compact space.

viii. Hausdorff space.

CHECK YOUR PROGRESS

1. A locally compact at a point space is a space that contains a compact subspace containing a ………………………..of the point
2. A Hausdorff space is locally compact iff ……..neighborhood of any point contains a compact closure of a neighborhood of the point.
3. A closed subspace of a ………………is locally compact.
4. An ………………….of a locally compact Hausdorff space is locally compact
5. The ……………………….. of a locally compact space is locally compact.
6. In general, the continuous image of a locally compact space does not have to be locally compact. **True/False**
7. A homeomorphism of two locally compact Hausdorff spaces cannot be extended to a homeomorphism of their one-point compactifications. **True/False**
8. The finite product of locally compact spaces is locally compact. **True/False**

9. The rationals Q is locally compact **True/False**

10. The product of any family of spaces is locally compact ⇔ all but finitely many of them are compact and those which are not compact are locally compact. **True/False**

8.7 REFERENCES

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8.8 SUGGESTED READINGS

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iii. **https://archive.nptel.ac.in/noc/courses/noc22/SEM1/noc22-ma36/**

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8.9 TERMINAL QUESTIONS

TQ1:What is a local compactness?..

TQ2: Is local compactness hereditary?...

TQ3: Why Q is not locally compact?..

TQ4: Is Z locally compact?...

TQ5: Is local compactness a topological property?...

TQ6: What is a one-point compactification?..

TQ7: What is the one-point compactification of positive integers?................

8.10 *ANSWERS*

CHECK YOUR PROGRESS

CHQ 1: neighborhood.

CHQ2: any.

CHQ3: locally compact space.

CHQ4: open subspace

CHQ5: open continuous image.

CHQ6: True.

CHQ7: False.

CHQ8: True.

CHQ9: False.

CHQ10: True.

TERMINAL QUESTIONS

TQ2: Local compactness is not hereditary.

TQ 4: The order topology on Z is indeed the discrete topology, and it is locally compact.

TQ5: There are several slightly different definitions of local compactness; they are equivalent for Hausdorff spaces but not necessarily for non-Hausdorff spaces. All of them define topological properties.

TQ7: The one-point compactification of the set of positive integers is homeomorphic to the space consisting of $A = \{1/n \mid n \in Z\}$ with the order topology.

BLOCK III :

COUNTABILITY AND SEPARATION AXIOMS

UNIT 9: COUNTABLITY AXIOMS

CONTENTS:

9.1 INTRODUCTION

Before this the concept of Topological Spaces, Continuous Functions, Connectedness and Compactness has defined. In this unit we are explaining about the concept of countablity axioms. The countability axioms in general topology are making generalizations. Countable local basis. The concept of First countable space, Second Countable Topological Space. Seprable Topological Space, Lindelöf space are explained in this unit. The examples and results are also discussed here.

9.2 OBJECTIVES

After completion of this unit learners will be able to

i. Define the concept of Countability axioms.

ii. Describe the notion of First countable space and Second Countable Space

iii. Explain the concept of Lindelöf space.

10.3 COUNTABLE LOCAL BASIS

A topological space $(X, \mathcal{J})$ is said to have a ***countable local basis*** (or countable basis) at a point $x \in X$ if there exists a countable collection say $\mathscr{B}_x$ of open sets containing x such that for each open set U containing x there exists $V \in \mathscr{B}_x$ with $V \subseteq U$.

10.4 FIRST COUNTABLE SPACE

A topological space $(X, \mathcal{J})$ is said to be ***first countable*** or said to satisfy the first countability axiom if for each $x \in X$ there exists a countable local base at x.

9.4.1 EXAMPLES

Example 1:

(i) Let (X, d) be a metric space then for each $x \in X$, $\mathscr{B}_x = \{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local basis at x. Hence $(X, \mathcal{J}_d)$ is a first countable space. So, we say that every metric space (X, d) is a first countable space.

(ii) Let $X = \mathbb{N}$ and $\mathcal{J} = \{\phi, X, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}, \ldots, \}$ then obviously $(X, \mathcal{J})$ is a first countable topological space.

Note that this is not an interesting example of a first countable topological space. Once the topology $\mathcal{J}$ is a countable collection then $(X, \mathcal{J})$ is a first countable space.

Example 2:

Let $X = \mathbb{R}$ and $\mathcal{J}_l$ be the lower limit topology on $\mathbb{R}$ generated by $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. For each $x \in X$, $\mathscr{B}_x = \{[x, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local base at x. Hence $(\mathbb{R}, \mathcal{J}_l) = \mathbb{R}_l$ is a first countable topological space. Now let us see a stronger version of first countable topological space.

10.5 SECOND COUNTABLE TOPOLOGICAL SPACE

If a topological space $(X, \mathcal{J})$ has a countable basis $\mathscr{B}$ then we say that $(X, \mathcal{J})$ is a ***second countable topological space*** or it satisfies the second countability axiom.

Though it is trivial from the definition, prove that every second countable topological space $(X, \mathcal{J})$ is a first countable topological space. □

What about the converse?

Let X be any uncountable set and $\mathcal{J}_D$ be the discrete topology on X. Then $(X, \mathcal{J}_D)$ is first countable, but it is not second countable. In fact, for each $x \in X$, $\mathscr{B}_x = \{\{x\}\}$ is a countable local base at x. Take any open set U containing x then there exists $V = \{x\} \in \mathscr{B}_x$ such that $x \in V \subseteq U$. Hence $\mathscr{B}_x$ is a local base at x.

How to prove that $(X, \mathcal{J})$ is not a second countable topological space ? Well we use the method of proof by contradiction.

Suppose there exists a countable basis say $\mathscr{B} = \{B_1, B_2, \ldots, \}$ for $(X, \mathcal{J})$. Let us assume that each $B_k \neq \phi$. For each $k \in \mathbb{N}$, let $x_k \in B_k$. Since X is an uncountable set we can select an $x \in X$ such that $x \neq x_k$ for all $k \in \mathbb{N}$. Now $\{x\}$, the singleton set containing x, is an open set and $\mathscr{B}$ is a basis for $(X, \mathcal{J})$ implies there exists $k \in \mathbb{N}$ such that $x \in B_k \subseteq \{x\}$ this implies $B_k = \{x\}$. But $x_k \in B_k$ implies $x = x_k$, a contradiction to our assumption that $x \in X$ such that $x \neq x_k$ for all $k \in \mathbb{N}$. Hence if X is an uncountable set then the discrete topological space $(X, \mathcal{J}_D)$ is first countable but not second countable.

Also we have seen that the lower limit topological space $\mathbb{R}_l$ is first countable. Now let us prove that $\mathbb{R}_l = (\mathbb{R}, \mathcal{J}_l)$ is not a second countable topological space. That

is we will have to prove that if $\mathscr{B}$ is a basis for $(\mathbb{R}, \mathcal{J}_l)$ then $\mathscr{B}$ is not a countable collection. So, fix a basis say $\mathscr{B}$ for $(\mathbb{R}, \mathcal{J}_l)$. For each $x \in \mathbb{R}$, $[x, x+1) \in \mathcal{J}_l$. Hence $\mathscr{B}$ is a basis for $(\mathbb{R}, \mathcal{J}_l)$ implies there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq [x, x+1)$.

For $x, y \in \mathbb{R}$, $x \neq y$ we have $[x, x+1) \neq [y, y+1)$. Also $B_x \subseteq [x, x+1)$ implies $\inf B_x \geq \inf[x, x+1) = x$. Also $x \in B_x$ implies $x \geq \inf B_x$. Hence $x = \inf B_x$. Now define $f : \mathbb{R} \to \mathscr{B}$ as $f(x) = B_x$. Then $x \neq y$ implies $B_x \neq B_y$. ($B_x = B_y$ implies $\inf B_x = \inf B_y$) That is $f(x) \neq f(y)$. Hence f is an one-one function. This implies that $f : \mathbb{R} \to f(\mathbb{R}) \subseteq \mathscr{B}$ is a bijective function. Therefore $f(\mathbb{R})$ is an uncountable set and hence $\mathscr{B}$ is an uncountable set. We have proved that if $\mathscr{B}$ is a basis for $(\mathbb{R}, \mathcal{J}_l)$ then $\mathscr{B}$ is an uncountable set. Hence $(\mathbb{R}, \mathcal{J}_l)$ cannot have a countable basis and therefore $(\mathbb{R}, \mathcal{J}_l)$ is not a second countable topological space.

It is a simple exercise to check $\overline{\mathbb{Q}} = \mathbb{R}$ in $(\mathbb{R}, \mathcal{J}_l)$. That is $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$ with respect to $(\mathbb{R}, \mathcal{J}_l)$. Such a topological space is known as a separable topological space.

10.6 SEPRABLE TOPOLOGICAL SPACE

A topological space $(X, \mathcal{J})$ is said to be a ***separable topological space*** if there exists a countable subset say A of X such that $\overline{A} = X$.

10.7 Lindelöf Space

A topological space $(X, \mathcal{J})$ is said to be a ***Lindelöf*** space if for any collection $\mathcal{A}$ of open sets such that $X = \underset{A\in\mathcal{A}}{\cup} A$, there exists a countable subcollection

say $\mathscr{B} \subseteq \mathcal{A}$ such that $X = \underset{B\in\mathscr{B}}{\cup} B$. That is, a topological space $(X, \mathcal{J})$ is said to be a Lindelöf space if and only if every open cover of X has a countable subcover for X.

By definition every compact topological space $(X, \mathcal{J})$ is a Lindelöf space. But the converse need not be true. It is easy to prove that $\mathbb{R}$ ($\mathbb{R}$ with usual topology) is a Lindelöf space. But $\mathbb{R}$ is not compact space.

Now let us prove that every second countable topological space is a Lindelöf space.

10.8 THEOREMS

Theorem 9.8.1.

If $(X, \mathcal{J})$ is a second countable topological space then $(X, \mathcal{J})$ is a Lindelöf space.

Proof. Let $\mathscr{B} = \{B_1, B_2, B_3, \ldots\}$ be a countable basis for $(X, \mathcal{J})$ and $\mathcal{A}$ be an open cover for X. Let us assume that, $X \neq \phi$, $A \neq \phi$ for each $A \in \mathcal{A}$ and $B \neq \phi$, for each $B \in \mathscr{B}$. Fix $A \in \mathcal{A}$ and $x \in A$. Now $x \in A$, A is an open set implies there exists $B \in \mathscr{B}$ such that

$$x \in B \subseteq A. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

For each $n \in \mathbb{N}$, let $\mathcal{F}_n = \{A \in \mathcal{A} : B_n \subseteq A\}$. Here it is possible that $\mathcal{F}_n = \phi$, for some $n \in \mathbb{N}$. At the same time note that, from Eq. (5.1), $\{n \in \mathbb{N}\colon \mathcal{F}_n \neq \phi\}$ is a nonempty set. Let $\{n \in \mathbb{N}\colon \mathcal{F}_n \neq \phi\} = \{n_1, n_2, \ldots, n_k \ldots, \}$ (it may be a finite set) and for each such k take $A_{n_k} \in \mathcal{F}_{n_k}$. This will give us $B_{n_k} \subseteq A_{n_k} \in \mathcal{A}$.

Let us prove that $\overset{\infty}{\underset{k=1}{\cup}} A_{n_k} = X$. So, let $x \in X$. Now $\mathcal{A}$ is an open cover for X implies $x \in A$ for some A. Now $x \in A$, $\mathscr{B}$ is a basis for $(X, \mathcal{J})$ implies there exists $k \in \mathbb{N}$ such that $x \in B_{n_k} \subseteq A$. This implies that $A \in \mathcal{F}_{n_k}$. Also $A_{n_k} \in \mathcal{F}_{n_k}$. Hence by our definition of $\mathcal{F}_{n_k}$, $B_{n_k} \subseteq A_{n_k}$. Hence $x \in X$ implies $x \in A_{n_k}$, for some $k \in \mathbb{N}$.

This implies that $X \subseteq \bigcup_{k=1}^{\infty} A_{n_k}$. That is $\{A_{n_k}\}_{k=1}^{\infty}$ is a countable subcover for $\mathcal{A}$. Therefore every open cover $\mathcal{A}$ of X has a countable subcover. Hence $(X, \mathcal{J})$ is a Lindelöf space. ■

Note. Recall that, for $1 \leq p < \infty$, $l_p = \{x = (x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ is a second countable metric space, where for $x = (x_n) \in l_p$, $y = (y_n) \in l_p$,

$$d_p((x_n), (y_n)) = d_p(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}.$$

Also note that $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ is a second countable metric space with respect to any of the metric given by $d_p(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $1 \leq p < \infty$ or $d_\infty(x, y) = \max\{|x_k - y_k| : k = 1, 2, \ldots, n\}$. So, all the above mentioned metric spaces are all Lindelöf spaces. But none of these metric spaces is a compact space. ⋆

Now let us prove that a second countable topological space is a separable space.

Theorem 9.8.2:

Every second countable topological space $(X, \mathcal{J})$ is a separable space.

Proof. Given that $(X, \mathcal{J})$ is a second countable topological space. Hence there exists a countable basis say $\mathscr{B} = \{B_1, B_2, \ldots\}$ for $(X, \mathcal{J})$. When we write $\mathscr{B} = \{B_1, B_2, \ldots,\}$, it does not mean that $\mathscr{B}$ is a countably infinite set. It means that either for some $n \in \mathbb{N}$, $\mathscr{B} = \{B_1, B_2, \ldots, B_n\}$ or $\mathscr{B} = \phi$ or $\mathscr{B}$ is a countably infinite set. If $X \neq \phi$ then $\mathscr{B} \neq \phi$. If for some $k \in \mathbb{N}$, $B_k = \phi$, then $\mathscr{B}' = \{B_1, B_2, \ldots, B_{k-1}, B_{k+1}, \ldots,\}$ is also a basis for $(X, \mathcal{J})$.

So, let us assume that each $B_n \neq \phi$ for all n. Since $B_n \neq \phi$, for each $n \in \mathbb{N}$, let $x_n \in B_n$ (note that by axiom of choice there exists a function $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} B_n$ such that $x_n = f(n) \in B_n$) and $A = \{x_1, x_2, x_3 \ldots, \}$. Here also it is quite possible that A is a finite set. Now let us prove that $\overline{A} = X$. So, take an $x \in X$ and an open set U containing x. Now $\mathscr{B}$ is a basis for $(X, \mathcal{J})$, U is an open set containing x implies there exists $B_n \in \mathscr{B}$ such that $x \in B_n$ and $B_n \subseteq U$. Also $x_n \in B_n$. Hence $x_n \in U \cap A$. This gives that $U \cap A \neq \phi$. That is we have proved that $U \cap A \neq \phi$ for each open set U containing x. Hence $x \in \overline{A}$. That is $x \in X$ and hence $x \in \overline{A}$ and hence $\overline{A} = X$. Therefore $(X, \mathcal{J})$ has a countable dense subset and therefore $(X, \mathcal{J})$ is a separable space. ■

Now let us prove that subspace of a separable metric space is separable.

Theorem 9.8.3:

Let (X, d) be a separable metric space and Y be a subspace of X (that is $Y \subseteq X$, and for $x, y \in Y$, $d_Y(x, y) = d(x, y)$). Then (Y, d_Y) is a separable space.

Proof. (X, d) is a separable metric space implies there exists a countable subset say $A = \{x_1, x_2, x_3 \ldots, \}$ of X such that $\overline{A} = X$ (here $\overline{A}$ denotes the closure of A with respect to (X, d)). We will have to find a countable subset say B of Y such that $\overline{B}_Y = Y$ (here $\overline{B}_Y = \overline{B} \cap Y$, the closure of B with respect to the subspace (Y, d_Y)). For $n \in \mathbb{N}$, let $A_{n,k} = B(x_n, \frac{1}{k}) \cap Y$. Here we do not know whether $A_{n,k} = \phi$ or $A_{n,k} \neq \phi$. If $A_{n,k} \neq \phi$ $(n, k \in \mathbb{N})$ let $a_{n,k} \in A_{n,k}$ be a fixed element. Then $B = \{a_{n,k} : a_{n,k} \in A_{n,k}$ whenever $A_{n,k} \neq \phi\}$ is a countable subset of Y. Now let us prove that $\overline{B}_Y = \overline{B} \cap Y = Y$. So let $x \in Y$ and U be an open set in X containing x. Hence there exists $k \in \mathbb{N}$ such that $B(x, \frac{1}{k}) \subseteq U$. Again $x \in \overline{A} = X$ implies $B(x, \frac{1}{2k}) \cap A \neq \phi$. Then there exists $x_n \in A$ such that $x_n \in B(x, \frac{1}{2k})$. Therefore $x \in B(x_n, \frac{1}{2k}) \cap Y = A_{n,2k}$. Hence $A_{n,2k} \neq \phi$. Now $A_{n,2k} \neq \phi$ implies $a_{n,2k} \in B$. Further $a_{n,2k} \in B(x_n, \frac{1}{2k})$. Now $d(x, a_{n,2k}) \leq d(x, x_n) + d(x_n, a_{n,2k}) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$.

Hence $a_{n,2k} \in B(x, \frac{1}{k}) \cap B \subseteq U \cap B$. That is $U \cap B \neq \phi$ for each open set U containing x. This implies $x \in \overline{B}$. Also $x \in Y$. Therefore $x \in \overline{B} \cap Y = \overline{B}_Y$. This implies $Y \subseteq \overline{B}_Y \subseteq Y$ and hence $\overline{B}_Y = Y$. That is B is a countable dense subset of Y. This proves that the subspace (Y, d_Y) is a separable metric space. ■

Note:

Subspace of a separable topological space need not be separable.

We give an example to show that subspace of a separable topological space need not be separable.

Let $X = \{(x,y) : x \in \mathbb{R}, y \geq 0\}$. Basic open sets are of the type: (i) for $(x,y) \in \mathbb{R}^2$, $x \in \mathbb{R}, y > 0$ basic open sets containing (x,y) are of the form $B((x,y),r), 0 < r < y$, and (ii) for $(x,0) \in \mathbb{R}^2, x \in \mathbb{R}$, basic open sets are of the form $(B(x,0),r) \cap X) \setminus \{(y,0) : 0 < |y-x| < r\}, r > 0$. Here $B((x,y),r) = \{(a,b) \in \mathbb{R}^2 : d((x,y),(a,b)) = \sqrt{(x-a)^2 + (y-b)^2} < r\}$, the open ball centered at (x,y) and radius r with respect to the Euclidean metric d on $\mathbb{R}^2$.

It is easy to see that the above collection of sets will form a basis for a topology on X.

Let $\mathcal{J}$ be the topology on X induced by the collection of basic open sets described as above and if $A = \{(x,y) : x, y \in \mathbb{Q}, y \geq 0\}$ then A is a countable subcollection of X such that $\overline{A} = X$. That is A is a countable dense subset of X, and hence $(X, \mathcal{J})$ is a separable topological space.

Now for $Y = \{(x,0) : x \in \mathbb{R}\}$ (an uncountable set), $\mathcal{J}_Y = \mathcal{P}(Y)$. That is the subspace $(Y, \mathcal{J}_Y)$ of $(X, \mathcal{J})$ is the discrete topological space. That is every subset U of Y is both open and closed in $(Y, \mathcal{J}_Y)$. Therefore if B is a countable subset of Y then $\overline{B} = B \neq Y$ here $(\overline{B} = \overline{B}_{\mathcal{J}_Y}$, the is closure of B in $(Y, \mathcal{J}_Y))$. This proves that $(Y, \mathcal{J}_Y)$ is not a separable subspace, though $(X, \mathcal{J})$ is a separable topological space.

Exercise:

Prove that every separable metric space is second countable. $\square$

Exercise:

Prove that $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ is a separable metric space for $1 \leq p < \infty$, $d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $d_\infty(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$. $\square$

It is easy to prove that if $\mathcal{J}_2$ is the topology induced by d_2 then $\mathcal{J}_p = \mathcal{J}_2$, $\forall\, p \geq 1$. That is all these metrics $d_p, 1 \leq p \leq \infty$ will induce the same topology on $\mathbb{R}^n$. So if we want to prove that $(\mathbb{R}^n, d_p)$ is a separable metric space, it is enough to prove that $(\mathbb{R}^n, \mathcal{J}_1)$ (or say $(\mathbb{R}^n, \mathcal{J}_2)$) is separable.

For $1 \leq p < \infty$, let $l_p = \{x = (x_n) : x_n \in \mathbb{R}$ for all n and $\sum_{n=1}^{\infty} |x_n|^p < \infty\}$. If we define, $d_p(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$, then d_p is a metric on l_p. Now let us see how to prove that (l_p, d_p) is a separable metric space.

Step 1: For each $n \in \mathbb{N}$, let $A_n = \{(r_1, r_2, \ldots, r_n, \ldots, 0, 0, \ldots) : r_i \in \mathbb{Q}, i = 1, 2, \ldots, n\}$.

If we define $f : \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ (n times) $= \mathbb{Q}^n \to A_n$ as $f(r_1, r_2, \ldots, r_n) = (r_1, r_2, \ldots, r_n, \ldots, 0, 0, \ldots)$. Then f is a bijective function. Now $\mathbb{Q}$ is a countable set implies $\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ (finite product of countable sets is countable) is countable. Hence there is a bijection between $\mathbb{Q}^n$ and A_n implies A_n is a countable set. Now each A_n is a countable set implies $\overset{\infty}{\cup} A_n$ is also countable.

We leave it as an exercise to prove that $\bigcup_{n=1}^{\infty} A_n = l_p$. That is $\bigcup_{n=1}^{\infty} A_n$ is a countable dense subset of l_p. Hence (l_p, d_p) is a separable metric space. These spaces are important examples of Banach spaces. If $l_\infty = \{x = (x_n) : (x_n)$ is a bounded real sequence $\}$ and $d_\infty(x, y) = \sup\{|x_n - y_n| : n \geq 1\}$, then (l_∞, d_∞) is also a metric space. Let $X = \{x = (x_n) : x_n = 0$ or $x_n = 1\}$. For $x, y \in X$, $x \neq y$, $d(x, y) = 1$. Hence (X, d_∞) (that is d_∞ is restricted to the subspace X of l_∞) is a metric space. Now the topology $\mathcal{J}$ on X induced by the metric d_∞ is the discrete topology on X.

10.9 PROPERTIES OF FIRST COUNTABLE SPACE

If $(X, \mathcal{J})$ is a first countable topological space then for each $x \in X$ there exists a countable local base say $\{V_n(x)\}_{n=1}^{\infty}$ such that $V_{n+1}(x) \subseteq V_n(x)$.

Proof. Fix $x \in X$. Now $(X, \mathcal{J})$ is a first countable topological space implies there exists a countable local base say $\{U_n\}_{n=1}^{\infty}$ at x. Let $V_n(x) = U_1 \cap U_2 \cap \cdots \cap U_n$ then $\{V_n(x)\}_{n=1}^{\infty}$ is a collection of open sets such that $V_{n+1}(x) \subseteq V_n(x)$ for all $n \in \mathbb{N}$. So, it is enough to prove that $\{V_n(x)\}_{n=1}^{\infty}$ is a local base at x. So start with an open set V containing x. Now $\{U_n\}_{n=1}^{\infty}$ is a local base at x and V is an open set containing x implies there exists $n_0 \in \mathbb{N}$ such that $U_{n_0} \subseteq V$. By the definition of $V_n(x)'s$ we have $V_{n_0}(x) \subseteq U_{n_0}$. Hence we have the following: for each open set V containing x there exists $n_0 \in \mathbb{N}$ such that $V_{n_0}(x) \subseteq V$. This implies that $\{V_n(x)\}$ is a local base at x satisfying $V_{n+1}(x) \subseteq V_n(x)$ for all $n \in \mathbb{N}$. ■

Let us use the above characterization of a first countable base to show that, in some sense, first countable topological spaces behave like metric spaces.

10.10 THEOREMS

Theorem 9.10.1:

Let $(X, \mathcal{J})$ be a first countable topological space and A be a nonempty subset of X. Then for each $x \in X$, $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $x_n \to x$ as $n \to \infty$.

Proof. First let us assume that $x \in \overline{A}$. Now $(X, \mathcal{J})$ is a first countable topological space implies there exists a countable local base say $\mathscr{B} = \{V_n\}_{n=1}^{\infty}$ such that $V_{n+1} \subseteq V_n$, for all $n \in \mathbb{N}$. Hence $x \in \overline{A}$ implies $A \cap V_n \neq \phi$, for each $n \in \mathbb{N}$. Let $x_n \in A \cap V_n$.

Claim: $x_n \to x$ as $n \to \infty$.

So start with an open set U containing x (enough to start with V_n) then there exists $n_0 \in \mathbb{N}$ such that $x \in V_{n_0} \subseteq U$. Hence $x_n \in V_n \subseteq V_{n_0} \subseteq U$ for all $n \geq n_0$. That is $x_n \in U$ for all $n \geq n_0$. This means $x_n \to x$ as $n \to \infty$.

Conversely, suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $x_n \to x$. Then for each open set U containing x there exists a positive integer n_0 such that $x_n \in U$ for all $n \geq n_0$. In particular $x_{n_0} \in U \cap A$. Hence for each open set U containing x, $U \cap A \neq \phi$ and this implies $x \in \overline{A}$. ■

Theorem 9.10.2:

Let X and Y be topological spaces and further suppose X is a first countable topological space. Then a function $f : X \to Y$ is continuous at a point $x \in X$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in X, $x_n \to x$ as $n \to \infty$, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$ in Y.

Proof. Assume that $f : X \to Y$ is continuous at a point $x \in X$. Also assume that $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$.

To prove: $f(x_n) \to f(x)$ in Y.

So start with an open set V in Y containing $f(x)$. Since f is continuous at x, $U = f^{-1}(V)$ is an open set in X. Now $f(x) \in V$ implies $x \in f^{-1}(V) = U$. That is

U is an open set containing x. Hence $x_n \to x$ implies there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. This implies $f(x_n) \in V$ for all $n \geq n_0$. That is, whenever $x_n \to x$ as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$.

Conversely, suppose that $\{x_n\}$ is a sequence in X, $x_n \to x$ as $n \to \infty$ implies $f(x_n) \to f(x)$. Now we will have to prove that f is continuous at x. It is to be noted that to prove this converse part we will make use of the fact that X is a first countable space. Now X is a first countable space implies there exists a local base $\{V_n\}_{n=1}^{\infty}$ at x such that $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. We will use the method of proof by contradiction. If f is not continuous at x then there should exist an open set W containing $f(x)$ such that $f(U) \nsubseteq W$ for any open set U containing x. In particular for such an open set W, $f(V_n) \nsubseteq W$ for all $n = 1, 2, 3, \ldots$. Hence there exists $x_n \in V_n$ such that $f(x_n) \notin W$.

Claim: $x_n \to x$ as $\to \infty$. So start with an open set V in X containing x.

Now $\{V_n\}_{n=1}^{\infty}$ is a local base at x implies there exists $n_0 \in \mathbb{N}$ such that $V_{n_0} \subseteq V$. Hence $x_n \in V_n \subseteq V_{n_0} \subseteq V$ for all $n \geq n_0$. That is for each open set V containing x there exists $n_0 \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_0$. Hence $x_n \to x$ as $n \to \infty$. But this sequence $\{x_n\}$ in X is such that $f(x_n) \notin W$, where W is an open set containing $f(x)$. So we have arrived at a contradiction to our assumption namely $x_n \in X$, $x_n \to x \in X$ implies $f(x_n) \to f(x)$. We arrived at this contradiction by assuming f is not continuous at x. Therefore our assumption is wrong and hence f is continuous at x. ■

9.10.1 EXAMPLES

Example1:

Let $\mathcal{J}_c = \{A \subseteq \mathbb{R} : A^c \text{ is countable or } A^c = \mathbb{R}\}$, the co-countable topology on $\mathbb{R}$, and $X = (\mathbb{R}, \mathcal{J}_c), Y = (\mathbb{R}, \mathcal{J}_s)$, where $\mathcal{J}_s$ is the standard topology on $\mathbb{R}$. Define $f : X \to Y$ as $f(x) = x$ for all $x \in X$. Suppose $\{x_n\}$ is a sequence in X such that $\{x_n\}$ converges to $x \in X = \mathbb{R}$. Then it is easy to prove that there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for all $n \geq n_0$. (If this statement is not true then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \neq x$ for all $k \in \mathbb{N}$. Then $U = \mathbb{R} \setminus \{x_{n_k} : k \in \mathbb{N}\}$ is an open set in X containing x. Hence $\{x_n\}$ converges to x in X implies there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. In particular for $k \geq n_0, n_k \geq k \geq n_0$ and this implies $x_{n_k} \in U$.) So we have the following: $x_n \to x$ in X implies $f(x_n) \to f(x)$ in Y. But the given function $f : X \to Y$ is not a continuous function (note: $f^{-1}(0,1) = (0,1)$ is not an open set in $(\mathbb{R}, \mathcal{J}_c)$). This example does

This example does not give any contradiction to theorem (9.8.3) From this example.

$X = (\mathbb{R}, \mathcal{J}_c)$ is not a first countable topological space.

Example 2: $\mathbb{R}$ with usual topology is first countable.

Take $B_x = \{A_n : n \in N\}$.

Evidently, B_x is a local base at $x \in X$ for the usually topology on R.

Clearly, $B_x \sim N$ under the map $A_n \to n$.

Therefore, B_x is a countable local base at $x \in X$. But $x \in X$ is arbitrary.

Hence R with usual topology is first countable.

Consider $x \in R$

$$A_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \forall\ x \in N$$

Example 3: (R, U) is second countable.

The set of all open intervals (r, s) and r with s as rational numbers forms a base, say B for the usual topology $\bigcup$ of R. Since Q, Q × Q are countable sets and so B is a countable base for $\bigcup$ on R.

$\therefore$ (R, $\bigcup$) is second countable.

Example 4: Prove that (R^2, U) is second countable.

Solution: If we write

$$\mathcal{B} = \{S_r(x) : x, r \in Q\}$$

then $\mathcal{B}$ forms a countable base for the usual topology $\bigcup$ on R^2. Hence $(R^2, \cup)$ is second countable space.

10.11SUMMARY

This unit is an explanation of

- **i.** concept of First countable space,
- **ii.** Concept of Second Countable Topological Space.
- **iii.** Concept of Seprable Topological Space,
- **iv.** Concept of Lindelöf space.
- **v.** Important Results and examples are also given here.

10.12GLOSSARY

- **i.** Topology
- **ii.** Basis for a Topology.
- **iii.** First and second Countable space.
- **iv.** Local base
- **v.** Open sphere.

CHECK YOUR PROGRESS

Problem 1:

A space is said to be second countable if and only if
a. it has countable elements
b. it has countable sub base
c. either (a) or (b)

Problem 2:

A topological space is said to be second countable if
a. it has a countable base
b. it has countable elements
c. it has a finite base

Problem 3:

A subset A of a space X is said to be a Lindeloff subset of X if
a. every cover of A by open subsets of X has a countable subcover
b. every cover of A by open subsets of X has a finite subcover
c. there exists cover of A by open subsets of X which has countable subcover

Problem4: Subspace of a separable topological space is separable. True\False

Problem 5: Every separable metric space is second countable. True\False

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9.15 TERMINAL QUESTIONS

TQ1.

Prove that the property of being a first axiom space is a topological property.

TQ2:

Prove that the property of being a second axiom space is a topological property.

TQ3:

Give an example of a separable space which is not second countable.

TQ4:

Show that in a second axiom space, every collection of non empty disjoint open sets is countable.

TQ5:

Give an example of a separable space which is not second countable.

TQ6:

Show that every separable metric space is second countable. Is a separable topological space is second countable? Justify your answer.

TQ7:

Every sub-space of a second countable space is second countable and hence show that it is also separable.

9.16 ANSWERS

CHECK YOUR PROGRESS

CHQ 1: b

CHQ 2: a

CHQ 3: a

CHQ 4: False

CHQ5: True.

UNIT 10: SEPRATION AXIOMS.

CONTENTS:

10.1 INTRODUCTION

In previous unit countablity axiom defined in easy manner. The separation axioms which will be used to define the types of topological spaces in this chapter may be stated as follows:

i. If X and y are distinct points of a topological space, then there exists an open set U which contains one of the points but not the other.

ii. If X and y are distinct points of a topological space, then there exists an open set U which contains X but not y and an open set V which contains y but not x.

iii. If X and y are distinct points of a topological space, then there exist disjoint open sets.

iv. If X is a closed set in a topological space and y is a point, then there exist disjoint open sets.

v. If X and T are disjoint closed sets in a topological space, then there exist disjoint open sets U and V such that X and Y contained in open sets.

Topological spaces may be defined which satisfy one or more axioms. In this unit several types of spaces will be defined and the relationships among them will be discussed.

$$\boxed{\text{normal} \implies \text{regular} \implies \text{Hausdorff}}$$

Here \regular" comes from the Latin \regula", originally meaning a straight piece of wood, asin a ruler. Similarly, \normal" comes from \norma", a carpenter's square with four right angles. Its edges are normal, or perpendicular, to one another. The pendulum (of \perpendicular") is another tool for the recognition of vertical lines. We can strengthen the Hausdorff property ($\boldsymbol{\mathcal{T}_2}$)by demanding to be able to separate not onlypairs of points, but pairs of points and closed sets, or pairs of closed sets. This leads to regular ($\boldsymbol{\mathcal{T}_3}$)and normal ($\boldsymbol{\mathcal{T}_4}$)spaces.

10.2 OBJECTIVES

After completion of this unit learners will be able to

i. Explained the concept of Hausdorff Topological spaces.

ii. Defined the Regular Topological spaces.

iii. Discussed the Normal Topological spaces.

10.3 $\boldsymbol{\mathcal{T}_0}$ –SPACES

Definition:

A topological space $(X,\mathcal{T})$, is said to be a $\mathcal{T}_0$- space or $\mathcal{T}_0 -$ topology iff for all pair of distinct points $x, y \epsilon X$, there is a neighborhood of at least one to which the other does not belong or in other words,$\forall\, x, y \in X, x \neq y$, either $\exists$ open set $O \in \mathcal{T}$, such that $x \in O, y \notin O$ or $\exists$ open set $H \in \mathcal{T}$ such that $x \notin H, y \in H$.

- Every discrete space is a $\mathcal{T}_0$-space.
- Any indiscrete space containing not more than one point is a $\mathcal{T}_0$-space.

10.4 $\mathcal{T}_1$ –SPACES

Definition:

A topological space $(X, \mathcal{J})$ is called a T_1 ***space*** if for each $x \in X$, the singleton set $\{x\}$ is a closed set in $(X, \mathcal{J})$.

- Every metric space is $\mathcal{T}_1$-space.
- Every Cofinite (or Coarest) topology on X is a $\mathcal{T}_1$- space.

10.5 $\mathcal{T}_2$ –SPACES OR HAUSDORFF SPACES

Definition:

A topological space $(X, \mathcal{J})$ is said to be a ***Hausdorff topological space*** (or Hausdorff space) if for $x, y \in X$, $x \neq y$, there exist $U, V \in \mathcal{J}$ such that (i) $x \in U, y \in V$, (ii) $U \cap V = \phi$.

- Every discrete topological space is a $\mathcal{T}_2$-space.
- Every metric space is $\mathcal{T}_2$-space.

Note: In above definition in place of if it is also absolutely correct to use if and only if . That is, above definition can also be read as:

A topological space $(X, \mathcal{J})$ is said to be a Hausdorff topological space (or Hausdorff space) if and only if (iff) for $x,\ y\ \in\ X,\ x\ \neq y,$ there exist $U,\ V\ \in\ \mathcal{J}$ such that (i) $x \in U,\ \ y \in V$, (ii) $U \cap V = \phi$.

What is important to note here (that is while giving a definition) is one can use interchangeably "if" and "if and only if". ⋆

- A Hausdorff Topological Space is also called T_2 space.
-

If $X = \mathbb{R}$, and $\mathcal{J}_s$ is the standard topology on $\mathbb{R}$, then $(\mathbb{R}, \mathcal{J}_s)$ is a Hausdorff space.

-

If X is a set containing at least two elements and $\mathcal{J} = \{\phi, X\}$ then $(X, \mathcal{J})$ is not a Hausdorff space.

-

If $X = \mathbb{R}, \mathscr{B} = \{(a, \infty) : a \in \mathbb{R}\}$ then $\mathscr{B}$ is a basis for a topology $\mathcal{J}_{\mathscr{B}}$ on $\mathbb{R}$. It is easy to see that $(\mathbb{R}, \mathcal{J}_{\mathscr{B}})$ is not a Hausdorff space.

10.6 $\mathcal{T}_3$ − SPACES OR REGULAR SPACES

Definition:

A T_1-topological space $(X, \mathcal{J})$ is called a ***regular space*** if for each $x \in X$ and for each closed subset A of X with $x \notin A$, there exist open sets U, V in X satisfying the following:

(i) $x \in U, A \subset V$, (ii) $U \cap V = \phi$.

- Every discrete space is regular.
- Every indiscrete space is regular
- The space of real or complex numbers with usual topology of open sets is regular.

10.7 $\mathcal{T}_4$ − SPACES OR NORMAL SPACES

Definition:

A topological space $(X, \mathcal{J})$ is said to be a ***normal space*** if and only if it satisfies:

(i) $(X, \mathcal{J})$ is a T_1-space,

(ii) A, B closed sets in X, $A \cap B = \phi$ implies there exist open sets U, V in X such that $A \subseteq U, B \subseteq V$ and $U \cap V = \phi$.

Remark It is to be noted that every normal space is a regular space. ♦

- Every metric space is normal.
- Usual topological space $(\mathbb{R}, U)$ is a $\mathcal{T}_4$ spaces.

10.8 EXAMPLES

Example 1:

$\mathscr{B}_l = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, $\mathcal{J}_l = \mathcal{J}_{\mathscr{B}_l}$ is known as the lower limit topology on $\mathbb{R}$ and $\mathcal{J}_s \subseteq \mathcal{J}_l$. Hence $(\mathbb{R}, \mathcal{J}_l)$ is a Hausdorff space.

Note. Weaker topology is Hausdorff implies stronger is also Hausdorff. ⋆

Let X be an infinite set and $\mathcal{J}_f$ be the cofinite topology on X. Also let $x, y \in X$, $x \neq y$. If $U \in \mathcal{J}_f$ and $x \in U$ then U^c is finite, because $U^c \neq X$. Also $y \in V \in \mathcal{J}_f$ implies V^c is finite. If $U \cap V = \phi$, then $X = (U \cap V)^c = U^c \cup V^c$ and hence X is a finite set. Which gives a contradiction. Therefore $U \cap V \neq \phi$. Hence $\mathcal{J}_f$ is not a Hausdorff space.

Example 2:

$\mathscr{B} = \{U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$: each U_i is open in $\mathbb{R}$, $i = 1, 2, \ldots, n, n \in \mathbb{N}\}$ is a basis for a topology $\mathcal{J}$ (known as product topology) on $\mathbb{R}^w$, where $\mathbb{R}^w = \{x = (x_n)_{n=1}^{\infty} : x_n \in \mathbb{R} \ \forall \ n\}$. Now $X = \mathbb{R}^w$, $x = (x_n) \in X$ and $y = (y_n) \in X$ such that $x \neq y$. Therefore there exists $k \in \mathbb{N}$ such that $x_k \neq y_k$. Let $\epsilon = \frac{|x_k - y_k|}{2} > 0$ and $U_k = (x_k - \epsilon,\ x_k + \epsilon)$, $V_k = (y_k - \epsilon,\ y_k + \epsilon)$. Let $U = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times U_k \times \mathbb{R} \times \mathbb{R} \cdots$ and $V = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times V_k \times \mathbb{R} \times \mathbb{R} \cdots$. Clearly, $x \in U$, $y \in V$ and $U \cap V = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \phi \times \mathbb{R} \times \mathbb{R} \times \cdots = \phi$. Hence $X = \mathbb{R}^w$ is a Hausdorff space.

Note. $\prod_{n=1}^{\infty} (\frac{-1}{n}, \frac{1}{n})$ is not an open set in the product topology on $\mathbb{R}^w$. ⋆

Example 3:

Let $X = \{a, b, c\}$ and $\mathcal{J} = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $(X, \mathcal{J})$ is not a Hausdorff space.

Example 4:

Let X = {1, 2, 3} be a non-empty set with topology T = P(X) (all the subsets of X, powers set or discrete topology). Hence

For 1, 2 $\quad 1 \in \{1\}, 2 \notin \{1\}$

For 2, 3 $\quad 2 \in \{2\}, 3 \notin \{2\}$

For 3, 1 $\quad 3 \in \{3\}, 1 \notin \{3\}$ and (X, T) is a T_2-space

For 1, 2 $\quad 1 \in \{1\}, 2 \in \{2\} \Rightarrow \{1\} \cap \{2\} = \phi$

For 2, 3 $\quad 2 \in \{2\}, 3 \in \{3\} \Rightarrow \{2\} \cap \{3\} = \phi$

For 3, 1 $\quad 3 \in \{3\}, 1 \in \{1\} \Rightarrow \{3\} \cap \{1\} = \phi$

Example 5:

Let X be an infinite set and $\mathcal{J}_f$ be the cofinite topology on X. Then $(X, \mathcal{J}_f)$ is a T_1-space, but $(X, \mathcal{J}_f)$ is not a Hausdorff space. For each $x \in X$, $U = X \backslash \{x\}$ is an open set. Hence $U^c = X \backslash U = \{x\}$ is a closed set in X. That is for each $x \in X$, the singleton set $\{x\}$ is a closed set. Therefore $(X, \mathcal{J})$ is a T_1 -space. Take any $x, y \in X, x \neq y$. Suppose there exist open sets U, V in X such that $x \in U, y \in V$ and $U \cap V = \phi$. Now U, V are nonempty open subsets of the cofinite topological space $(X, \mathcal{J}_f)$ implies U^c, V^c are finite sets. Hence $X = \phi^c = (U \cap V)^c = U^c \cup V^c$ is a finite set. Therefore there cannot exist any open sets U, V in $(X, \mathcal{J}_f)$ satisfying $x \in U, y \in V$ and $U \cap V = \phi$. This means $(X, \mathcal{J}_f)$ is not a Hausdorff space.

Now let us give an example of a topological space which is Hausdorff but not regular. Take $X = \mathbb{R}$ and $\mathscr{B}_K = \{(a,b), (a,b)\backslash K : a,b \in \mathbb{R}, a < b\}$, where $K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Now it is easy to prove that (left as an exercise) $\mathscr{B}_K$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{J}_K$ be the topology on $\mathbb{R}$ generated by $\mathscr{B}_K$. If $\mathcal{J}$ is the usual topology on $\mathbb{R}$ then we know that $\mathcal{J}$ is generated by $\mathscr{B} = \{(a,b) : a,b \in \mathbb{R}, a < b\}$. Since we have $\mathscr{B} \subseteq \mathscr{B}_K$ and this implies that $\mathcal{J} = \mathcal{J}_{\mathscr{B}} \subseteq \mathcal{J}_{\mathscr{B}_K} = \mathcal{J}_K$.

From this, it is clear that $(\mathbb{R}, \mathcal{J}_K)$ is a Hausdorff space. For $x, y \in \mathbb{R}, x \neq y$, $(\mathbb{R}, \mathcal{J})$ is a Hausdorff space implies there exist open sets U and V in $(\mathbb{R}, \mathcal{J})$ such that $x \in U, y \in V$ and $U \cap V = \phi$. But $\mathcal{J} \subseteq \mathcal{J}_K$. Hence $U, V \in \mathcal{J}_K$ are such that $x \in U, y \in V$ and $U \cap V = \phi$ and this shows that $(X, \mathcal{J}_K)$ is a Hausdorff topological space.

Is $K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ a closed set? Here K is a subset of $\mathbb{R}$ and $\mathcal{J}$, $\mathcal{J}_K$ are two different topologies on $\mathbb{R}$, $0 \in \overline{K}$ and $0 \notin K$ with respect to $(\mathbb{R}, \mathcal{J})$. Hence K is not a closed set in $(\mathbb{R}, \mathcal{J})$. But $\mathbb{R}\backslash K = \bigcup\limits_{n=1}^{\infty} A_n$, where $A_n = (-n, n)\backslash K$ for each $n \in \mathbb{N}$. Each A_n is an open set in $(\mathbb{R}, \mathcal{J}_K)$ implies $\mathbb{R}\backslash K$ is an open set in $(\mathbb{R}, \mathcal{J}_K)$. This implies K is a closed set in $(\mathbb{R}, \mathcal{J}_K)$. Also $0 \notin K$. What are the open sets containing K? If V is an open set containing K, then for each $n \in \mathbb{N}$, $\frac{1}{n} \in V$, there exists a basic open set say (a_n, b_n) such that $\frac{1}{n} \in (a_n, b_n) \subseteq V$ ($\frac{1}{n} \notin (a_n, b_n)\backslash K$) and $0 < a_n < b_n$ implies $K \subseteq \bigcup\limits_{k=1}^{\infty} (a_k, b_k) \subseteq V$.

Suppose U, V are open sets such that $0 \in U$ and $K \subseteq V$. Since $0 \in U$, there exists a basic open set B such that $0 \in B \subseteq U$. If B is of the form $(a,\ b)$ then $(a,\ b) \cap K \neq \phi$. So $U \cap V \neq \phi$. If B is of the form $(a,\ b)\backslash K$, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < b$. Since $\frac{1}{n_0} \in V$, there exists an open interval (c, d) such that $\frac{1}{n_0} \in (c, d) \subseteq V$. Now since $(a, b) \cap (c, d)$ is not empty (it contains $\frac{1}{n_0}$), it is an interval and hence uncountable. As K is countable, $((a, b) \cap (c, d))\backslash K \neq \phi$, i.e, $((a, b)\backslash K) \cap (c, d) \neq \phi$. Therefore $U \cap V \neq \phi$.

Example of a topological space which is regular but not normal:

Let $\mathcal{J}_l$ be a lower limit topology on $\mathbb{R}$. That is $\mathbb{R}_l = (\mathbb{R}, \mathcal{J}_l)$. Now let us prove that the product space $\mathbb{R}_l \times \mathbb{R}_l$ is a regular space. (If X, Y are regular topological spaces then the product space $X \times Y$ is a regular space. Hence it is enough to prove that $\mathbb{R}_l$ is a regular space.) For $(x, y) \in \mathbb{R}^2$, each basic open set U of the form $U = [x, a) \times [y, b)$ is both open and closed. Hence for each basic neighbourhood U of (x, y) in $\mathbb{R}_l \times \mathbb{R}_l$ there exists a neighbourhood $V = U$ of (x, y) such that $\overline{V} = \overline{U} \subseteq U$. Now if U' is any open set containing (x, y) then there exists a basic open set U as given above such that $(x, y) \in U = [x, a) \times [y, b) \subseteq U'$. Therefore $V = U$ is an open set containing (x, y) and $\overline{V} = \overline{U} = U \subseteq U'$. Also $\mathbb{R}_l \times \mathbb{R}_l$ is a Hausdorff space. Hence $\mathbb{R}_l \times \mathbb{R}_l$ is a regular space. Now let us take $Y = \{(x, y) \in \mathbb{R}^2 : y = -x\}$ then for each $(x, y) \in Y$ there exists $a, b \in \mathbb{R}$, $x < a$, $y < b$ such that $([x, a) \times [y, b)) \cap Y = \{(x, y)\}$.

Then we can observe that $U \cap V \neq \phi$. Therefore the product space $\mathbb{R}_l \times \mathbb{R}_l$ is not a normal space.

Remark:

We can prove that $(\mathbb{R}, \mathcal{J}_l) = \mathbb{R}_l$ is a normal space. So, $\mathbb{R}_l \times \mathbb{R}_l$ is a regular space but it is not a normal space. ♦

10.9 THEOREMS

Theorem 1:

Every regular topological space $(X, \mathcal{J})$ is a Hausdorff space.

Proof. Let $x, y \in X, x \neq y$. By definition every regular space is a T_1-space. Hence $\{y\}$ is a closed set. Also $x \neq y$ implies $x \notin A = \{y\}$. Now $\{y\}$ is a closed set which does not contain x. Since $(X, \mathcal{J})$ is a regular space, there exist open sets U, V in X satisfying the following:

(i) $x \in U, A = \{y\} \subseteq V$,

(ii) $U \cap V = \phi$ that is U, V are open sets in X such that $x \in U, y \in V$ and $U \cap V = \phi$. Hence $(X, \mathcal{J})$ is a Hausdorff topological space. ■

Exercise:

Prove that every Hausdorff space is a T_1-space. □

Theorem 2:

Every metric space (X, d) is a normal space, That is if $\mathcal{J}_d$ is the topology induced by the metric then the topological space $(X, \mathcal{J}_d)$ is a normal space.

Proof. Let A, B be disjoint closed subsets of X. Then for each $a \in A$, $a \notin B = \overline{B}$ implies $d(a, B) = \inf\{d(a, b) : b \in B\} > 0$. If $r_a = d(a, B) > 0$ then $B(a, r_a) \cap B = \phi$ (if there exists $b_0 \in B$ such that $d(b_0, a) < r_a$, then $r_a = d(a, B) \leq d(a, b_0) < r_a$ a contradiction). Similarly for each $b \in B$ there exists $r_b > 0$ such that $B(b, r_b) \cap A = \phi$. Let $U = \underset{a \in A}{\cup} B(a, \frac{r_a}{3})$, $V = \underset{b \in B}{\cup} B(a, \frac{r_b}{3})$. Now it is easy to prove that $U \cap V = \phi$. Hence if A, B are disjoint closed subsets of X then there exist open sets U, V in X such that $A \subseteq U, B \subseteq V$ and $U \cap V = \phi$. This implies $(X, \mathcal{J}_d)$ is a normal space. ■

Theorem 3:

A T_1-topological space $(X, \mathcal{J})$ is regular if and only if whenever x is a point of X and U is an open set containing x then there exists an open set V containing x such that $\overline{V} \subseteq U$.

Proof. Assume that $(X, \mathcal{J})$ is a normal topological space. Now take a closed set A and an open set U in X such that $A \subseteq U$. Now $A \subseteq U$ implies $U^c \subseteq A^c$. Here $A, U^c = B$ are closed sets such that $A \cap B = A \cap U^c \subseteq U \cap U^c = \phi$. That is A, B are disjoint closed subsets of the normal space $(X, \mathcal{J})$. Hence there exist open sets U, W in X such that $A \subseteq V, B = U^c \subseteq W$ and $V \cap W = \phi$. Further $\overline{V} \subseteq W^c$ (note: $V \subseteq W^c$ implies $\overline{V} \subseteq \overline{W^c} = W^c$). Now $\overline{V} \subseteq W^c \subseteq U$. Hence whenever A is a closed set and U is an open set containing A then there exists an open set V such that $A \subseteq V, \overline{V} \subseteq U$.

Now let us assume that the above statement is satisfied. So our aim is to prove that $(X, \mathcal{J})$ is a normal space. So start with disjoint closed subsets say A, B of X. Now $A \cap B = \phi$ implies $A \subseteq B^c = U$. That is U is an open set containing the closed set A. Hence by our assumption there exists an open set V such that $A \subseteq V, \overline{V} \subseteq U$. Now $\overline{V} \subseteq U$ implies $U^c \subseteq (\overline{V})^c$ implies $B \subseteq (\overline{V})^c$. Further $V \cap (\overline{V})^c \subseteq V \cap V^c = \phi$. That is whenever A, B are closed subsets of X, then there exist open sets V and $(\overline{V})^c = W$ such that $A \subseteq V, B \subseteq W$ and $V \cap W = \phi$. Therefore by definition $(X, \mathcal{J})$ is a normal space. ■

Theorem 4:

Every compact Hausdorff topological space $(X, \mathcal{T})$ is a regular space.

Proof. Let A be a closed subset of X and $x \in X \setminus A$, then for each $y \in A, x \neq y$. Hence X is a Hausdorff space implies that there exist open sets U_y, V_y in X satisfying $x \in U_y, y \in V_y$ and $U_y \cap V_y = \phi$. We know that closed subset of a compact space is compact. Here $A \subseteq \bigcup_{y \in A} V_y$. That is $\{V_y : y \in A\}$ is an open cover for the compact space A. Therefore there exists $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Then U, V are open sets in X satisfying $x \in U, A \subseteq V$ and $U \cap V \subseteq U \cap (V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}) = (U \cap V_{y_1}) \cup (U \cap V_{y_2}) \cup \cdots \cup (U \cap V_{y_n})$

$\subseteq (U_{y_1} \cap V_{y_1}) \cup (U_{y_2} \cap V_{y_2}) \cup \cdots \cup (U_{y_n} \cap V_{y_n}) = \phi$. Hence by definition $(X, \mathcal{J})$ is a regular space. ■

Now let us prove that every compact Hausdorff space is a normal space.

Theorem 5: Every compact Hausdorff topological space $(X, \mathcal{T})$ is a normal space.

Proof. Let A, B be disjoint closed sets in X. Then for each $x \in A, x \notin B$. Now $(X, \mathcal{J})$ is a regular space implies there exist open sets U_x, V_x satisfying: $x \in U_x$; $B \subseteq V_x$ and $U_x \cap V_x = \phi$. Now $\{U_x : x \in A\}$ is an open cover for A implies there exists $n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{x_i}$. Let $U = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$ and $V = V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n}$. Then U, V are open sets in X satisfying $A \subseteq U, B \subseteq V$ and $U \cap V = \phi$. Hence by definition $(X, \mathcal{J})$ is a normal space. ■

Theorem 6:

Closed subspace of a normal topological space $(X, \mathcal{J})$ is normal.

Proof. Let Y be a closed subspace of $(X, \mathcal{J})$. That is Y is a closed subset of $(X, \mathcal{J})$ and $\mathcal{J}_Y = \{A \cap Y : A \in \mathcal{J}\}$ is a topology on Y. So we will have to prove that $(Y, \mathcal{J}_Y)$ is a normal space. To prove this, take a closed set $A \subseteq Y$ and an open set U in $(Y, \mathcal{J}_Y)$

such that $A \subseteq U$. Now U is an open set in $(Y, \mathcal{J}_Y)$ implies there exists $V \in \mathcal{J}$ such that $U = V \cap Y$. Also A is a closed set in the subspace implies $A = \overline{A}_Y = \overline{A} \cap Y$ (here $\overline{A}_Y$ denotes the closure of A in $(Y, \mathcal{J}_Y)$ and $\overline{A}$ denotes the closure of A in $(X, \mathcal{J})$). Now $\overline{A}, Y$ are closed sets in X implies $\overline{A} \cap Y$ is also a closed set in X. Hence A is a closed set in $(X, \mathcal{J})$ and V is an open set in $(X, \mathcal{J})$ containing A and $(X, \mathcal{J})$ is a normal topological space implies there exists an open set W in $(X, \mathcal{J})$ such that

$A \subseteq W$ and $\overline{W} \subseteq V$. Now $W \cap Y$ is an open set in $(Y, \mathcal{J}_Y)$ and $A \subseteq W \cap Y$ and $\overline{W \cap Y} \subseteq \overline{W} \cap \overline{Y} \subseteq V \cap Y \subseteq U$. We started with a closed set A in $(Y, \mathcal{J}_Y)$ and an open set U in $(Y, \mathcal{J}_Y)$ such that $A \subseteq U$. Now we have proved that there exists an

open set $W \cap Y$ in $(Y, \mathcal{J}_Y)$ satisfying $A \subseteq W \cap Y$ and $(\overline{W \cap Y})_Y = \overline{W \cap Y} \cap Y = \overline{W \cap Y} \subseteq U$. That is $W \cap Y$ is an open set in the subspace containing A and closure of this open set with respect to the subspace $(Y, \mathcal{J}_Y)$ is contained in U. Hence $(Y, \mathcal{J}_Y)$ is a normal space. ■

10.10 SUMMARY

This unit complete combination of concept of Hausdorff Topological spaces, Topological spaces, Normal Topological spaces. The definitions, examples, remark and important theorems are explained in the detailed manner.

10.11 GLOSSARY

- **i.** Topological space.
- **ii.** Open set and Closed set.
- **iii.** Basis for a Topology
- **iv.** Compact space.

CHECK YOUR PROGRESS

1.

A space X is said to be regular if
a. every two mutually disjoint closed subsets can be separated from each other by disjoint open sets
b. every distinct points can be separated from each other by disjoint open sets c. every point can be separated from every closed subset not containing it by disjoint open sets

2.

An indiscrete space is
a. regular but not normal
b. normal but not regular
c. both regular and normal

3

Let I be the topology on $\boldsymbol{R}$ whose members are $\varphi, \boldsymbol{R}$ and all sets of the form (a, ∞) for $a \in \boldsymbol{R}$. Then $(\boldsymbol{R}, I)$ is
a. regular but not normal
b. normal but not regular
c. both regular and normal

4

The number of points of a finite Hausdorff space is always a prime power.
true
a. True
b. False

5

All Hausdorff spaces with countably many points are compact.
a. True
b. False

10.12 REFERENCES

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10.14 TERMINAL QUESTIONS

TQ1: Show that any finite **-**space is a discrete space. Is a discrete space T_1 space? Justify your answer.

TQ2: Prove that A finite subset of a T1-space has no cluster point.

TQ3: Show that one-to-one continuous mapping of a compact topological space onto a Hausdorff space is a homeomorphism.

TQ4: Prove that **t**he product of any non-empty class of Hausdorff spaces is a Hausdorff space.

TQ5: Show that if (X, T) is a Hausdorff space and T* is finer than T_2 then (X, T*) is a T_2 -space.

TQ6: Show that every finite Hausdorff space is discrete.

TQ7: Show that every infinite Hausdorff space contains an infinite isolated set.

10.15 ANSWERS

CHECK YOUR PROGRESS

CHQ1: b

CHQ2: c

CHQ3: b

CHQ4: False

CHQ5: False

BLOCK- IV:

TYCHONOFF THEOREM

UNIT 11: THE URYSOHN LEMMA AND TIETZE EXTENSION THEOREM

CONTENTS:

11.1 INTRODUCTION

In previous unit separation axioms defined in easy manner. Urysohn's lemma is sometimes called "the first non-trivial fact of point set topology" and is commonly used to construct continuous functions with various properties on normal spaces. It is widely applicable since all metric spaces and all compact Hausdorff spaces are normal. The lemma is used for the proof of Tietz Extension Theorem. The next concept The Tietz Extension Theorem is also explained here. The Tietz Extension Theorem solved the problem of extending a continuous real-valued function that is defined on a subspace of a space X to a continuous function defined on all of X. This theorem is important in many of the applications of topology.

Pavel Samuilovich Urysohn (3,February,1898- 17August,1924) **Ref:** https://mathshistory.st-andrews.ac.uk/Biographies/Urysohn/#:~:text=He%20came%20from%20a%20family,first%20paper%20in%20this%20year. ***Fig: 11.1.1***	

11.2 OBJECTIVES

After completion of this unit learners will be able to

i. Explained the concept of Urysohn Lemma.

ii. Understand the Tietze Extension Theorem.

iii. Discussed the Comletely Regular

11.3 URYSOHN'S LEMMA

Statement:

Let $(X, \mathcal{J})$ be a normal space and A, B be disjoint nonempty closed subsets of X. Then there exists a continuous function $f : X \to [0,1]$ such that

$f(x) = 0$ for every x in A, and $f(x) = 1$ for every x in B.

Proof. $A \cap B = \phi$ implies $A \subseteq B^c = X \backslash B$. Hence B^c is an open set containing the closed set A. Now X is a normal space implies there exists an open set U_0 such that $A \subseteq U_0$ and $\overline{U}_0 \subseteq B^c = U_1$. Now $[0,1] \cap \mathbb{Q}$ is a countable set implies there exists a bijective function say $f : \mathbb{N} \to [0,1] \cap \mathbb{Q}$ satisfying $f(1) = 1, f(2) = 0$ and $f(\mathbb{N} \backslash \{1,2\}) = (0,1) \cap \mathbb{Q}$. That is $[0,1] \cap \mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ such that $r_1 = 1, r_2 = 0$ and $f(k) = r_k$ for $k \geq 3$.

Aim: To define a collection $\{U_p\}_{p \in [0,1] \cap \mathbb{Q}}$ of open sets such that for $p, q \in [0,1] \cap \mathbb{Q}$, $p < q$ implies $\overline{U}_p \subseteq U_q$.

Let $P_n = \{r_1, r_2, \ldots, r_n\}$. Assume that U_p is defined for all $p \in P_n$, where $n \geq 2$ and this collection satisfies the property namely $p, q \in [0,1] \cap \mathbb{Q}, p < q$ implies $\overline{U}_p \subseteq U_q$. Note that this result is true when $n = 2$. Now let us prove this result for P_{n+1}. Here $P_{n+1} = P_n \cup \{r_{n+1}\}$.

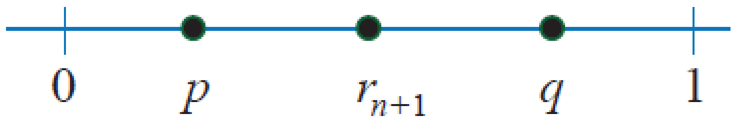

Fig 11.3.1

Let $p, q \in P_{n+1}$ be such that $p = \max\{r \in P_{n+1} : r < r_{n+1}\}$ and $q = \min\{r \in P_{n+1} : r > r_{n+1}\}$. Now $p, q \neq r_{n+1}$ implies $p, q \in P_n$. By our assumption U_p, U_q are known and $\overline{U}_p \subseteq U_q$. Now U_q is an open set containing the closed set $\overline{U}_p$ and X is a normal space. Hence there exists an open set say $U_{r_{n+1}}$ such that $\overline{U}_p \subseteq U_{r_{n+1}}$ and $\overline{U}_{r_{n+1}} \subseteq U_q$.

If $r, s \in P_n$ then we are through.

Suppose $r \in P_n$ and $s = r_{n+1}$ then $r \leq p$ or $r \geq q$. If $r \leq p$, $\overline{U}_r \subseteq U_p \subseteq \overline{U}_p \subseteq U_s$. If $r \geq q$, $U_s \subseteq U_q \subseteq U_q \subseteq U_r$ and therefore by induction U_p is defined for all $p \in [0,1] \cap \mathbb{Q}$ and $p, q \in [0,1] \cap \mathbb{Q}, p < q$ implies $\overline{U}_p \subseteq U_q$.

Now define $U_p = \phi$, if $p \in \mathbb{Q}$, $p < 0$ and $U_p = X$ if $p \in \mathbb{Q}, p > 1$. Then $p, q \in \mathbb{Q}$, $p < q$ implies $\overline{U}_p \subseteq U_q$. Define $f : X \to [0,1]$ as $f(x) = \inf\{p \in \mathbb{Q} : x \in U_p\}$. Now $x \in A$, then $x \in U_0$. Hence $x \in U_p$ for all $p \geq 0$. In this case $\{p \in \mathbb{Q} : x \in U_p\} = [0, \infty) \cap \mathbb{Q}$. Hence $\inf\{p \in \mathbb{Q} : x \in U_p\} = 0$.

That is $x \in A$ implies $f(x) = 0$. Now suppose $x \in B = U_1^c$ then $x \notin U_p$ for all $p \leq 1$. Hence $\{p \in \mathbb{Q} : x \in U_p\} = [1, \infty) \cap \mathbb{Q}$ implies $f(x) = 1$ for all $x \in B$.

Now let us prove that f is a continuous function. $\mathcal{S} = \{[0,a), (a,1] : 0 < a < 1\}$ is a subbase for $[0, 1]$. Hence it is enough to prove that for each a, $0 < a < 1$, $f^{-1}([0,a))$ and $f^{-1}((a,1]))$ are open sets in X. For $0 < a < 1$, let us prove that $f^{-1}([0,a)) = \{x \in X : 0 \leq f(x) < a\} = \underset{p<a}{\cup} U_p$. Now $x \in f^{-1}([0,a))$ implies $f(x) < a$ implies there exists a rational number p such that $f(x) < p < a$. By the definition of $f(x)$, $x \in U_p$. Hence

$f^{-1}([0,a)) \subseteq \underset{p<a}{\cup} U_p$.

…………(10.2.1)

Now let $x \in U_p$ for $p < a$ implies $f(x) \leq p$ implies $x \in f^{-1}([0,a))$. Hence we have

$\underset{p<a}{\cup} U_p \subseteq f^{-1}([0,a))$.

……………….(10.2.2)

From equation (10.2.1) and (10.2.2) we have,

$f^{-1}([0,a)) = \underset{p<a}{\cup} U_p$. Now each U_p is an open set implies that $\underset{p<a}{\cup} U_p$ is an open set in X. In a similar way we can prove that $f^{-1}((a,1])$ is also an open subset of X for each $0 < a < 1$. Now $f : X \to [0,1]$ such that inverse image of each subbasic open set is an open set implies that $f : X \to [0,1]$ is a continuous function. ■

11.3.1 THEOREM

Let $(X, \mathcal{J})$ be a normal space and A, B be disjoint nonempty closed subsets of X. Then for $a, b \in \mathbb{R}, a < b$ there exists a continuous function $f : X \to [a,b]$ such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Proof. Define $g : [0,1] \to [a,b]$ as $g(t) = a + (b-a)t$ then g is continuous. Now by theorem 5.4.1 there is a continuous function $f_1 : X \to [0,1]$ such that $f_1(x) = 0$, for all $x \in A$ and $f_1(x) = 1$ for all $x \in B$. The function $f = g \circ f_1 : X \to [a,b]$ is a continuous function and further $f(x) = g(f_1(x)) = g(0) = a$ for all $x \in A$ and $f(x) = g(f_1(x)) = g(1) = b$ for all $x \in B$. ■

Remark:

Let A, B be nonempty disjoint closed subsets of a metric space (X, d). Define $f : X \to \mathbb{R}$ as $f(x) = \dfrac{d(x, A)}{d(x, A) + d(x, B)}$. Observe that f is a continuous function satisfying the condition that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. It shows that the proof of Urysohn lemma is trivial (or say simple) if our topological space is a metrizable topological space. ♦

11.4 TIETZ EXTENSION THEOREM

Tietze Extension Theorem. *Let A be a nonempty closed subset of a normal space X and let f: $A \to [-1, 1]$ be a continuous function. Then there exists a continuous function $g : X \to [-1, 1]$ such that $g(x) = f(x)$ for all x in A.*

Proof. The sets $\left[-1, \frac{-1}{3}\right]$, $\left[\frac{1}{3}, 1\right]$ are closed subsets of $[-1, 1]$ and $f : A \to [-1, 1]$ is a continuous function implies $A_1 = f^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$, $B_1 = f^{-1}\left(\left[-1, \frac{-1}{3}\right]\right)$ are closed subsets of the subspace A. (Here consider A as a subspace of X.) Now $x \in A_1 \cap B_1$ implies $f(x) \in \left[-1, \frac{-1}{3}\right] \cap \left[\frac{1}{3}, 1\right]$ a contradiction. Hence $A_1 \cap B_1 = \phi$. Now A_1, B_1 are closed in A and A is closed in X implies A_1, B_1 are closed in the normal space X. Hence by Urysohn's lemma there exists a continuous function $f_1 : X \to \left[\frac{-1}{3}, \frac{1}{3}\right]$ such that $f_1(A_1) = \frac{1}{3}$ and $f_1(B_1) = -\frac{1}{3}$ then $|f(x) - f_1(x)| \leq \frac{2}{3}$ for all $x \in A$. Now consider the function $f - f_1 : A \to \left[\frac{-2}{3}, \frac{2}{3}\right]$ then $A_2 = (f - f_1)^{-1}\left(\left[\frac{2}{9}, \frac{2}{3}\right]\right)$ and $B_2 = (f - f)^{-1}\left(\left[\frac{-2}{3}, \frac{-2}{9}\right]\right)$ are disjoint closed subsets of X. By Urysohn lemma there exists a continuous function $f_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ such that $f_2(A_2) = \frac{2}{9}$ and $f_2(B_2) = -\frac{2}{9}$. Also $|f(x) - (f_1(x) + f_2(x))| \leq \frac{4}{9}$ for all $x \in A$. By proceeding as above by induction for each $n \in \mathbb{N}$ there exists a continuous function $f_n : X \to \left[\frac{-2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}\right]$ such that

$$\left|f(x) - \sum_{i=1}^{n} f_i(x)\right| \leq \left(\frac{2}{3}\right)^n \quad \text{for all } x \in A. \qquad \textbf{......(11.3.1)}$$

That is $f_n : X \to [-1, 1]$ is a sequence of continuous functions such that $|f_n(x)| \leq \frac{2^{n-1}}{3^n} = M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$. By Weierstrass M-test, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X. That is, if $s_n(x) = \sum_{i=1}^{n} f_i(x),\ x \in X$ then $s_n(x)$ converges uniformly on X. Also each $s_n : X \to \mathbb{R}$ is continuous. We know, from analysis, if a sequence $s_n : X \to \mathbb{R}$ of continuous functions converges uniformly to a function $g : X \to \mathbb{R}$ then g is also a continuous function. Hence $g : X \to \mathbb{R}$ be defined as $g(x) = \sum f_n(x)$ is continuous. Now for each $x \in A$, $\left| f(x) - \sum_{i=1}^{n} f_i(x) \right| \leq \left(\frac{2}{3}\right)^n$

(By using the equation 11.3.1). Therefore,

$|g(x) - f(x)| = \left| \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) - f(x) \right| = \lim_{n\to\infty} \left| \sum_{i=1}^{n} f_i(x) - f(x) \right| \leq \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$. This implies $g(x) = f(x)$ for all $x \in A$. ■

11.5 COMPLETLY REGULAR

A topological space $(X, \mathcal{J})$ is said to be ***completely regular*** if (i) for each $x \in X$, singleton $\{x\}$ is closed in $(X, \mathcal{J})$ (that is $(X, \mathcal{J})$ is a T_1-space),

(ii) for $x \in X$ and any nonempty closed set A with $x \notin A$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in A$.

Remark: Every normal space $(X, \mathcal{T})$ is completely regular.

Proof. Let $x \in X$ and A be a nonempty closed set with $x \notin A$. Now $\{x\}$, A are disjoint closed sets. Hence by Urysohn's lemma there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in A$. ■

Remark:

If Y is a subspace of a completely regular space $(X, \mathcal{J})$ then $(Y, \mathcal{J}_Y)$ is also a completely regular space.

Proof. Let $y \in Y$ and A be a closed set in $(Y, \mathcal{J}_Y)$ with $y \notin A$. Since A is a closed set in Y there exists a closed set F in $(X, \mathcal{J})$ such that $A = F \cap Y, y \notin F$, F is a closed set in the completely regular space $(X, \mathcal{J})$ implies there exists a continuous function $f : X \to [0, 1]$ such that $f(y) = 0$ and $f(a) = 1$ for all $a \in F$. Now $f : X \to [0, 1]$ is a continuous function implies $f|Y = g : (Y, \mathcal{J}_Y) \to [0, 1]$ (here $g(x) = (f|Y)(x) = f(x)$ for all $x \in Y$) is a continuous function. Now g: $(Y, \mathcal{J}_Y) \to [0, 1]$ is a continuous function such that $g(y) = f(y) = 0$ and $g(a) = f(a) = 1$ for all $a \in A = F \cap Y$. Also subspace of a T_1-space (do it as an exercise) is T_1-space. Hence the subspace $(Y, \mathcal{J}_Y)$ is a completely regular space. ■

11.6 EXAMPLES

Example 1: If F_1 and F_2 are T-closed disjoint subsets of a normal space (x, T), then there exist a continuous map g of X into [0, 1] such that

$$g(x) = \begin{cases} 0 & \text{if } x \in F_1 \\ 1 & \text{if } x \in F_2 \end{cases}$$

$f(F_1) = \{0\}$ and $g(F_2) = \{1\}$.

For proved the example the step of Urysohn's lemma can be used.

Example 2: If F_1 and F_2 are T-closed disjoint subsets of a normal space (X, T) and [a, b] is any closed interval on the real line, then there exists a continuous map f of X into [a, b] such that

$$f(x) = \begin{cases} a & \text{if } x \in F_1 \\ b & \text{if } x \in F_2 \end{cases}$$

i.e., $f(F_1) = \{a\}, f(F_2) = \{b\}$

This example is general type of Urysohn's lemma.

Solution: Let F_1 and F_2 be disjoint closed subset of (X, T).

To prove that ∃ a continuous map

$$f : X \to [a, b] \text{ s.t. } f(F_1) = \{a\}, f(F_2) = \{b\}$$

By Urysohn's lemma, ∃ a continuous map

$$g : X \to [0, 1] \text{ s.t. } g(F_1) = \{0\}, g(F_2) = \{1\}.$$

Define a map $h : [0, 1] \to [a, b]$ s.t.

$$h(x) = \frac{(b-a)x}{1-0} + a$$

i.e., $h(x) = x(b - a) + a$

[This is obtained by writing the equation of the straight line joining (0, a) and (1, b) and then putting $y = h(x)$].

Evidently $h(0) = a$, $h(1) = b - a + a = b$

Also h is continuous

Write $f = hg$

$g : X \to [0, 1]$, $h : [0, 1] \to [a, b]$

$\Rightarrow hg : X \to [a, b] \Rightarrow f : X \to [a, b]$

Product of continuous functions is continuous

Therefore $f(F_1) = (hg)(F_1) = h[g(F_1)] = h(\{0\}) = \{a\}$

$f(F_2) = (hg)(F_2) = h[g(F_2)] = h(\{1\}) = \{b\}$

Thus $\exists$ a continuous map.

$f : X \to [a, b]$ s.t. $f(F_1) = \{a\}, f(F_2) = \{b\}$

11.7 SUMMARY

In this unit is Urysohn's lemma is explained i.e. Urysohn's lemma is a lemma that states that a topological space is normal iff any two disjoint closed subsets can be separated by a function. Tietze Extension Theorem and Comletely Regular topological space are also disussed here.

11.8 GLOSSARY

i. Continuous map.

ii. Disjoint set.

iii. Normal space.

iv. Separated sets.

v. Closed set.

CHECK YOUR PROGRESS

1.

Let X be a normal Hausdorff space. Let A_1, A_2, and A_3 be closed subsets of X which are pairwise disjoint. Then there always exists a continuous real valued function f on X such that $f(x) = a_i$ if x belongs to A_i, $i = 1, 2, 3$

a.

iff each a_i is either 0 or 1.

b.

iff at least two of the numbers a_1, a_2, a_3 are equal.

c.

for all real values of a_1, a_2, a_3.

d.

only if one among the sets A_1, A_2, A_3 is empty.

2. For satisfying the Urysohn's lemma the topological space is normal space.

True\False

3: For satisfying the Urysohn's lemma the subset should not disjoint closed subsets.

True\False

4:For satisfying the Tietz Extension Theorem the topological space is normal space.

True\False

5: Every normal space $(X, \mathcal{T})$ is not completely regular. True\False

11.9 REFERENCES

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- **v.** G.F. Simmons (2017), *Introduction to Topology and Modern Analysis*, Mc. Graw Hill Education.
- **vi.** https://en.wikipedia.org/wiki/Topology

11.10 SUGGESTED READINGS

- **i.** K. Ahmad (2020), *Introduction to Topology*, Alpha Science International Ltd.
- **ii.** W. J. Pervin (1964) *Foundations of General Topology*, Academic Press.
- **iii.** https://archive.nptel.ac.in/noc/courses/noc22/SEM1/noc22-ma36/
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11.11 TERMINAL QUESTIONS

TQ1: Prove that every continuous image of a separable space is separable.

TQ2: Prove that the set of all isolated points of a second countable space is countable.

TQ3: Show that any uncountable subset A of a second countable space contains at least one point which is a limit point of A.

TQ4: Let f be a continuous mapping of a Hausdorff non-separable space $(X, \mathcal{T})$ onto itself. Prove that there exists a proper non-empty closed subset A of X such that f(A) = A. Is the this result true if $(X, \mathcal{T})$ is separable?

TQ5: Show that the Tietze extension theorem implies the Urysohn lemma.

11.11 ANSWERS

CHECK YOUR PROGRESS

CHQ1: c

CHQ2: True.

CHQ3: False.

CHQ4: True.

CHQ5: False.

UNIT 12: BAIRE CATEGORY THEOREM

CONTENTS:

12.1 INTRODUCTION

In previous unit Urysohn lemma and Tietze extension theorem state and proof in simple manner.

The **Baire category theorem** is an main result in general topology and functional analysis. The theorem has two forms, each of which gives sufficient conditions for a topological space to be a Baire space
(a topological space such that the intersection of countably many dense open sets is still dense).

It is used in the proof of results in many areas of analysis and geometry, including some of the fundamental theorems of functional analysis.

René-Louis Baire

(French: ; 21 January 1874 – 5 July 1932)

Ref:

https://en.wikipedia.org/wiki/Ren%C3%A9-Louis_Baire

Fig 13.1.1

12.2 OBJECTIVES

After completion of this unit learners will be able to

i. Explained the concept first category

ii. Discussed the definition of second category.

iii. Understand the Baire Category Theorem.

12.3 FIRST CATEGORY AND SECOND CATEGORY

The notion of category stems from countability. The subsets of metric spaces are divided into two categories: first category and second category. Subsets of the first category can be thought of as small, and subsets of category two could be thought of as large, since it is usual that asset of the first category is a subset of some second category set; the verse inclusion never holds.

Recall that a metric space is defined as a set with a distance function. Because this is the sole requirement on the set, the notion of category is versatile, and can be applied to various metric spaces, as is observed in Euclidian spaces, function spaces and sequence spaces. However, the Baire category theorem is used as a method of proving existence.

In this section we are defining first category and second category.

12.3.1 NO WHERE DENSE

A subset A of a topological space $(X, \mathcal{J})$ is said to be ***nowhere dense in*** X if and only if $(\overline{A})^\circ = int(\overline{A}) = \phi$.

12.3.2 EXAMPLE

(i) $\mathbb{N}$ is nowhere dense in $\mathbb{R}$ ($\mathbb{R}$ with standard topology).

(ii) $\mathbb{Q}$ is dense in $\mathbb{R}$, that is $\overline{\mathbb{Q}} = \mathbb{R}$, and hence $\mathbb{Q}$ is not nowhere dense in $\mathbb{R}$. Here we have $(\overline{\mathbb{Q}})^\circ = \mathbb{R}^\circ = \mathbb{R} \neq \phi$.

12.3.3 FIRST CATEGORY

A topological space $(X, \mathcal{J})$ is said to be of ***first category*** if and only if there exists a countable collection $\{E_n\}_{n=1}^{\infty}$ of subsets of X satisfying:

(i) for each $n \in \mathbb{N}$, $\left(\overline{E_n}\right)^\circ = \overline{E_n}^\circ = \phi$, and

(ii) $X = \bigcup\limits_{n=1}^{\infty} E_n$.

- If Y is a nonempty subset of a topological space $(X, \mathcal{J})$ then $(Y, \mathcal{J}_Y)$ $(\mathcal{J}_Y = \{U \cap Y : U \in \mathcal{J}\})$ is also a topological space. It is possible that a subset Y of a topological space $(X, \mathcal{J})$ is of first category in $(X, \mathcal{J})$ but the subspace $(Y, \mathcal{J}_Y)$ is not of first category.

12.3.4 EXAMPLE

Let $X = \mathbb{R}$ and $\mathcal{J}_s$ be the standard topology on $\mathbb{R}$. Then $Y = \mathbb{N}$, the set of all natural numbers, is of first category in $\mathbb{R}$, but the subspace $(\mathbb{N}, \mathcal{J}_{s/\mathbb{N}})$ is not of first category. For each $n \in \mathbb{N}$ let $E_n = \{n\}$. As E_n contains only one element namely n, $(\overline{E_n})^\circ = \{n\}^\circ = \phi$ in $\mathbb{R}$. Also $\mathbb{N} = \bigcup\limits_{n=1}^{\infty} \{n\} = \{1, 2, \ldots\} = \bigcup\limits_{n=1}^{\infty} E_n$. Hence $\mathbb{N}$ is of first category in $(\mathbb{R}, \mathcal{J}_s)$. But note that the subspace $(\mathbb{N}, \mathcal{J}_{s/Y})$ is the discrete topological space on $\mathbb{N}$. For $n \in \mathbb{N}$, $(n-1, n+1)$ is an open set in $\mathbb{R}$ and hence $(n-1, n+1) \cap \mathbb{N} = \{n\}$ is an open set in the subspace $(\mathbb{N}, \mathcal{J}_{s/\mathbb{N}})$. Now it is easy to see that there cannot exist any countable collection $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}$

satisfying $(\overline{A_n})^\circ = \phi$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. Note that for $A_n \subseteq \mathbb{N}$, $(\overline{A_n})^\circ = A_n$ with respect to $(\mathbb{N}, \mathcal{J}_{s/\mathbb{N}})$ and hence $(\mathbb{N}, \mathcal{J}_{s/\mathbb{N}})$ is not of first category.

12.3.5 SECOND CATEGORY

If a topological space $(X, \mathcal{J})$ is ***not of first category*** then we say that the topological space $(X, \mathcal{J})$ is of ***second category***.

Note. We have seen that $\mathbb{N}$ is of first category in $(\mathbb{R}, \mathcal{J}_s)$ but the topological space $(\mathbb{N}, \mathcal{J}_{s/Y})$ is of second category.

Now our main aim is to prove that every locally compact Hausdorff topological space is of second category.

-

First let us prove that every locally compact Hausdorff topological space $(X, \mathcal{J})$ is a regular space. So let us take a closed set A in $(X, \mathcal{J})$ and a point $x \in X \setminus A$.

We have seen that every compact Hausdorff space is a normal space and hence every compact Hausdorff space is a regular space. We know that the one point of compactification $(X^*, \mathcal{J}^*)$ of $(X, \mathcal{J})$ is a compact Hausdorff space and $\mathcal{J}^* |_X = \mathcal{J}$. That is $(X, \mathcal{J})$ is a subspace of the compact Hausdorff space $(X^*, \mathcal{J}^*)$. Also it is easy to prove that subspace of a regular space is regular (and it is to be noted that subspace of a normal space need not be a normal space) and hence $(X^*, \mathcal{J}^*)$ is a regular space implies the subspace $(X, \mathcal{J})$ of $(X^*, \mathcal{J}^*)$ is also a regular space.

Now we are in a position to state and prove the main theorem.

12.4 BAIRE CATEGORY THEOREM

12.4.1 STATEMENT AND PROOF

Let $(X, \mathcal{J})$ be a locally compact Hausdorff topological space and $\{E_n\}_{n=1}^{\infty}$ be a countable collection of open sets in $(X, \mathcal{J})$. Further suppose for each $n \in \mathbb{N}$, $\overline{E_n} = X$ (E_n is dense in X) then $\bigcap_{n=1}^{\infty} E_n$ is also dense in X. That is $\overline{\left(\bigcap_{n=1}^{\infty} E_n\right)} = X$.

Proof. We want to prove that $\bigcap_{n=1}^{\infty} E_n$ is dense in X.

So take $x \in X$ and an open set U containing x. Now $(X, \mathcal{J})$ is a locally compact Hausdorff space implies there exists an open set V containing x such that $\overline{V}$ is compact. Now let $U_0 = U \cap V$. Then U_0 is an open set containing x. Also $\overline{U_0} \subseteq \overline{V}$ implies $\overline{U_0}$ is a compact set (since closed subset of compact set is compact). Now our aim is to prove that $U \cap \left(\bigcap_{n=1}^{\infty} E_n\right) \neq \phi$. For each n, E_n is open and $\overline{E_n} = X$, that is each E_n is open and dense in X. Start with $n = 1$, now $x \in X = \overline{E_1}$ and U_0 is an open set containing x implies $U_0 \cap E_1 \neq \phi$. So take an element say $x_1 \in U_0 \cap E_1$. Now U_0, E_1 are open sets implies $U_0 \cap E_1$ is also an open set. Now $U_0 \cap E_1$ is an open set containing x_1 and $(X, \mathcal{J})$ is a regular space (every locally compact Hausdorff space is a regular space) implies there exists an open set U_1 in X satisfying $x_1 \in U_1, \overline{U_1} \subseteq U_0 \cap E_1$. Now $x_1 \in \overline{E_2} = X$ implies $U_1 \cap E_2 \neq \phi$. Let $x_2 \in U_1 \cap E_2$. Since X is a regular space implies there exists an open set U_2 in X satisfying $x_2 \in U_2, \overline{U_2} \subseteq U_1 \cap E_2$. Again $x_2 \in \overline{E_3} = X$

and U_2 is an open set containing x_2 implies $U_2 \cap E_3 \neq \phi$. Let $x_3 \in U_2 \cap E_3$. Choose an open set U_3 such that $x_3 \in U_3, \overline{U}_3 \subseteq U_2 \cap E_3$. Continuing in this way (that is using induction) we get a sequence $\{x_n\}_{n=1}^{\infty}$ in X and a sequence of open sets $\{U_n\}_{n=1}^{\infty}$ satisfying $x_n \in U_n, \overline{U}_n \subseteq U_{n-1} \cap E_n$ for all $n \in \mathbb{N}$. Note that $\overline{U}_n \subseteq U \cap \left(\bigcap_{k=1}^{n} E_k \right)$ for all $n \in \mathbb{N}$. Then $\{\overline{U}_k\}_{k=1}^{\infty}$ is a sequence of nonempty closed subsets of X and hence of the compact subspace $\overline{U}_0$. Further $\overline{U}_{k+1} \subseteq \overline{U}_k$ for any $k \in \mathbb{N}$ implies $\{\overline{U}_k\}_{k=1}^{\infty}$ has finite intersection property. That is $\{\overline{U}_k\}_{k=1}^{\infty}$ is a family of closed subsets of the compact topological space $\overline{U}_0$ and further $\{\overline{U}_k\}_{k=1}^{\infty}$ has finite intersection property. Therefore $\bigcap_{k=1}^{\infty} \overline{U}_k \neq \phi$. Let $a \in \bigcap_{k=1}^{\infty} \overline{U}_k$. Then $a \in \overline{U}_k$ for all $k \in \mathbb{N}$ and hence $a \in U$. Also $a \in E_n$ for all $n \in \mathbb{N}$. So $a \in \bigcap_{n=1}^{\infty} E_n$. Thus $a \in U \cap \left(\bigcap_{n=1}^{\infty} E_n \right)$. That is for each $x \in X$ and

for each open set U containing x, $U \cap \left(\bigcap_{n=1}^{\infty} E_n \right) \neq \phi$. Hence $x \in \overline{\left(\bigcap_{n=1}^{\infty} E_n \right)}$. This gives that $X \subseteq \overline{\left(\bigcap_{n=1}^{\infty} E_n \right)}$ and hence $X = \overline{\left(\bigcap_{n=1}^{\infty} E_n \right)}$, that is $\bigcap_{n=1}^{\infty} E_n$ is dense in X. ■

Exercise:

Prove that a subset E of a topological space $(X, \mathcal{J})$ is nowhere dense in X (that is $(\overline{E})^{\circ} = \phi$ if and only if $(\overline{E})^c$ is dense in X.

•

It is known that every complete metric space X is of second category. The notion of completeness cannot be defined in a topological space. So we give the following version of Baire Category theorem for a locally compact Hausdorff topological space.

12.4.2 THEOREM

Every nonempty locally compact Hausdorff topological space $(X, \mathcal{J})$ is of second category.

Proof. Proof by contradiction.

Suppose $(X, \mathcal{J})$ is of first category. Then there exists a countable collection $\{E_n\}_{n=1}^{\infty}$ of subsets of X satisfying $\overline{E_n}^{\circ} = \phi$ and $X = \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \overline{E}_n$. Therefore $X^c =$

$\phi = \bigcap_{n=1}^{\infty} \overline{E}_n^c$ and hence $\phi = \overline{\phi} = \overline{\bigcap_{n=1}^{\infty} \overline{E_n}^c}$. But $\{\overline{E}_n^c\}_{n=1}^{\infty}$ is a countable collection of dense open sets implies by, **Baire category theorem**

$$\overline{\bigcap_{n=1}^{\infty} \overline{E_n}^c} = X \neq \phi.$$

This contradicts,

$\phi = \overline{\phi} = \overline{\bigcap_{n=1}^{\infty} \overline{E_n}^c}$. Hence $(X, \mathcal{J})$ is of second category. ■

Now we are in a position to prove Urysohn metrization theorem that gives sufficient conditions under which a topological space is metrizable. Also it is interesting to note that the well known Nagata-Smirnov metrization theorem gives a set of necessary and sufficient conditions for metrizability of a topological space.

12.4.3 EXAMPLES

Example 1:

Let $q \in \mathcal{Q}$ be arbitrary.

$$\overline{\{q\}} = \{q\} \cup D(\{q\}), \qquad [\because\ \overline{A} = A \cup D(A)]$$

$$= \{q\} \cup \phi = \{q\}$$

$\therefore \quad \text{int}\ \overline{\{q\}} = \text{int}\ \{q\}$

$= \cup \{G \subset \mathcal{R} : G \text{ is open}, G \subset \{q\}\} = \phi.$

For every subset of $\mathcal{R}$ contains rational as well irrational numbers.

Thus, $\quad \text{int}\ \overline{\{q\}} = \phi.$

This proves that $\{q\}$ is a non-dense subset of $\mathcal{Q}$.

$$\mathcal{Q} = \cup \{\{q\} : a \in \mathcal{Q}\}.$$

Furthermore $\mathcal{Q}$ is enumerable.

$\therefore$ $\mathcal{Q}$ is an enumerable union of non-dense sets.

From what has been done it follows that $\mathcal{Q}$ is of the first category.

Example 2:

Consider a sequence $<f_n(x)>$ of continuous functions defined from $I = [0, 1]$

into $\mathcal{R}$ s.t. $f_n(x) = x_n\ \forall\ x \in N$.

Then $<f_n>$ converges pointwise to $g : \mathcal{I} \to \mathcal{R}$ s.t.

$$g(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

Evidently g is not continuous.

12.5 SUMMARY

In first section we have defined the nowhere dense and Second Category. In second section Statement and Proof of Baire Category Theorem defined in easy manner. Some theorems and examples also explained.

12.6 GLOSSARY

- **i.** Continuous map.
- **ii.** Normal space.
- **iii.** Regular space.
- **iv.** Hausdorff space.
- **v.** Compact space.
- **vi.** Complete metric space.
- **vii.** No where dense.
- **viii.** Dense.

CHECK YOUR PROGRESS

1. Euclidean space $\mathbb{R}^n$ is a Baire space. **True/False.**
2. Complete metric spaces are not Baire spaces. **True/False.**
3. Let X be a nonempty Baire space, and suppose that $X = \bigcup_{n\in\mathbb{N}} B_n$ for some countable collection of subsets B_n. Then $\overline{B_n}$ must have nonempty interior for some n.

   True/False.
4. The set of real number $\mathbb{R}$ is second category **True/False.**
5. Any countable intersection of open dense sets is not dense **True/False.**
6. Any countable union of closed sets with empty interior has an empty interior.

   True/False.

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12.9 TERMINAL QUESTIONS

TQ1: Show that every locally compact Hausdorff space is a Baire space.

TQ2: Show that the irrationals are a Baire space.

TQ3: Let B be a Banach space where the dimension of the underlying vector space is countable.

TQ4: Using the Baire Category Theorem, prove that the dimension of the underlying vector space is, in fact, finite.

TQ5: State and proof Baire category theorem.

TQ6: How do you define an open dense set in terms of Baire Category?

TQ7: Describe the Baire's characterization of complete metric spaces..

TQ8: What is a "nowhere dense" set in the context of Baire category and give an example?

TQ9: What are the applications of the Baire category theorem in Functional Analysis?..............

TQ10: How can the Baire category theorem be used to prove that any linear subspace of finite co-dimension is of the first category?..

TQ11: Explain how the Baire category theorem applies to a countable intersection of dense open subsets in a complete metric space…………………………………………………….

12.10 ANSWERS

CHECK YOUR PROGRESS

CHQ 1: True.

CHQ 2: False.

CHQ3: True.

CHQ4: True

CHQ5: False

CHQ6: True

UNIT 13: THE URYSOHN METRIZATION THEOREM, PARTITIONS OF UNITY.

CONTENTS:

13.1 INTRODUCTION

In previous unit Baire category theorem state and proof in simple manner. The Urysohn Metrization Theorem tells us under which conditions a topological space X is metrizable, i.e. when there exists a metric on the underlying set of X that induces the topology of X. One of the first widely recognized metrization theorems was Urysohn's metrization theorem. This states that every Hausdorff second-countable regular space is metrizable. The form of the theorem shown here was in fact proved by Tikhonov in 1926. What Urysohn had shown, in a paper published posthumously in 1925. In this unit Urysohn's metrization theorem and partitions of unity explained in the simple manner.

13.2 OBJECTIVES

After completion of this unit learners will be able to

i. Explained the concept of Urysohn's metrization theorem.

ii. Discussed the partitions of unity.

iii. Defined the manifold.

13.3 URYSOHN'S METRIZATION THEOREM

13.3.1 METRIZATION

Given any topological space $(X, \mathcal{T})$,if it is possible to find a metric ρ on X which induces the topology $\mathcal{T}$ i.e. the open sets determined by the metric ρ are precisely the members of $\mathcal{T}$, then X is said to the metrizable.

- The set $\mathbb{R}$ with usual topology is metrizable. For the usual metric on $\mathbb{R}$ induces the usual topology on . Similarly $\mathbb{R}^2$ with usual topology is metrizable.
- A discrete space $(X, \mathcal{T})$ is metrizable. For the trivial metric induces the discrete topology $\mathcal{T}$ on X
- if a set is metrizable, then it is metrizable in an infinite number of different ways.

13.3.2 URYSOHN'S METRIZATION THEOREM

The statement of Urysohn's metrization theorem don't feel that it needs much motivating. Having studied metric spaces in detail and having convinced ourselves of how nice they are, a theorem that gives conditions implying that a space is metrizable should seem innately useful.

Statement : Every normal space $(X, \mathcal{T})$ with a countable basis is metrizable.

Proof. Let $\mathscr{B} = \{B_1, B_2, \ldots,\}$ be a countable basis for $(X, \mathcal{J})$. Suppose $n, m \in \mathbb{N}$ are such that $\overline{B}_n \subseteq B_m$ then $\overline{B_n} \cap B_m^c = \phi$. Hence by Urysohn's lemma there exists a continuous function say $g_{n,m} : X \to \mathbb{R}$ such that

$$g_{n,m}(x) = 0 \quad \text{for all } x \in B_m^c,$$

........................(13.3.1)

and

$$g_{n,m}(x) = 1 \quad \text{for all } x \in \overline{B_n}.$$

......................................(13.3.2)

Now take $x_0 \in X$ and an open set U containing x_0. Since $\mathscr{B}$ is a basis for $(X, \mathcal{J})$ there exists $B_m \in \mathscr{B}$ such that $x_0 \in B_m \subseteq U$. Now B_m is an open set containing x_0 implies there exists an open set V containing x_0 such that $\overline{V} \subseteq B_m$. Hence there exists a basic open set B_n containing x_0 such that $\overline{B}_n \subseteq \overline{V} \subseteq B_m$. Hence for such pair (n, m) we have a continuous function $g_{n,m} : X \to \mathbb{R}$ satisfying equation (12.3.1).

So if $x_0 \in X$ and U is an open set containing x_0 then there exists a continuous function $g_{n,m} : X \to \mathbb{R}$ such that $g_{n,m}(x_0) = 1$ and $g_{n,m}(x) = 0$ for all $x \in U^c \subseteq B_m^c$. So we have proved that there exists a countable collection of continuous functions

$f_n : X \to [0, 1]$ such that for $x_0 \in X$ and open set U containing x_0, there exists $n \in \mathbb{N}$ such that $f_n(x_0) = 1 > 0$ and $f_n(x) = 0$ for all $x \in U^c$.

It is to be noted that $\{(n, m): n, m \epsilon \mathbb{N}\}$ is a countable set. We know that,

$\mathbb{R}^w = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ with product topology is metrizable. That is there is a metric d on $\mathbb{R}^w$ such that $\mathcal{J}_d$, the topology on $\mathbb{R}^w$ induced by d, coincides with the product topology on $\mathbb{R}^w$.

Now let us define a map $T : X \to \mathbb{R}^w$ as $T(x) = (f_1(x), f_2(x), \ldots,)$ and using this map we define $d_1(x, y) = d(T(x), T(y))$ and conclude that $\mathcal{J}_{d_1} = \mathcal{J}$. This will prove that $(X, \mathcal{J})$ is a metrizable topological space. Now let us prove that $(X, \mathcal{J})$ is homeomorphic to a subspace of $\mathbb{R}^w$. Each $f_n : X \to \mathbb{R}$ is a continuous function implies $T(x) = (f_1(x), f_2(x), \ldots)$ is a continuous function.

To prove T is injective (one-one).

Let $x, y \in X$ be such that $x \neq y$. Then there exist open sets $U, V \in X$ such that $x \in U, y \in V$ and $U \cap V = \phi$. Now U is an open set containing x implies there exists $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f_n(y) = 0$ (note that $y \in U^c$). This implies $f_n(x) \neq f_n(y)$ for this particular $n \in \mathbb{N}$ and hence $(f_1(x), f_2(x), \ldots, f_n(x), \ldots) \neq$

$(f_1(y), f_2(y), \ldots, f_n(y), \ldots)$. This means $Tx \neq Ty$. That is $x, y \in X, x \neq y$ implies $Tx \neq Ty$. This implies T is 1-1.

Now it is enough to prove that T maps open set A in X to an open set $T(A)$ in $Y = T(X)$. Let A be an open set and $y_0 \in T(A)$. Now $y_0 \in T(A)$ implies there exists $x_0 \in A$ such that $T(x_0) = y_0$. Now $x_0 \in A$, A is an open set implies there exists $n_0 \in \mathbb{N}$ such that $f_{n_0}(x_0) = 1$ and $f_{n_0}(x) = 0$ for all $x \in A^c$. We know that for each $n \in \mathbb{N}$ the projection map $p_n : \mathbb{R}^w \to \mathbb{R}$ defined as $p_n\left((x_k)_{k=1}^{\infty}\right) = x_n$ is a continuous map. Hence $(0, \infty)$ is an open set implies $V = p_{n_0}^{-1}((0, \infty))$ is an open subset of $\mathbb{R}^w$.

This implies $V \cap Y$ is an open set in Y.

Now let us prove that $y_0 \in V \cap Y$ and $V \cap Y \subseteq T(A)$. $p_{n_0}(y_0) = (p_{n_0} \cdot T)(x_0) = f_{n_0}(x_0) = 1 > 0$ implies $y_0 \in V$. Also $y_0 \in Y$. Hence $y_0 \in V \cap Y$. That is $V \cap Y$ is an open set in Y containing the point y_0.

Now we claim that $V \cap Y \subseteq T(A)$. So, let $y \in V \cap Y$. Then there exists $x \in X$ such that $y = Tx$. This implies $p_{n_0}(y) \in (0, \infty)$ and $p_{n_0}(y) = p_{n_0}(T(x)) = f_{n_0}(x) \in (0, \infty)$. Hence $x \in A$ ($f_{n_0}(x) = 0$ for $x \in A^c$). So we have proved that $y = Tx \in V \cap Y$

implies $y = Tx \in T(A)$. Hence $V \cap Y$ is an open set in Y containing Tx and this set is contained in $T(A)$. Therefore $T(A)$ is open in Y. Hence we have proved that $T : (X, \mathcal{J}) \xrightarrow{onto} (Y, d_Y)$ is a homeomorphism. (Here (Y, d_Y) is a subspace of $(\mathbb{R}^w, d)$.)

Now $d_1(x, y) = d(Tx, Ty)$ for all $x, y \in X$ implies d_1 is a metric on X. Also it is easy to see that a subset A of X is open in $(X, \mathcal{J})$ if and only if A is open in $(X, \mathcal{J}_{d_1})$. Therefore $\mathcal{J}_{d_1} = \mathcal{J}$. ■

13.4 PARTITIONS OF UNITY

13.4.1 SUPPORT

If $\phi: X \to R$, then the support of ϕ is defined to be closure of the set $\phi^{-1}(R - \{0\})$. Thus if x lies outside the support of ϕ, there is some neighborhood of x on which ϕ vanishes.

13.4.2 PARTITION OF UNITY

Let $\{U_1, , , , , , , U_n\}$ be finite indexed open covering of the space X. An induced family of continuous functions $\phi: X \to [0,1]$ for $i = 1, \dots \dots n$ is said to be a partition of unity dominated $[U_i]$ if :

1. (Support ϕ_i) $\subset$ U_i for each i

2. $\sum_{i=1}^{n} \phi_i\,(x) = 1$ for each x.

13.4.3 THEOREM

Theorem (Existence of finite partitions of unity). Let $\{U_1, , , , , , , U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.

13.4.4 m −Manifold

An m −Manifold is a Hausdorff space X with a countable basis such that each point x of X has a neighbourhood that is homeomorphic with an subset of R^m.

A 1- manifold is often called a curve, and a 2-manifold is called a surface.

13.4.5 THEOREM

Theorem. If X is a compact m −Manifold, then X can be imbedded in $R^{\mathbb{N}}$ for some positive integer $\mathbb{N}$.

13.5 SUMMARY

In this unit is Urysohn metrization theorem is explained i.e. Every second countable normal space is metrizable. Here partitions of unity are also discussed.

13.5 GLOSSARY

- **i.** Continuous map.
- **ii.** Normal space.
- **iii.** Regular space.
- **iv.** Hausdorff space.
- **v.** Compact space.

CHECK YOUR PROGRESS

1. Every normal space $(X, \mathcal{T})$ with a countable basis is not metrizable. **True/False.**
2. The set $\mathbb{R}$ with usual topology is metrizable. **True/False**
3. A discrete space $(X, \mathcal{T})$ is metrizable **True/False**
4. **1-** manifold is often called a curve, and a 2-manifold is called a surface. **True/False**
5. An $m-$Manifold is a Hausdorff space X with not a countable basis **True/False**

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13.9 TERMINAL QUESTIONS

TQ1: Give an example showing that a Hausdorff space with a countable basis need not be metrizable.

TQ2: Prove that the topological product of a finite family of metrizable spaces is metrizable.

TQ3: Prove that every metrizable space is first countable.

TQ4: Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

13.10 ANSWERS

CHECK YOUR PROGRESS

CHQ1. False.

CHQ2. True.

CHQ3. True.

CHQ4. True.

CHQ5. False.

UNIT 14: TYCHONOFF'S THEOREM FOR PRODUCT SPACES

CONTENTS:

14.1 INTRODUCTION

Tychonoff' theorem states that the with respect to the product topology product of any collection of compact topological spaces is compact. The theorem is named after Andrey Nikolayevich Tikhonov who proved it first in 1930 for powers of the closed unit interval and in 1935 stated the full theorem along with the remark that its proof was the same as for the special case. The earliest known published proof is contained in a 1935 article by Tychonoff, "Über einen Funktionenraum".The theorem is also valid for topological spaces based on fuzzy sets.

14.2 OBJECTIVES

After completion of this unit learners will be able to

i. Explained the concept of Zorn's lemma

ii. State and prove the Tychonoff theorem.

iii. Defined the definitions of Net and other important concepts.

iv. Understand the concepts ,Relation, partially ordered set (PO set), least upper bound (lub), greatest lower bound (glb), maximal element ,Chain, Net, Accumulation point of the given net.

14.3 *DEFINITIONS*

14.3.1 PARTIALLY ORDERED SET (PO Set)

Let X be a nonempty set and $R \subseteq X \times X$, that is R is a relation on X. If $(x, y) \in R$ then we say that x is related to y and write $x \leq y$. The pair (X, R) is said to be a partially ordered set if and only if

(i) $x \leq x$ ($\leq$ is a reflexive),

(ii) for $x, y \in X$, $x \leq y$ and $y \leq x \Rightarrow x = y$. (That is $\leq$ is against symmetry in the sense that $x \leq y$ and $y \leq x$ can happen only when $x = y$.) In this case we say that $\leq$ is antisymmetry,

(iii) for $x, y, z \in X$ $x \leq y$ and $y \leq z \Rightarrow x \leq z$. ($\leq$ is transitive.)

In this case we say that $(X, \leq)$ is a partially ordered set (PO set).

14.3.2 LEAST UPPER BOUND AND GREATEST LOWER BOUND (GLB)

Let $(X, \leq)$ be a partially ordered set and A be a nonempty subset of X. Then an element $x \in X$ (note: x need not be in A) is called an ***upper bound*** of A if and only if $a \leq x$ for all $a \in A$. An element $y \in Y$ is called a ***lower bound*** of

A if and only if $y \leq a$ for all $a \in A$. If there exists an $x_0 \in X$ such that (i) x_0 is an upper bound of A, (ii) $x \in X$ is an upper bound of A implies $x_0 \leq x$ then such an upper bound x_0 is called the ***least upper bound*** (lub) of A and we can easily show that l.u.b of A is unique, when it exists. An element $x_0 \in X$ is called the ***greatest lower bound*** (glb) of A if it satisfies the following: (i) x_0 is a lower bound of A, (ii) if $y_0 \in X$ is a lower bound of A implies $y_0 \leq x_0$.

14.3.4 MAXIMAL ELEMENT

An element $x_0 \in X$ of a partially ordered set is called a ***maximal element*** of X if $x \in X$ is such that $x_0 \leq x$ then $x = x_0$. An element $y_0 \in X$ is called a minimal element of X if $y \in X$ is such that $y \leq y_0$ then $y = y_0$.

14.3.5 EXAMPLE

Let $X = \{1, 2, 3, 4, 5\}$, $R = \{(1, 2), (3, 4), (n, n) : n \in \{1, 2, 3, 4, 5\}\}$. If $(x, y) \in R$ then we say that $x \leq y$. Here 2,4, 5 $\in X$ and they are maximal elements of X. Note that (2,3) $\notin R$ and hence 2 is not related to 3. That is $2 \leq 3$ is not true.

Similarly 2 is not related to 4 and 2 is not related to 5. So 2 is not smaller than other elements of X and hence 2 is a maximal element of X. Since $3 \leq 4$ and $3 \neq 4$, 3 is not maximal element of X. If $y_0 \in X$ is such that y_0 is not larger than any other element of X then we say that y_0 is a minimal element of X. That is if there exists $y \in X$ such that $y \leq y_0$ then $y = y_0$.

14.3.6 CHAIN

A nonempty subset A of X is said to be a chain (also known as totally ordered set) if for $x, y \in A$, $x \leq y$ or $y \leq x$. That is any pair of elements x, y in A are comparable.

14.4 ZORN'S LEMMA

Zorn's Lemma. *Let $(X, \leq)$ be a partially ordered set. Further suppose every chain $C \subseteq X$ has an upper bound in X. Then X will have at least one maximal element.*

We observe the following: A topological space $(X, \mathcal{J})$ is compact if and only if whenever $\mathcal{A}$ is a collection of subsets of X which has finite intersection property (f.i.p) then $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \phi$.

14.5 TYCHONOFF'S THEOREM

14.5.1 STATEMENT AND PROOF

Tychonoff theorem. *Let $(X_\alpha, \mathcal{J}_\alpha)$, $\alpha \in J$ be a collection of compact topological spaces. Then the product space $(\prod_{\alpha \in J} X_\alpha, \mathcal{J})$ is also a compact space.*

Proof. Start with a collection $\mathcal{A}$ of subsets of $X = \prod_{\alpha \in J} X_\alpha$ which has f.i.p. Then we aim to prove that $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \phi$.

Step 1:

Let $\mathcal{F} = \{\mathcal{D} : \mathcal{D}$ is a collection of subsets of X containing $\mathcal{A}$ and $\mathcal{D}$ has f.i.p $\}$.

For $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{F}$, define $\mathcal{D}_1 \leq \mathcal{D}_2$ if $\mathcal{D}_1 \subseteq \mathcal{D}_2$. Then $(\mathcal{F}, \leq)$ is a partially ordered set. Now let $\mathcal{C}$ be a chain in $\mathcal{F}$ and $\mathcal{A}_0 = \underset{\mathcal{D}\in\mathcal{C}}{\cup} \mathcal{D}$ (here $\mathcal{C} \subseteq \mathcal{F}$ and $\mathcal{D} \in \mathcal{F}$). It is easy to prove that $\mathcal{A}_0$ is an upper bound for $\mathcal{C}$. For this, we will have to prove that $\mathcal{A}_0 \in \mathcal{F}$ and $\mathcal{D} \leq \mathcal{A}_0$ for all $\mathcal{D} \in \mathcal{C}$. First let us prove that $\mathcal{A}_0$ has f.i.p. Let $A_j \in \mathcal{A}_0$ for $j = 1, 2, \ldots, n$. Then $A_j \in \mathcal{D}_j$, for some $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \in \mathcal{C}$. As $\mathcal{C}$ is a chain for $j \in \{1, 2, \ldots, n\}$ either $\mathcal{D}_i \subseteq \mathcal{D}_j$ or $\mathcal{D}_j \subseteq \mathcal{D}_i$. Hence there exists k, $1 \leq k \leq n$ such that $\mathcal{D}_j \subseteq \mathcal{D}_k$ for all $j \in \{1, 2, \ldots, n\}$. Then $A_j \in \mathcal{D}_k$ for all j and $\mathcal{D}_k$ has f.i.p implies $\overset{n}{\underset{j=1}{\cap}} A_j \neq \phi$. Also $\mathcal{A} \subseteq \mathcal{A}_0$. Hence $\mathcal{A}_0 \subseteq \mathcal{F}$. By the definition of $\mathcal{A}_0$, $\mathcal{D} \subseteq \mathcal{A}_0$ for all $\mathcal{D} \subseteq \mathcal{C}$. This proves that $\mathcal{A}_0 \in \mathcal{F}$ is an upper bound for $\mathcal{C}$.

Now we have proved that every chain $\mathcal{C}$ in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Therefore by Zorn's lemma $\mathcal{F}$ will have a maximal element say $\mathcal{B} \in \mathcal{F}$. This $\mathcal{B} \in \mathcal{F}$ is

such that (i) $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{B}$ has f.i.p, (ii) whenever $\mathcal{A}'$ is a collection of subsets of X such that $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}'$ has f.i.p then $\mathcal{A}' \subseteq \mathcal{B}$.

Step 2: Now let us prove that $\mathcal{B}$ has the following properties:

(i) For $n \in \mathbb{N}, A_1, A_2, \ldots, A_n \in \mathcal{B}$ implies $A_1 \cap A_2 \cap \cdots \cap A_n \in \mathcal{B}$.

(ii) If A is subset of X such that $A \cap B \neq \phi$, for all $B \in \mathcal{B}$ then $A \in \mathcal{B}$.

To prove (i), let $A_0 = A_1 \cap A_2 \cap \cdots \cap A_n$ and $\mathcal{B}_0 = \mathcal{B} \cup \{A_0\}$. Then $\mathcal{B}_0 \in \mathcal{F}$ and $B \subseteq \mathcal{B}_0$. Since B is maximal, $B = \mathcal{B}_0$. This proves that $A_0 \in \mathcal{B}$.

To prove (ii), take $\mathcal{B}_0 = \mathcal{B} \cup \{A\}$. Then $\mathcal{B}_0 \in \mathcal{F}$ and hence by step 1, $A \in \mathcal{B}$.

Step 3: Let us prove that $\underset{A\in\mathcal{B}}{\cap} \overline{A} \neq \phi$.

For each $\alpha \in J$, $\{P_\alpha(A) : A \in \mathcal{B}\}$ is a collection of subsets of $(X_\alpha, \mathcal{J}_\alpha)$. If $A_1, A_2, \ldots, A_n \in \mathcal{B}$, then $\mathcal{B}$ has f.i.p and $\overset{n}{\underset{j=1}{\cap}} A_j \neq \phi$. Let $x \in \overset{n}{\underset{i=1}{\cap}} A_j$. Now $P_\alpha(x) \in P_\alpha(A_j)$ for all $j = 1, 2, \ldots, n$. Hence $\{P_\alpha(A) : A \in \mathcal{B}\}$ is a collection of

subsets of the compact topological space $(X_\alpha, \mathcal{J}_\alpha)$. Further this collection has f.i.p. This gives that $\underset{A\in\mathcal{B}}{\cap} \overline{P_\alpha(A)} \neq \phi$. Let $x_\alpha \in \underset{A\in\mathcal{B}}{\cap} \overline{P_\alpha(A)}$ and $x = (x_\alpha)_{\alpha\in J}$. (That is, we define $f : J \to \underset{\alpha\in J}{\cup} X_\alpha$ as $f(\alpha) = x_\alpha \in X_\alpha$ and we identify f with x.) Now we aim to prove that $x \in \overline{A}$, for each $A \in \mathcal{B}$. So fix $A \in \mathcal{B}$ and let $P_\beta^{-1}(V_\beta)$ be a subbasic open set containing x. Now $x = (x_\alpha) \in P_\beta^{-1}(V_\beta)$ implies $x_\beta \in V_\beta$. We have $x_\beta \in \overline{P_\beta(A)}$ and hence V_β is an open set in $(X_\alpha, \mathcal{J}_\alpha)$ containing x_β implies $V_\beta \cap P_\beta(A) \neq \phi$ implies there exists $y \in A$ such that $P_\beta(y) \in V_\beta$. This gives that $y \in P_\beta^{-1}(V_\beta) \cap A$. Hence $P_\beta^{-1}(V_\beta) \cap A \neq \phi$ for all $A \in \mathcal{B}$ implies $P_\beta^{-1}(V_\beta) \in \mathcal{B}$. Again if B is a basic open set containing x in the product space $(X, \mathcal{J})$ then $B = P_{\beta_1}^{-1}(V_{\beta_1}) \cap P_{\beta_2}^{-1}(V_{\beta_2}) \cap \cdots \cap P_{\beta_n}^{-1}(V_{\beta_n})$ for some $V_{\beta_i} \in \mathcal{J}_{\beta_i}$, $i = 1, 2, 3, \ldots, n$. We have proved that each $P_{\beta_i}^{-1}(V_{\beta_i}) \in \mathcal{B}$ and hence $B \in \mathcal{B}$. Hence whenever B is a basic open set containing x, then $B \cap A \neq \phi$ $(A \in \mathcal{B})$ implies $x \in \overline{A}$, for all $A \in \mathcal{B}$ implies $x \in \bigcap_{A \in \mathcal{B}} \overline{A} \neq \phi$. Now $\mathcal{A} \subseteq \mathcal{B}$ implies $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \phi$. That is, whenever $\mathcal{A}$ is a collection of closed subsets of the product space $(X, \mathcal{J})$ and further $\mathcal{A}$ has f.i.p then $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \phi$. This proves that $(X, \mathcal{J})$ is a compact topological space. ■

Now let us introduce the notion of a generalized sequence, known as net and convergence of a net in a topological space.

Let $(X, \leq)$ be a partially ordered set. Further suppose for $\alpha, \beta \in X$ there exist $\gamma \in X$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Then we say that $(X, \leq)$ is a directed set. (In the above case if $\alpha \leq \gamma$ then we also say $\gamma \geq \alpha$.)

14.5.2 NET

Let X be a nonempty set and $(D, \leq)$ be a directed set. Then any function $f : D \to X$ is called a ***net*** in X. For each $\alpha \in D$, $f(\alpha) = x_\alpha \in X$ and we say that $\{x_\alpha\}_{\alpha \in D}$ is a net in X.

14.5.3 EXAMPLES

Example 1:

Let $D = \mathbb{N}$ and $\leq$ be the usual relation on $\mathbb{N}$. Then $(\mathbb{N}, \leq)$ is a directed set. If X is a nonempty set and $f : \mathbb{N} \to X$ then for each $n \in \mathbb{N}$, $f(n) = x_n \in X$. Hence our net $\{x_n\}_{n\in\mathbb{N}}$ is the well known concept namely sequence in X. In this sense we say that every sequence is a net. Now take $D = [0, 1]$. Then $(D, \leq)$ is also a directed set. Define $f : [0, 1] \to \mathbb{R}$ as $f(\alpha) = \alpha + 3$, $\forall\ \alpha \in [0, 1]$. Here $f = \{f(\alpha)\}_{\alpha\in D} = \{\alpha + 3\}_{\alpha\in[0,1]}$ is a net (generalized sequence) in $\mathbb{R}$.

It is intuitively clear that the net $\{\alpha + 3\}_{\alpha\in[0,1]}$ approaches to 4. What do we mean by saying that the net $\{x_\alpha\}_{\alpha\in D}$ approaches to 4? Can we also say that the net $\{x_\alpha\}_{\alpha\in D}$ approaches 3? Well, in $\mathbb{R}$ consider a sequence $\{x_n\}_{n\in\mathbb{N}} = \{x_n\}_{n=1}^{\infty}$. We know that $\lim\limits_{n\to\infty} x_n = x$ (i.e $x_n \to x$ as $n \to \infty$) if and only if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \geq n_0$. Note that $x_n \to x$ as $n \to \infty$ if and only if for each open set U containing x there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.

Keeping this in mind, we define:

Let $(X, \mathcal{J})$ be a topological space and $\{x_\alpha\}_{\alpha\in D}$ be a net in X. Then we say that the net $\{x_\alpha\}_{\alpha\in D}$ converges to an element $x \in X$ if and only if for each open set U containing x there exists $\alpha_0 \in D$ such that $x_\alpha \in U$, $\forall\ \alpha \geq \alpha_0$ (that is $\alpha \in D$ with $\alpha_0 \leq \alpha$). If $\{x_\alpha\}_{\alpha\in D}$ converges to x then we write $x_\alpha \to x$.

In a metric space (X, d) we know that a sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to at most one element x in X.

What about in a topological space ? Whether a net $\{x_\alpha\}_{\alpha\in D}$ in a topological space converges to at most one element in X. Obviously the answer is no. For example, let X be any set containing at least two elements and $\mathcal{J} = \{\phi, X\}$. Take $x_1, x_2 \in X$, $x_1 \neq x_2$. Now with usual $\leq$, $(\mathbb{N}, \leq)$ is a directed set.

Define $f : \mathbb{N} \to X$ as

$$f(n) = \begin{cases} x_1 & \text{when } n \text{ is odd} \\ x_2 & \text{when } n \text{ is even} \end{cases}$$

Here our net is $\{x_1, x_2, x_1, x_2, \ldots\}$ that is our net is a sequence in X. Let $x = x_1$. Then the only open set U containing x_1 is X and hence $n_0 = 1 \in \mathbb{N}$. Then for all $n \geq n_0$, $x_n \in X = U$. Hence $x_n \to x$ for any $x \in X$. Also nothing special about the net $\{x_1, x_2, x_1, x_2, \ldots\}$. In fact if D is a directed set and $\{x_\alpha\}_{\alpha \in D}$ is an arbitrary net in X then for each $x \in X$, $x_\alpha \to x$.

Example 2:

Now consider $X = \mathbb{R}$ and $\mathcal{J}_f$, the cofinite topology on $\mathbb{R}$. $D = \mathbb{R}$ and $\leq$ is our usual relation. Then $(D, \leq)$ is a directed set. Define $f : D \to \mathbb{R}$ as $f(\alpha) = \alpha$ for $\alpha \in D = \mathbb{R}$. Then $\{\alpha\}_{\alpha \in \mathbb{R}}$ is a net in $\mathbb{R}$. Fix an element say $x \in \mathbb{R}$ Whether $x_\alpha \to x$? How to start? Start with an open set U containing x in our topological space $(\mathbb{R}, \mathcal{J})$. Now $U \in \mathcal{J}_f$, $x \in U$ (that is $U \neq \phi$) implies U^c is a finite subset of $\mathbb{R}$.

Case (i). $U^c = \phi$ $(\Rightarrow U = X)$.

Case (ii). $U^c \neq \phi$.

That is U^c is a nonempty finite subset of $\mathbb{R}$. Hence there exists $n_0 \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{n0} \in \mathbb{R} = D$ such that $U^c = \{\alpha_1, \alpha_2, \ldots, \alpha_{n_0}\}$. Now take a real number say α_0 such that $\alpha_0 > \alpha_i$ for all $i = 1, 2, \ldots, n_0$. This $\alpha_0 \in D$ is such that $x_\alpha = \alpha \in U$ $\forall\, \alpha \geq \alpha_0$, $(\alpha \geq \alpha_0, \alpha_0 > \alpha_i \Rightarrow \alpha > \alpha_i \Rightarrow \alpha \notin U^c \Rightarrow \alpha \in U)$.

Conclusion: We started with an open set U containing x and we could get an $\alpha_0 \in D$ (α_0 depends on U) such that $x_\alpha \in U$, $\forall \alpha \geq \alpha_0$. Hence by our definition $x_\alpha \to x$. That this net $\{x_\alpha\} = \{\alpha\}_{\alpha \in D}$ converges to every element x of the given topological space $(\mathbb{R}, \mathcal{J})$.

(iii) $D = \{1, 2, \ldots, p\}$ and $\leq$ is our usual relation. $(D, \leq)$ is a directed set (check).

What about $\{x_\alpha\}_{\alpha \in D}$. Here $D = \{1, 2, \ldots, 10\}$ implies $\{x_\alpha\}_{\alpha \in D} = \{1, 2, \ldots, 10\}$. Now for any open set U containing 10 there exists $\alpha_0 = 10 \in D$ is such that $\alpha \in D$, $\alpha \geq \alpha_0 = 10 \Rightarrow \alpha = 10$ and $x_\alpha = \alpha = 10 \in U$. Hence $\{x_\alpha\}_{\alpha \in D} \to 10$.

14.5.3 THEOREMS

Theorem 1:

In a Hausdorff topological space $(X, \mathcal{J})$ a net $\{x_\alpha\}_{\alpha \in D}$ in X cannot converge to more than one element.

Proof. Suppose a net $\{x_\alpha\}_{\alpha \in D}$ converge to say $x, y \in X$, where $x \neq y$. Now $x \neq y$, $(X, \mathcal{J})$ is a Hausdorff topological space implies there exist open sets U, V in X such that (i) $x \in U, y \in U$, (ii) $U \cap V = \phi$. Now $x_\alpha \to x$, U is an open set containing x implies

there exists $\alpha_1 \in D$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_1$.

…………..(14.5.3.1)

Also $y_\alpha \to y$, V is an open set containing y implies

there exist $\alpha_2 \in D$ such that $y_\alpha \in V$ for all $\alpha \geq \alpha_2$.

…………..(14.5.3.2)

Note that D with a relation $\leq$ is a directed set and hence for $\alpha_1, \alpha_2 \in D$ there exists $\alpha_0 \in D$ such that $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$ (that is $\alpha_1 \leq \alpha_0$ and $\alpha_2 \leq \alpha_0$). Now $\alpha_0 \geq \alpha_1$

implies $x_{\alpha_0} \in U$ from

equation (14.5.3.1)

and $\alpha_0 \geq \alpha_2$ implies $x_{\alpha_0} \in V$.

and from equation (14.5.3.2). Hence,

$x_{\alpha_0} \in U \cap V$, a contradiction to $U \cap V = \phi$. We arrived at this contradiction by assuming $x_\alpha \to x$, $x_\alpha \to y$ and $x \neq y$. This means $\{x_\alpha\}_{\alpha \in D}$ cannot converge to more than one element. ■

Note. In a Hausdorff topological space a net $\{x_\alpha\}_{\alpha \in D}$ may not converge. If a net converges then it converges to a unique limit. ⋆

Theorem 2:

Let $(X, \mathcal{J})$ be a topological space and $A \subseteq X$. Then an element x of X is in $\overline{A}$ if and only if there exists a net $\{x_\alpha\}_{\alpha \in D}$ in A such that $x_\alpha \to x$.

Proof. Let us assume that $x \in \overline{A}$. Our tasks are the following: (i) using the fact that $x \in \overline{A}$ construct a suitable directed set, $(D, \leq)$, (ii) and then define a net $\{x_\alpha\}_{\alpha \in D}$ that converges to x. Now $x \in \overline{A}$ implies for each open set U containing x,

$U \cap A \neq \phi$. (If our topology $\mathcal{J}$ is induced by a metric d on X then $\mathcal{J} = \mathcal{J}_d$. In this case $B(x, \frac{1}{n}) \cap A \neq \phi$ for each $n \in \mathbb{N}$. So take $x_n \in B(x, \frac{1}{n}) \cap A$. Then $d(x_n, x) < \frac{1}{n}$ and $\frac{1}{n} \to 0$ as $n \to \infty$. Hence $x_n \to x$). Take $D = \mathcal{N}_x = \{U \in \mathcal{J}\colon\ x \in U\}$ that is $\mathcal{N}_x$

is the collection of all open sets containing x. For $U, V \in \mathbb{N}_x$, define $U \leq V$ if and only if $V \subseteq U$ (reverse set inclusion is our relation $\leq$). Now define $f : \mathbb{N}_x \to X$ as $f(U) = x_U \in U \cap A$ ($U \cap A \neq \phi$ for each $U \in \mathbb{N}_x$ implies by axiom of choice such a function exists). Now we have a net $\{x_U\}_{U \in \mathbb{N}_x}$.

Claim: $x_U \to x$.

Take an open set U_0 containing x, then such an $U_0 \in \mathbb{N}_x$ implies $f(U_0) \in U_0 \cap A$. Now $U \in \mathbb{N}_x$ (our directed set) and $U \geq U_0$ implies $U \subseteq U_0$ implies $x_U \in U \subseteq U_0$. Now $U \geq U_0$ implies $x_U \in U_0$. Hence by definition of convergence of a net, $x_U \to x$.

Conversely, assume that there is a net say $\{x_\alpha\}_{\alpha \in D}$ in A such that $x_\alpha \to x$. Now we will have to prove that $x \in \overline{A}$. So start with an open set U containing x. Hence $x_\alpha \to x$ implies

there exists $\alpha_0 \in D$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. ……………(14.5.3.3)

(D is a directed set means $(D, \leq)$ is a directed set). In particular when $\alpha = \alpha_0$, $\alpha \geq \alpha_0$ and therefore

From equation (14.5.3.3),

$x_{\alpha_0} \in U$. Also $x_{\alpha_0} \in A$. Hence $x_{\alpha_0} \in U \cap A$.

That is for each open set U containing x, $U \cap A \neq \phi$. This implies $x \in \overline{A}$. ■

Theorem 3:

Let X, Y be topological spaces and f: $X \to Y$. Then f is continuous if and only if for every net $\{x_\alpha\}_{\alpha \in J}$ converging to an element $x \in X$ the net $\{f(x_\alpha)\}_{\alpha \in J}$ converges to f(x).

Proof. Assume that $f : X \to Y$ is a continuous function. Now let $\{x_\alpha\}_{\alpha \in J}$ be a net in X such that $x_\alpha \to x$ for some $x \in X$. We will have to prove that $f(x_\alpha) \to f(x)$. Let V be an open set containing $f(x)$ in Y. Now V is an open set containing $f(x)$ and $f : X \to Y$ is a continuous function implies

there exists an open set U containing x such that $f(U) \subseteq V$.

……………………….(14.5.3.4)

Now U is an open set containing x and $x_\alpha \to x$ implies there exists an $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Hence from (14.5.3.4) ,

$f(x_\alpha) \in V$ for all $\alpha \geq \alpha_0$. That is, for each open set V containing $f(x)$ there exists $\alpha_0 \in J$ such that $f(x_\alpha) \in V$ for all $\alpha \geq \alpha_0$. This in turn implies $f(x_\alpha) \to f(x)$.

Now let us assume that whenever a net $\{x_\alpha\}_{\alpha\in J}$ converges to an element x in X then $f(x_\alpha) \to f(x)$ in Y. In this case we will have to prove that $f : X \to Y$ continuous. We know that f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

(An element z of X is closer to A, that is if $z \in \overline{A}$ then the image $f(z)$ is closer to $f(A)$.) So start with $A \subseteq X$ and an element $y \in f(\overline{A})$ $(f(\overline{A}) = \phi \Rightarrow f(\overline{A}) \subseteq \overline{f(A)})$. Now $y \in f(\overline{A})$ implies there exists $x \in \overline{A}$ such that $y = f(x)$. Hence $x \in \overline{A}$ implies there exists a net $\{x_\alpha\}_{\alpha\in J}$ in A such that $x_\alpha \to x$ (refer the previous theorem) this implies by our assumption, $f(x_\alpha) \to f(x)$. Now $f(x_\alpha) \in f(A)$ and $f(x_\alpha) \to f(x)$ implies $f(x) \in \overline{f(A)}$ (again refer the previous theorem). So we have proved that $f(\overline{A}) \subseteq \overline{f(A)}$ whenever $A \subseteq X$. This implies $f : X \to Y$ is a continuous function. ■

14.5.4 ACCUMULATION POINT OF THE GIVEN NET

Let $(X, \mathcal{J})$ be a topological space and $\{x_\alpha\}_{\alpha\in J} = (x_\alpha)_{\alpha\in J}$ be a net in X. An element $x \in X$ is said to be an ***accumulation point*** of the given net $(x_\alpha)_{\alpha\in J}$ if and only if for each open set U containing x, the set $K_U = \{\alpha \in J : x_\alpha \in U\}$ is cofinal in J. Now K_U is cofinal in J means for each $\alpha \in J$ there exists $\beta \in K_U$ such that $\beta \geq \alpha$ (it is like saying that $k \to \infty$ implies $n_k \to \infty$).

14.5.5 THEOREMS

Theorem 4:

Let $(x_\alpha)_{\alpha \in J}$ be a net in a topological space. Then a point x in X is an accumulation point of the given net $(x_\alpha)_{\alpha \in J}$ if and only if $(x_\alpha)_{\alpha \in J}$ has a subnet and that subnet converges to x.

Proof. $\Rightarrow$ Assume that x is an accumulation point of $(x_\alpha)_{\alpha \in J}$. By the definition of accumulation point of a net we have for each open set U containing x

$$K_U = \{\alpha \in J : x_\alpha \in U\} \text{ is cofinal in } J.$$

…………………………………..(14.5.5.1)

Using equation (14.5.5.1),

Let $K = \{(\alpha, U) \in J \times \mathcal{N}_x : x_\alpha \in U\}$, where $\mathcal{N}_x$ is the collection of all open set containing x. $K_U \neq \phi$. (Fix $\alpha \in J$. Now K_U is cofinal in J implie

there exists $\beta \in K_U$ such that $\beta \geq \alpha$.) For $(\alpha, U), (\beta, V) \in K$ define $(\alpha, U) \leq (\beta, V$ if and only if $\alpha \leq \beta$ and $V \subseteq U$ (reverse set inclusion). It is easy to see that $(K, \leq$

is a directed set. It is given that $(x_\alpha)_{\alpha \in J}$ is a net in X. Hence $(J, \leq)$ is a directed se and $f : J \to X$ is such that $f(\alpha) = x_\alpha$. Now define $g : K \to J$ as $g(\alpha, U) = \alpha$

Claim: $g(K)$ is cofinal in J.

Using equation (14.5.5.1),

So take $\alpha \in J$. Now K_U is cofinal in J there exists $\beta \in K_U$ such that $\beta \geq \alpha$. Now $\beta \in K_U$ implies $x_\beta \in U$ that is $(\beta, U) \in K$ is such that

$g(\beta, U) = \beta \geq \alpha$ implies $g(K)$ is cofinal in J. Also $(\alpha, U), (\beta, V) \in K, (\alpha, U) \leq (\beta, V)$ implies $g(\alpha, U) = \alpha \leq \beta = g(\beta, V)$. Hence $f \circ g : K \to X$ is a subnet of f (or say $f(\alpha) = (x_\alpha)$). Now let us prove that this subnet converges to x. So take an open set U containing x. This implies K_U is cofinal in J. Fix $(\alpha_0, U) \in K$. Now $\alpha_0 \in J$, K_U is cofinal in J implies $\beta_0 \in K_U$ such that $\beta_0 \geq \alpha_0$. Hence $(\alpha, V) \in K$, $(\alpha, V) \geq (\alpha_0, U)$ implies $(f \circ g)(\alpha, V) = f(\alpha) = x_\alpha \in V \subseteq U$. That is for each open set U containing x there exists $(\alpha_0, U) \in K$ such that $(\alpha, V) \in K$ $(\alpha, V) \geq (\alpha_0, U)$ implies $(f \circ g)(\alpha, V) \in U$. This proves that $f \circ g \to x$.

Conversely, suppose there is a subnet of $(f(\alpha))_{\alpha \in J} = (x_\alpha)_{\alpha \in J}$ which converge to an element $x \in X$. A subnet of f converges to x means there exists a directed set say $(K, \leq)$ and a function say $g : K \to J$ such that $i, j \in K, i \leq j$ implies $g(i) \leq g(j)$, $g(K)$ is cofinal in J, and $(f \circ g)(i) = f(g(i)) \to x$. Now let us prove that x is an accumulation point of the net f. So take an open set U containing x.

Claim: $\{\alpha \in J : f(\alpha) = x_\alpha \in U\}$ is cofinal in J.

Let $\alpha_0 \in J$. Now $f \circ g : K \to X$ is a subnet such that $f \circ g \to x$. Hence for this given $\alpha_0 \in J$ there exists $\beta \in K$ such that $g(\beta) \geq \alpha_0$ (note $g(K)$ is cofinal in J).

Now $f \circ g \to x$, U is an open set containing x implies there exists $\beta_0 \in K$ such that $\alpha \in K, \alpha \geq \beta_0 \Rightarrow f(g(\alpha)) \in U$, $\beta \in J$ is such that $g(\beta) \geq \alpha_0$. Take $\gamma_0 \in K$ such that $\alpha_0 \geq \beta, \beta_0$. Then $(f \circ g)(\gamma_0) \in U$ and $g(\gamma_0) \geq g(\beta) \geq \alpha_0$. That is for $\alpha_0 \in J$, there exists $g(\gamma_0) \in J$ such that $f(g(\gamma_0)) \in U$ implies $\{\alpha \in J : f(\alpha) = x_\alpha \in U\}$ is cofinal in J. Hence x is an accumulation point. ■

Recall that a metric space (X, d) is a compact metric space if and only if every sequence $\{x_n\}_{n=1}^{\infty}$ in X has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to an element in X. It is to be noted that this result is not true for an arbitrary topological space. For a topological space we have the following theorem.

Theorem 5:

A topological space $(X, \mathcal{J})$ is compact if and only if every net in X has a subnet that converges to an element in X.

Proof. Assume that $(X, \mathcal{J})$ is a compact topological space and $f : J \to X$ is a net in X. We will have to prove that f has a subnet that converges to an element in X. So it is enough to prove that f has an accumulation point.

For each $\alpha \in J$, let $A_\alpha = \{x_\beta : \alpha \leq \beta\}$ (note: $f : J \to X$ is a net means with respect to a relation $\leq$, $(J, \leq)$ is directed set). Now $\{A_\alpha\}_{\alpha \in J}$ is a collection of sets which has finite intersection property. For $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}$ if we take $\alpha \in J$ such

that $\alpha \geq \alpha_j$ for all $j = 1, 2, \ldots, k$, that is $\alpha_j \leq \alpha$, then $x_\alpha \in A_{\alpha_j}, \forall\, j = 1, 2, \ldots, k$ and hence $x \in \bigcap\limits_{j=1}^{k} A_{\alpha_j}$. Now $(X, \mathcal{J})$ is a compact topological space $\{\overline{A}_\alpha\}_{\alpha \in J}$ is a collection of closed subsets of X which has finite intersection property implies $\bigcap\limits_{\alpha \in J} \overline{A}_{\alpha_j} \neq \phi$. Let $x \in \bigcap\limits_{\alpha \in J} \overline{A}_{\alpha_j}$.

Now we aim to prove that x is an accumulation point of f. So, start with an open set U containing x, and we will have to prove that $\{\alpha \in J : x_\alpha \in U\}$ is cofinal in J. Take $\alpha_0 \in J$. Now U is an open set containing x, $x \in \overline{A}_{\alpha_0}$ implies $U \cap A_{\alpha_0} \neq \phi$. Hence there exists $\alpha \geq \alpha_0$ such that $x_\alpha \in U$. This proves that $\{\alpha \in J : x_\alpha \in U\}$ is cofinal in J. Hence we have proved that x is an accumulation point of the stated net f. This implies there exists a subnet of f which converges to f.

To prove the converse part let us assume that every net in X has convergent subnet in X. By assuming this, we aim to prove that $(X, \mathcal{J})$ is a compact topological space.

To prove that $(X, \mathcal{J})$ is a compact topological space, let us prove: if $\mathcal{A}$ is a collection of closed subsets of X which has finite intersection property then $\bigcap\limits_{A \in \mathcal{A}} A \neq \phi$. So, we have a collection $\mathcal{A}$ of closed subsets of X which has finite intersection property.

Let $\mathcal{B} = \{A \subseteq X : A = A_1 \cap A_2 \cap \cdots \cap A_k, k \in \mathbb{N}, A_1, \ldots, A_k \in \mathcal{A}\}$. That is $\mathcal{B}$ is the collection of finite intersection of members of $\mathcal{A}$. (Note. $\bigcap\limits_{A \in \phi} A = X$ and hence we do not require to consider this case.) For $A, B \in \mathcal{B}$ define $A \leq B$, whenever $B \subseteq A$.

Then $(\mathcal{B}, \leq)$ is a directed set. Now define $f : \mathcal{B} \to X$ as $f(A) = f(A_1 \cap A_2 \cap \cdots \cap A_k) = x_A$, where $x_A \in A_1 \cap A_2 \cap \cdots \cap A_k$ is fixed $(A_1 \cap A_2 \cap \cdots \cap A_k)$ may contains more than one element and in that case first take any one element form $A_1 \cap A_2 \cap \cdots \cap A_k$. Hence $f = (f(A))_{A \in \mathcal{B}}$ is a net in X. By our assumption this net f will have a subnet

that will converge to an element say x in X. So there will exists a directed set K and a function $g : K \to \mathcal{B}$ satisfying $f \circ g$ is a subnet of f and $f \circ g$ converges to x.

Now we claim that $x \in A$ for each $A \in \mathcal{A}$. Suppose for some $A \in \mathcal{A}, x \notin A$. Then $x \in A^c = U$, an open set. Since $f \circ g \to x$ and U is an open set containing x there exists $\alpha_0 \in K$ such that $(f \circ g)(\alpha) \in U$ for all $\alpha \geq \alpha_0$. Now $\alpha_0 \in K$ implies $g(\alpha_0) \in \mathcal{B}$

implies there exists $k \in \mathbb{N}$ and $A_1, A_2, \ldots, A_k \in \mathcal{A}$ such that $g(\alpha_0) = A_1 \cap A_2 \cap \cdots \cap A_k$. $A_1 \cap A_2 \cap \cdots \cap A_k \in \mathcal{B}$ is such that $A_1 \cap A_2 \cap \cdots \cap A_k \geq g(\alpha_0)$. We have $f \circ g(\alpha_0) = f(g(\alpha_0)) \in U = A^c$. Now K is a directed set and $g(K)$ is cofinal in $\mathcal{B}$ implies there

exists $\alpha \in K$ such that (i) $\alpha \geq \alpha_0$, (ii) $g(\alpha) \geq A_1 \cap A_2 \cap \cdots \cap A_k$. Now $\alpha \geq \alpha_0$ implies $(f \circ g)(\alpha) \in U = A^c$ but by the definition of f, $f(g(\alpha)) \in g(\alpha) \subseteq A_1 \cap A_2 \cap \cdots \cap A_k \subseteq A$. So we get a contradiction. Therefore $x \notin A$ for some $A \in \mathcal{A}$ cannot happen. We have proved that if $\mathcal{A}$ is a collection of closed subsets of X which has finite intersection property then $\bigcap_{A \in \mathcal{A}} A \neq \phi$. Hence $(X, \mathcal{J})$ is a compact topological space. ■

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A topological property is any property so that if $(X, \mathcal{J})$, $(Y, \mathcal{J}')$ are topological spaces and $f : (X, \mathcal{J}) \to (Y, \mathcal{J}')$ is a homeomorphism (that is $(X, \mathcal{J})$ is homeomorphic to $(Y, \mathcal{J}')$) then $(X, \mathcal{J})$ has the property if and only if $(Y, \mathcal{J}')$ has the same property.

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Compactness, connectedness, local compactness are all topological properties.

14.6 SUMMARY

In this unit we explained the Definitions of Relation, partially ordered set (PO set) least upper bound (lub), greatest lower bound (glb), maximal element ,Chain, Net and accumulation point of the given net. The statement proof of Zorn's lemma and Tychonoff theorem also explained. Different examples and theorems also discussed in easy manner.

14.4 GLOSSARY

i. Relation
ii. Function
iii. Partially order set
iv. Least upper bound
v. Greatest lower bound
vi. Maximal element
vii. Net

CHECK YOUR PROGRESS

1. Tychnoff's Theorem proves that the product (even infinite) of compact spaces is also compact. **True/False.**
2. In Zorn's lemma the set is partially ordered set is used. **True/False.**
3. Compactness, connectedness, local compactness are not topological properties. **True/False.**
4. In a Hausdorff topological space a net converges at most one point in set.. **True/False.**
5. Let $D = \mathbb{N}$ and $\leq$ be the usual relation on $\mathbb{N}$. Then $(\mathbb{N}, \leq)$ is not a directed set. **True/False.**

14.5 REFERENCES

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14.7 TERMINAL QUESTIONS

TQ1: State and proof Tychnoff's Theorem?

TQ2: Proof that a topological space $(X, \mathcal{T})$is compact if and only if every net in X has a subset that converges to an element in X.

TQ3: Proof that in a Hausdorff topological space $(X, \mathcal{T})$a net $\{X_\alpha : \alpha \epsilon D\}$ in X cannot converge to more than one element.

14.8 ANSWERS

CHECK YOUR PROGRESS

CHQ1. True.

CHQ2. True.

CHQ3. False.

CHQ4. True.

CHQ5. False.

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