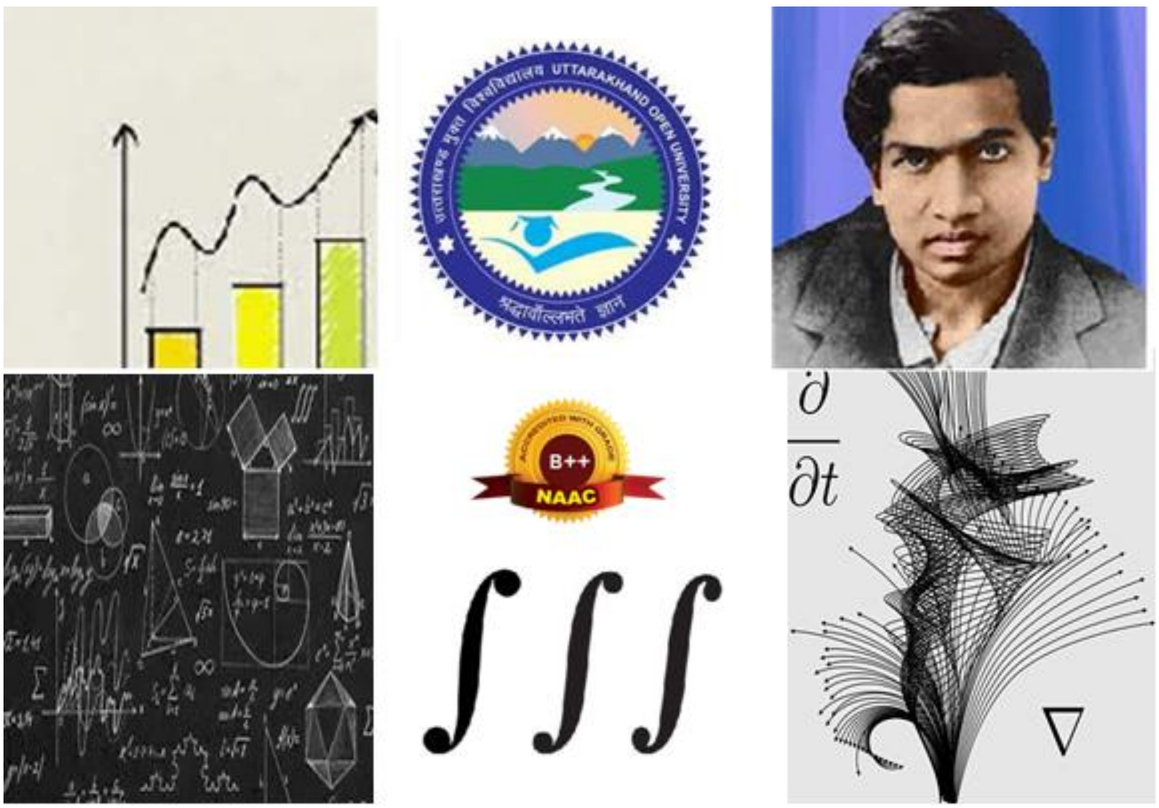
# MASTER OF SCIENCE
# (FIRST SEMESTER)

## MAT502
# ADVANCED REAL ANALYSIS



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# COURSE NAME: ADVANCED REAL ANALYSIS

## COURSE CODE: MAT502

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## ADVANCED REAL ANALYSIS
## MAT502

# COURSE INFORMATION

The present self learning material **“Advanced Real Analysis”** has been designed for M.Sc. (First Semester) learners of Uttarkhand Open University, Haldwani. This self-study material was created to increase learners access to excellent learning materials.. There are 14 units in this course. Real Number System and Countable Set is the focus of the first unit. Sequences, Series and properties and Limit and Continuity is covered in Unit 2 and Unit 3. Derivatives and the mean Value Theorem are the main topics of Unit 5. The aim of Unit 5 and Unit 6 are to analyze Riemann Integral briefly and introduce Riemann-Stieltjes Integral. Unit 7 explained Improper integral. Units 8 and 9 each provided an explanation of Pointwise Convergence and Uniform Convergence. Lebesgue integral is the topic of unit 10. The concepts of completeness, continuous function, and compactness are presented in Units 11, 12, and 13 together with the concept of the metric space. Discussion of fixed Point theorems in the last unit. This subject matter is also employed in competitive exams. Simple, succinct, and clear explanations of the fundamental ideas and theories have been provided. The right amount of relevant examples and exercises have also been added to help learners to understand the material.

# BLOCK I: REAL NUMBER SYSTEM

# UNIT 1: INTRODUCTION OF REAL NUMBER SYSTEM AND COUNTABLE SET

**CONTENTS**



## *1.1 INTRODUCTION*

The Mathematical analysis concepts mostly related to the real numbers, so first we will discussed about real numbers system. Several methods are used to introduce real number. From which in one method the set of positive integers are used to build large of numbers of system i.e. positive rational number ($p/q$, $p, q \in \mathbb{Z}^+$). Then set of rational number was used to construct irrational number for eg. $\sqrt{2}$. Then the system of rational number and irrational number both defined real numbers system.
For convenience we will recall about some concepts and terminology. Let $A$ be a Set (collection of well defined objects). The notation $a \in A$ means that the object $a$ is in the set $A$ while $a \notin A$ means that the object $a$ is not in the set $A$. If set A has a subset of $A_1$ then we write $A_1 \subseteq$ A. A set is said to be empty set if no element is present in set A and can be denoted by the notation $\emptyset$. The set of real number, rational number and integers can be denoted by the notation $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ respectively.

In this unit we will study about the real number system and countable set.

Georg Cantor, in full Georg Ferdinand Ludwig Philipp Cantor, (born March 3, 1845, St. Petersburg, Russia—died January 6, 1918, Halle, Germany), German mathematician who founded set theory and introduced the mathematically meaningful concept of transfinite numbers, indefinitely large but distinct from one another.

In a series of 10 papers from 1869 to 1873, Cantor dealt first with the theory of numbers; this article reflected his own fascination with the subject, his studies of Gauss, and the influence of Kronecker.

In 1873 Cantor demonstrated that the rational numbers, though infinite, are countable (or denumerable) because they may be placed in a one-to-one correspondence with the natural numbers (i.e., the integers, as 1, 2, 3,…). He showed that the set (or aggregate) of real numbers (composed of irrational and rational numbers) was infinite and uncountable.

***Fig 1. Georg Cantor Reference(https://www.britannica.com/biography/Georg-Ferdinand-Ludwig-Philipp-Cantor )***

## *1.2 OBJECTIVES*

After reading this unit learners will be able to

1. construct the basic concept of real numbers and its properties
2. comprehend the basic concept of countable and uncountable
3. study about cardinality of infinite set
4. analyze cantor set.

## *1.3 ORDERED SETS*

Let $A$ be a set. An **order on $A$** is a relation, denoted by $<$, with the following properties

i. If $a_1 and\ a_2 \in A$ then one and only one of the statements $a_1 < a_2, a_1 = a_2$ and $a_1 > a_2$ is true.

ii. If $a_1, a_2, a_3 \in A$ if $a_1 < a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

**NOTE:** the statement "$a_1 < a_2$" may be read as "$a_1$is less than $a_2$". The notation $a_1 \leq a_2$" may be read as "$a_1$is less than equal to $a_2$".

**Ordered Set :** An ordered set is a set $A$ in which an order is defined.
For example: $\mathbb{R}$ is an ordered set if we defined order " $<$ " on $\mathbb{R}$.

**Upper bound:** Consider $A$ be an ordered set and $X \subseteq$ A. If there exists a $c \in A$ such that $x_1 \leq c$ for every $x_1 \in X$ then $X$ is bonded above and $c$ is an upper bound of $X$.

**Lower Bound:** Consider $A$ be an ordered set and $X \subseteq$ A. If there exists a $c \in A$ such that $x_1 \geq c$ for every $x_1 \in X$ then $X$ is bonded below and $c$ is a lower bound of $X$.

**Least upper bound (Supremum):** Consider $A$ be an ordered set, $X \subseteq$ A and $X$ is bounded above. Let there exists $\vartheta \in A$ with the following properties:

(i) $\vartheta$ is an upper bound of $X$.

(ii) If $\omega < \vartheta$, then $\vartheta$ is not an upper bound of $X$.

Then $\vartheta$ is called the least upper bound (Supremum) of $X$. It can be written as $\vartheta = \sup X$.

**Greatest lower bound (Infimum):** Consider $A$ be an ordered set, $X \subseteq$ A and $X$ is bounded below. Let there exists $\mu \in A$ with the following properties:

(i) $\mu$ is a lower bound of $X$.

(ii) If $\mu < \rho$, then $\mu$ is not a lower bound of $X$.

Then $\vartheta$ is called the least upper bound (Infimum) of $X$. It can be written as $\mu = \inf X$.

**Least Upper bound Property:** An ordered set A is said to have the least upper bound property if $X \subseteq$ A, X is not empty and $X$ is bounded above.

**Theorem 1.1 Let $A$ is an ordered set with the least upper bound property and $A_1 (\neq \emptyset) \subseteq$ A which is bounded below. Let $L$ be the set of all lower bounds of $A_1$. Then $\mu = sup\ L$ exists in $A$ and $\mu = \inf A_1$.**

**Proof.** As we see that $A_1$ is bounded below, it implies that $L$ is not empty. Since L consists exactly those $a_1\ in\ A$ which satisfy the inequality

$a_1 \leq a$, for every $a \; in \; A_1$, we know that every $a \; in \; A_1$ is an upper bound of L. Hence $L$ is bounded above.

As it is given that A satisfy least upper bound property. Thus there exists $\mu$ such that $\mu = \sup$ L.

If $\omega < \mu$, then $\omega$ is not an upper bound of L, which implies $\omega \notin \mathrm{A}_1$. Hence $\mu \leq a$ for every a in B. Thus $\mu \in L$.

If $\vartheta > \mu$ then $\vartheta \notin$ L, because $\mu$ is an upper bound of L. It implies that $\mu$ is a lower bound of $\mathrm{A}_1$, but $\vartheta$ is not if $\vartheta > \mu$. Hence $\mu = \inf A_1$.

**Fields:** A field is a set F with two operations, called addition and multiplication, which satisfy the following axioms:

(F1) Axiom for addition

(F11) Closed under addition: If $x_1$ and $x_2 \; in \; F$, then $x_1 + x_2 \; in \; F$.

(F12) Commutative under addition: $x_1 + x_2 = x_2 + x_1$ for all $x_1, x_2 \in F$

(F13) Associative under addition: $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$ for all $x_1, x_{2,} x_3 \in F$

(F14) Additive identity: There exists $0$ such that $0 + x_1 = x_1$ for every $x_1 \in F$.

(F15) Additive inverse: For every $x_1 \; in \; F$ there exists ($-x_1$) such that $x_1 + (-x_1) = 0$.

(F2) Axiom for multiplication

(F21) Closed under multiplication: If $x_1$ and $x_2 \; in \; F$, then $x_1 x_2 \; in \; F$.

(F22) Commutative under multiplication: $x_1 x_2 = x_2 x_1$ for all $x_1, x_2 \in F$

(F23) Associative under multiplication: $(x_1 x_2) x_3 = x_1 (x_2 x_3)$ for all $x_1, x_{2,} x_3 \in F$

(F24) Multiplicative identity: There exists 1 such that $1. x_1 = x_1$ for every $x_1 \in F$.

(F25) Multiplicative inverse: For every $x_1 \; in \; F$ there exists $(\frac{1}{x_1})$ such that $x_1 \left(\frac{1}{x_1}\right) = 1$.

(F3) Distributive Law: $x_1(x_2 + x_3) = x_1 x_2 + x_1 x_3$ for all $x_1, x_{2,} x_3 \in F$

**Ordered Field:** An ordered field is a field F which is also an ordered set such that

(i) $x_1 + x_2 < x_1 + x_3 \; if x_1, x_{2,} x_3 \in F \; and \; x_2 < x_3$

(ii) $x_1 x_2 > 0$ if $x_1, x_2 \in F \; and \; x_1, x_2 > 0$

For ex. $\mathbb{Q}$ is an ordered field.

**<u>Real field</u>:**

**Theorem 1.2. There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.**

The proof of given theorem is long and laborious.

**Theorem 1.3. If $\boldsymbol{x_1, x_2 \in \mathbb{R}\ and\ x_1 > 0}$, then there exists a positive integer $\boldsymbol{n}$ such that $\boldsymbol{nx_1 > x_2}$.**

**If $\boldsymbol{x_1, x_2 \in \mathbb{R}\ and\ x_1 < x_2}$, then there exists a $\boldsymbol{q \in \mathbb{R}}$ such that $\boldsymbol{x_1 < p < x_2}$.**

**Proof.** (i) Let $X$ be the set of all $nx_1$, where $n \in \mathbb{Z}^+$. Let $nx_1 \leq x_2$, for all $n \in \mathbb{Z}^+$. It implies that y could be an upper bound of $X$.

But $X$ has a least upper bound in $R$. Let $\mu = \sup X$.

Because $x_1 > 0$ implies $\mu - x_1 < mx_1$ for some positive integer $m$.

But then $\mu < (m+1)x \in X$, which cannot be possible because $\mu$ is an upper bound of $X$. Thus our assumption is wrong. Thus $nx_1 > x_2$.

(ii) Now $x_1 < x_2$ implies $x_2 - x_1 > 0$ and $1 \in \mathbb{R}$.

Now from (i), we conclude that there exists a positive integer $n$ such that $n(x_2 - x_1) > 1$.

Again using (i) , to obtain integers $n_1$ and $n_2$ such that $n_1 > nx_1$, $n_2 > -nx_1$ .

Then $-n_2 < nx_1 < n_1$. Hence there is an integer $m$ (with $-n_2 \leq m \leq n_1$ ) such that $m - 1 \leq nx_1 \leq m$.

If we combine these inequalities, we obtain $nx_1 < m \leq 1 + nx < ny$. Since $n > 0$, it follows that $x_1 < \frac{m}{n} < x_2$. If $p = \frac{m}{n}$, we obtain $x_1 < p < x_2$.

*The Completeness Axiom: Every nonempty set that is bounded above has a supremum.*

The Archimedean Property: The property of the real numbers described in the next theorem is called the Archimedean property. Intuitively, it states that it is possible to exceed any positive number, no matter how large, by adding an arbitrary positive number, no matter how small, to itself sufficiently many.

**Ex. 1.1 The rational number system is not complete.**

**Proof.** We have to prove that a set of rational numbers may be bounded above (by rationals), but not have a rational upper bound less than any other rational upper bound.

To see this, let $S = \{r | r\ is\ rational\ and\ r^2 < 2\}$
If $r \in\ S$, then $r\ <\ \sqrt{2}$.
if $\alpha > 0$ there is a rational number $r'$ such that $\sqrt{2} - \alpha < r'\ <\ \sqrt{2}$, which implies that $\sup S = \sqrt{2}$. However, $\sqrt{2}$ is irrational; i.e. cannot be written as the ratio of integers. Therefore, if $q_1$ is any rational upper bound of $S$, then $\sqrt{2} < q_1$. There is a rational number $q_2$ such that $\sqrt{2}\ <\ q_2\ <\ q_1$. Since $q_2$ is also a rational upper bound of S, this shows that S has no rational

**The Extended real number system**

The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. we preserve the original order in R and define $+\infty < a < -\infty$ for every $a \in \mathbb{R}$

**CHECK YOUR PROGRESS**

(CQ 1) The real number system is not complete. (T/F)
(CQ 2) The rational number system is not complete. (T/F)
(CQ 3) 3 is upper bound of interval (1,3). (T/F)
(CQ 4) Lower bound of set $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots \dots \frac{.1}{n}\right\}$ is ________________.

## 1.4 FINITE COUNTABLE AND UNCOUNTABLE SETS

Let two sets $X$ and $Y$ and let with each element $x$ of $X$ there is associated, in some manner, an element of $Y$, which we denote by $f(x)$. Then $f$ is said to be a function from $X$ to $Y$. The set $X$ is called the domain of $f$ and the elements $f(x)$ are called the values of $f$. The set of all values of $f$ is said to be range of $f$.
Consider $X$ and $Y$ be two sets and let $f$ be a mapping of $X$ into $Y$, If $A \subset X, f(A)$ is defined to be the set of all elements $f(a)$, for $a \in A$. Here $f(A)$ be the image of $A$ under $f$. In this notation $f(X)$ is the range of $f$. It is clear that $f(X) \subset Y$. If $f(X) = Y$, then $f$ maps $X$ onto $Y$.
If $A \subset Y, f^{-1}(A)$ denotes the set of all $c \in X$ such that $f(c) \in A$. Here $f^{-1}(E)$ the inverse image of $A$ under $f$. If $y \in Y, f^{-1}(y)$ is the set of all $c\ \in X$ such that $f(c) = y$. If for each $y \in X, f^{-1}(y)$ conisists of at most one element of $X$, then f is said to be injective (one-one) mapping of $X$

into $Y$. or we can say, f is injective if $f(a) \neq f(b)$ whenever $a \neq b, a, b \in X$

Let $X$ and $Y$ are two sets. Then $X$ and $Y$ are equivalent $(X \sim Y)$ if it satisfy following properties:

i) Reflexive: $X \sim X$

ii) Symmetric: If $X \sim Y$, then $Y \sim X$

iii) Transitive: If $X \sim Y$ and $Y \sim Z$ then $X \sim Z$

For any positive integer n, let $A_n$ be the set whose elements are the integers $1,2, \dots, n$; let $A$ be the set consisting of all positive integers. For any set $X$, we say:

(i) $X$ is finite if $X \sim A_n$ for some $n$.

(ii) $X$ is infinite if $X$ is not finite

(iii) $X$ is countable if $X \sim A$

(iv) $X$ is uncountable if $X$ is neither finite nor countable.

(v) $X$ is at most countable if $X$ is finite or countable.

**NOTE:**

(i) Contable sets are also called enumerable or denumerable.

(ii) For two finte sets X and Y, $X \sim Y$ if and only if $X$ and $Y$ contain the same number of elements.

**Theorem 1.4. Let $\mathbb{Z}$ be the set of all integers then $\mathbb{Z}$ is countable.**

**Proof.** Consider the following arrangements of the sets $\mathbb{Z}$ and A:

$\mathbb{Z}$: 0,1,-1,2,-2,3,-3,…….

A: 1,2,3,4,5,6,7,………

Let f be function from $A$ to $\mathbb{Z}$:

$$f(n) = \begin{cases} \frac{n}{2}; & n = even \\ -\frac{n-1}{2}; & n = odd \end{cases}$$

Here we can see that explicit formula for a function which setup a one-one correspondence.

By a sequence, we mean a function f defined on the set A of all positive integers. If $f(n) = x_n$, for $n \in A$, then it is used to denote the sequence f by the symbol $\{x_n\}$.

The values of f, that is, the elements $x_n$, are called terms of the sequence.

If $B$ is a set and if $x_n \in B \ \forall n \in A$, then $\{x_n\}$ is said to be sequence in $B$.

**Theorem 1.5. Every natural number can be expressed in the form $m = 2^a b$, where $a$ is a nonnegative integer and $b$ is an odd natural number.**

**Proof.** We will prove this theorem using mathematical induction.
For the base case $n = 1, n = 20 \cdot 1$. Now let $k \in N$, and suppose that every natural number less than k can be written in the desired form. If k is odd, we just write $k = 2^0 k$.
If $k$ is even, then there is an integer $l$ such that $k = 2l$, and $k$ is positive implies $l$ is positive.
Since $l < k$, the inductive hypothesis implies that there exist a nonnegative integer p and an odd natural number q such that $l = 2^p q$, and then $k = 2l = 2^{p+1}q$, which satisfies the conclusion.

**Ex 1.2. The set $\mathbb{N} \times \mathbb{N}$ is countably infinite**.

**Proof.** Let a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $g(m,n) = 2^{m-1}(2n - 1)$.
To show that $g$ is injective, suppose $(m_1, n_1), (m_2, n_2)$ are elements of $\mathbb{N} \times \mathbb{N}$ such that $g(m_1, n_1) = gh(m_2, n_2)$, which is to say
$2^{m_1-1}(2n_1 - 1) = 2^{m_2-1}(2n_2 - 1)$ ……………………………(1)
We will first prove by contradiction that $m_1 = m_2$.
Suppose not; then one is larger, and we may assume without loss of generality that $m_2 > m_1$.
Multiplying both sides of (1) by $2^{1-m_1}$ , we obtain
$2n_1 - 1 = 2^{m_2-m_1}(2n_2 - 1)$ ……………………….…………(2)
The fact that $m_2 > m_1$ implies that the right-hand side is even, while the left-hand side is odd; this is a contradiction, so we can conclude that $m_1 = m_2$.
Then simple algebra shows that $n_1 = n_2$ as well, so $(m_1, n_1) = (m_2, n_2)$.
To prove surjectivity, let $x \in \mathbb{N}$ be arbitrary. Previous theorem shows that we can write $x = 2^a b$ for some nonnegative integer $a$ and some odd natural number $b$.
The fact that q is odd means that $a = 2j + 1$ for some integer $j$, and the fact that $b \geq 1$ means $j \geq 0$.
Therefore, $(a + 1, j + 1) \in \mathbb{N} \times \mathbb{N}$, and
$g(a + 1, j + 1) = 2^{(a+1)-1}(2(j + 1) - 1) = 2(2j + 1) = 2^a b = x$. Thus we have shown that $g$ is bijective, so $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

**Theorem 1.6. Every infinite subset of a countable set $B$ is countable.**

**Proof.** Suppose $X \subset B$ and $X$ is infinite. Arrange the elements $x$ of $B$ in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:
Let $m_1$ be the smallest positive integer such that $x_{m_1} \in X$.
Now choose $m_1, m_2, \dots \dots \dots .., m_{k-1}$ such that $m_k$ be the smallest integer greater than $n_{k-1}$ such that $x_{m_k} \in X$.
Putting $f(k) = x_{m_k}$, we obtain a 1-1 correspondence between $X$ and $A$.
The union of set $X_n$ is defined to by the set $S$ such that $x \in S$ if and only if $x \in X_n$ for at least one $\delta \in B$. It can be written as $S = \bigcup_{\delta \in B} X_\delta$
The intersection of set $X_n$ is defined to by the set P such that $x \in P$ if and only if $x \in X_n$ for every $\delta \in B$. It can be written as $\bigcap_{\delta \in B} X_\delta$ .

**Theorem Let $\{X_n\}$, n=1,2,3,……be a sequence of countable sets and $S = \bigcup_{n=1}^{\infty} X_n$. Then $S$ is countable.**

**Proof.** Let every set $X_n$ be arranged in a sequence $\{a_{nm}\}, m = 1,2,3, \dots$.. and consider the infinite array

$$
\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

In which the elements of $X_n$ form the $n^{th}$ row. The array contain all elements of S. As indicated by the arrows, these elements can be arranged in a sequence as following:

$a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}, a_{51}, a_{42}, a_{33}, a_{42}, a_{15}, \dots.$

If any two of the sets $X_n$ have elements in common, these will appear more than once in above sequence.
Hence there is a subset Y of the set of all positive integer such that $S \sim Y$, which shows that S is at most countable. (because Every infinite subset of a countable set B is countable)
Because $X_1 \subset S$ and $X_1$ is infinite, S is infinite and thus countable.

**NOTE:** Suppose X is at most countable and for every $\delta \in X, X_n$is at most countable then $T = \bigcup_{\delta \in X} X_n$ is at most countable.

**Theorem 1.7. Let $X$ be a countable set, and let $X_n$ be the set of all n-tuples $(x_1, \dots, x_n)$ where $x_k \in X$(k = 1, 2, ..., n) and the elements $x_1, \dots, x_n$ need not be distinct. Then $X_n$ is countable.**

**Proof.** That $X_1$ is countable is evident, since $X_1 = X$. Suppose $X_{n-1}$ is countable $(n = 2,3,4,\dots)$. The elements of $X_n$ are of the form $(\beta,\alpha)$ $(\beta \in X_{n-1}, \alpha \in X)$ . For ever fixed $\beta$, the set of pairs $(\beta,\alpha)$ is equivalent to $X$, and thus countable.

Thus $X_n$ is the union of a countable set of countable sets; thus, $X_n$ is countable, and the proof follows by induction on $n$.

**Corollary. The set of all rational numbers is countable.**

**Proof.** We apply the previous theorem with n = 2, noting that every rational number can be written as $\frac{p}{q}$, where p and $q \neq 0$ are integers. Since the set of pairs $(\beta,\alpha)$ is countable, the set of quotients $\frac{p}{q}$, and Thus the set of rational numbers, is countable.

**Ex 1.3. The set *E* of positive even integers is countably infinite.**

**Proof.** Let $f: N \to E$ be $f(n) = 2n$. We can easily see that $f$ is bijection.

**Theorem 1.8. (Cantor's Theorem) If X is any set, then there is no surjection of X onto the set P(X) of all subset of X.**

**Proof.** Let $g$ be map from set $X$ to $P(X)$ and it is surjective.

Now if $x \in X$ then $g(x) \in P(X)$ which implies that $g(x)$ is subset of X.

Therefore, either x belong to g(x) or it does not belong to g(x).

Let the set $Y = \{x \in X \ \mid x \notin g(x)\}$.

Since $Y$ is a subset of $X$, if g is surjection then $Y = g(y)$ for some $y \in X$.

Now either $y \in Y$ or $y \notin Y$.

If $y \in Y$, then as above $Y = g(y)$, then there must have $y \in g(y)$, but it contradict the definition of $Y$.

Similarly, if $y \notin Y$ then $y \notin g(y)$ which implies that $y \in Y$, a contradiction.

Therefore, $g$ cannot be a surjection.

**NOTE:** Cantor's Theorem implies that there is an unending progression of larger and larger sets. In particular, it implies that the collection $P(X)$ of all subsets of natural numbers N is uncountable.

**Ex 1.4 Let $A, B, C, D$ be sets. If $A \sim C$ and $B \sim D$, then $A \times B \sim C \times D$.**

**Proof.** By definition of equipotent, we know there exist bijections f : A → C and g : B → D. It is natural to define a function $h : A \times B \rightarrow C \times D$ by $h(a, b) = (f(b), g(c))$. Clearly it is 1-1 and onto.

**Theorem. (G.Cantor, 1874). The set $\{x \in \mathbb{R} | 0 < x \leq 1\}$ is uncountable.**

**Proof.** Let a bijection $f : N \rightarrow (0, 1]$ exists.
Listing the $f(n)$ by their nonterminating decimal expansions, we build a bi-infinite array:

$f(1) = 0. a_{11}a_{12}a_{13}a_{14}a_{15}...$
$f(2) = 0. a_{21}a_{22}a_{23}a_{24}a_{25}...$
$f(3) = 0. a_{31}a_{32}a_{33}a_{34}a_{35}...$
$f(4) = 0. a_{41}a_{42}a_{43}a_{44}a_{45}...$
$f(5) = 0. a_{51}a_{52}a_{53}a_{54}a_{55}$ ...
.....................................

Given the array we can explicitly exhibit a real number $x \in (0, 1]$ that it can't possibly include. Namely, let $x$ be the number with nonterminating decimal expansion:
$x = 0. d_1d_2d_3d_4d_5...$ where the $d_n$ are defined using the diagonal entries of the array, modified as follows:
$d_n = a_{nn} + 1$ if $a_{nn} \in \{0, 1, ..., 8\}$; $dn = 8$ if $a_{nn} = 9.5$
Here $d_n \neq 0$ for all $n$, so this nonterminating decimal expansion of the allowed kind, and defines a real number in (0, 1].
We claim that for all $n \in N$ $f(n) \neq x$, contradicting the fact that f is onto.
To see this, observe that the $n^{th}$ digits in the decimal expansion of $x$ is $d_n$, and in the expansion of f(n) is $d_{nn}$; these are different (from the construction above). This concludes the proof.

**Cardinality of a Set:** Georg Cantor (Germany, 1845-1918) helped to establish the theory of sets as a fundamental topic in modern mathematics. He provided a new way of thinking about the "size" of a set, which allows us to describe infinite sets with more nuance than just saying that they are

infinite. Instead of counting the number of elements in a set to determine its size, Cantor suggested the following definition:
Two sets $A$ and $B$ have the same cardinality if there is a one-to-one mapping between their elements; if such a mapping exists, we write $|A| = |B|$.
The symbol $\aleph_0$ (aleph-null) is **standard for the cardinal number of** $\mathbb{N}$ (sets of this cardinality are called denumerable).

**The Continuum Hypothesis:** Every uncountable subset of $\mathbb{R}$ is of the same cardinality as $\mathbb{R}$.
The cardinality of the set of all real numbers, denoted by $\mathfrak{c}$ and called the cardinality of the continuum, is strictly greater than the cardinality of the set of all natural numbers (denoted by $\aleph_0$).

**Cantor set:** We will define the Cantor set on the real line. We start with the unit interval [0,1], which we denote by $I_0$.
We remove the open interval of length 1/3 from the center of $I_0$ and we denote the remaining set by $I_1, I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
We continue the process of remoying from the center of each newly created subinterval, the open interval whose length is one-third of the subinterval to define inductively the $k^{th}$ set $I_k$
$I_k$ is a union of $2^{th}$ subintervals of length $3^{-k}$ and $\{I_k\}$ is a monotone decreasing sequence of compact sets.
The limit of this sequence $I = \bigcap_{k=1}^{\infty} I_k \neq \emptyset$ is called the Cantor set.
Since the Cantor set is a limit of nested non-empty compact sets, it is compact.

**Ex. 1.5 The Cantor Set is uncountable.**

**Sol.** To show that the Cantor set is uncountable,we assign to each element of the Cantor set a "label" consisting of a sequence of $1^s$ and $2^s$ that identifies its location in the set.
Fix an element $y$ in the Cantor set.
Then certainly $y$ is in $C_1$. If $y$ is in the left half of $C_1$,then the first digit in the "label" of $y$ is 1; otherwise it is 2.
Likewise $y \in C_2$ By the first part of this argumeht, it is either in the left half, $C_{21}$,of $C_2$ or the right half, $C_{22}$, of $C_2$ (when the first digit of the label is 2).

Whichever of these is correct,that half will consist of two intervals of length $3^{-2}$ .

If $y$ is in the leftmost of these two intervals then the second digit of the "label" of $y$ is 1. Otherwise the second digit is 2.

Continuing in this fashion, we may assign to x an infinite sequence of I's and 2's.

Conversely,if $a, b, c, \ldots$ is asequence of I's and 2's, then we may locate a unique corresponding element $z$ of the Cantor set.

If the first digit is a 1 then $z$ is in the left half of $C_1$; otherwise $z$ is in the right half of $C_1$,

Likewise the second digit locates $z$ within $C_2$ and so forth. Thus we have a one-to-one correspondence between the Cantor set and the collection of all infinite sequences of ones and twos.

In fact, we are thinking of the point assigned to the sequence $C_1, C_2, C_3, \ldots\ldots\ldots\ldots$ of I's and 2's as the limit of the points assigned to $C_1, C_2, C_3, \ldots\ldots\ldots\ldots$

Thus we are using the fact that $C$ is closed. However, the set of ail infinite sequences of ones and twos is uncountable. Thus the Cantor set is uncountable.

**CHECK YOUR PROGRESS**

(CQ 5) Set of irrational number is countable. (T/F)

(CQ 6) Infinite set always uncountable. (T/F)

(CQ 7) $\mathbb{Z} \times \mathbb{Z}$ is ________________________.

(CQ 8) Cantor set is ____________________________.

## *1.5 SUMMARY*

In this unit we discussed Sets, Algebra of Sets and Countable and Uncountable sets by proving some important theorems and giving illustrative examples.

## *1.6 GLOSSARY*

1. Set- a well defined collection of elements
2. Countable set-set having one-one onto mapping with set of Natural Number

3. Cantor set-set of points lying on a single line segment that has number of unintutive properties.
4. Lower bound-lowest possible value
5. Upper bound-largest possible value

## *1.7 REFERENCES*


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## *1.8 SUGGESTED READINGS*


1. W. Rudin (2019) Principles of Mathematical Analysis, McGraw-Hill Publishing, 1964.
2. Tom M. Apostol (1996). Mathematical Analysis (2nd edition), Narosa Book Distributors Pvt Ltd-New Delhi.
3. Pawan K. Jain and Khalil Ahmad (2005). Metric spaces, 2nd Edition, Narosa.


## *1.9 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Suppose $A$ and $B$ are countable sets.

(a) Every subset of $A$ is countable.

(b) $A \cup B$ is countable.

(c) $A \times B$ is countable.

(TQ 2) Prove that a nonempty set $T_1$ is finite iff there is a bijection from $T_1$ onto finite set $T_2$.

(TQ 3) Prove that the collection $F(\mathbb{N})$ of all finite subsets of $\mathbb{N}$ is countable.

(TQ 4) State and Prove Cantour’s theorem

(TQ 5) Is set of rational number Q is complete ordered set. Justify.

**Fill in the blanks**

(TQ 6) The unit interval [0,1] is________________________.

(TQ 7) If $I_n = \left[0, \frac{1}{n}\right]$ $for\ n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} I_n =$ _________.

(TQ 8) The set of all irrational number is_________________

(TQ 9) The union of two disjoint countably infinite set is

## *1.10 ANSWERS*

| | | |
|---|---|---|
| (CQ 1) F | (CQ 2) T | (CQ 3) T |
| (CQ 4) 0 | (CQ 5) F | (CQ 6) F |
| (CQ 7) countable | (CQ 8) uncountable | |
| (TQ 6) not countable | (TQ 7) {0} | (TQ 8) uncountable |
| (TQ 9) Countable | | |

# UNIT 2: SEQUENCES, SERIES AND PROPERTIES

**CONTENTS**



## *2.1 INTRODUCTION*

Now the foundation of the real number system $\mathbb{R}$ has been laid in previous unit, now we will study about the convergence of sequence.
First we will try to introduced the meaning of sequence nad convergence of sequence in real numbers and disussed some basic but useful results about convergent sequences i.e Nonotonic convergence theorem, Bolzanno weiesrstrass theorem and Cauchy Criterion for convergence of sequences.

A brief introduction to infinite series and some results in infinite series will be studied in this unit.

French mathematician, philosopher, and author Jean Le Rond d'Alembert was born in Paris on November 17, 1717, and he passed away there on October 29, 1783. He first rose to prominence as a mathematician and scientist before establishing a solid name as a contributor to and editor of the renowned Encyclopédie.

In a message to the Académie des Sciences in July 1739, he made his first contribution to mathematics by pointing out the mistakes he had found in Charles-René Reynaud's Analyse démontrée, which was first published in 1708.

He also developed the ratio test, a method for figuring out whether a series converges.In contemporary theoretical physics, the D'Alembert operator, which was initially introduced in D'Alembert's study of vibrating strings, is crucial.

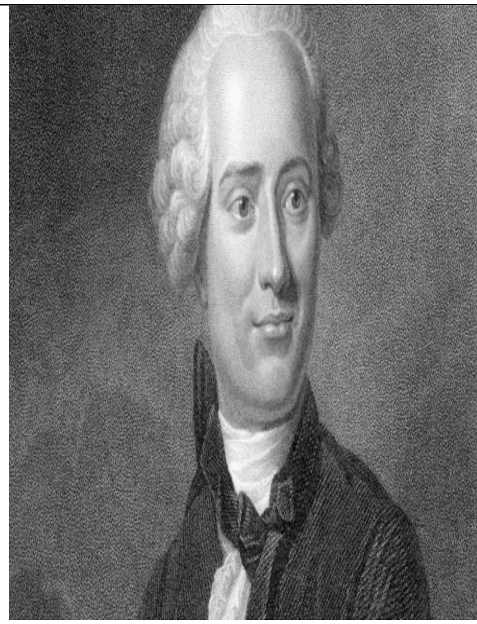


***Fig 1. Jean Le Rond d'Alembert (Source:https://www.britannica.com /biography/Jean-Le-Rond-dAlembert)***

## *2.2 OBJECTIVES*

After reading this unit learners will be able to

1. recognized the basic concept of sequences
2. construct the basic concept of limit supremum and limit infremum
3. analyze about infinite series and rearrangenet of terms in series
4. learned some important theorem like bozanno weierstrass theorem with proof.

## *2.3 SEQUENCES*

**Sequence:** A sequence of real numbers (or sequence in R) is a function defined on the set $\mathbb{N}$ of natural numbers and whose range is contained in the set R of real numbers.

If $X: \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of X at n by the symbol $x_n$ rather than using the function notation $X(n)$. The values $x_n$ are also called the terms or the elements of the sequence. We will denote this sequence by the notations $X, \{y_n\}, \{y_n: n \in \mathbb{N}\}$.
Of course, we will often use other letters, such as $X = \{x_k\}, Z = \{z_i\}$ and so on to denote sequences.

We purposely use parentheses to emphasize that the ordering induced by the natural order $\mathbb{N}$ is a matter of importance. Thus, we distinguish notationally between the sequence $\{y_n: n \in \mathbb{N}\}$ whose infinitely many terms have an ordering and the set of values $\{x_n: n \in \mathbb{N}\}$ in the range of the sequence which are not ordered. For example, the sequence $X = \{(-1)^n: n \in \mathbb{N}\}$ has infinitely many terms that alternate between -1 and 1, whereas the set of values $\{(-1)^n: n \in \mathbb{N}\}$ is equal to the set $\{-1,1\}$, which has only two elements.
Sequences are often defined by giving a formula for the $n^{th}$ term $x_n$. Frequently, it is convenient to list the terms of a sequence in order, stopping when the rule of formation seems evident. For example, we may define the sequence of reciprocals of the even numbers by writing
$X = \left\{\frac{1}{2n}: n \in \mathbb{N}\right\}$.

**Examples:**
**(a)** If $a \in \mathbb{R}$, the sequence $A = \{a, a, \dots \dots\}$ all of whose terms equal $a$, is called the **constant sequence $a$.**
**(b) The Fibonacci sequence** $F = \{f_n\}$ is given by the inductive definition
$f_1 = 1, f_2 = 1,\ \ f_{n+1} = f_{n-1} + f_n \ \ (n \geq 2)$

**Range**
The range set consisting of all distinct elements of a sequence, without repetition and without regard to all position of a term.
**Example:** The Range of sequence $\{y_n\}$, where $y_n = 1 + (-1)^n$ is $\{0,2\}$.

**<u>Bounds of a sequence</u>**

**Bounded above sequence:** A sequence $\{y_n\}$ is said to be bounded above if there exists $\alpha \in \mathbb{R}$ such that $y_n \leq \alpha \quad for\ all\ n \in \mathbb{N}$.

**Bounded below sequence:** A sequence $\{y_n\}$ is said to be bounded below if there exists $\alpha \in \mathbb{R}$ such that $y_n \geq \alpha \quad for\ all\ n \in \mathbb{N}$

**Bounded below sequence:** A sequence $\{y_n\}$ is bounded if there exists a $X \in \mathbb{R}$ such that $|y_n| \leq X$ for all $n \in \mathbb{N}$.

## *2.4 LIMIT OF A SEQUENCE*

**Converge to a point:** A sequence $X = \{x_n\}$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ and 'x' is said to be a limit of {x_n}, if for every $\in > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms $x_n$ satisfy $|x_n - x| < \varepsilon$.

**Convergent sequence:** A sequence is said to be convergent if it has a limit.

**Divergent sequence:** A sequence is said to be divergent if it has no limit.

**Theorem 2.1. (Uniqueness of Limits) A sequence in $\mathbb{R}$ can have at most one limit.**

**Proof.** Suppose that $a$ and $b$ are both limits of $\{x_n\}$.
For each $\varepsilon > 0$ there exists $K_0$ such that
$|x_n - a| < \frac{\varepsilon}{2}$ for all $n \geq K_0$ ……………………………………………(1)
and there exists $K_0'$ such that
$|x_n - b| < \frac{\varepsilon}{2}$ for all $n \geq K_0'$. ……………………………………………(2)
Now we consider $K$ such that $K = sup\ \{K_0, K_0'\}$.
Therefore for any $n \geq K$, we apply the triangle inequality to get
$|a - b| = |a - x_n + x_n - b| \leq |x_n - a| + |x_n - b|$
Using (1) and (2), we get
$|a - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
because $\varepsilon > 0$ is an arbitrary positive number, Hence $|a - b| = 0 \Rightarrow a = b$.

**Theorem 2.2. Let $\boldsymbol{X = \{x_n\}}$ be a sequence of real numbers and let $\boldsymbol{x} \in \mathbb{R}$. The following statements are equivalent**

(i) **$\boldsymbol{X}$ converges to $\boldsymbol{a}$.**

(ii) **For every $\boldsymbol{\varepsilon > 0}$, there exists a natural number $\boldsymbol{K}$ such that for all $\boldsymbol{n \geq K}$, the terms $\boldsymbol{x_n}$ satisfy $\boldsymbol{|x_n - a| < \varepsilon}$.**

(iii) **For every $\boldsymbol{\varepsilon > 0}$, there exists a natural number K such that for all $\boldsymbol{n \geq K}$, the terms $\boldsymbol{x_n}$ satisfy $\boldsymbol{a - \varepsilon < x_n < a + \varepsilon}$.**

(iv) **For every $\varepsilon$ −neighborhood $V_{\varepsilon}(x)$ of $a$, there exists a natural number $K$ such that for all $n \geq K$, the terms $x_n$ belong to $V_{\varepsilon}(x)$ .**

**Proof.** The equivalence of (i) and (ii) is just the definition.
The equivalence of (ii), (iii) and (iv) follows from the following implications:
$|v-a| < \varepsilon \; iff - \varepsilon < v - x < \varepsilon \; iff \; a - \varepsilon < a < a + \varepsilon \; iff \; v \in V_{\varepsilon}(x)$.

**Ex. 2.1. Prove that $lim_{n\to\infty} \left\{\frac{1}{n}\right\} = 0$.**

**Proof.** If $\varepsilon > 0$ is given then $\frac{1}{\varepsilon} > 0$.

Now applying Archimedean Property, we get

There is a natural number $K = K(\varepsilon)$ such that $\frac{1}{K} < \varepsilon$.

Then, if $n \geq K$, then

$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$.

Hence sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

**Ex. 2.2. Prove that $lim_{n\to\infty} \left\{\frac{4n+5}{n+1}\right\} = 4$.**

**Proof.** If $\varepsilon > 0$ is given then we will prove

$\left|\frac{4n+5}{n+1} - 4\right| < \varepsilon$

……………………………………………………………….(1)

$\left|\frac{4n+5-4n-4}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}$ .

Now if the inequality $\frac{1}{n} < \varepsilon$ is satisfied, then the inequality (1) holds.

Thus if $\frac{1}{K} < \varepsilon$, then for any $n \geq K$, we also have $\frac{1}{n} < \varepsilon$ and hence (1) holds.

Therefore the sequence $lim_{n\to\infty} \left\{\frac{4n+5}{n+1}\right\} = 4$

**Ex. 2.3. The sequence $\{(-1)^n\}$ is divergent.**

**Proof:** Let $y$ be the limit of given sequence, then for $\varepsilon = \frac{1}{2}$, $\boldsymbol{\varepsilon}$ satisfies the definition.

Suppose there exists $\boldsymbol{\varepsilon}$ such that for an even $n \geq \boldsymbol{\varepsilon}$.

We get $\frac{1}{2} > |y_n - y| = |1 - y|$ and

$\frac{1}{2} > |y_{n+1} - y| = |-1 - y|$.

But 2 $= |1 - y - (-1 - y)| \leq |1 - y| + |-1 - y|$

$< 1/2 + 1/2 = 1$, and that is a contradiction.

Therefore, $\{(-1)^n\}$ is divergent.

**CHECK YOUR PROGRESS**

(CQ 1) The sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$ is divergent. (T/F)

(CQ 2) $\lim_{n\to\infty} \left\{\frac{3n+7}{n+2}\right\} = 3$. (T/F)

(CQ 3) A sequence in $\mathbb{R}$ can have at most one limit. (T/F)

(CQ 4) $\lim_{x\to\infty} \sin\frac{1}{x} =$ ____________ .

(CQ 5) $\lim_{x\to\infty} \frac{\log x}{x} =$ ____________ .

## 2.5 MONOTONE SEQUENCES AND SUBSEQUENCES

**Monotone increasing**: A sequence $\{y_n\}$ is said to be monotone increasing if $y_n \leq y_{n+1}$ for all $n \in N$.

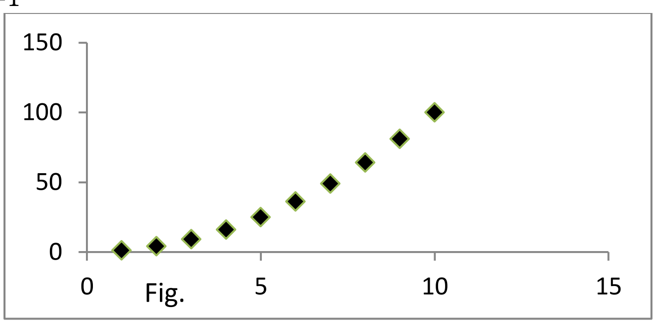
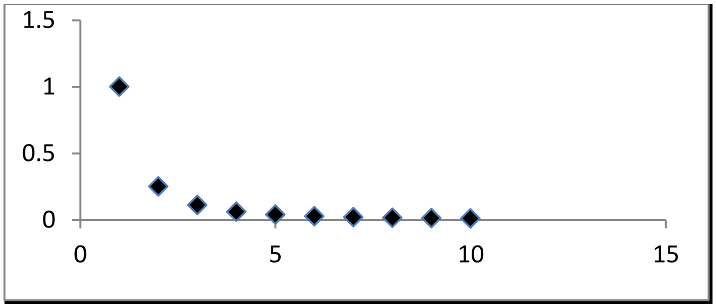


Fig.

**Monotone decreasing**: A sequence $\{y_n\}$ is monotone decreasing if $y_n \geq y_{n+1}$ for all $n \in N$.




If a sequence is either monotone increasing or monotone decreasing, we can simply say the sequence is monotone.

**Example:**

1. $y_n = \{n^2;\ n \geq 1\}$, is monotone increasing
2. $y_n = \left\{\frac{1}{n^2}; n \geq 1\right\}$ is monotone decreasing,
3. the constant sequence $y_n = \{1\}$ is both monotone increasing and monotone decreasing,
4. $\{(-1)^n\}$ is not monotone.

**Theorem 2.3 A monotone sequence $\{y_n\}$ is bounded if and only if it is convergent. Also**

**(i) if $\{y_n\}$ is monotone increasing and bounded, then $\lim_{n\to\infty} y_n = \sup\{y_n : n \in N\}$.**

**(ii) If $\{y_n\}$ is monotone decreasing and bounded, then $\lim_{n\to\infty} y_n = \inf\{y_n : n \in N\}$.**

**(iii)**

**Proof.** Suppose a monotone increasing sequence $\{y_n\}$.
Let the sequence is bounded ⇒the set $\{y_n : n \in N\}$ is bounded.
Suppose $y = \sup\{y_n : n \in N\}$.
Let $k > 0$ be an arbitrary. As $y$ be the supremum, then there exists $m \in N$ such that $y_m > y - k$.
As $\{y_n\}$ is monotone increasing, then by mathematical induction we get $y_n > y_m$ for all $n \geq m$.
$|y_n - y| = y - y_n \leq y - y_m < k$.
Therefore, the sequence converges to $y$.
Hence, bounded monotone increasing sequence converges.
For the other direction, we have already proved that a convergent sequence is bounded.

**Tail of a sequence**

Let $\{y_n\}$ be a sequence then $\lambda$ −tail (where $\lambda \in N$), or just the tail, of $\{y_n\}$, is defined as the sequence starting at $\lambda + 1$. It can be written as$\{y_{n+\lambda}\}_{n=1}^{\infty}$

For example, the 3-tail of $\left\{\frac{1}{n^2}\right\}$ is $\frac{1}{16}, \frac{1}{25}, \frac{1}{49}, \frac{1}{64}, \ldots$..

**NOTE:** The 0-tail of a sequence is the sequence itself. The the limit and convergence of a sequence depends only on its tail.

**Subsequences**

Consider $\{y_n\}$ be a sequence and $\{n_k\}$ be a strictly increasing sequence of natural numbers, i.e., $n_k < n_k + 1$ for all i or we can say $n_1 < n_2 < n_3 < \cdots$). The sequence $\{y_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{y_n\}$.

**Example:** $\left\{\frac{1}{2n}\right\}$ is a subsequence of $\left\{\frac{1}{n}\right\}$

**Theorem 2.4. Every subsequence of convergent sequence is also convergent and convergent to the same limit as the sequence.**

**Proof.** Consider $\{y_n\}$ be a convergent sequence, and $\{z_n\}$ is any subsequence of $\{y_n\}$.

Now we will prove $\lim_{n\to\infty} y_n = \lim_{k\to\infty} z_n$.

Let $\lim_{n\to\infty} y_n = y$. Now by the definition of convergence, for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$|y_n - y| < \varepsilon$, for all $n \geq M$.

Now if $n \geq M$ then $z_n = y_m$ for some $m \geq n \geq N$.

It implies that $|z_n - y| = |y_m - y| < \varepsilon$.

Hence $|z_n - y| < \varepsilon$ for all $n \geq M$.

Therefore, $\lim_{n\to\infty} y_n = \lim_{k\to\infty} z_n = y$ .

**CHECK YOUR PROGRESS**

(CQ 6) $y_n = \left\{(\frac{n}{2})^2;\ n \geq 1\right\}$, is monotone increasing. (T/F)

(CQ 7) $y_n = \{n^2; n \geq 1\}$ is monotone decreasing. (T/F)

(CQ 8) Every subsequence of convergent sequence need not be convergent. (T/F)

(CQ 9) A monotone sequence $\{y_n\}$ is bounded iff it is ___________.

## *2.6 LIMIT AND INEQUALITIES*

**Theorem 2.5. (Squeeze lemma). If $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ converge and $\lim_{n\to\infty} x_n = \lim_{k\to\infty} z_n$. Then $\{y_n\}$ converges and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{k\to\infty} z_n$.**

**Proof.** Suppose $\lim_{n\to\infty} x_n = \lim_{k\to\infty} z_n = l$..

Let ε > 0 and there exists $N_1$ such that $|x_n - l| < \varepsilon$ for all $n \geq N_1$, and there exists $N_2$ such that $|z_n - l| < \varepsilon$ for all $n \geq N_1$,

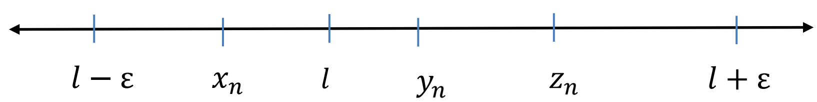
Now let $N = \max\,\{N_1, N_2\}$

As $n \geq N_1$ and $n \geq N_1$, hence for all $n \geq N$

$|x_n - l| < \varepsilon \Rightarrow -\varepsilon < x_n - l < \varepsilon \Rightarrow \mathrm{l} - \varepsilon < x_n < l + \varepsilon$ and

Similarly, $\mathrm{l} - \varepsilon < z_n < l + \varepsilon$.

Now, $x_n \leq y_n \leq z_n \Rightarrow \mathrm{l} - \varepsilon < x_n \leq y_n \leq z_n < l + \varepsilon$.




Hence, $|y_n - l| < \varepsilon$ for all $n \geq N$.

Therefore, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{k\to\infty} z_n$

**Ex. 2.4. Find $\lim_{n\to\infty} \frac{1}{n\sqrt{n}}$**

Proof. The given equation is $\left\{\frac{1}{n\sqrt{n}}\right\}$

Now as $\sqrt{n} \geq 1\ for\ all\ n \in \mathbb{N}$, we have

$0 \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}\ for\ all\ n \in \mathbb{N}$

We already know that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore, using squeeze lemma with constant sequence $\{0\}$ and the sequence $\left\{\frac{1}{n}\right\}$, we get

$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$

**CHECK YOUR PROGRESS**

(CQ 10) $\lim_{x\to 0} x^2 e^{\sin\frac{1}{x}} = 1$. (T/F)

(CQ 11) $\lim_{x\to\infty} \frac{[ax+b]}{x} = a$. (T/F)

(CQ 12) $\lim_{n\to 0} n^2 \sin\frac{1}{n} =$ ____________ .

## 2.6.1 ALGEBRIC OPERATION IN LIMITS

**Theorem 2.6. Consider $\{y_n\}$ and $\{z_n\}$ be convergent sequences.Then**

**(i)** **Sequence $\{x_n\}$ such that $x_n = y_n + z_n$ converges and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n + \lim_{n\to\infty} z_n$**

**(ii)** **sequence $\{x_n\}$ such that $x_n = y_n - z_n$ converges and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n - \lim_{n\to\infty} z_n$**

**(iii)** **sequence $\{x_n\}$ such that $x_n = y_n z_n$ converges and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \,.\, \lim_{n\to\infty} z_n$**

**(iv)** **sequence $\{x_n\}$ converges if $x_n = \frac{y_n}{z_n}$ such that $z_n \neq 0$ and $\lim_{n\to\infty} x_n = \frac{\lim_{n\to\infty} y_n}{\lim_{n\to\infty} z_n}$**

**Proof.** (i) Let $\{y_n\}$ and $\{z_n\}$ be convergent sequences and $\{x_n\}$ is a sequence such that $x_n = y_n + z_n$.

Let $\lim_{n\to\infty} y_n = l_1$ and $\lim_{n\to\infty} z_n = l_2$ and $l = l_1 + l_2$

Let $\varepsilon > 0$ and there exist $N_1$ such that

$|y_n - l_1| < \frac{\varepsilon}{2}, for\ all\ n \geq N_1$……………………………….(i)

Similarly

there exist $N_2$ such that

$|z_n - l_2| < \frac{\varepsilon}{2}, for\ all\ n \geq N_2$…………………………………..(ii)

Now we choose $N$ such that $N = \{N_1, N_2\}$.

Therefore, for all $n \geq N$

$$|x_n - l| = |(y_n + z_n) - (l_1 + l_2)| = |(y_n - l_1) + (z_n - l_2)|$$
$$\leq |y_n - l_1| + |z_n - l_2|$$

Using (i) and (ii), we get

$$|x_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Hence we conclude that

$|x_n - l| < \varepsilon$, for all $n \geq N$ or we can say that $x_n$ converges to $l$.

i.e. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n + \lim_{n\to\infty} z_n$

(ii) Similarly, $\{x_n\}$ is a sequence such that $x_n = y_n - z_n$, and $l = l_1 + l_2$

Let $\varepsilon > 0$ and for all $n \geq N$

$|x_n - l| = |(y_n - z_n) - (l_1 - l_2)| = |(y_n - l_1) + (l_2 - z_n)|$

$\leq |y_n - l_1| + |l_2 - z_n| \leq |y_n - l_1| + |z_n - l_2|$

Using (i) and (ii), we get

$|x_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

Hence we conclude that

$|x_n - l| < \varepsilon$, for all $n \geq N$ or we can say that $x_n$ converges to $l$.

i.e. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n - \lim_{n\to\infty} z_n$

(iii) Let $\{x_n\}$ is a sequence such that $x_n = y_n z_n$, and $l = l_1 l_2$

Now we take $\varepsilon > 0$ and choose $N$ such that $K = \max\ \{|l_1|, |l_2|, \frac{\varepsilon}{3}, 1\}$ .

Now there exist $N_1$ such that

$|y_n - l_1| < \frac{\varepsilon}{3K}, for\ all\ n \geq N_1$……………………………….(iii)

Similarly, there exist $N_2$ such that

$|z_n - l_2| < \frac{\varepsilon}{3K}, for\ all\ n \geq N_2$…………………………………....(iv)

Let $N = \{N_1, N_2\}$ and $for\ all\ n \geq N$

$|x_n - l| = |y_n z_n - l_1 l_2| = |(y_n - l_1 + l_1)(z_n - l_2 + l_2) - l_1 l_2|$

$= |(y_n - l_1)(z_n - l_2) + (y_n - l_1)l_2 +$

$l_1(z_n - l_2) + l_1 l_2 - l_1 l_2|$

$= |(y_n - l_1)l_2 + l_1(z_n - l_2) + (y_n - l_1)(z_n - l_2)|$

$\leq |(y_n - l_1)l_2| + |l_1(z_n - l_2)| + |(y_n - l_1)(z_n - l_2)|$

$\leq |(y_n - l_1)||l_2| + |l_1||(z_n - l_2)| + |(y_n - l_1)||(z_n - l_2)|$

Using (iii) and (iv), we get

$|x_n - l| < \frac{\varepsilon}{3K}|l_2| + |l_1|\frac{\varepsilon}{3K} + \frac{\varepsilon}{3K}|\frac{\varepsilon}{3K}$

$< \frac{\varepsilon}{3K}K + K\frac{\varepsilon}{3K} + \frac{\varepsilon}{3K}K$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$

Therefore, $|x_n - l| < \varepsilon$, for all $n \geq N$ or we can say that $x_n$ converges to $l$.

i.e. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \lim_{n\to\infty} z_n$

(iv) First, we prove that if $\{z_n\}$ is convergent sequence, $\lim_{n\to\infty} z_n = l_2 \neq 0$ and $z_n \neq 0$ then for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{l_2}$

Let $\varepsilon > 0$ and as $l_2 \neq 0 \Rightarrow \min\left\{\frac{\varepsilon}{2}|y|^2, \frac{|y|}{2}\right\}$

Now we choose $N$ such that

$|z_n - l_2| < \min\left\{\frac{\varepsilon}{2}|l_2|^2, \frac{|l_2|}{2}\right\}$ …………………………………………..(v)

Now

$|l_2| = |l_2 - z_n + z_n| \leq |l_2 - z_n| + |z_n|$

$\Rightarrow |l_2| < \frac{|l_2|}{2} + |z_n| \Rightarrow |l_2| - \frac{|z_n|}{2} < |y_n| \Rightarrow \frac{|l_2|}{2} < |z_n| \Rightarrow \frac{2}{|l_2|} > \frac{1}{|z_n|}$

Now, for all $n \geq N$

$\left|\frac{1}{z_n} - \frac{1}{l_2}\right| = \frac{|l_2 - z_n|}{|z_n||l_2|} < \frac{2}{|l_2|}\frac{|l_2 - z_n|}{|l_2|}$

Using (v) in above equation , we get

$\left|\frac{1}{z_n} - \frac{1}{l_2}\right| < \frac{2}{|l_2|}\frac{\frac{|l_2|^2\varepsilon}{2}}{|l_2|} < \varepsilon$

Therefore, $\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{l_2}$

Hence using property (iii), we get

sequence $\{x_n\}$ converges, if $x_n = \frac{y_n}{z_n}$ such that $z_n \neq 0$ and $\lim_{n\to\infty} x_n = \frac{\lim_{n\to\infty} y_n}{\lim_{n\to\infty} z_n}$

**NOTE:**

If $c \in R$ and $\{x_n\}$ is a convergent sequence, then $\lim_{n\to\infty}(c\, y_n) = c \lim_{n\to\infty} y_n$

**Theorem 2.7. (the Bolzano-Weierstrass Theorem) Every bounded sequence of real numbershas a convergent subsequence.**

**Proof.** Consider a a bounded sequence of real numbers $\{y_n\}$.

Let $\alpha > 0$ such that $|y_n| < \alpha$ for all $n$. Now we define $S_n = closure\{y_j | \, j \geq n\} \; for \, n \in \mathbb{N}$. Then $S_n \subseteq [-\alpha, \alpha]$ and $S_n$ is closed. (because closure of a set is closed)

Thus $\{S_n\}$ is a descending sequence of nonempty closed bounded subsets of $\mathbb{R}$.

Therefore $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$ (Nested Set Theorem)

Let $\in \bigcap_{n=1}^{\infty} S_n$ . For each $k \in \mathbb{N}$, $y$ is a point of closure of $\{ y_i |\, i \geq \beta \}$.

Therefore, for infinitely many $i > n$, $y_i$ belongs to $(y - \frac{1}{\beta}, y + \frac{1}{\beta})$.

Now we inductively choose a strictly increasing sequence of natural numbers $\{n_\beta\}$ such that

$\left|y - y_{n_\beta}\right| < \frac{1}{\beta}$ for all $\beta$.

The subsequence $y_{n_\beta}$ converges to $y$. (By Archimedean Property of $\mathbb{R}$)

## 2.6.2 CONVERGENCE TESTS

**Cauchy sequence:** A sequence of real numbers $\{y_n\}$ is said to be Cauchy if for each $\varepsilon > 0$, there is an index $\mathbb{N}$ for which if $n, m \geq \mathbb{N}$, then $|y_n - y_m| < \varepsilon$.

**Cauchy Convergence Criterion for Real Sequences**

**Theorem 2.8. A sequence of real numbers converges iff it is Cauchy.**

**Proof.** Let sequence $\{y_n\}$ converges to $y$.

Now consider $\epsilon > 0$. Because $\{y_n\}$ converges to $y$, we can choose a $N \in \mathbb{N}$ such that

$|y_n - y| < \frac{\epsilon}{2} \quad for\ all\ \ n \geq N$ ……………(1)

Now for all $n, m \in \mathbb{N}\ and\ \ n, m \geq N$

$$|y_n - y_m| = |(y_n - y) + (y - y_m)| \leq |y_n - y| + |y_m - y|$$

Using (1), we get

$|y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \quad \forall\ n, m \geq N$ …………………….(2)

Therefore, $\{y_n\}$ is a Cauchy sequence.

*Converse*

Consider $\{y_n\}$ be a Cauchy sequence. Now first we try to prove that it is bounded.

For $\epsilon = 1$, we may choose N such that

$|y_n - y_m| < 1 \quad for\ all\ \ n, m \geq N$ …………………………….(3)

Now $for\ all\ \ n \geq N$

$|y_n| = |(y_n - y_N) + y_N| \leq |y_n - y_n| + |y_N| < 1 + |y_N|$

Let $\alpha = 1 + \max\{|a_1|, |a_2|, \dots \dots \dots |a_n|\}$

Thus, $|y_n| < \alpha$ for all n.

Hence , $\{y_n\}$ is bounded sequence.

Now Bolzano-Weierstrass Theorem state that every bounded sequence of real numbers has a convergent subsequence

Therefore there exists a subsequence $\{y_{n_k}\}$ converges to $y$.

It implies that we can choose a natural number $N$ such that

$|y_{n_k} - y| < \frac{\epsilon}{2} \quad if\ \ n_k \geq N$ ……………………………..(4)

As we know $\{y_n\}$ is Cauchy sequence. Let $\epsilon > 0$ and we can choose a natural number $N$ such that

$|y_n - y_m| < \frac{\epsilon}{2} \quad if\ \ n, m \geq N$ ……………………………..(5)

Therefore, for all $n \geq N$

$|y_n - y| = |(y_n - y_{n_k}) + (y_{n_k} - y)| \leq |y_n - y_{n_k}| + |y_{n_k} - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$

Hence, $\{y_n\}$ is convergent sequence.

**Convergence test**

**Theorem 2.9. Consider a sequence $\{y_n\}$ and suppose $\{z_n\}$ is a converegent sequence such that**
**$\lim_{n\to\infty} z_n = 0$ and $y \in \mathbb{R}$ such that $|y_n - y| \le z_n \quad for\ all\ n \in \mathbb{N}$. Then $\{y_n\}$ coverges to $y$. i.e. $\lim_{n\to\infty} y_n = y$**

**Proof.** It is given that $|y_n - y| \le z_n\ for\ all\ n \in \mathbb{N}$.
Therefore $z_n \ge 0$ for all n.
Let ε > 0 and we choose $N$ such that
$z_n = |z_n - 0| < \varepsilon\ for\ all\ n \ge N$
Hence $|y_n - y| \le z_n < \varepsilon\ \ for\ all\ n \ge N$
Therefore $\{y_n\}$ coverges to $y$.

**Theorem 2.10.Let $y > 0$ and**
**(i) If $y < 1$, then $\lim_{n\to\infty} y^n = 0$**
**(ii) If $y > 1$, then sequence $\{y^n\}$ is unbounded.**

**Proof.** (i) It is given that $y > 0$ implies $y^n > 0\ for\ all\ n\ in\ \mathbb{N}$.
Let $y < 1$, Using induction we get $y^{n+1} < y^n\ for\ all\ n$ .
Which implies $\{y^n\}$ is decreasing function ⇒ $\{y^n\}$ is bounded below⇒ $\{y^n\}$ is convergent.
Let $l = \lim_{n\to\infty} y^n$. Then 1-tail $\{y^{n+1}\}$ is also converges to $l$. i.e. $\lim_{n\to\infty} y^{n+1} = l$
Now $y^{n+1} = y^n . y$
Taking limit of both sides we get
$\lim_{n\to\infty} y^{n+1} = \lim_{n\to\infty} y^n . y \Rightarrow \lim_{n\to\infty} y^{n+1} = y\ \lim_{n\to\infty} y^n$
Therefore $l = yl$
$\Rightarrow l - yl = 0 \Rightarrow l(1-y) = 0 \Rightarrow l = 0\ or\ 1 - y = 0$
$\Rightarrow l = 0$ (because $y \ne 1$)
Thus we proved that $\lim_{n\to\infty} y^n = 0$
(ii) Let $y > 1$ and $\alpha > 0$ be an arbitrary.
Now $\frac{1}{y} < 1$ , then sequence $\{\left(\frac{1}{y}\right)^n\}$ converges to 0.
Therefore $\frac{1}{y^n} = \left(\frac{1}{y}\right)^n < \frac{1}{\alpha}$ for some large value of $n$
$\Rightarrow y^n > \alpha$, therefore $\alpha$ is not an upper bound for $\{y^n\}$.
Therefore $\{y^n\}$ is unbounded (as $\alpha$ is arbitrary)

**Theorem 2.11. (Ratio test for sequences) Let $\{y_n\}$ be a sequence such that $y_n \neq 0\ \ \forall n$ and $y = \lim_{n\to\infty} \frac{|y_{n+1}|}{|y_n|}$ exists. If**

- **(i) If $y < 1$, then $\lim_{n\to\infty} y_n = 0$**
- **(ii) If $y > 1$, then sequence $\{y_n\}$ is unbounded.**
- **(iii)**

**Proof.** Let $y < 1$. As $\frac{|y_{n+1}|}{|y_n|} \geq 0\ \forall n$ then $y \geq 0$.

Assume $\alpha$ such that $y < \alpha < 1$ and it is given that $y = \lim_{n\to\infty} \frac{|y_{n+1}|}{|y_n|}$

Now $\alpha - y > 0$ and there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$\left|\frac{|y_{n+1}|}{|y_n|} - y\right| < \alpha - y.$

Hence $\frac{|y_{n+1}|}{|y_n|} - y < \alpha - y \Rightarrow \frac{|y_{n+1}|}{|y_n|} < \alpha \quad \forall n \geq N$

Now for $n > N$ $(n >= N + 1)$

$|y_n| = |y_N|.\frac{|y_{N+1}|}{|y_N|}.\frac{|y_{N+2}|}{|y_{N+1}|}.\frac{|y_{N+3}|}{|y_{N+2}|} \dots\dots\dots\dots\dots\dots\dots\dots.\frac{|y_n|}{|y_{n-1}|}$

$< |y_N|.\alpha.\alpha.\alpha \dots\dots..\alpha = |y_N|\alpha^{n-M} = (|y_M|\alpha^{-M})\alpha^n.$

$\{\alpha^n\}$ converges to 0 (If $0 < y < 1$, then $\lim_{n\to\infty} y^n = 0$)

$\Rightarrow |y_N|\alpha^M\alpha^n$ converges to zero.

Therefore M-tail of $\{y_n\}$ converges to 0 $\Rightarrow \{y_n\}$ converges to 0.

Now let y > 1 and we choose $\alpha$ such that 1 < $\alpha$ < L. As y− $\alpha$ > 0, there exists an N ∈ ℕ such that $\left|\frac{|y_{n+1}|}{|y_n|} - y\ \right| < y - \alpha \quad for\ all\ n \geq$ N.

Hence $-(y - \alpha) < \frac{|y_{n+1}|}{|y_n|} - y < y - \alpha \Rightarrow -(y - \alpha) < \frac{|y_{n+1}|}{|y_n|} - y$

$\Rightarrow \frac{|y_{n+1}|}{|y_n|} > \alpha$

Again for $n > N$, we can write

$$|y_n| = |y_N|.\frac{|y_{N+1}|}{|y_N|}.\frac{|y_{N+2}|}{|y_{N+1}|}.\frac{|y_{N+3}|}{|y_{N+2}|} \dots\dots\dots\dots\dots\dots\dots\dots.\frac{|y_n|}{|y_{n-1}|}$$

$$< |y_N|.\alpha.\alpha.\alpha \dots\dots..\alpha = |y_N|\alpha^{n-N} = (|y_N|\alpha^{-N})\alpha^n$$

$\Rightarrow \{\alpha^n\}$ is unbounded (As $\alpha > 1$)

Let $\{y_n\}$ is bounded $\Rightarrow$ $L > 0$ such that $|y_n| \leq L$ for all $n$ then $(|y_M|\alpha^{-M})\alpha^n \leq$ L

$\Rightarrow \alpha^n \leq \frac{\text{L}}{|y_M|\alpha^{-M}}$, a contradiction (As $\alpha^n$ is unbounded)

$\Rightarrow \{y_n\}$ is unbounded.

Therefore $\{y_n\}$ cannot be converges.

**Ex. 2.5. Prove that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$**

**Proof. .** Now $\frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{2^n}.\frac{n!}{(n+1)!} = \frac{2}{n+1}$

And we already know that $\left\{\frac{2}{n+1}\right\}$ converges to 0.

Therefore $\lim_{n\to\infty}\frac{2^n}{n!}=0$

**Limit superior and limit inferior**

**Limit superior:** Let $\{y_n\}$ be a sequence of real numbers. The limit superior of $\{y_n\}$, denoted by $\limsup\{y_n\}$ and defined by
$\limsup\{y_n\}=\lim_{n\to\infty}[\sup\{\,y_k \mid k\geq n\}]$

**Limit inferior:** Let $\{y_n\}$ be a sequence of real numbers. The limit inferior of $\{y_n\}$, denoted by lim sup{y_n } and defined by
$\liminf\{y_n\}=\lim_{n\to\infty}[\inf\{\,y_k \mid k\geq n\}]$

**Ex.2.6. Find Limit superior, limit inferior of sequence $\{\boldsymbol{y_n=1/n}\ |\boldsymbol{n}\in\mathbb{N}\}$.**

**Sol.** Given $y_n=1/n\ for\ all\ n\in\mathbb{N}$

If $n=1$, then we get highest value i.e. $\sup y_n\ =1$ therefore $\limsup\{y_n\}=1$

Now if $n$ gets largear then $\frac{1}{n}$ gets smaller or we can say when we increased the value of n y-n approaching to 0. Hence $\liminf\{y_n\}=0$.

**Ex.2.7. Find Limit superior, limit inferior of sequence**

$\{\boldsymbol{y_n=n}\ |\boldsymbol{n}\in\mathbb{N}\}$**.**

Sol. Given $\boldsymbol{y_n=n}\ for\ all\ n\in\mathbb{N}$

Let $\underline{Y_n}=\inf\{y_n,y_{n+1},\dots\dots\dots.\}=$ n and

$\overline{Y_n}=\inf\{y_n,y_{n+1},\dots\dots\dots.\}=\infty$

Therefore

$\liminf\{y_n\}=\infty$ and $\limsup\{y_n\}=\infty$

**CHECK YOUR PROGRESS**

(CQ 13) Limit superior $\{y_n=2n\ |n\in\mathbb{N}\}$ is 2. (T/F)

(CQ 14) Let $y>0$ and If $y<1$, then $\lim_{n\to\infty}y^n=0$. (T/F)

(CQ 15) $\lim_{n\to\infty}\frac{3}{2n}+\sin\frac{1}{n}$=________________________.

## *2.7 SERIES*

The sum of the terms of a sequence is said to be a **series**. Thus if $y_1, y_2, y_3, ....$is a sequence then the sum $y_1 + y_2 + \ y_3 + \cdots$ of all the terms is called an infinite series and is expressed by $\sum_{n=1}^{\infty} y_n$ or $\sum y_n$.

Evidently we cannot just add up all the infinite number of terms of the series in ordinary way and in fact it is not obvious that this kind of sum has any meaning. Therefore we start by associating with the given series, a sequence $\{S_n\}$, where $S_n$ denotes the sum of the first $n$ terms of the series.

Hence $S_n = y_1 + y_2 + \cdots . + y_n \qquad \forall n$

and this sequence $\{S_n\}$ is said to be the sequence of partial sums of the series.

The **partial sums**

$\boldsymbol{S_1 = y_1; \ S_2 = y_1 + y_2; \ S_3 = y_1 + y_2 + y_3 + \cdots ..}$ and so on.

The series is **convergent** if the sequence $\{S_n\}$ of partial sums converges and lim $S_n$ is called the sum of the series.

If $\{S_n\}$ does not tend to a limit then the sum of the infinite series does not exist or we can say that the series does not converges.

An infinite series is converge, diverge or oscillate according as its sequence of partial sums $\{S_n\}$ converges, diverges and oscillates.

N**ecessary condition of convergences of an infinite series**

**Theorem A** N**ecessary condition of convergences of an infinite series** $\boldsymbol{\sum y_n}$ **is** $\boldsymbol{\lim_{n\to\infty} y_n = 0}$.

**Proof.** Let $S_n = y_1 + y_2 + \cdots \dots \dots . + y_n$, so that $\{S_n\}$ is the sequence of partial sums.

It is given that series is converges

Thus, the sequence $\{S_n\}$ is also converges.

Let $\lim_{n\to\infty} S_n = t$. Now $y_n = S_n - S_{n-1}, \quad n > 1$.

Therefore, $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} (S_n - S_{n-1}) = \lim\limits_{n\to\infty} S_n - \lim\limits_{n\to\infty} S_{n-1} = t - t = 0$

Thus $\lim_{n\to\infty} y_n = 0$.

**NOTE:**

A series cannot converges if $n^{th}$ term does not tends to zero.

**<u>Cauchy's General Principle of Convergence for Series</u>**

**Theorem 2.12. A necessary and sufficient condition for the convergence of an infinite series $\sum y_n$ is that the sequence of its partial sums $\{S_n\}$ is convergent**

**Or**

**An infinite series $\sum y_n$ converges iff for every $\varepsilon > 0$ there exists a positive integer M such that $|y_1 + y_2 + Y_3 + \cdots \ldots + y_n| < \varepsilon$ whenever $m \geq n \geq M$**

**Proof.** Let $S_n = \sum y_n = y_1 + y_2 + y_3 + \cdots \ldots + y_n$ and $S_m = \sum y_m = y_1 + y_2 + y_3 + \cdots \ldots + y_m$ be the $n^{th}$ and $m^{th}$ partial sum of series respectively, where $m \geq n$.

$\Rightarrow |S_m - S_n| = |(y_1 + y_2 + y_3 + \cdots + y_m) - (y_1 + y_2 + y_3 + \cdots + y_n)|$

$= |y_{m+1} + y_{m+2} + \cdots + y_n|$ .

Let $\varepsilon > 0$ and for every $\varepsilon$ the series $\sum y_n$ converges iff the sequence of partial sums {S_n} converges

$\Leftrightarrow |S_m - S_n| < \varepsilon \quad \forall\, m \geq n\, for\ some\ M \in \mathbb{N}$

$\Leftrightarrow |y_{m+1} + y_{m+2} + \cdots + y_n| < \varepsilon \quad \forall\, m \geq n\, for\ some\ M \in \mathbb{N}$.

**Ex. 2.8. Prove that $\sum \frac{1}{n}$ does not converge.**

**Proof.** Let the given series be converges.

Therefore, for any given $\varepsilon > 0$, there exists a positive integer $m$ such that

$\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right| < \varepsilon \quad \forall\, n \geq m\ and\ p \geq 1$.

If $n = m$ and $p = m$, we get

$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+m}$

$= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$

$> m.\frac{1}{2m} > \frac{1}{2} > \varepsilon$

i.e. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} > \varepsilon$, a contradiction.

Therefore $\sum \frac{1}{n}$ does not converge

**NOTE:**

We can see that $lim_{n\to\infty}\left(\frac{i}{n}\right) = 0$ but $\sum \frac{1}{n}$ does not converge.

If $\sum y_n = y$ then $\sum c y_n = cy$ independent of n.

**Ex. 2.9. If $y_n > 0$ and $\sum y_n$ is convergent with the sum $S$, then prove that $\frac{y_n}{y_1+y_2+\cdots+y_n} < \frac{2y_n}{S}$, when $n$ is sufficiently large. Also prove that $\sum \frac{y_n}{y_1+y_2+\cdots+y_n}$ is convergent.**

**Proof.** It is given that $\sum y_n$ is convergent with the sum S.

Hence for $\varepsilon > 0 \ \ \exists\, m \in \mathbb{Z}^+$

$|S_n - S| < \varepsilon \ \forall\, n \geq m$ where $S_n = y_1 + y_2 + \cdots + y_n$,

or $\varepsilon < S_n - S < \varepsilon \Rightarrow S - \varepsilon < S_n < S + \varepsilon, \forall\, n \geq m$

For $\varepsilon = \frac{1}{2}S > 0$

$S - \frac{1}{2}S < S_n < S + \frac{1}{2}S \Rightarrow \frac{S}{2} < S_n < \frac{3S}{2} \Rightarrow \frac{2}{S} > \frac{1}{S_n} > \frac{2}{3S} \ \ , \forall\, n \geq m$

or $\ \frac{2}{S} > \frac{1}{S_n}, \forall\, n \geq m \Rightarrow \frac{2y_n}{S} > \frac{y_n}{S_n}, \forall\, n \geq m$ .

Now , $\forall\, n \geq m, p \geq 1$

$\frac{y_{n+1}}{S_{n+1}} + \frac{y_{n+2}}{S_{n+2}} + \frac{y_{n+3}}{S_{n+3}} + \cdots + \frac{y_{n+p}}{S_{n+p}}$

$< \frac{2}{S}\left(y_{n+1} + y_{n+2} + y_{n+3} + \cdots + y_{n+p}\right)$

$\Rightarrow \frac{y_{n+1}}{S_{n+1}} + \frac{y_{n+2}}{S_{n+2}} + \frac{y_{n+3}}{S_{n+3}} + \cdots + \frac{y_{n+p}}{S_{n+p}} < \frac{2}{S}(S_{n+p} - S_n), \forall\, n \geq m, p \geq 1.$

As $\sum y_n$ is convergent, then given $\varepsilon > 0$, there exists a positive integer $m_1$, such that

$S_{n+p} - S_n < \frac{\varepsilon S}{2}, \forall\, n \geq m_1$

Therefore,

$\frac{y_{n+1}}{S_{n+1}} + \frac{y_{n+2}}{S_{n+2}} + \frac{y_{n+3}}{S_{n+3}} + \cdots + \frac{y_{n+p}}{S_{n+p}} < \frac{2}{S}\frac{\varepsilon S}{2} < \varepsilon, \forall\, n \geq \max(m_1, m)\,, p \geq 1$

Therefore by Cauchy's General Principle of convergence, $\sum \frac{y_n}{y_1 + y_2 + \cdots + y_n}$ is convergent.

<u>**Positive term Series**</u>

Let $\sum y_n$ be an infinite series of positive term series of positive terms $(y_n \geq \ \ 0)$ and $\{S_n\}$ be the sequence of its partial sums such that

$S_n = y_1 + y_2 + \cdots + y_n \geq 0, \qquad \forall\, n$

$\Rightarrow S_n - S_{n-1} = y_n \geq 0 \Rightarrow S_n \geq S_{n-1}, \ \forall\, n > 1$

Therefore the sequence $\{S_n\}$ of partial sums of a series of positive terms is a monotonic increasing sequence.

Hence $\{S_n\}$ can either converge or diverge to $+\infty$.

**Theorem 2.13. A positive term series converges if and only if the sequence of its partial sums is bounded above**.

**Proof.** Let $\sum y_n$ and $\{S_n\}$ be positive term series and a sequence of its partial sums respectively.

$\Rightarrow \{S_n\}$ be a monotonic increasing sequence.

As we know that monotonic increasing sequence converges iff it is bounded above.
Hence $\{S_n\}$ converges if and only if the sequence of its partial sums is bounded above.

**<u>Necessary Conditions for convergence of positive term series</u>**

**Theorem 2.14. (Pringsheim's theorem) If a series** $\sum y_n$ of positive monotonic decreasing terms converges then $y_n \to 0$ and also $\lim_{n\to\infty} ny_n = 0$.

**Proof.** Let $\sum y_n$ be the convergent series of positive monotonic decreasing terms.
By the definition of convergent series, for any $\varepsilon > 0$, there exists a positive integer $M$ such that
$\left|y_{m+1} + y_{m+2} + \cdots + y_{m+p}\right| < \frac{\varepsilon}{2}, \ \ \forall m \geq M, p \geq 1$
Let $m + p = n > 2M$
and $m = \left[\frac{n}{2}\right]$ $i.e. m = greatest\ integer\ not\ greater\ than \frac{n}{2}$.
Hence
$y_{m+1} + y_{m+2} + \cdots + y_n < \frac{\varepsilon}{2}$
But $\sum y_n$ is positive monotonic decreasing.
i.e.
$y_{m+1} > y_{m+2} > \cdots > y_n \Rightarrow y_{m+1} + y_{m+2} + \cdots + y_n >$
$\underbrace{y_n + y_n + \cdots .. + y_n}_{(n-m) times}$
$\Rightarrow y_{m+1} + y_{m+2} + \cdots + y_n > (n - m)y_n$
Therefore $(n - m)y_n < y_{m+1} + y_{m+2} + \cdots + y_n < \frac{\varepsilon}{2}$
$\left(n - \frac{n}{2}\right) y_n < \frac{\varepsilon}{2}$ $\quad$ $because\ m = \left[\frac{n}{2}\right]$
$\Rightarrow \frac{n}{2} < \frac{\varepsilon}{2} \Rightarrow \ ny_n < \varepsilon$
Hence $\lim_{n\to\infty} ny_n = 0$

**NOTE:**
$\lim_{n\to\infty} ny_n = 0$ is only necessary not sufficient condition.If $\lim_{n\to\infty} ny_n \neq 0$ then the series $\sum y_n$ is obviously divergent..
Example $\sum \frac{1}{n}$ diverges because $\lim_{n\to\infty} ny_n = 1 \neq 0$ and positive monotonic decreasing terms.

**Theorem 2.15. Let $\sum \frac{1}{n^p}$ be positive term series then it is convergent iff $p > 1$.**

**Proof.** Let $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$

Case 1. When $p > 1$

Now

$\frac{1}{1^p} = 1$ ……………………………………………………………(1)

$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} < \frac{2}{2^p} = \frac{1}{2^{p-1}}$……………………………………….(2)

$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} < \frac{4}{4^p} = \frac{1}{4^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^2$ …….(3)

$\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} < \underbrace{\frac{1}{8^p} + \frac{1}{8^p} + \cdots + \frac{1}{8^p}}_{8\ times} < \frac{8}{8^p} = \frac{1}{8^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^3$…….(4)

… … … … … … … … … … … … … … … … … … … … … … … … … … … …

… … … … … … … … … … … … … … … … … … … … … … … … … … … …

… … … … … … … … … … … … … … … … … … … … … … … … … … … …

$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \cdots + \frac{1}{(2^{n+1}-1)^p} < \underbrace{\frac{1}{(2^n)^p} + \frac{1}{(2^n)^p} + \cdots + \frac{1}{(2^n)^p}}_{2^n\ times}$

$= \left(\frac{1}{2^{p-1}}\right)^{p-1}$ ..(n)

Adding (1), (2),…..,(n), we get

$$\frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{(2^{n+1}-1)^p} = S_{2^{n+1}-1} < 1 + \left(\frac{1}{2}\right)^{p-1} + \cdots + \left(\frac{1}{2^{p-1}}\right)^{p-1}$$

**CHECK YOUR PROGRESS**

(CQ 16) $\sum \frac{1}{2^p}$ is convergent iff $p < 1$. (T/F)

(CQ 17) $\sum 2^p$ is convergent iff $p < 1$. (T/F)

(CQ 18) A positive term series converges if and only if the sequence of its partial sums is bounded above. (T/F)

(CQ 19) $x_n = \sum_{k=1}^{n} \frac{3k^2+2k}{2^k}$ , then series ________________.

## 2.8 SUMMARY

In this Unit we discussed about sequence and series and proved some important test for convergence with illustrative examples.

## 2.9 GLOSSARY

1. Set- a well defined collection of elements

2. Sequence-a function whose domain is set of natural number and range is set of real number
3. Series-sum of the term of sequences

## *2.10 REFERENCES*

1. R.G. Bartley and D.R. Sherbert (2000) Introduction of real analysis, John Wiley and Sons (Asia) P. Ltd., Inc.
2. Gerald G. Bilodeau, Paul R. Thie & G. E. Keough (2015). An Introduction to Analysis (2nd edition), Jones and Bartlett India .
3. K. A. Ross (2013). Elementary Analysis: The Theory of Calculus ($2^{nd}$ edition). Springer.

## *2.11 SUGGESTED READINGS*

1. W. Rudin (2019) Principles of Mathematical Analysis, McGraw-Hill Publishing, 1964.
2. Tom M. Apostol (1996). Mathematical Analysis (2nd edition), Narosa Book Distributors Pvt Ltd-New Delhi.
3. Pawan K. Jain and Khalil Ahmad (2005). Metric spaces, 2nd Edition, Narosa.

## *2.12 TERMINAL QUESTION*

**<u>Long Answer Questions</u>**

(TQ 1) State and Prove Bolzanno-Weierstrass theorem.

(TQ 2) Prove that a Cauchy sequence of real numbers is bounded.

(TQ 3) Prove that a monotonic sequence of real numbers is properly divergent iff it is bounded

(TQ 4) Prove that the series $\sum y_n$ converges then lim $(y_n) = 0$

(TQ 5) Prove that $\sum \frac{1}{n}$ is diverges.

**<u>Fill in the blanks</u>**

(TQ 6) If $\sum x_n \, and \; \sum y_n$ is convergent then $\sum(x_n + y_n)$ is ______________________.

(TQ 7) $\sum \frac{1}{n+1} \; is$_________.

(TQ 8) A sequence $\left\{y_n = \frac{n}{n+1}\right\}$ is _________________

(TQ 9) A sequence $\{y_n = (-1)^n\}$ is__________________

## *2.13 ANSWERS*

| | | |
|---|---|---|
| (CQ 1) F | (CQ 2) T | (CQ 3) F |
| (CQ 4) 0 | (CQ 5) 0 | (CQ 6) T |
| (CQ 7) F | (CQ 8) F | (CQ 9) Convergent |
| (CQ 10) F | (CQ 11) T | (CQ 12) 0 |
| (CQ 13) F | (CQ 14) T | (CQ 15) 0 |
| (CQ 16) F | (CQ 17) T | (CQ 18) T |
| (CQ 19) convergent | | |
| (TQ 6) convergent | (TQ 7) divergent | (TQ 8) Cauchy sequence |
| (TQ 9) not a Cauchy sequence | | |

# UNIT 3: LIMIT AND CONTINUITY

**CONTENTS**



## *3.1 INTRODUCTION*

In previous unit we discussed about sequence and series. In this unit we will discussed about limit and continuity.

The rudimentary notion of a limiting process emerged in the 1680s as Isaac Newton(1642–1727) and Gottfried Leibniz (1646–1716) struggled with the creation of the calculus. Though each person’s work was initially unknown to the other and their creative insights were quite different, both realized the need to formulate a notion of function and the idea of quantities being ‘‘close to’’ one another. Newton used the word ‘‘fluent’’ to denote a relationship between variables, and in his major work Principia in 1687 he discussed limits ‘‘to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in infinitum.’’ Leibniz introduced the term ‘‘function’’ to indicate a quantity that depended on a variable, and he invented ‘‘infinitesimally small’’ numbers as a way of handling the concept of a limit. The term ‘‘function’’ soon became standard terminology, and Leibniz also introduced the term ‘‘calculus’’ for this new method of calculation. In 1748, Leonhard Euler (1707–1783) published his two-volume treatise Introduction in Analys in Infinitorum, in which he discussed power series, the exponential and logarithmic functions, the trigonometric functions, and many related topics. This was followed by Institutiones Calculi Differentialis in 1755 and the three-volume Institutiones Calculi Integralis in 1768–1770. These works remained the standard textbooks on calculus for many years. But the concept of limit was very intuitive and its looseness led to a number of problems. Verbal descriptions of the limit concept were proposed by other mathematicians of the era, but none was adequate to provide the basis forrigorous proofs.

In 1821, Augustin-Louis Cauchy (1789–1857) published his lectures on analysis in his Cours d'Analyse, which set the standard for mathematical exposition for many years. He was concerned with rigor and in many ways raised the level of precision in mathematical discourse. He formulated definitions and presented arguments with greater care than his predecessors, but the concept of limit still remained elusive. In an early chapter he gave the following definition: If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others. The final steps in formulating a precise definition of limit were taken by Karl Weierstrass (1815–1897). He insisted on precise language and rigorous proofs, and his definition of limit is the one we use today.

We now begin the study of the most important class of functions that arises in real analysis: the class of continuous functions. The term ‘‘continuous’’ has been used since the time of Newton to refer to the motion of bodies or to describe an unbroken curve, but it was not made precise until the nineteenth century. Work of Bernhard Bolzano in 1817 and Augustin-Louis Cauchy in 1821 identified continuity as a very significant property of functions and proposed definitions, but since the concept is tied to that of limit, it was the careful work of Karl Weierstrass in the 1870s that brought proper understanding to the idea of continuity.

We will first define the notions of continuity at a point and continuity on a set, and then show that various combinations of continuous functions give rise to continuous functions.

## 3.2 OBJECTIVES

After reading this unit learners will be able to

1. recognized the basic concept of limit
2. construct the basic concept of continuity
3. learned some important theorems.

## 3.3 FUNCTIONS

First we will try to recall some basic definition which will be used in this unit and which also discussed in graduate level.

**Cartesian Product:** Let $A$ and $B$ are two nonempty sets, then the Cartesian product $A \times B$ of $A$ and $B$ is defined as the set of all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. i.e.
$A \times B = \{(a, b): a \in A \text{ and } b \in B.$

**Function:** Consider $A$ and $B$ be sets. Then a function from $A$ to $B$ is a set $f$ of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.
The notation $f : A \to B$ is often used to indicate that $f$ is a function from $A$ into $B$.

**Domain of a function:** The first elements of set $A$ of a function $f$ is called the domain of $f$ and it is denoted by $D(f)$.

**Range of a function:** The set of all second elements in $f$ is called the range of $f$ and denoted by $R(f)$.

**NOTE:**
In geometrical terms we can say every vertical line $x = a$ with $a \in A$ intersects the graph of f exactly once.

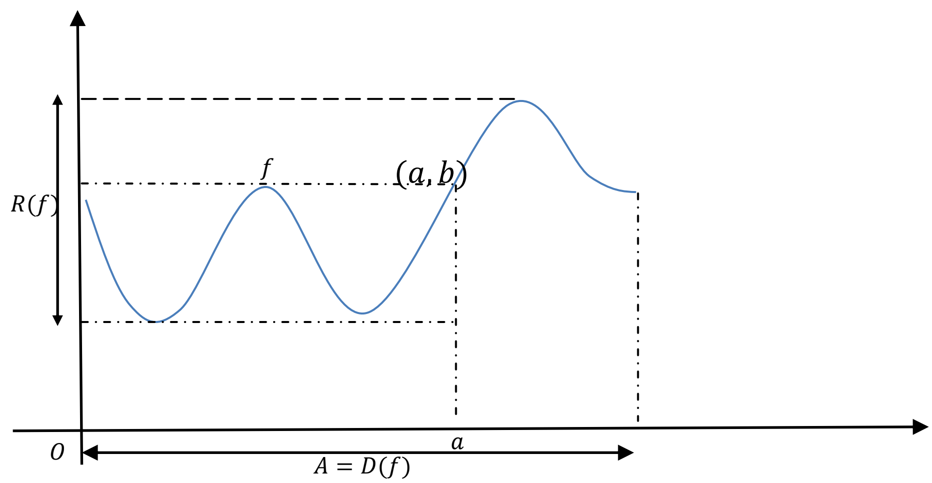



***Fig 3.1.Function***

**Inverse function:** If $f : A \to B$ is a bijection of $A$ onto $B$, then $f' = \{(b,a) \in B \times A : (a,b) \in f\}$ is a function on $B$ into $A$. Then function $f'$ is known as the inverse function of $f$, and is denoted by $f^{-1}$. The function $f^{-1}$ is also called the inverse of $f$.

**Composite function**: If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq d(g) = b$, then the composite function $gof(x)$ is the function from $A$ into $C$ defined by $gof(x) = g\big(f(x)\big)\ \forall x \in A$.

| **CHECK YOUR PROGRESS** |
|---|
| (CQ 1) A relation $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(x) = \sqrt{x}$ is a function. (T/F) |
| (CQ 2) The first elements of set $A$ of a function $f$ is called the range of $f$. (T/F) |
| (CQ 3) Every Cartesian Product is a function (T/F). |
| (CQ 4) Range of function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(x) = x^2$ is ________ |

## *3.4 LIMIT*

Now we will introduce the important notion of the limit of a function. The intuitive idea of the function f having a limit $L$ at the point $c$ is that the values f (x) are close to L when x is close to (but different from) $c$. But it is necessary to have a technical way of working with the idea of ‘‘close to’’ and this is accomplished in the $\delta - \varepsilon$ definition given below.

In order for the idea of the limit of a function $f$ at a point $c$ to be meaningful, it is necessary that $f$ be defined at points near c. It need not be defined at the point $c$, but it should be defined at enough points close to $c$ to make the study interesting. This is the reason for the following definition.

**Cluster point:** Consider a set $A \in \mathbb{R}$ then a point $c \in \mathbb{R}$ is a cluster point of $A$ if for every d > 0 there exists at least one point $x \in A$; $x = c$ such that $|x - c| < \delta$.

**Theorem 3.1. A point $c \in R$ is a cluster point of a subset $A$ of $\mathbb{R}$ if and only if there exists a sequence $\{a_n\}$ in $A$ such that $\lim_n a_n = c$ and$a_n \neq c$ for all $n \in \mathbb{N}$.**

**Proof.** Let $c$ is a cluster point of $A$, then for any $m \in N$ then neighbourhood $V_{1/m}(c)$ contains atleast one point $a_n$ in $A$ distinct from c.

*Converse*

Let $\{a_n\}$ be a sequence in $A\backslash\{c\}$ with $\lim(a_n) = c$, then for any $\delta > 0$ there exists $M$ such that if $n \geq M,$ then $a_n \in V_\delta(c)$.

Hence the $\delta$ −neighborhood $V_\delta(c)$ of $c$ contains the points $a_n$, for $n \geq M$, which belongs to $A$ and are distinct from $c$.

**Limit**

Let $A \subseteq R$, and let $c$ be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number $l$ is said to be a limit of $f$ at $c$ if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $x \in A$ and $0 < |x - c| < d$, then $|f(x) - l| < \varepsilon$.

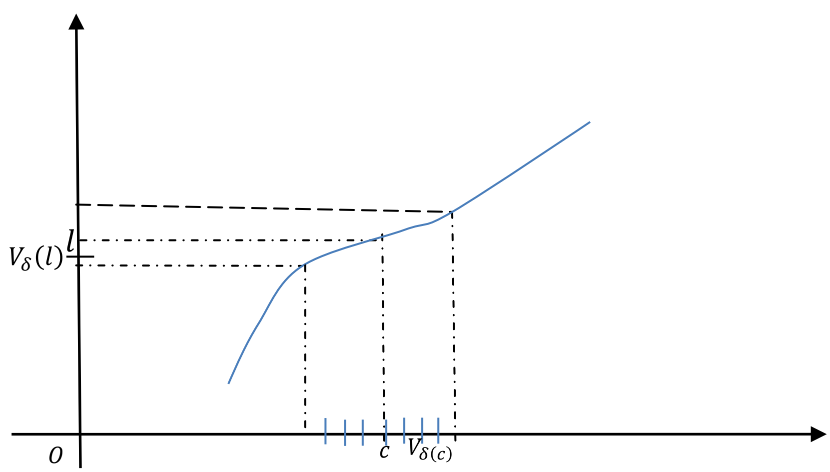



***Fig 3.2. The limit of $f$ at $c$ is $l$***

**Theorem 3.2. If $f : A \to \mathbb{R}$ and if $c$ is a cluster point of $A$, then $f$ can have only one limit at $c$.**

**Proof.** Let $l$ and $l'$ be the limits of function $f$.

For any $\varepsilon > 0$, there exists $\delta_1\left(\frac{\varepsilon}{2}\right) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_1\left(\frac{\varepsilon}{2}\right)$, then

$|f(x) - l| < \frac{\varepsilon}{2}$. and

there exists $\delta_2\left(\frac{\varepsilon}{2}\right) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_2\left(\frac{\varepsilon}{2}\right)$, then

$|f(x) - l'| < \frac{\varepsilon}{2}$.

Let $\delta = \inf\left\{\delta_1\left(\frac{\varepsilon}{2}\right), \delta_2\left(\frac{\varepsilon}{2}\right)\right\}$. Then if $x \in A$ and $0 < |x - c| < \delta$.

The Triangle Inequality implies that

$|l - l'| = \left|l + \left(-f(x) + f(x)\right) - l'\right| \le |l - f(x)| + |f(x) - l'|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

Because $\varepsilon > 0$ is arbitrary, Therefore $l - l' = 0 \Rightarrow l = l'$.

**Ex.3.1. Prove that $\lim_{x\to a} c = c$**

**Proof.** Let $f(x) = c \; for \; all \; x \in \mathbb{R}$.

Now we will try to prove that $\lim_{x\to a} f(x) = c$.

Let $\varepsilon > 0$ and $\delta = 1$.

Then if $0 < |x - a| < 1$, we have

$|f(x) - c| = |c - c| = 0 < \varepsilon$.

As $\varepsilon > 0$ is arbitrary, by definition of limit we get

$\lim_{x\to a} f(x) = c$ .

**Ex.3.2. Prove that $\lim_{x\to b} x^2 = b^2$**

**Proof.** Let $f(x) = \boldsymbol{x^2} \; for \; all \; x \in \mathbb{R}$.

Now we will try to prove that $\lim_{x\to a} f(x) = b^2$.

Now we try to prove that $|f(x) - b^2| = |x^2 - b^2|$ less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $b$.

Now

$x^2 - b^2 = (x - b)(x + b)$.

If $|\,x - b| < 1$, then

$|x| < |b| + 1$

Hence $|x + b| \le |x| + |b| < |b| + 1 + |b| < 2|b| + 1$

Thus, if $|x - b| < 1$ then

$|x^2 - b^2| \leq |x-b||x+b| < (2|b|+1)|x-$
$b|$…………………………(1)
Let $|x-b| < \frac{\varepsilon}{2|c|+1}$ and we choose $\delta(\varepsilon) = \inf\left\{1, \frac{\varepsilon}{2|c|+1}\right\}$,
Then if $0 < |x-b| < \delta(\varepsilon)$,
Now if $|x-b| < 1$, then equation (1) is valid.
If $|x-b| < \frac{\varepsilon}{2|b|+1}$ then $|x^2-b^2| < (2|b|+1)|x-b| < (2|b|+1).\frac{\varepsilon}{2|c|+1} < \varepsilon$
As we have choice to choose $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$.
We deduce that $\lim_{x\to b} x^2 = b^2$

**Ex.3.3. Prove that $\lim_{x\to b} \frac{1}{x} = \frac{1}{b}$ *if* $b > 0$**

**Proof.** Let $f(x) = \frac{1}{x}\ for\ all\ x > 0$ and assume $\boldsymbol{b} > \mathbf{0}$
Now we will try to prove that $\lim_{x\to a} f(x) = \frac{1}{b}$.
Therefore we will try to prove that the difference $\left|f(x) - \frac{1}{b}\right| = \left|\frac{1}{x} - \frac{1}{b}\right|$ less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $b > 0$.
Now
$\left|\frac{1}{x} - \frac{1}{b}\right| = \left|\frac{1}{bx}(b-x)\right| = \frac{1}{bx}|x-b|\ \ for\ \ x > 0.$
Now if $|\,x-b| < \frac{1}{2}b$ then $-\frac{1}{2}b < x-b < \frac{1}{2}b \Rightarrow \frac{1}{2}b < x < \frac{3}{2}b \Rightarrow \frac{1}{2}b^2 < bx \Rightarrow \frac{2}{b^2} > \frac{1}{bx}$.
Therefore
$0 < \frac{1}{bx} < \frac{2}{b^2}$ for $|\,x-b| < \frac{1}{2}b$
Hence, for these values of $x$ we have
$\left|f(x) - \frac{1}{b}\right| < \frac{2}{b^2}|x-b|$ ………………………………………(1)
In order to make this last term less than $\varepsilon$ it suffices to take $|x-b| < \frac{1}{2}b^2\varepsilon$. Consequently, if we choose $\delta(\varepsilon) = \inf\left\{\frac{1}{2}b, \frac{1}{2}b^2\varepsilon\right\}$,
Then if $0 < |x-b| < \delta(\varepsilon)$,
Now if $|x-b| < \frac{1}{2}b$, then equation (1) is valid.
Therefore, since $|x-b| < \frac{1}{2}b^2\varepsilon$, that
$\left|f(x) - \frac{1}{b}\right| = \left|\frac{1}{x} - \frac{1}{b}\right| < \varepsilon$
Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we conclude that

$\lim_{x \to b} \frac{1}{x} = \frac{1}{b}$ $if\ b > 0$.

**Theorem 3.3. (Sequential Criterion): Let $f : X \to \mathbb{R}$ and let $c$ be a cluster point of $X$. Then the following are equivalent.**

**(i) $\lim_{x \to b} f = l$**

**(ii) For every sequence $\{y_n\}$ in $X$ that converges to $b$ such that $\{y_n\}$ for all $n \in \mathbb{N}$, the sequence $\{f(y_n)\}$ converges to $l$.**

**Proof.** Let $\lim_{x \to b} f = l$

Let $\{y_n\}$ be a sequence in $X$ such that $\lim_n y_n = b$ $and\ y_n \neq b$ $for\ all\ n$.

Now we will try to prove that the sequence $\{ f(y_n)\}$ converges to $l$.

Consider $\varepsilon > 0$ be given. Then by the definition of limit of function, there exists $\delta > 0$ such that if $y \in X$ and $0 < |y - c| < \delta$ then

$|f(y) - l| < \varepsilon$.

By the definition of convergent sequence, for the given $\delta$ there exists a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|y_n - b| < \delta$.

But for each such $y_n$ we have $|f(y_n) - l| < \varepsilon$

Therefore, we get

$|f(y_n) - l| < \varepsilon$ if $n > K(\delta)$.

Hence the sequence $\{ f(y_n)\}$ converges to $l$.

It implies (i)⇒(ii)

Let be the sequence $\{ f(y_n)\}$ not converges to $l$., then there exists an $\varepsilon'$ −neighborhood $V_{\varepsilon'}$ such that there exists atleast one number $y_\delta$ in $X \cap V_\delta(b)\ with\ y_\delta \neq b$ such that $f(y_\delta) \neq V_{\varepsilon'}(L)$.

Therefore for every $n \in \mathbb{N}$, the $\left(\frac{1}{n}\right)$ −neigbourhood of $b$ contains number $y_n$ such that

$0 < |y_n - b| < \frac{1}{n}$ and $y_n \in X$

such that $|f(y_n) - l| < \varepsilon$ for all $n \in \mathbb{N}$.

Therefore, we conclude that the sequence $\{y_n\}$ in $X - \{b\}$ converges to $b$ but the sequence $\{ f(y_n)\}$ not converges to $l$.

Therefore we have shown that if (i) is not true, then (ii) is not true. We conclude that (ii) implies (i).

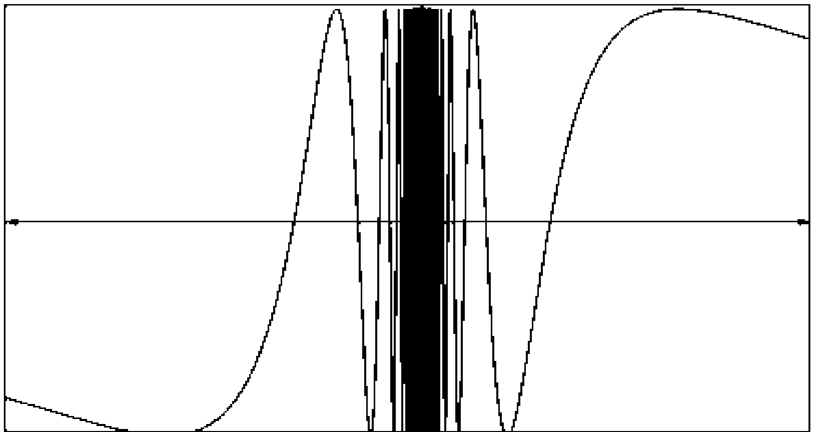


*Fig 3.3.* $\boldsymbol{f(x) = \sin\frac{1}{x}}$

**Ex.3.4. Prove that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist in $\mathbb{R}$.**

**Sol.** Let $f(x) = \sin\frac{1}{x}\ for\ all\ x \neq 0$

Now we try to prove that f does not have a limit at 0.

Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi n+\frac{\pi}{2}}$ for all $\in \mathbb{N}$ .

Now

$\lim_n a_n = \lim_n \frac{1}{2\pi n} = 0$ and and $\lim_n b_n = \lim_n \frac{1}{2\pi n+\frac{\pi}{2}} = 0$ but

$\lim_n f(a_n) = \lim_n f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0$

$\lim_n f(b_n) = \lim_n f\left(\frac{1}{2\pi n+\frac{\pi}{2}}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = \cos(2\pi n) = 1$ are different.

Therefore, $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist in $\mathbb{R}$.

**Ex.3.5. Let the signum function sgn be defined by**

$$\boldsymbol{sgn(x) = \begin{cases} 1, & for\ x > 0 \\ 0, & for\ x = 0 \\ -1, & for\ x < 0 \end{cases}}$$

**Prove that $sgn(x)$ does not have limit at 0.**

**Proof.** Let $\{a_n\}$ be a sequence such that $a_n = \frac{(-1)^n}{n}$ for $n \in \mathbb{N}$.

Now $\lim_n a_n = \frac{(-1)^n}{n} = 0$.

However since $sgn(a_n) = (-1)^n$ for $n \in \mathbb{N}$

And we know that $\{(-1)^n\}$ is divergent.

Therefore $\{sgn(a_n)\}$ does not converge.
Therefore $\lim_n sgn(x)$ does not exist.

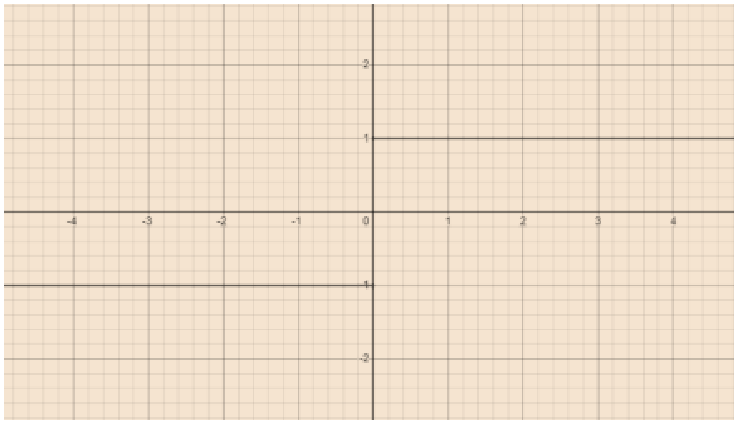



***Fig 3.3.*** $\boldsymbol{f(x) = sgn(x)}$

**CHECK YOUR PROGRESS**

(CQ 5) Signum function does not have limit at 0.(T/F)
(CQ 6) $\lim_{x\to 0} \log x = 1$ (T/F)
(CQ 7) $\lim\limits_{x\to 0}[x] =$ ________________

## *3.5 IMPORTANT LIMIT THEOREMS AND EXTENSION OF LIMIT CONCEPTS*

**Bounded function:** Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$, and consider $b \in \mathbb{R}$ be the cluster point of $X$. Then $f$ is said to be bounded on a neighborhood of $b$ if there exists a $\delta$-neighborhood $V_\delta(b)$ of and a constant $K > 0$ such that $|f(x)| \le M \quad for\ all\ x \in X \cap V_\delta(b)$.

**Theorem 3.4. If $\boldsymbol{X} \subseteq \mathbb{R}$ and $\boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}$ has a limit point at $\boldsymbol{b} \in \mathbb{R}$, then $\boldsymbol{f}$ is bounded on some neighborhood of $\boldsymbol{b}$.**

Proof. Let $l$ be the limit point of $f$ at $b$.
Then for $\varepsilon = 1$, there exists $\delta > 0$ such that
$|f(x) - l| < 1$ if $0 < |x - b| < \delta$
Now $|f(x) - l| \ge |f(x)| - |l|$ .
i.e. $|f(x)| - |l| \le |f(x) - l| < 1$ .
Hence, if $x \in X \cap V_\delta(b)$ and $x \neq b$, then
$|f(x)| - |l| < 1$ or $|f(x)| < 1 + |l|$.
If $b \notin X$, we take $K = 1 + |l|$ and

If $b \in X$, we take $K = \sup\{|f(b)|,\ 1 + |l|\}$
Hence if $\in X \cap V_\delta(b)$, then $|f(x)| \leq K$.
Therefore $f$ is bounded on neighborhood $V_\delta(b)$ of $b$.
Let $X \subseteq \mathbb{R}$ and $f, g: X \to \mathbb{R}$ then
(i) sum $f + g: X \to \mathbb{R}$ is given by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$
(ii) difference $f - g: X \to \mathbb{R}$ is given by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$
(iii) Product $fg: X \to \mathbb{R}$ is given by $(fg)(x) = f(x)g(x)$, for all $x \in X$
(iv) $h(x) \neq 0$ for $x \in X$, the quotient $\frac{f}{h}$ be the function given by $\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$ for all $x \in X$.

**Theorem 3.4. Let $X \subseteq \mathbb{R}$ and $f, g: X \to \mathbb{R}$ and let $b \in \mathbb{R}$ be a cluster point of $X$ and $\in \mathbb{R}$ .**
**(a) If $\lim_{x\to b} f = l_1$ and $\lim_{x\to b} g = l_2$, then**
**(i) $\lim_{x\to b} f + g = l_1 + l_2$ (ii) $\lim_{x\to b} f - g = l_1 - l_2$**
**(iii) $\lim_{x\to b} fg = l_1 l_2$**
**(b) If $h: X \to \mathbb{R}$ and $h(x) \neq 0$ for all $x \in X$, if $\lim_{x\to b} h = l_3 \neq 0$, then $\lim_{x\to b} \frac{f}{h} = \frac{l_1}{l_3}$**

**Proof.** (a) It is given that $\lim_{x\to b} f = l_1$ and $\lim_{x\to b} g = l_2$, Hence for any $\varepsilon > 0$ there exists a positive numbers $\delta_1$ and $\delta_2$ such that
$|f(x) - l_1| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta_1$ and $|g(x) - l_2| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta_2$
Let $\delta = \min(\delta_1,\ \delta_2)$, then
$|f(x) - l_1| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta$…………………………(1) and
$|g(x) - l_2| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta$…………………………(2)
Now, when $0 < |x - b| < \delta$

$$\begin{aligned} |(f+g)(x) - (l_1 + l_2)| &= |f(x) - l_1 + g(x) - l_2| \\ &\leq |f(x) - l_1| + |g(x) - l_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore
$|(f+g)(x) - (l_1 + l_2)| < \varepsilon$ when $0 < |x - b| < \delta$
Thus, $\lim_{x\to b} f + g = l_1 + l_2$
(ii) When $0 < |x - b| < \delta$
$|(f-g)(x) - (l_1 - l_2)| = |f(x) - l_1 + g(x) - l_2|$

$$\leq |f(x) - l_1| + |l_2 - g(x)|$$
$$= |f(x) - l_1| + |g(x) - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(From (1) and (2))

$|(f-g)(x) - (l_1 - l_2)| < \varepsilon$ when $0 < |x-b| < \delta$

Thus, $\lim_{x\to b} f - g = l_1 - l_2$

(iii) $|(fg)(x) - (l_1 l_2)| = |f(x)g(x) - l_1 l_2|$
$$= |f(x)g(x) - f(x)l_2 + f(x)l_2 - l_1 l_2|$$
$$= |f(x)(g(x) - l_2) + l_2(f(x) - l_1)|$$
$$\leq |f(x)||g(x) - l_2| + |l_2||f(x) - l_1| \;\ldots(3)$$

As we know that $\lim_{x\to b} f = l_1$ . Hence for any $\varepsilon = 1$ there exists a positive number $\delta'_1$ such that

$|f(x) - l_1| < 1$ when $0 < |x-b| < \delta'_1$.

Now

$|f(x)| = |f(x) - l_1 + l_1| \leq |f(x) - l_1| + |l_1|$

$< 1 + |l_1|$ , when $0 < |x-b| < \delta'_1$………………….(4)

$\lim_{x\to b} g = l_2$ , there exists a positive number $\delta'_2$ such that

$|g(x) - l_2| < \frac{\frac{\varepsilon}{2}}{1+|l_1|}$ when $0 < |x-b| < \delta'_2$. …………………….(5)

Again $\lim_{x\to b} f = l_1$, there exists a positive number $\delta'_3$ such that

$|f(x) - l_1| < \frac{\frac{\varepsilon}{2}}{|l_2|}$ when $0 < |x-b| < \delta'_3$……………...(6)

Let $\delta' = \min\{\delta'_1, \delta'_2, \delta'_3\}$. Then from (3), (4), (5) and (6), when $0 < |x-b| < \delta'$

$|(fg)(x) - (l_1 l_2)| < (1 + |l_1|)\frac{\frac{\varepsilon}{2}}{1+|l_1|} + |l_2|\frac{\frac{\varepsilon}{2}}{|l_2|} < \varepsilon.$

Hence $\lim_{x\to b} fg = l_1 l_2$.

(b) $\lim_{x\to b} h = l_3 \neq 0$ therefore for $\varepsilon = \frac{|m|}{2} > 0$ there exists $\delta_3 > 0$ such that

$|h(x) - l_3| < \frac{|m|}{2}$ when $0 < |x-b| < \delta_3$

Now

$|l_3| = |l_3 - h(x) + h(x)| \leq |l_3 - h(x)| + |h(x)| = |h(x) - l_3| + |h(x)|$

or

or $|l_3| < \frac{|l_3|}{2} + |h(x)| \Rightarrow |h(x)| > |l_3| - \frac{|l_3|}{2} = \frac{|l_3|}{2}. or \;\; \frac{1}{|h(x)|} < \frac{2}{|l_3|}$

It implies that there exists a deleted neighbourhood of $b$ on which $h(x)$ does not vanish.

Now, when $0 < |x-b| < \delta_3$

$\left|\left(\frac{f}{h}\right)(x) - \frac{l_1}{l_3}\right| = \left|\frac{f(x)}{h(x)} - \frac{l_1}{l_3}\right| = \left|\frac{f(x)l_3 - h(x)l_1}{h(x)l_3}\right| = \left|\frac{f(x)l_3 - l_1l_3 + l_1l_3 - h(x)l_1}{h(x)l_3}\right| = \left|\frac{l_3(f(x)-l_1)+l_1(l_3-h(x))}{h(x)l_3}\right|$

$\leq \frac{1}{|h(x)|}|f(x) - l_1| + \frac{|l_1|}{|l_3||h(x)|}|h(x) - l_3|$

$< \frac{2}{|l_3|} \cdot |f(x) - l_1| + \frac{2}{|l_3|}\frac{|l_1|}{|l_3|}|h(x) - l_3| = \frac{2}{|l_3|} \cdot |f(x) - l_1| + \frac{2|l_1|}{|l_3|^2}|h(x) - l_3|$ ..........................(7)

Let $\varepsilon > 0$ be given.

It is given that $\lim_{x\to b} f = l_1$ and $\lim_{x\to b} h = l_3$, hence there exists positive numbers $\delta_1''$ and $\delta_2''$ such that

$|f(x) - l_1| < \frac{1}{4}\varepsilon|l_3|$, when $0 < |x - b| < \delta_1''$ ……………………..(8)

$|h(x) - l_3| < \frac{1}{4}\varepsilon \frac{|l_3|^2}{|l_1|}$, when $0 < |x - b| < \delta_2''$……………………..(9)

Let $\delta'' = \min\{\delta_3, \delta_1'', \delta_2''\}$ . Then from (7), (8) and (9), we get

$\left|\left(\frac{f}{h}\right)(x) - \frac{l_1}{l_3}\right| < \frac{2}{|l_3|} \cdot \frac{1}{4}\varepsilon|l_3| + \frac{2|l_1|}{|l_3|^2} \cdot \frac{1}{4}\varepsilon \frac{|l_3|^2}{|l_1|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ when $0 < |x - b| < \delta''$

Therefore $\left|\left(\frac{f}{h}\right)(x) - \frac{l_1}{l_3}\right| < \varepsilon$ when $0 < |x - b| < \delta''$

Hence $\lim_{x\to b} \frac{f}{h} = \frac{l_1}{l_3}$

**Ex.3.6. Find $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$**

**Proof.** It is given that $lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

$lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}-2} = lim_{x\to 0} \cdot \frac{4+x-4}{\sqrt{4+x}+2} = lim_{x\to 0} \frac{x}{\sqrt{4+x}+2} = \frac{1}{4}$.

**One-Sided Limits:** In certain situations when a function f might not have a limit at a point c, still a limit does exist when the function is limited to an interval on one side of the cluster point $b$.

For example, the signum function has no limit at $b = 0$. However, if we restrict the signum function to the interval $(0, \infty)$, the function has a limit of 1 at $b = 0$. Likewise, if we establish the signum function to the interval $(-\infty, 0)$,, the function has a limit of $-1$ at $b = 0$. These are simple illustrations of the right- and left-hand limits at $b = 0$.

**Right hand limit:** Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ and if $b \in \mathbb{R}$ is a cluster point of the set $X \cap (b, \infty) = \{x \in X: x > b\}$ then $l \in \mathbb{R}$ is said to be

right-hand limit of $f$ at $b$ if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ with $0 < x - b < \delta$, then $|f(x) - l| < \varepsilon$.
It can be written as $\lim_{x \to b^+} f = l$.

**Left hand Limit:** Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ and if $b \in \mathbb{R}$ is a cluster point of the set $X \cap (\infty, -b) = \{x \in X: x < b\}$ then $l \in \mathbb{R}$ is said to be right-hand limit of $f$ at $b$ if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ with $0 < b - x < \delta$, then $|f(x) - l| < \varepsilon$.
It can be written as $\lim_{x \to b^-} f = l$.

**Infinite Limits:** (i) Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ and if $b \in \mathbb{R}$ is a cluster point of the set $X$ then $f$ tends to $\infty$ as $x \to b$, If for every $\mu \in \mathbb{R}$ there exists $\delta > 0$ such that for all $x \in X$ with $0 < |x - b| < \delta$, then $f(x) > \mu$
It can be written as $\lim_{x \to b} f = \infty$.
(ii) Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ and if $b \in \mathbb{R}$ is a cluster point of the set $X$ then $f$ tends to $-\infty$ as $x \to b$, If for every $\vartheta \in \mathbb{R}$ there exists $\delta > 0$ such that for all $x \in X$ with $0 < |x - b| < \delta$, then $f(x) < \vartheta$
It can be written as $\lim_{x \to b} f = -\infty$.
For example $\lim_{x \to \infty} \frac{1}{x} = 0$.

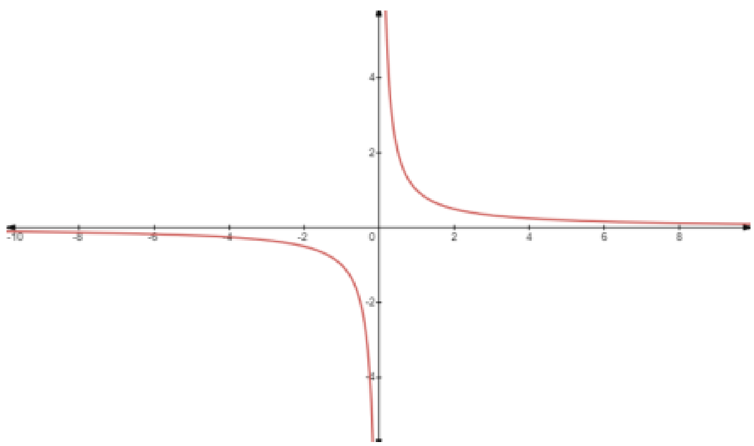



***Fig 3.4..*** $\boldsymbol{f(x) = \frac{1}{x}}$

**CHECK YOUR PROGRESS**

(CQ 8) If $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ has a limit point at $b \in \mathbb{R}$, then $f$ is bounded on some neighborhood of $b$. (T/F)
(CQ 9) When left hand limit=right hand limit then limit does not exist. (T/F)
(CQ 10) $f(x) = [x]$ , then limit at $x = 1$ ________________

## 3.6 *CONTINUOUS FUNCTION*

One of the fundamental ideas in mathematical analysis is the idea of continuity. We shall define what it means to say that a function is continuous at a point or on a set.

**Continuous at a point:** Let $\subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ and $c \in \mathbb{R}$ . f is said to be continuous at point c if given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if $x$ is any point of $X$ satisfies

$|x - b| < \delta$ then $|f(x) - f(b)| < \varepsilon$.

**Discontinuous at a point**: If $f$ fails to be continuous at $b$, then we say that $f$ is discontinuous at $b$.

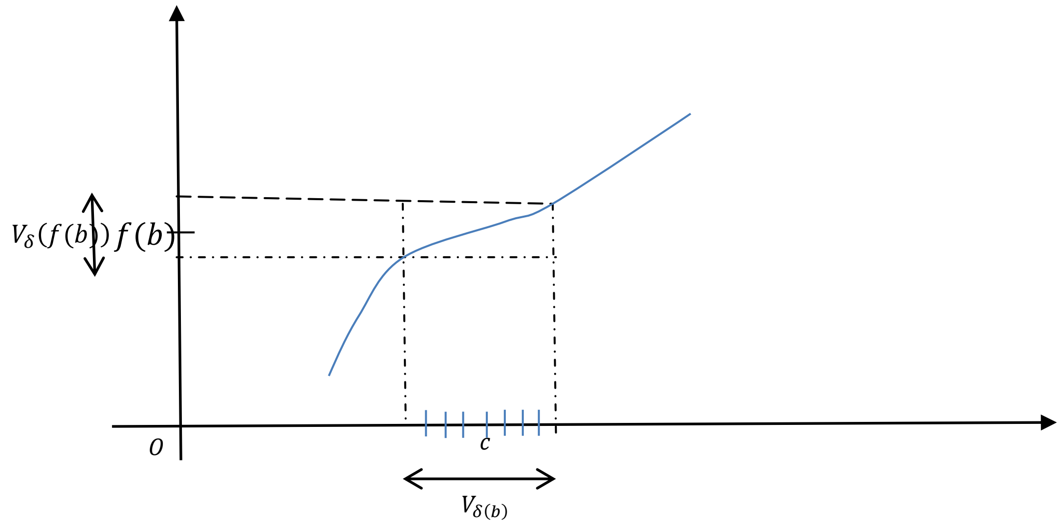



**Fig 3.5. $f(x)$ continuous at b**

**Sequential Criterion for Continuity:** Let $\subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ and $b \in \mathbb{R}$ . f is said to be continuous at point $b$ if and only if for every sequence $\{x_n\}$ in $X$ that converges to $b$ the sequence $\{f(x_n)\}$ converges to $f\ (b)$.

**Discontinuity Criterion:** Let $\subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ and $b \in \mathbb{R}$ . $f$ is said to be discontinuous at point $b$ if there exists atleast a sequence $\{x_n\}$ in $X$ that converges to $b$ but the sequence $\{f(x_n)\}$ does not converges to $f\ (b)$.

**Continuous on set $Y$:** Let $X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ and $Y \subseteq X$ .Then $f$ is said to be continuous on the set $Y$ if $f$ is continuous at every point of $Y$.

**Example:**

**2 )** $f(x) = x$ is continuous on $\mathbb{R}$.

3 $f(x) = \frac{1}{x}$ is continuous on $X = \{x \in \mathbb{R}: x > 0$

**Ex. 3.7. Dirichlet Function (This function was introduced in 1829 by P. G. L. Dirichlet)**
**Let $f(x)$ defined by**

$$\boldsymbol{f(x) = \begin{cases} 1 & if\ x\ is\ rational \\ 0 & if\ x\ is\ irrational \end{cases}}$$

**Then prove that $f$ is discontinuous at any point of $\mathbb{R}$.**

**Proof.** Let $a$ be a rational number and $\{x_n\}$ be a sequence of irrational numbers such that given sequence converges to $a$.
Now $f(x_n) = 0\ for\ all\ n \in \mathbb{N}$. Therefore $\lim_n f(x_n) = 0$ but $f(a) = 1$.
Hence $f$ is not continuous at the rational number $a$.
Let $b$ be a irrational number and $\{y_n\}$ be a sequence of rational numbers such that given sequence converges to $b$.
Now $f(x_n) = 0\ for\ all\ n \in \mathbb{N}$. Therefore $\lim_n f(y_n) = 1$ but $f(c) = 0$.
Hence $f$ is not continuous at the irrational number $b$.
Because every real number is either rational or irrational, we conclude that $f$ is not
continuous at any point in $\mathbb{R}$.

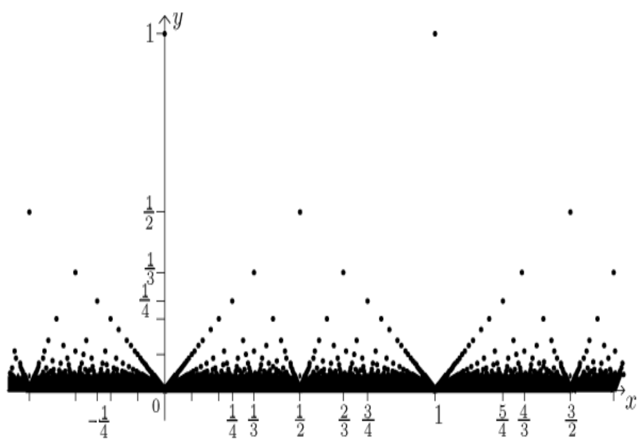



***Fig 3.6.Diritclet function***
**(Source: https://math.fel.cvut.cz/mt/txtb/4/txe3ba4s.htm)**

**CHECK YOUR PROGRESS**

(CQ 11) $f(x) = \begin{cases} \sin 1/x\,, & x = 0 \\ 1, & x \neq 0 \end{cases}$ is a continuous function. (T/F)

(CQ 12) $f(x) = \begin{cases} 1, & x = rational\ number \\ 0, & x = irrational\ number \end{cases}$ is not continuous function. (T/F)

## *3.7 SOME IMPORTANT THEOREMS*

**Theorem 3.5. Let $\subseteq \mathbb{R}$, $f, g: X \to \mathbb{R}$. Let $b \in X$ and $f$ and $g$ are continuous at $b$. Then $f + g, f - g$ and $f\,g$ are continuous at $b$.**

**Proof.** If $b \in X$ is not a cluster point of $X$, then the conclusion is inevitable.
Therefore, let $b$ is a cluster point of $X$.
It is given that $f$ and $g$ are continuous at $b$.
Therefore $\lim_{x\to b} f(x) = f(b)$ and $\lim_{x\to b} g(x) = g(b)$
From Theorem $(f + g)(b) = f(b) + g(b) = \lim_{x\to b}(f + g)$.
Therefore $f + g$ is continuous at $b$.
Similarly $f - g$ and $fg$ is continuous at $b$.

**Theorem 3.5. Let $\subseteq \mathbb{R}$, $f, h: X \to \mathbb{R}$ , $h(x) \neq 0$ $for\ all$ $x \in X$ and $b \in X$. Let $b \in X$ and $f$ and $h$ are continuous at $b$. Then $\frac{f}{h}$ is continuous at $b$.**

**Proof.** It is given that $b \in X$ which implies$h(b) \neq 0$.
$h$ is continuous at $b$. Therefore $\lim_{x\to b} h(x) = h(b)$
Hence $\frac{f}{h}(b) = \frac{f(b)}{h(b)} = \frac{\lim_{x\to b} f(x)}{\lim_{x\to b} h(x)} = \lim_{x\to b} \frac{f}{h}$.
Hence $\frac{f}{h}$ is continuous at c.

**Theorem 3.6. Let $\subseteq \mathbb{R}$, $f, g, h: X \to \mathbb{R}$ , $h(x) \neq 0$ $for\ all$ $x \in X$ and $b \in \mathbb{R}$ .**
**(i) If $f$ and $g$ are continuous at $b$ then $f + g, f - g$ and $f\,g$ are continuous on $X$**
**(ii) If $f$ and $h$ are continuous at $b$. Then $\frac{f}{h}$ is continuous on $X$.**
**Example:** sine and cosine function is continuous on $\mathbb{R}$.

**<u>Composition of Continuous Functions</u>**
**Theorem 3.7. Let $X, Y \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ and $h: Y \to \mathbb{R}$ be functions such that $f(X) \subseteq Y$. If $f$ is continuous at a point $b \in X$ and $g$ is continuous at $a = f(b) \in Y$ then the composition $g\,o\,f : X \to \mathbb{R}$ is continuous at $b$.**

**Proof.** Let $W$ be an $\varepsilon$ −neighborhood of $g(a)$. Because $g$ is continuous at $a$, there exists a $\delta$ −neighborhood $V$ of $a = f(b)$ such that if $y \in Y \cap V$ then $g(y) \in W$.
As $f$ is continuous at $b$, there exists a $\alpha$-neighborhood $U$ of $b$ such that if $x \in X \cap U$ , then $f(x) \in V$.

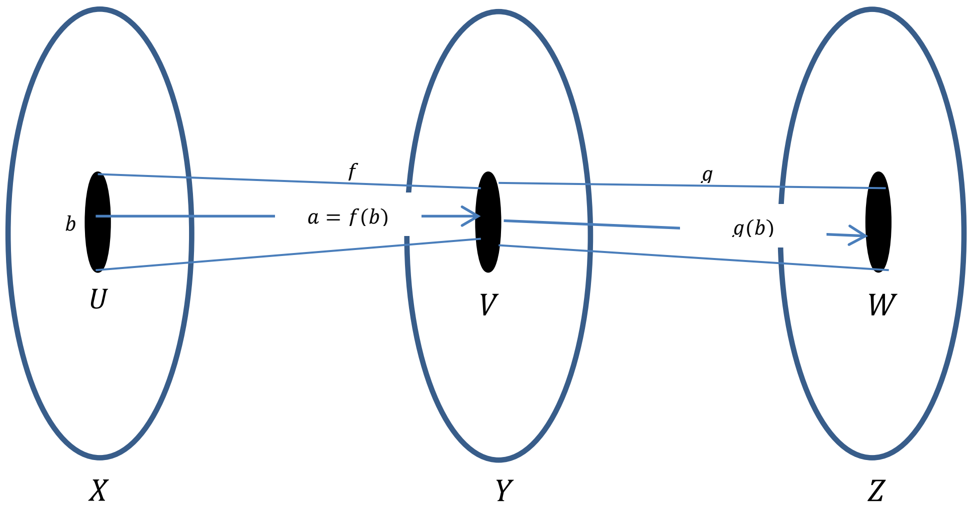



***Fig. 3.7. Composition of function $f$ and $g$***

Now
$f(X) \subseteq \boldsymbol{Y} \Rightarrow$ if $x \in X \cap U$, then $f(x) \in Y \cap V$ so that
$gof(x) = g\big(f(x)\big) \in W$.
Since $W$ is an arbitrary $\varepsilon$-neighborhood of $g(a)$.
Therefore $gof$ is continuous at $b$.

If $f$ and $g$ are continuous at every point of $X$ and $Y$ respectively then using above theorem we get following results.

**Theorem 3.8. Let $X, Y \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ and $h: Y \to \mathbb{R}$ be functions such that $f(X) \subseteq Y$. If $f$ is continuous on $X$ and $g$ is continuous on $Y$ then the composition $g\ o\ f\ :\ X \to \mathbb{R}$ is continuous on $X$.**

**CHECK YOUR PROGRESS**

(CQ 13) If $f$ and $g$ are continuous at $b$ then $f+g, f-g$ and $f\,g$ are continuous on $X$. (T/F)
(CQ 14) $\sin x + \cos x$ is continuous over $\mathbb{R}$. (T/F)
(CQ 15) If $f$ is continuous at a point $b \in X$ and $g$ is continuous at $a = f(b) \in Y$ then the composition $g\ o\ f\ :\ X \to \mathbb{R}$ is __________at $b$.

## *3.8 CONTINUOUS FUNCTIONS ON INTERVALS*

**Function bounded on Set:** A function $\boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}$ is said to be bounded on $X$ if there exists a constant $K > 0$ such that $|f(x)| < K \quad for\ all\ x \in X$. In other words, a function is bounded on a set if its range is a bounded set in $\mathbb{R}$.

**Function unbounded on Set:** A function $\boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}$ is said to be unbounded on $X$ if given any $K > 0$, there exists a point $\alpha \in X$ such that $|f(\alpha)| > K$

**<u>Boundedness Theorem</u>**

**Theorem 3.9. Let $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$ be a closed bounded interval and let $\boldsymbol{f} : \boldsymbol{I} \to \mathbb{R}$ be continuous on $\boldsymbol{I}$. Then $\boldsymbol{f}$ is bounded on $\boldsymbol{I}$.**

**Proof.** Let $f$ is not bounded on $I$.

So, for any $m \in \mathbb{N}$ there exists a number $x_n \in I$ such that $|f(x_n)| > m$.

As $I$ is bounded, the sequence $\{x_n\}$ is bounded.

Hence, By Bolzano-Weierstrass Theorem,

there exists a subsequence $\{x_{m_r}\}$ of $X$ that converges to a number $x$.

Since $I$ is closed and the elements of $\{x_{m_r}\}$ belong to $I$. i.e. $a \le x_{m_r} \le b$

Hence $\le x \le b \Rightarrow x \in I$. (because If $\{x_n\}$ is a convergent sequence and $a \le x_n \le b$ for all $n \in \mathbb{N}$ then $a \le \lim_n x_n \le b$)

$f$ is continuous at $x \Rightarrow f(x_{m_r})$ converges to $f(x)$.

As we know that a convergent sequence of real numbers is bounded.

Therefore convergent sequence $f(x_{m_r})$ should be be bounded, which is a contradiction .

As $\left|f\left(x_{m_r}\right)\right| > n_r \ge r \qquad for\ r \in \mathbb{N}$

Therefore our assumption is wrong.

Hence $f$ is bounded on $I$.

**Absolute maximum:** Let $X \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$. then $f$ has an absolute maximum on $X$ if there exists a point $a \in X$ such that $f(a) \ge f(x) \quad for\ all\ \in X$. $a$ is an absolute maximum point for $f$ on $X$.

**Absolute minimum:** Let $X \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$. then $f$ has an absolute minimum on $X$ if there exists a point $b \in X$ such that $f(b) \le f(x) \quad for\ all\ \in X$. $b$ is an absolute minimum point for $f$ on $X$.

**NOTE:** A continuous function on a set $X$ need not have an absolute maximum or an absolute minimum point. For example, $f(x) = \frac{1}{x}$ has neither an absolute maximum nor an absolute minimum point.

**Maximum-Minimum Theorem**

**Theorem 3.10. Consider $\boldsymbol{I = [a, b]}$ be a closed bounded interval and $\boldsymbol{f : I \to \mathbb{R}}$ be continuous on $\boldsymbol{I}$. Then $\boldsymbol{f}$ has an absolute maximum and an absolute minimum point on $\boldsymbol{I}$.**

**Proof.** Consider the nonempty set $f(I) = \{f(x) = x \in I\}$ of values off on $I = [a, b]$.

Therefore $f(I)$ is a bounded subset of $\mathbb{R}$. (By Bounded theorem)

Let $a = \sup f(I)$ and $b = \inf f(I)$.

Now we will try to prove that there exist points $\alpha$ and $\beta$ in $I$ such that $a = f(\alpha)$ and $b = f(\beta)$

Because $a = \sup f(I)$ then for $n \in \mathbb{N}$

Then $a - \frac{1}{n}$ is not an upper bound of the set $f(I)$.

Hence for all $n \in \mathbb{N}$, there exits an $x_n \in I$ such that $a - \frac{1}{n} < f(x_n) \leq a$.

As it is given that $I = [a, b]$ is a bounded set and $x_n \in I$ for all $n \in \mathbb{N}$,

Therefore the sequence $\{x_n\}$ is also bounded. (By The Bolzano-Weierstrass Theorem)

$\{x_n\}$ is a bounded sequence $\Rightarrow$ there exists a convergent subsequence $\{x_{n_m}\}$ converges to $a$.

i.e. $\lim_{m\to\infty} x_{n_m} \to a$.

Because $x_{n_m} \in I$ for all $n_m \in \mathbb{N} \Rightarrow a \in I$.

Now $a \in I$ and $\boldsymbol{f : I \to} \mathbb{R}$ is a continuous function, then $f$ is continuous at $a$.

Therefore $\lim_{m\to\infty} x_{n_m} = a \Rightarrow \lim_{m\to\infty} f(x_{n_m}) = f(a)$ (By Sequential Criterion for Continuity)

Now $a - \frac{1}{n} < f(x_n) \leq a.$ for all $n \in N \Rightarrow a - \frac{1}{n} < f(x_{n_m}) \leq a$ for all $m \in N$.

Since $\lim_{m\to\infty} 1 - \frac{1}{n_m} = a$ and $\lim_{m\to\infty} a = a$.

Then by the Squeeze Theorem we get

$\lim_{m\to\infty} f(x_{n_m}) = a$. Therefore there exists $\alpha \in I$ such that $f(\alpha) = a$ (absolute maximum)

Therefore $a = f(\alpha) \geq f(x)$.

Because $b = \inf f(I)$ then for $n \in \mathbb{N}$

Then $b+\frac{1}{n}$ is not a lower bound of the set $f(I)$.

Hence for all $n \in \mathbb{N}$, there exits an $y_n \in I$ such that $b \leq f(x_n) < b+\frac{1}{n}$.

As it is given that $I=[a,b]$ is a bounded set and $x_n \in I$ for all $n \in \mathbb{N}$,

Therefore the sequence $\{y_n\}$ is also bounded. (By The Bolzano-Weierstrass Theorem)

$\{y_n\}$ is a bounded sequence $\Rightarrow$ there exists a convergent subsequence $\{y_{n_k}\}$ converges to $b$.

i.e. $\lim_{k\to\infty} y_{n_k} \to b$.

Because $y_{n_k} \in I$ for all $n_k \in \mathbb{N} \Rightarrow b \in I$.

Now $b \in I$ and $\boldsymbol{f} : \boldsymbol{I} \to \mathbb{R}$ is a continuous function, then $f$ is continuous at $a$.

Therefore $\lim_{k\to\infty} y_{n_k} = b \ \Rightarrow \ \lim_{k\to\infty} f(x_{n_k}) = f(b)$ (By Sequential Criterion for Continuity)

Now $b \leq f(x_n) < b+\frac{1}{n}$. for all $n \in N \Rightarrow b < f(x_n) < b+\frac{1}{n}$ for all $m \in N$.

Since $\lim_{k\to\infty} 1+\frac{1}{n_k} = b$ and $\lim_{m\to\infty} b = b$.

Then by the Squeeze Theorem we get

$\lim_{m\to\infty} f(x_{n_k}) = b$. Therefore there exists $\beta \in I$ such that $f(\beta)=b$ (absolute minimum)

Therefore $b = f(\beta) \leq f(x)$.

**CHECK YOUR PROGRESS**

(CQ 16) Let $I=[a,b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ need not be bounded on $I$.(T/F)

(CQ 17) A function is bounded on a set if its range is a bounded set in $\mathbb{R}$. (T/F)

(CQ 18) A function $f:[0,1] \to [0,1]$ such that $f(x)=x^2$ then $f$ is _________.

## *3.9 UNIFORM CONTINUITY*

**Uniform continuity:** Let $X$ be a nonempty subset of $\mathbb{R}$. A function $f: X \to \mathbb{R}$ is said to be uniformly continuous on $X$if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ and $|x-y| < \delta$, then

$|f(x)-f(y)| < \varepsilon$

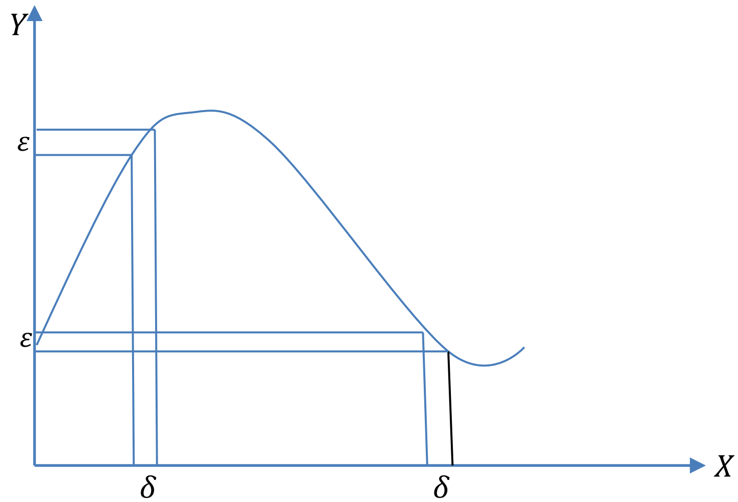



***Fig. 3.8. Uniform continuity***

**Example:**

$f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

$f(x) = x^2$ is uniformly continuous on $(-1,1)$.

**Theorem 3.12. A function which is uniformly continuous on an interval $\boldsymbol{I}$ is also continuous on interval $\boldsymbol{I}$.**

**Proof.** Let $f$ be a uniformly continuous on given interval $I$.

Hence for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

where $x_1, x_2 \in I$ any point.

Let $x \in I$, assume $x_1 = x$, then we see that

given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - f(x_2)| < \varepsilon$ whenever $|x - x_2| < \delta$.

Therefore the function is continuous at every point $x_2 \in I$.

Hence the function is continuous on interval $I$.

**Theorem 3.13. Let $\boldsymbol{f}$ be a continuous function defined on closed interval $[\boldsymbol{a}, \boldsymbol{b}]$. Then $\boldsymbol{f}$ is also uniformly continuous on $[\boldsymbol{a}, \boldsymbol{b}]$.**

**Proof.** It is given that function $f$ is continuous on $[a, b]$.

In contrary, let $f$ be not uniformly continuous on $[a, b]$.

Hence there exists an $\varepsilon > 0$ such that any $\delta > 0$

$|f(x) - f(y)| \geq \varepsilon$ whenever $|x - y| < \delta$

where $x, y \in [a. b]$.

Particularly for each positive integer $n$, we have $x_n, y_n \in \mathbb{R}$ in $[a, b]$ such that

$|f(x_n) - f(y_n)| \geq \varepsilon$ whenever $|x_n - y_n| < 1/n$…………………..(1)

Now $\{x_n\}$ and $\{y_n\}$ be sequences on a closed interval [a,b] and are bounded .

Hence $\{x_n\}$ and $\{y_n\}$ have at least one limit point say $l_1$ and $l_2$ respectively.

As we know that closed interval is closed set.

$\Rightarrow l_1, l_2 \in [a, b]$.

Also $l_1$ is a limit point of $\{x_n\}$

$\Rightarrow$there exists a convergent subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$x_{n_m} \to l_1$ when $m \to \infty$

Likewise there exists a convergent subsequence $\{y_{n_m}\}$ of $\{y_n\}$ such that

$y_{n_m} \to l_2$ when $m \to \infty$

From condition (1), we get

$\left|f(x_{n_m}) - f(y_{n_m})\right| \geq \varepsilon$ whenever $|x_n - y_n| < \frac{1}{n_m} \leq \frac{1}{m}$…………..(2)

From above condition we conclude that

$\lim_{m\to\infty} x_{n_m} = \lim_{m\to\infty} y_{n_m}$

Hence $l_1 = l_2$……………………………….(3)

From condition (1), we conclude that if sequences $\{f(x_{n_m})\}$ and $\{f(y_{n_m})\}$ converges then the limit they converege are distinct.

i.e. . $\{x_{n_m}\}$ and $\{y_{n_m}\}$ converges to $l_1$ but $\{f(x_{n_m})\}$ and $\{f(y_{n_m})\}$ do not converges to same limit.

Hence $f$ is not continuous, a contradiction.

Thus $f$ is uniformly continuous on $[a, b]$.

**CHECK YOUR PROGRESS**

(CQ 19) $f(x) = x^2$ is uniformly continuous on interval [0,1]. (T/F)

(CQ 20) $f(x) = x$ is not uniformly continuous on $\mathbb{R}$. (T/F)

## *3.10 SUMMARY*

In this unit we introduced the important notion of the limit of a function and discussed $\varepsilon - \delta$ definiotion of limits. We also established the fundamental properties that make continuous functions so important. For instance, we will prove that a continuous function on a closed bounded interval must attain a maximum and a minimum value as various examples illustrate, and thus they distinguish continuous functions as a very special class of functions.

## *3.11 GLOSSARY*

1. Set- a well defined collection of elements
2. Sequence-a function whose domain is set of natural number and range is set of real number
3. Series-sum of the term of sequences
4. Limit- numerical values get closer and closer to a given value
5. Continuity- a function that varies with no abrupt breaks or jumps

## *3.12 REFERENCES*

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## *3.12 SUGGESTED READINGS*

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2. W. Rudin (2019) Principles of Mathematical Analysis, McGraw-Hill Publishing, 1964.
3. Tom M. Apostol (1996). Mathematical Analysis (2nd edition), Narosa Book Distributors Pvt Ltd-New Delhi.
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## 3.13 TERMINAL QUESTION

**Long Answer Questions**

(TQ 1) Prove that $\lim_{x\to 0}\frac{1}{x^2}$ does not exist using $\varepsilon - \delta$ definition.

(TQ 2) Show by example that if $l \neq 0$ then $f$ may not have a limit at $b$.

(TQ 3) Find functions $f$ and $g$ defined on $(0, 1)$ such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty}(f(x) - g(x)) = \infty$.

(TQ 5) Let $f$ and $g$ be defined on $(a, \infty)$ and suppose $\lim_{x\to\infty} f(x) = l$ and $\lim_{x\to\infty} g(x) = \infty$. Prove that $\lim_{x\to\infty} fog(x) = l$.

**Fill in the blanks**

(TQ 6) $\lim_{x\to 0} x\, sin\frac{1}{x}$ is ____________________.

(TQ 7) $f(x) = \frac{1}{x}$ has ____________absolute maximum.

(TQ 8) $lim_{x\to 0}\ \frac{\sqrt{9+x}-3}{x}$

(TQ 9) A sequence $\{y_n = (-1)^n\}$ is_________________

## 3.14 ANSWERS

| | | |
|---|---|---|
| (CQ 1) F | (CQ 2) F | (CQ 3) F |
| (CQ 4) $\mathbb{R}^+$ | (CQ 5) T | (CQ 6) F |
| (CQ 7) does not exist | (CQ 8) T | (CQ 9) F |
| (CQ 10) does not exist | (CQ 11) F | (CQ 12) T |
| (CQ 13) T | (CQ 14) T | (CQ 15) continuous |
| (CQ 16) F | (CQ 17) T | (CQ 18) bounded |
| (CQ 19) T | (CQ 10) F | |
| (TQ 6) 0 | (TQ 7) divergent | (TQ 8) 0 |

# UNIT 4: DERIVATIVE AND MEAN VALUE THEOREM

**CONTENTS**



## *4.1 INTRODUCTION*

In previous unit we discussed about limit and continuity. In this unit we will discussed about differentiability and mean value theorem.

Before the seventeenth century, a curve was typically thought of as a collection of points that satisfied some kind of geometric requirement, and tangent lines were created using geometric operations. With the development of analytic geometry in the 1630s by Rene Descartes (1596-1650) and Pierre de Fermat (1601-1665), this perspective underwent a significant shift. In this new environment, algebraic expressions rather than geometric criteria were used to define new classes of curves and to recast geometry issues. In this new setting, the derivative concept developed. The problem of finding tangent lines and the seemingly unrelated. Fermat was the first to recognise a problem of finding maximum or minimum values connection by in the 1630s. And the relation between tangent lines to curves and Isaac Newton discovered in the late 1660s the connection between tangent lines to curves and the velocity of a moving particle. Once certain vocabulary and notational modifications are made, any contemporary student of differential calculus will be familiar with Newton's theory of "fluxions," which was founded on an intuitive notion of limit.

But the crucial finding was that areas under curves could be determined by reversing the differentiation process. This discovery was achieved independently by Gottfried Leibniz and Newton in the 1680s. This innovative method, which made it simple to address previously challenging area problems.

We will develop the theory of differentiation in this Unit. The following unit will cover integration theory, including the fundamental theorem connecting differentiation and integration. As a result we will discuss its mathematical characteristics.

## *4.2 OBJECTIVES*

After reading this unit learners will be able to

1. recognized the basic concept of derivative
2. analyze about Mean value theorem
3. learned some important theorems.

## *4.3 DERIVATIVE*

We begin with the definition of the derivative of a function.

**Derivative: Let $I \subseteq \mathbb{R}$** be an interval, $f:(x,y) \to \mathbb{R}$ and $b \in I$. Then $l \in \mathbb{R}$ is said to be derivative of $f$ at $b$ if for any given $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that if $x \in I$ then

$\left|\frac{f(x)-f(b)}{x-b} - l\right| < \varepsilon$ whenever $0 < |x-b| < \delta$.

We can also say that $f$ is differentiable at $b$, and we write $f'(c)$

Or

The derivative of $f$ at $b$ is given by$f'(c) = \lim_{h\to 0} \frac{f(x)-f(b)}{x-h}$ provided this limit exists.

We now show that continuity of $f$ at a point $b$ is a necessary (but not sufficient) condition for the existence of the derivative at $b$.

**Theorem 4.1. If $f : I \to \mathbb{R}$ has a derivative at $b \in I$, then $f$ is continuous at $b$.**

**Proof.** We have

$f(x) - f(b) = \left(\frac{f(x)-f(b)}{x-b}\right)(x-b)$ For all $x \in I; x \neq b$

Because $f'(b)$ exists, Therefore

$\lim_{x\to b}(f(x) - f(b)) = \lim_{x\to b}\left(\left(\frac{f(x)-f(b)}{x-b}\right)(x-b)\right) =$

$\lim_{x\to b}\left(\frac{f(x)-f(b)}{x-b}\right)\lim_{x\to b}(x-b)$

$= f'(b).0 = 0$

Therefore, $\lim_{x\to b}(f(x) - f(b)) = 0 \Rightarrow \lim_{x\to b} f(x) - \lim_{x\to b} f(b) \Rightarrow$

$\lim_{x\to b} f(x) = f(b)$

Hence $f$ is continuous at $b$.

**NOTE:** The continuity of $f : I \to \mathbb{R}$ at a point does not promise the existence of the derivative at that point.

**Theorem 4.2. Let $I \subseteq \mathbb{R}$ be an Interval and $f, g: X \to \mathbb{R}$ be functions that are differentiable at $b \in \mathbb{R}$ Then**

**(i) If $\alpha \in \mathbb{R}$, then the function $\alpha f$ is differentiable at $b$ and** $(\alpha f)'(b) = \alpha f'(b)$

**(ii) The function $f+g$ is differentiable at $b$ and** $(f+g)'(b) = f'(b) + g'(b)$

**(iii) The function $f$ and $g$ is differentiable at $b$ and** $(fg)'(b) = f'(b)g(b) + f(b)g'(b)$

**(iv) If $g(b) \neq 0$, then the function $f$ and $g$ is differentiable at $b$ and**

$$\left(\frac{f}{g}\right)'(b) = \frac{f'(b)g(b) - f(b)g'(b)}{\left(g(b)\right)^2}$$

**Proof.** (i) Let $h_1 = \alpha f$, then for $x \in I \; and \; x \neq b$, we have

$$\frac{h_1(x) - h_1(b)}{x-b} = \frac{(\alpha f)(x) - (\alpha f)(b)}{x-b} = \alpha \frac{f(x) - f(b)}{x-b}$$

Since $f$ is differentiable at $b$ implies $f'(b)$ exists. Therefore

$$\lim_{x\to b} \frac{h_1(x) - h_1(b)}{x-b} = \lim_{x\to b} \alpha \frac{f(x) - f(b)}{x-b} = \alpha \lim_{x\to b} \frac{f(x) - f(b)}{x-b} = \alpha f'(c)$$

Hence $(\alpha f)'(b) = \alpha f'(b)$

(ii) Let $h_2 = f + g$, then for $x \in I \; and \; x \neq b$, we have

$$\frac{h_2(x) - h_2(b)}{x-b} = \frac{(f+g)(x) - (f+g)(b)}{x-b} = \frac{f(x) + g(x) - f(b) - g(b)}{x-b} = \frac{f(x) - f(b) + g(x) - g(b)}{x-b}$$

$$= \frac{f(x) - f(b)}{x-b} + \frac{g(x) - g(b)}{x-b}$$

Since $f$ and $g$ are differentiable at $b$ implies $f'(b)$ and $g'(b)$ exists. Therefore

$$\lim_{x\to b} \frac{h_2(x) - h_2(b)}{x-b} = \lim_{x\to b} \frac{f(x) - f(b)}{x-b} + \lim_{x\to b} \frac{g(x) - g(b)}{x-b} = f'(b) + g'(b)$$

Hence $(f+g)'(b) = f'(b) + g'(b)$

(iii) Let $h_3 = fg$, then for $x \in I \; and \; x \neq b$, we have

$$\frac{h_3(x) - h_3(b)}{x-b} = \frac{(fg)(x) - (fg)(b)}{x-b} = \frac{f(x)g(x) - f(b)g(b)}{x-b}$$

$$= \frac{f(x)g(x) - f(b)g(x) + f(b)g(x) - f(b)g(b)}{x-b}$$

$$= \frac{g(x)(f(x) - f(b)) + f(b)(g(x) - g(b))}{x-b} = g(x)\frac{f(x) - f(b)}{x-b} + f(b)\frac{g(x) - g(b)}{x-b}.$$

It is given that $f$ and $g$ is differentiable at $b$ and

$g$ is differentiable at $b \Rightarrow g$ is continuous i.e. $\lim_{x\to b} g(x) = g(b)$ (by previous theorem)

Therefore

$$\lim_{x\to b} \frac{h_3(x) - h_3(b)}{x-b} = \lim_{x\to b} \left\{ g(x)\frac{f(x) - f(b)}{x-b} + f(b)\frac{g(x) - g(b)}{x-b} \right\}$$

$= \lim_{x\to b} g(x)\frac{f(x)-f(b)}{x-b} + \lim_{x\to b} f(b)\frac{g(x)-g(b)}{x-b} =$
$\lim_{x\to b} g(x) \lim_{x\to b}\frac{f(x)-f(b)}{x-b} + f(b)\lim_{x\to b}\frac{g(x)-g(b)}{x-b} = f'(b)g(b) + f(b)g'(b)$

Hence $(fg)'(b) = f'(b)g(b) + f(b)g'(b)$

(iv) Let $h_4 = \frac{f}{g}$, since $g$ is differentiable at $b \Rightarrow$ since $g$ is continuous at $b$. It is given that $g(b) \neq 0$, therefore there exists an interval $I_1 \subseteq I$ with $b \in I_1$ such that

$g(x) \neq 0 \;\; for\; all\; x \in I_1$.

Now for $x \in I_1, x \neq b$, we get

$\frac{h_4(x)-h_4(b)}{x-b} = \frac{\frac{f}{g}(x) - \frac{f}{g}(b)}{x-b} = \frac{\frac{f(x)}{g(x)} - \frac{f(b)}{g(b)}}{x-b} = \frac{f(x)g(b)-f(b)g(x)}{g(b)g(b)(x-b)}$

$= \frac{f(x)g(b)-f(b)g(x)}{g(b)g(b)(x-b)} = \frac{f(x)g(b)-f(b)g(b)+f(b)g(b)-f(b)g(x)}{g(b)g(b)(x-b)}$

$\frac{\left(f(x)-f(b)\right)g(b)-f(b)\left(g(x)-g(b)\right)}{g(x)g(b)(x-b)} =$

$\frac{1}{g(x)g(b)}\left[\frac{f(x)-f(b)}{x-b}.\, g(b) - f(b).\frac{g(x)-g(b)}{x-b}\right]$

Therefore

$\lim_{x\to b}\frac{h_4(x)-h_4(b)}{x-b} = \lim_{x\to b}\frac{1}{g(x)g(b)}\left[\frac{f(x)-f(b)}{x-b}.\, g(b) - f(b).\frac{g(x)-g(b)}{x-b}\right]$

$=$

$\lim_{x\to b}\frac{1}{g(x)g(b)}\left[\lim_{x\to b}\left(\frac{f(x)-f(b)}{x-b}\right).\, g(b) - f(b).\lim_{x\to b}\left(\frac{g(x)-g(b)}{x-b}\right)\right]$

$= \frac{1}{g'^2(b)}.\,[f'(b)g(b) - f(b)g'(b)]$

Hence

$\left(\frac{f}{g}\right)'(b) = \frac{f'(b)g(b)-f(b)g'(b)}{\left(g(b)\right)^2}$

**NOTE:**

➢ If $f_1,\ f_2, \dots \dots, f_n$ are functions on an interval $I$ to $\mathbb{R}$ that are differentiable at $b \in I$, then:

(a) The function $f_1 + f_2 + \cdots \dots + f_n$ is differentiable at $b$ and
$(f_1 + f_2 + \cdots \dots + f_n)'(b) = f_1'(b) + f_2'(b) + \cdots \dots + f_n'(b)$

(b) The function $f_1\, f_2, \dots \dots f_n$ is differentiable at $b$, and
$(f_1\, f_2 \dots \dots f_n)'(b) = f_1'(b)f_2(b) \dots \dots f_n(b) + f_1(b)f_2'(b) \dots \dots f_n(b) + \cdots + f_1(b)\, f_2(b) \dots \dots f_n'(b)$

➢ If $f_1, = f_2 = \cdots \dots = f_n = f$ then $(f^n)'(b) = f'(b)\left(f(b)\right)^{n-1} + f'(b)\left(f(b)\right)^{n-1} + \dots + f'(b)\left(f(b)\right)^{n-1} = n\left(f(b)\right)^{n-1} f'(b)$

**<u>Theorem on the differentiation of composite functions</u>**

The theorem on the differentiation of composite functions known as the ‘‘Chain Rule.’’ It provides a formula for finding the derivative of a composite function $gof$ in terms of the derivatives of $g$ and $f$.

**Caratheodory’s Theorem**

**Theorem 4.3. Let $f$ be defined on an interval $I$ containing the point $b$. Then f is differentiable at $b$ if and only if there exists a function $h$ on $I$ that is continuous at $b$ and satisfies**
$\boldsymbol{f(x) - f(b) = h(x)(x - b) \quad for\ x \in I}$ **.**
**Here $\boldsymbol{h(b) = f'(b)}$**

**Proof.** It is given that $f$ is differentiable at $b$.
It implies that $f'(b)$ exists, hence we define $h(x)$ by

$$h(x) = \begin{cases} \frac{f(x)-f(b)}{x-b} & for\ x \neq b, x \in I \\ f'(b) & for\ x = c \end{cases}$$

Now we can easily see that $\lim_{x\to b} h(x) = \lim_{x\to b} \frac{f(x)-f(b)}{x-b} = f'(b) = h(b)$.
Therefore, $h$ is continuous at $b$.
$f(x) - f(b) = h(x)(x - b) \quad for\ x \in I$ ……….(1)
*Converse*
Now if $x = b$ then both side of equation (1) equal to 0.
If $x \neq b$, then multiplied $(x - b)$ in $h(x)$, we get
$(x - b)h(x) = f(x) - f(b)$ for $x \neq b, x \in I$
Hence $h(x)$ satisfies equation (1).
Now if we divide equation (1) by $(x - b) \neq 0$, then the continuity of $h$ implies that
$h(b) = \lim_{x\to b} h(x) = \lim_{x\to b} \frac{f(x)-f(b)}{x-b}$ exists.
Therefore $f$ is differentiable at $b$ and $f'(b) = h(b)$.

**Example**
Let we defined a function $f(x) = x^2\ for\ x \in \mathbb{R}$.
For $c \in \mathbb{R}$, Now
$f(x) - f(b) = x^2 - b^2 = (x + b)(x - b)$
Let $h(x) = x + b$ and we can see that $f(x) - f(b) = h(x)(x - b)$
Hence it satisfies the Caratheodory’s Theorem
Therefore $f$ is differentiable at $b \in \mathbb{R}$ and $f'(b) = h(b) = 2b$

**Chain Rule**
**Theorem 4.4. Let $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ be intervals in $\mathbb{R}$, let $\boldsymbol{g : I_1 \to \mathbb{R}}$ and $\boldsymbol{f : I_2 \to \mathbb{R}}$ be functions such that $\boldsymbol{f(I_2) \subseteq I_1}$, and let $\boldsymbol{b \in I_2}$. If $f$ is differentiable at $b$ and if $g$ is differentiable at $f(b)$, then the composite function $gof$ is differentiable at $b$ and**

$(\boldsymbol{gof})'(\boldsymbol{b}) = \boldsymbol{g}'\big(\boldsymbol{f}(\boldsymbol{b})\big).\boldsymbol{f}'(\boldsymbol{b})$ **……………….(1)**

**Proof.** It is given that $f'(b)$ exists.
So, by Caratheodory's Theorem
There exists a function $h$ on $I_2$ such that $h$ is continuous at $b$ and
$f(x) - f(b) = h(x)(x - b) \quad for\ x \in I_2,$
Where $h(b) = f'(b)$.
Also because $g'\big(f(b)\big)$ exists, there is a function $h$ defined on $I_1$ such that $h$ is continuous at $\alpha = f(b)$ and
$g(y) - g(\alpha) = h(y)(y - \alpha) \ \ for\ y \in I_1,\ where\ h(d) = g'(d)$
Substitution of $y = f(x)$ and $d = f(b)$ then
$g\big(f(x)\big) - g\big(f(b)\big) = h\big(f(x)\big)\big(f(x) - f(b)\big) = \big[\big(hof(x)\big).h(x)\big](x - b) for\ all\ x \in I_2\ such\ that\ f(x) \in I_1.$
Since the function $(hof).h$ is continuous at $b$ and
its value at $b$ is $g'\big(f(b)\big).f'(b)$, Caratheodory's Theorem gives (1).

**Example:**
If $f\ :\ I \to\ \mathbb{R}$ is differentiable on $I$ and $(y) = y^2\ for\ y \in \mathbb{R}$ .
Since $g'(y) = 2y$.
By chain rule $(gof)'(x) = g'\big(f(x)\big).f'(x) \quad for\ x \in I$
$(f^2)'(x) = 2f(x)f'(x)\ for\ x \in I$

**<u>Inverse Functions</u>**

We will now relate the derivative of a function to the derivative of its inverse function, when this inverse function exists

**Theorem 4.5. Let $\boldsymbol{I}$ be an interval in $\mathbb{R}$ and let $\boldsymbol{f}\ :\ \boldsymbol{I_1} \to\ \mathbb{R}$ be strictly monotone and continuous on $\boldsymbol{I}$. Let $\boldsymbol{I_2} = \boldsymbol{f}(\boldsymbol{I_1})$ and let $\boldsymbol{g}: \boldsymbol{I_2} \to \mathbb{R}$ be the strictly monotone and continuous**
**function inverse to $\boldsymbol{f}$. If $\boldsymbol{f}$ is differentiable at $\boldsymbol{b} \in \boldsymbol{I_1}$ and $\boldsymbol{f}'(\boldsymbol{b}) \neq \boldsymbol{0}$, then $\boldsymbol{g}$ is differentiable at $\boldsymbol{\alpha} = \boldsymbol{f}(\boldsymbol{b})$ and $\boldsymbol{g}'(\boldsymbol{\alpha}) = \frac{\boldsymbol{1}}{\boldsymbol{f}'(\boldsymbol{b})} = \frac{\boldsymbol{1}}{\boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{\alpha}))}$**

**Proof.** It is given that $b \in \mathbb{R}$, and $f$ is differentiable at $b \in I_1$. From Caratheodory's Theorem there exists a function $h$ on $I_1$ with properties that $h$ is continuous at $b$
$f(x) - f(b) = h(x)(x - b)\ \ for\ x \in I_1$ and $h(b) = f'(b)$.
Since $h(b) \neq 0$ by hypothesis, there exists a neighborhood $V = (b - \delta, b + \delta)$ such that $h(x) \neq 0\ for\ all\ x \in V \cap I$.
If $U = f(V \cap I)$, then the inverse function $g$ satisfies $f\big(g(y)\big) = y \quad for\ all\ y \in U,$ so that
$y - \alpha = f\big(g(y)\big) - f(c) = h\big(g(y)\big).\big(g(y) - g(\alpha)\big).$
Since the function $h\big(g(y)\big) \neq 0\ for\ y \in U.$
We can divide to get
$g(y) - g(\alpha) = \frac{1}{h\big(g(y)\big)}.(y - \alpha).$

As the function $\frac{1}{hog}$ is continuous at $\alpha$, By Catheodary theorem we get $g'(\alpha)$ exists and $g'(\alpha) = \frac{1}{h(g(\alpha))} = \frac{1}{h(b)} = \frac{1}{f'(b)}$.

**CHECK YOUR PROGRESS**
(CQ 1) State and prove Caratheodory's Theorem
(CQ 2) Define derivative of a function.

## *4.4 MEAN VALUE THEOREM*

The Mean Value Theorem, which relates the values of a function to values of its derivative, is one of the most useful results in real analysis
We begin by looking at the relationship between the relative extrema of a function and the values of its derivative.

**Relative Maximum:** The function $f : I \to \mathbb{R}$ is said to have a relative maximum at $b \in I$ if there exists a neighborhood $V = V_\delta(b)$ of $b$ such that $f(x) \leq f(b)$, for all $x \; in \; V \cap I$.

**Relative Minimum:** The function $f : I \to \mathbb{R}$ is said to have a relative minimum at $b \in I$ if there exists a neighborhood $V' = V'_{\delta'}(b)$ of $b$ such that $f(x) \geq f(b)$, for all $x \; in \; V' \cap I$.

**Relative Extremum:** $f$ has a relative extremum at $b \in I$ if it has either a relative maximum or a relative minimum at $b$.

**Interior Extremum Theorem**

**Theorem 4.6. Let $b$ be an interior point of the interval $I$ at which $f : I \to R$ has a relative extremum. If the derivative of $f$ at $b$ exists, then $f'(b) = 0$.**
**Proof.** If $f'(b) > 0$, Then there exists a neighborhood $V \subseteq I$ of $b$ such that
$\frac{f(x)-f(b)}{x-b} > 0$ for $x \in V, x \neq b$
$If \;\; x \in V, x > b$, then we get
$f(x) - f(b) = (x - b).\frac{f(x)-f(b)}{x-b} > 0$
But this contradicts the hypothesis that $f$ has a relative maximum at $b$.
Hence we cannot have $f'(b) > 0$.
Similarly we cannot have $f'(b) < 0$.
Therefore, $f'(b) = 0$.

**Rolle's Theorem**

**Theorem 4.7. Consider that $f$ is continuous on a closed interval $I = [a, b]$ and the derivative $f'(0)$ exists at every point of the open interval $(a, b)$, and $f(a) = f(b) = 0$.Then there exists at least one point $c$ in $(a, b)$ such that $f'(c) = 0$**

**Proof.** If $f(x) = 0\ for\ all\ x\ in\ I$ or vanishes identically on $I$, then any $c$ in $(a, b)$ will satisfy the result of the theorem.
Let $f$ does not vanish identically or $f \neq 0$.
Now replacing $f$ by $(-f)$ and Consider $f$ assumes some positive values.
So by the Maximum Minimum Theorem,
The function $f$ attains the value $\sup\{f(x): x \in I\} > 0$ at some point $c$ in $I$.
Since $f(a) = f(b) = 0$. the point c must lie in $(a, b)$.
Hence $\boldsymbol{f'(c)}$ exists.
Since $f$ has a relative maximum at $c$.
By the Interior Extremum Theorem, we get
$f'(c) = 0$

**Geometrical Representation of Rolle's theorem**

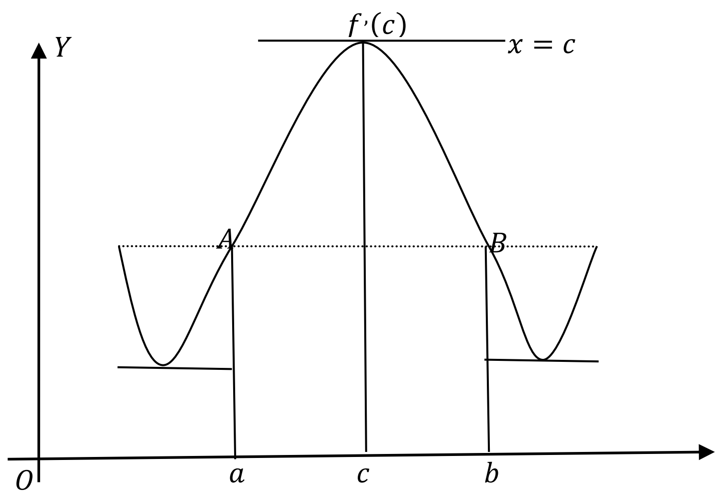


***Fig.4.1. Rolle's theorem***

In the given graph, the curve $y = f(x)$ is continuous between $x = a$ and $x = b$ and at every point, within the interval, it is possible to draw a tangent and ordinates corresponding to the abscissa and are equal then there exists at least one tangent to the curve which is parallel to the $x$-axis. Algebraically, this theorem tells us that if $f(x)$ is representing a polynomial function in $x$ and the two roots of the equation $f(x) = 0$ are $x = a$ and $x = b$, then there exists at least one root of the equation $f'(x) = 0$ lying between these values. The converse of Rolle's theorem is not true and it is also possible that there exists more than one value of $x$, for which the theorem holds good but there is a definite chance of the existence of one such value.

**NOTE:**

➢ Rolle's theorem does not hold good if

(i) f (x) is discontinuous in the closed interval [a, b].

(ii) f (x) does not exists at some point in (a, b).

(iii) f (a) ≠ f (b).

**Example:** Rolle's Theorem can be used for the location of roots of a function.

**NOTE:**

For, if a function $g$ can be identified as the derivative of a function $f$, then between any two roots of $f$ there is at least one root of $g$. For example, let $g(x) = \cos x$ then $g$ is known to be the derivative of $f(x) = \sin x$. Hence, between any two roots of $sin\, x$ there is at least one root of $\cos x$ . On the other hand $g'(x) = -sinx = -f(x)$ .

Another application of Rolle's Theorem informed us that between any two roots of $cos$ there is at least one root of $sin$.Therefore, we conclude that the roots of sin and cos interlace each other.

**Mean Value Theorem**

**Theorem 4.7. Suppose that $\boldsymbol{f}$ is continuous on a closed interval $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$and $\boldsymbol{f}$ has a derivative in the open interval $(\boldsymbol{a}, \boldsymbol{b})$. Then there exists atleast one point $\boldsymbol{c}$ in $(\boldsymbol{a}, \boldsymbol{b})$ such that**

$$\boldsymbol{f(b) - f(a) = f'(c)(b - a)}$$

**Proof.** Assume the function $\Phi$ defined on $I$ such that

$$\Phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$$

We can easily see that The Conditions of Rolle's Theorem are satisfied by $\Phi$ since $\Phi$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $\Phi(a) = \Phi(b)$.

Therefore, there exists a point $b$ in $(a, b)$ such that

$$0 = \Phi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} .$$

Therefore $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow f(b) - f(a) = f'(c)(b - a)$

### Geometrical Interpretation

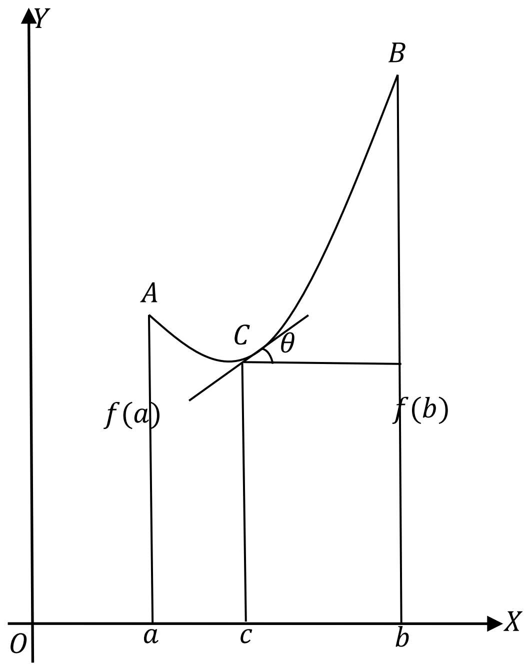


***Fig.4.2. Lagrange theorem***

The geometric view of the Mean Value Theorem is that there is some point on the curve $y = f(x)$ at which the tangent line is parallel to the line segment through the points $(a, f\ (a))\ and\ (b, f\ (b))$. Thus it is easy to remember the statement of the Mean Value Theorem by drawing appropriate diagrams. While this should not be discouraged, it tends to suggest that its importance is geometrical in nature, which is quite misleading. In fact the Mean Value Theorem is a wolf in sheep's clothing and is the Fundamental Theorem of Differential Calculus. The Mean Value Theorem permits one to draw conclusions about the nature of a function f from information about its derivative $f^{'}$ . The following results are obtained in this manner.

**Theorem 4.8. Suppose that $\boldsymbol{f}$ is continuous on the closed interval $\boldsymbol{I = [a, b]}$ that $\boldsymbol{f}$ is differentiable on the open interval $\boldsymbol{(a, b)}$, and that $\boldsymbol{f'(x) = 0\ for\ x \in (a, b)}$. Then $\boldsymbol{f}$ is constant on $\boldsymbol{I}$.**

**Proof.** Let $x \in I$ and $x > a$
Then by mean value theorem to $f$ on the closed interval $[a, x]$.
There exists a point c( depending on x) between a and x such that
$f(x) - f(a) = f'(c)(x - a)$.
Because $f'(c) = 0$ (given)
Hence we conclude that
$f(x) - f(a) = 0$ .
Therfore $f(x) = f(a)\ for\ any\ x \in I$

$f$ is constant on $I$.

**Theorem 4.9. Let $f : I \to \mathbb{R}$ be differentiable on the interval $I$. Then**
**(i) $f$ is increasing on $I$ if and only if $f'(x) \geq 0$ $for\ all\ x \in I$.**
**(ii) $f$ is decreasing on $I$ if and only if $f'(x) \leq 0$ $for\ all\ x \in I$.**

**Proof.** (i) Suppose that $f'(x) \geq 0$ $for\ all\ x \in I$.
If $x_1$ and $x_2$ in $I$ satisfy $x_1 < x_2$,
then by Mean Value Theorem to $f$ on the closed interval $I_2 = [x_1, x_2]$ to obtain a point $c$ in
$(x_1, x_2)$ such that
$f(x_2) - f(x_1) = f'(b)(x_2 - x_1)$ ....…………………………(1)
Since $f'(b) \geq 0$ and $x_2 - x_1 > 0$, using this values in equation (1) we get
$f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_2) \geq f(x_1)$
Hence, $f(x_1) \leq f(x_2)$ $where\ x_1 and\ x_2 are\ arbitrary\ points$.
Therefore, f is increasing on I.
*Converse*
We suppose that $f$ is differentiable and increasing on $I$.
Thus, for any point $x \neq b$ in I, we have $\frac{f(x)-f(b)}{x-b} \geq 0$.
Hence, $f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b} \geq 0$.

(ii) Suppose that $f'(x) \leq 0$ $for\ all\ x \in I$.
If $x_1$ and $x_2$ in $I$ satisfy $x_1 < x_2$,
then by Mean Value Theorem to $f$ on the closed interval $I_2 = [x_1, x_2]$ to obtain a point $c$ in
$(x_1, x_2)$ such that
$f(x_2) - f(x_1) = f'(b)(x_2 - x_1)$ ....…………………………(1)
Since $f'(b) \leq 0$ and $x_2 - x_1 > 0$, using this values in equation (1) we get
$f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_2) \leq f(x_1)$
Hence, $f(x_1) \geq f(x_2)$ $where\ x_1 and\ x_2 are\ arbitrary\ points$.
Therefore, $f$ is decreasing on $I$.
*Converse*
We suppose that $f$ is differentiable and decreasing on $I$.
Thus, for any point $x \neq b$ in I, we have $\frac{f(x)-f(b)}{x-b} \leq 0$.
Hence, $f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b} \leq 0$.

**Strictly Increasing:** A function $f$ is said to be strictly increasing on an interval $I$ if for any points $x_1 and\ x_2$ in I such that $x_1 < x_2$, we have $(x_1) < f(x_2)$ .

**Strictly decreasing:** A function $f$ is said to be strictly increasing on an interval $I$ if for any points $x_1 and\, x_2$ in I such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

## The Intermediate Value Property of Derivatives

**Lemma Let $I \subseteq R$ be an interval, let $f : I \to \mathbb{R}$, let $b \in I$ and let $f$ has a derivative at $b$. Then:**
**(i) If $f'(b) > 0$, then there is a number $\delta_2 > 0$ such that $f(x) > f(b)$ for $x \in I$ such that $b < x < b + \delta_2$**
**(ii) If $f'(b) < 0$, then there is a number $\delta_1 > 0$ such that $f(x) < f(b)$ for $x \in I$ such that $b - \delta_2 < x < b$.**

**Proof.** (i) let $f$ has a derivative at $b$.
Therefore $lim_{x\to b} \frac{f(x)-f(b)}{x-b} = f'(b) > 0$ (Given)
Therefore that there exists a number $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - b| < \delta_1$ then
$\frac{f(x)-f(b)}{x-b} > 0$
If $x \in I$ and $x > b \Rightarrow x - b > 0$, then
$f(x) - f(b) = (x - b)\frac{f(x)-f(b)}{x-b} > 0$
Hence, if $x \in I$ and $b < x < b + \delta_1$, then $f(x) > f(b)$

(ii) If $f'(b) > 0$, then there is a number $\delta_2 > 0$ such that $f(x) > f(b)$ for $x \in I$ such that $b < x < b + \delta_2$
(ii) $lim_{x\to b} \frac{f(x)-f(b)}{x-b} = f'(b) < 0$ (Given)
Therefore that there exists a number $\delta_2 > 0$ such that if $x \in I$ and $0 < |x - b| < \delta_2$ then
$\frac{f(x)-f(b)}{x-b} < 0$
If $x \in I$ and $x > b \Rightarrow x - b > 0$, then
$f(x) - f(b) = (x - b)\frac{f(x)-f(b)}{x-b} < 0$
Hence, if $x \in I$ and $b - \delta_2 < x < b$, then $f(x) < f(b)$

## Darboux's Theorem

**Theorem 4.10. If $f$ is differentiable on $I = [a, b]$ and if $k$ is a number between $f'(a)$ and $f'(b)$, then there is at least one point $c$ in $(a, b)$ such that $f'(c) = k$.**

**Proof.** Let $f'(a) < \alpha < f'(b)$.
Now we define $g$ on $I$ by $g(x) = \alpha x - f(x)\, for\; x \in I$.
As $g$ is continuous, therefore it attains a maximum value on $I$.
Now $g'(a) = \alpha - f'(a) > 0$
Therefore from previous lemma we conclude that

the maximum of $g$ does not occur at $x = a$.
Similarly, maximum does not occur at $x = b$.
Hence, $g$ attains its maximum at some $c$ in $(a, b)$.
Then from Interior Extremum Theorem, we have
$0 = g'(c) = \alpha - f'(c)$
Hence $f'(c) = \alpha$.

**<u>Cauchy Mean Value Theorem</u>**

**Theorem 4.11. Let $f$ and $g$ be continuous on $[a, b]$ and differentiable on $(a, b)$, and assume that $g(x) \neq 0$ $for\ all\ x\ in$ $(a, b)$. Then there exists $c$ in $(a, b)$ such that**

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

**Proof.** Since $g'(x) \neq 0$ for all $x$ in $(a, b)$, therefore
Using Rolle's Theorem we get
$g(a) \neq g(b)$.
For $x$ in $[a, b]$, now new define

$$\varphi(x) = \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x) - g(a)\big) - \big(f(x) - f(a)\big)$$

Then $h$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $\varphi(a) = \varphi(b) = 0$ .
Therefore, According to Rolle's Theorem
there exists a point $c$ in $(a, b)$ such that

$$0 = \varphi'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) - f'(c)$$

As we know $g'(c) \neq 0$ , we obtain required result that is

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

**Geometrical Interpretation**

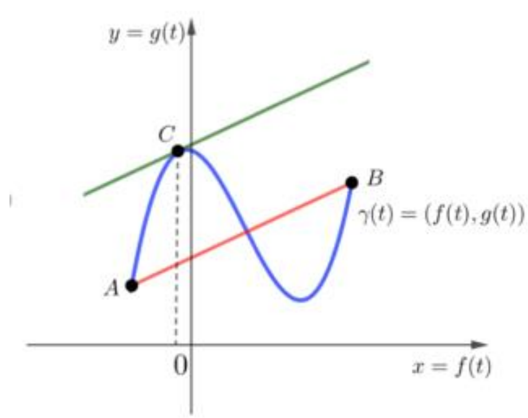


***Fig. 4.3. Cauchy Mean Value theorem***

According to the Cauchy's mean value theorem, there is a point $C = (f\ (\eta),\ g(\eta))$ on the curve $\gamma$ where the tangent is parallel to the chord joining the points $A = (f\ (a), g(a))$ and $B = (f\ (b), g(b))$ of the curve

**Taylor's Theorem**

A very useful technique in the analysis of real functions is the approximation of functions by polynomials. In this section we will prove a fundamental theorem in this area that goes back to Brook Taylor (1685–1731), although the remainder term was not provided until much later by Joseph-Louis Lagrange (1736–1813). Taylor's Theorem is a powerful result that has many applications. We will illustrate the versatility of Taylor's Theorem by briefly discussing some of its applications to numerical estimation, inequalities, extreme values of a function, and convex functions.

**Theorem 4.12. Let $\boldsymbol{n \in \mathbb{N}}$ and $\boldsymbol{I = [a, b]}$ and let $\boldsymbol{f : I \to \mathbb{R}}$ be such that $\boldsymbol{f}$ and its derivatives $\boldsymbol{f', f'', \dots\dots.., f^n}$ are continuous on $\boldsymbol{I}$ and that $\boldsymbol{f^{n+1}}$ exists on (a, b). If $\boldsymbol{x_0 \in I}$, then for any $\boldsymbol{x}$ in $\boldsymbol{I}$ there exists a point $\boldsymbol{c}$ between $\boldsymbol{x}$ and $\boldsymbol{x_0}$ such that**

$$\boldsymbol{f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + \frac{f^{n+1}(x_0)}{(n+1)!}(x - x_0)^n}$$

**Proof.** Consider $x_0$ and $x$ be given and let $I_1$ denote the closed interval with endpoints $x_0$ and $x$.

Now we define the function $F$ on $I_1$ by

$$F(t) = f(x) - f(t) - (x - t)f'(t) - \frac{(x-t)^2}{2!}f''(t) \dots - \frac{(x-t)^{n-1}}{(n-1)!}f^{(n-1)}(t) - \frac{(x-t)^n}{n!}f^{(n)}(t) \text{ for } t \in I_1 \dots\dots\dots\dots\dots\dots\dots(1)$$

Now by differentiate equation (1) w.r.t to $t$, we get

$$F'(t) = -f'(t) + f'(t) - (x - t)f''(t) + (x - t)f''(t) - \frac{(x-t)^2}{2!}f'''(t) - \cdots + \frac{(x-t)^{n-2}}{(n-2)!}f^{(n-1)}(t) - \frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t) - \frac{(x-t)^n}{n!}f^{(n+1)}(t)$$

Therefore

$$F'(t) = -\frac{(x-t)^n}{n!}f^{(n+1)}(t) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Now we define $H$ on $I_1$ such that $H(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0)$ for $t \in I_1$, then

$$H(x_0) = F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = F(x_0) - F(x_0) = 0 \text{ and}$$

$H(x) = F(x) - \left(\frac{x-x}{x-x_0}\right)^{n+1} F(x_0) = F(x) = 0$ (by putting $t = x$ in equation (1))

Now $H(x_0) = H(x) = 0$, by Rolle's theorem there exists $\alpha$ between $x$ and $x_0$ such that $H'(\alpha) = 0$

$H'(t) = F'(t) + \frac{(n+1)}{(x-x_0)}\left(\frac{x-t}{x-x_0}\right)^n F(x_0) \Rightarrow H'(\alpha) = F'(\alpha) + (n+1)\frac{(x-\alpha)^n}{(x-x_0)^{n+1}} F(x_0)$

Now $H'(\alpha) = 0 \Rightarrow F'(\alpha) + (n+1)\frac{(x-\alpha)^n}{(x-x_0)^{n+1}} F(x_0) = 0 \Rightarrow (n+1)\frac{(x-\alpha)^n}{(x-x_0)^{n+1}} F(x_0) = -F'(\alpha)$

$\Rightarrow F(x_0) = -\frac{1}{(n+1)}\frac{(x-x_0)^{n+1}}{(x-\alpha)^n} F'(\alpha)$

Using equation (2) we get

$F(x_0) = -\frac{1}{(n+1)}\frac{(x-x_0)^{n+1}}{(x-\alpha)^n}\left(-\frac{(x-\alpha)^n}{n!} f^{(n+1)}(\alpha)\right) = \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-x_0)^{n+1}$

which implies the stated result.

**Convex Functions:** Let $I \to \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be convex on $I$ if for any $t$ satisfying $0 \le t \le 1$ and any points $x, y\ in\ I$, then

$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y)$

**Geometrical Representation**

If $x < y$, then as $t$ ranges from 0 to 1, the point $(1-t)x + ty$ traverses the interval from $x$ to $y$. Hence if $f$ is convex on $I$ and if $x, y \in I$ then the chord joining any two points $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$ on the graph of $f$ lies above the graph of $f$.

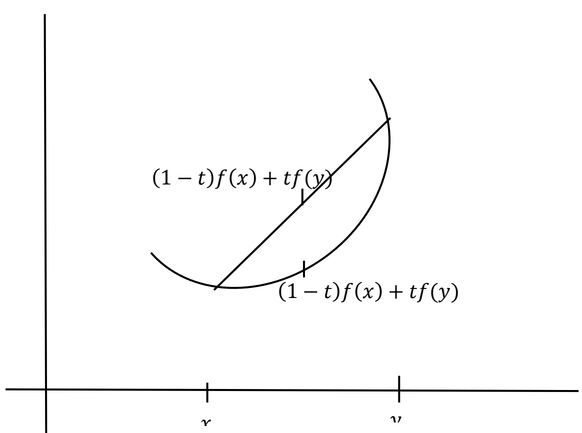


***Fig. 4.4 .Convex Function***

**NOTE:**

- A convex function need not be differentiable at every point, as the example $f(x) = |x|, x \in \mathbb{R}$.

- if I is an open interval and if $f : I \to \mathbb{R}$ is convex on $I$, then the left and right derivatives of $f$ exist at every point of $I$.
- We can also conclude that a convex function on an open interval is necessarily continuous.

**Theorem 4.13. Let $I$ be an open interval and let $f : I \to \mathbb{R}$ have a second derivative on $I$. Then $f$ is a convex function on $I$ if and only if $f''(x) \geq 0$ $for\ all\ x \in I$.**

**Proof.** $f$ have a second derivative on $I$ .
Hence second derivative of $f$ is given by

$f''(c) = \lim_{h\to 0} \frac{f'(x)-f'(c)}{x-h}$ $\quad for\ all\ c \in I$ ……………………….(1)

Now $f$ have a derivative on $c \in I$

Hence $f'(c) = \lim_{h\to 0} \frac{f'(x)-f'(c)}{x-h}$……………………………………..(2)

Similarly $f$ have a derivative on $c + k \in I$

Therefore $f'(c) = \lim_{h\to 0} \frac{f'(x)-f'(c+k)}{x-h}$………………………….(3)

Subtract equation (2) from (3), we get

$f'(c + h) - f'(c) = \lim_{h\to 0} \frac{f(x)-f(c+k)}{x-h} - \lim_{h\to 0} \frac{f(x)-f(c)}{x-h} =$
$\lim\limits_{h\to 0} \frac{f(x)-f(c+k)-(f(x)-f(c))}{x-h}$

Therefore

$f''(c) = \lim_{h\to 0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2}$ $\ for\ each\ c \in I$ ………………(2)

Now $c \in I$, let $h$ be such that $c + h$ and $c - h$ belong to $I$.
Then $c$ can be written as

$c = \frac{1}{2}\big((c + h) + (c - h)\big)$and since $f$ is convex on $I$, we have

$f(c) = f\left(\frac{1}{2}(c + h) + \frac{1}{2}(c - h)\right) \leq \frac{1}{2}f(c + h) + \frac{1}{2}f(c - h)$

$\Rightarrow 2f(c) \leq f(c + h) + f(c - h)$

Therefore, we have $f(c + h) - 2f(c) + f(c - h) \geq 0$.
Because $h^2 > 0$ for all $h \neq 0$, we observe that
$\lim_{h\to 0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2}$ must be nonnegative. Hence, we obtain $f''(c)$ for any $c \in I$.

*Converse*

Let $x_1$ and $x_2$ be any two points of $I$, assume $0 < t < 1$ and

$x_0 = (1 - t)x_1 + tx_2$ ....…………………………………..(1)

Now Applying Taylor's Theorem to $f$ at $x_0$, we get a point $\alpha_1$ between $x_0$ and $x_1$ such that

$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2}f''(\alpha_1)(x_1 - x_0)^2$

a point $\alpha_2$ between $x_0$ and $x_2$ such that

$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2}f''(\alpha_2)(x_2 - x_0)^2$

Hence, we obtain

$(1-t)f(x_1)+tf(x_2)=(1-t)\Big(f(x_0)+f'(x_0)(x_1-x_0)+\frac{1}{2}f''(\alpha_1)(x_1-x_0)^2\Big)+t\Big(f(x_0)+\ f'(x_0)(x_2-x_0)+\frac{1}{2}f''(\alpha_2)(x_2-x_0)^2\Big)$

$=(1-t)f(x_0)+(1-t)f'(x_0)(x_1-x_0)+\frac{(1-t)}{2}f''(\alpha_1)(x_1-x_0)^2+tf(x_0)+tf'(x_0)(x_2-x_0)+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2$

$=(1-t)f(x_0)+(1-t)f'(x_0)(x_1-x_0)+\frac{(1-t)}{2}f''(\alpha_1)(x_1-x_0)^2+tf(x_0)+tf'(x_0)(x_2-x_0)+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2$

$=f(x_0)+f'(x_0)\big((1-t)x_1+tx_2-x_0\big)+\frac{(1-t)}{2}f''(\alpha_1)(x_1-x_0)^2+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2$

From equation (1), we get

$(1-t)f(x_1)+tf(x_2)=f(x_0)+f'(x_0)(x_0-x_0)+\frac{(1-t)}{2}f''(\alpha_1)(x_1-x_0)^2+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2$

Therefore

$(1-t)f(x_1)+tf(x_2)=$
$f(x_0)+\frac{(1-t)}{2}\frac{f''(\alpha_1)}{2}(x_1-x_0)^2+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2$

Let $R=\frac{(1-t)}{2}f''(\alpha_1)(x_1-x_0)^2+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2$

$\Rightarrow(1-t)f(x_1)+tf(x_2)=f(x_0)+R$

If $f''(x)\geq 0$ for every $x\in I$, then term

$R=\frac{(1-t)}{2}f''(\alpha_1)(x_1-x_0)^2+\frac{t}{2}f''(\alpha_2)(x_2-x_0)^2\geq 0$

Hence $(1-t)f(x_1)+tf(x_2)\geq f(x_0)=f\big((1-t)x_1+tx_2\big)$

Or $f\big((1-t)x_1+tx_2\big)\leq(1-t)f(x_1)+tf(x_2)$

Therefore, $f$ is a convex function on $I$.

**CHECK YOUR PROGRESS**

(CQ 3) $f(x)=\sin x$ is a convex function. (T/F)
(CQ 4) $f(x)=x+1$ is strictly increasing function (T/F)
(CQ 5) $f(x)=\frac{1}{x^2}$ is strictly decreasing function (T/F)

## *4.5 SUMMARY*

The first section is devoted to a presentation of the basic results concerning the differentiation of functions. In next sections we discussed the fundamental Mean Value Theorem and some of its applications.

## *4.6 GLOSSARY*

Set- a well defined collection of elements
Sequence-a function whose domain is set of natural number and range is set of real number
Discontinuity-lack of continuity
Derivative- the rate of change of a function with respect to a variable

## *4.7 REFERENCES*


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5. https://www.researchgate.net/figure/Geometric-interpretation-of-Cauchys-mean-value-theorem_fig1_349314019


## *4.8 SUGGESTED READINGS*


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1 W. Rudin (2019) Principles of Mathematical Analysis, McGraw-Hill Publishing, 1964.

2 Tom M. Apostol (1996). Mathematical Analysis (2nd edition), Narosa Book Distributors Pvt Ltd-New Delhi.

3 Pawan K. Jain and Khalil Ahmad (2005). Metric spaces, 2nd Edition, Narosa.


## *4.9 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) State and Prove Rolle's Theorem
(TQ 2) State and Prove Mean Value Theorem
(TQ 3) Use the definition to find the derivative of $f(x) = x^3 \; for x \in \mathbb{R}$

(TQ 4) Stae and prove Darbaux's theorem.
(TQ 5) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function and has a derivative at every point, then the derivative $f'$ is an odd function

**Fill in the blanks**

(TQ 6) $f(x) = |x| + |x+1|$ is not differentiable at ________________.
(TQ 7) Let $b$ be an interior point of the interval $I$ at which $f : I \to R$ has a relative extremum. If the derivative of $f$ at $b$ exists, then ____________.
(TQ 8) If $f(x) = |x|$ on $= [-1,1]$ , then f has an interior minimum at ________

## *4.10 ANSWERS*

| | | |
|---|---|---|
| (CQ 3) F | (CQ 4) T | (CQ 4) F |
| (TQ 6) 0 and -1. | (TQ 7) $f'(b) = 0$ | (TQ 8) $x = 0$ |

# BLOCK II: RIEMANN INTEGRAL

# UNIT 5: RIEMANN INTEGRAL

**CONTENTS**



## *5.1 INTRODUCTION*

During a century and a half of development and refinement of techniques, calculus consisted of these paired operations and their applications, primarily to physical problems. In this chapter we will discussed about Riemann integral and some mean value theorems.

In the 1850s, Bernhard Riemann adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of ‘‘integration’’ could be defined: the class of ‘‘integrable’’ functions. The Fundamental Theorem of Calculus became a result that held only for a restricted set of integrable functions. The viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue’s theory of integration. But there have been some advances made in more recent times that extend even the Lebesgue theory to a considerable extent.

In previous unit we discussed about derivative and mean value theorem. In this unit we discussed about Riemann integral by using some examples and theorems.

German mathematician Georg Friedrich Bernhard Riemann (17 September 1826–20 July 1866) had a substantial impact on analysis, number theory, and differential geometry. He is well known for his work on the Fourier series and the Riemann integral, the first accurate statement of the integral in real analysis.
His development of Riemann surfaces, which pioneered a natural, geometric approach to complex analysis, is perhaps his most notable contribution to the field.

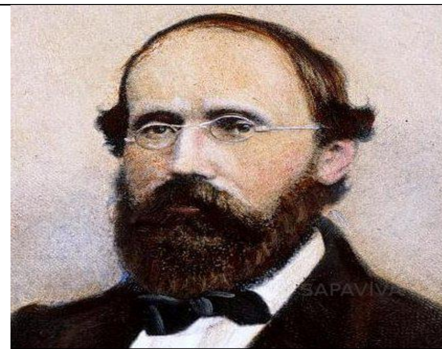

***Fig. 5.1. Bernhard Riemann (Source: https://www.sapaviva.com/georg-bernhard-riemann-2/***

## *5.2 OBJECTIVES*

In this Unit, we will
1. analyze about Riemann Integral
2. construct mean value theorem of calculus

## *5.3 RIEMANN INTEGRAL*

Now we will discuss the definition of Riemann integral of a function $f$ on an interval $[a, b]$.
We first define some basic terms that will be frequently used.

**Partition of $I$:** If $I = [a, b]$ is a closed bounded interval in $\mathbb{R}$, then a partition of $I$ is a finite, ordered set $P = (x_0, x_1, \ldots, x_{n-1}, x_n)$ of points in $I$ such that
$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.
The points of $P$ are used to divide $I = [a, b]$ into non-overlapping subintervals
$I_1 = [x_0, x_1],\ I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$

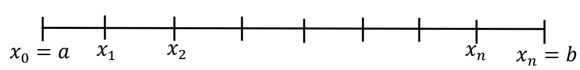


***Fig. 5.2. Partition of $I = [a, b]$***

Let $f$ be a bounded real function on $[a, b]$. Obviously f is bounded on each sub-interval corresponding to each partition P. Let $M_i$ and $m_i$ be the supremum and infimum respectively of $f$ in $\Delta x_i$. Then

**Upper Darboux Sums:** $\boldsymbol{U(P,f) = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n = \sum_{i=1}^{n} M_i}\,\Delta x_i$ is called Upper Darboux Sums of $f$ corresponding to the partition $P$.
**Lower Darboux Sums:** $\boldsymbol{L(P,f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n = \sum_{i=1}^{n} m_i}\,\Delta x_i$ is called Lower Darboux Sums of $f$ corresponding to the partition $P$.

**NOTE:** Let $M$ and $m$ are the bounds of $f$ in $[a,b]$. Then
$m \leq m_i \leq M_i \leq M \Rightarrow m\Delta x_i \leq m_i\Delta x_i \leq M_i\Delta x_i \leq M\Delta x_i$
$\Rightarrow \sum_{i=1}^{n} m\Delta x_i \leq \sum_{i=1}^{n} m_i\Delta x_i \leq \sum_{i=1}^{n} M_i\Delta x_i \leq \sum_{i=1}^{n} M\Delta x_i$
$\Rightarrow m\sum_{i=1}^{n} \Delta x_i \leq \sum_{i=1}^{n} m_i\Delta x_i \leq \sum_{i=1}^{n} M_i\Delta x_i \leq M\sum_{i=1}^{n} \Delta x_i$
$\Rightarrow m(a-b) \leq L(P,f) \leq U(P,f) \leq M(a-b)$

**Upper Integral:** The infimum of the set of upper sums is called Upper Integral.
i.e. $\int_a^{-b} f\,dx = \inf U = \inf\,\{U(P,f) : P$ is a partition of $[a,b]\}$
**Lower Integral:** The supremum of the set of lower sums is called Lower Integral.
i.e. $\int_{-a}^{b} f\,dx = \sup L = \sup\,\{L(P,f) : P$ is a partition of $[a,b]\}$

**Darboux's condition of integrability:** When Upper integral and lower integral are equal then $f$ is said to be Riemann Integral over $[\,a,b]$.
$\int_a^b f\,dx = \int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx$

**Another definition of Riemann Integrable:** A function $f : [\,a,b] \to \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $P'$ is any tagged partition of $[a,b]$ with $\left||P'|\right| > 0$, then $|S(f,P') - L| < \varepsilon$
The set of all Riemann integrable functions on $[a,b]$ will be denoted by $R[a,b]$.

**Ex. 5.1. Show that a constant function $\boldsymbol{\alpha}$ is integrable and $\boldsymbol{\int_a^b dx = \alpha(b-a)}$.**

**Proof.** Let P be any partiion of the interval [a,b], then
$L(P,f) = \alpha\Delta x_1 + \alpha\Delta x_2 + \cdots + \alpha\Delta x_n$
$= \alpha\,(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = \alpha(b-a)$
Similarly $U(P,f) = \alpha\Delta x_1 + \alpha\Delta x_2 + \cdots + \alpha\Delta x_n = \alpha(b-a)$
Therefore
$\int_{-a}^{b} \alpha dx = \sup L(P,f) = \alpha(b-a)$ and
$\int_{-a}^{b} \alpha dx = \inf U(P,f) = \alpha(b-a)$

$\Rightarrow \int_{-a}^{b} \alpha dx = \int_{-a}^{b} \alpha dx = \alpha(b-a)$

Therefore, the constant function is R-integrable and $\int_{a}^{b} \alpha dx = \alpha(b-a)$.

**Ex.5.2. Prove that function $f$ defines as**

$$\boldsymbol{f(x) = \begin{cases} 0, \text{when } x \text{ is rational} \\ 1, \text{ when } x \text{ is irrational} \end{cases}}$$

**is not integrable on any interval.**

**Proof.** Let P be any partiion of the interval [a,b], then

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = 0\Delta x_1 + 0\Delta x_2 + \cdots + 0\Delta x_n = 0$

Similarly $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = 1\Delta x_1 + 1\Delta x_2 + \cdots + 1\Delta x_n = b-a$

Therefore

$\int_{-a}^{b} \alpha dx = \sup L(P,f) = 0$ and

$\int_{-a}^{b} \alpha dx = \inf U(P,f) = b-a$

$\Rightarrow \int_{-a}^{b} \alpha dx \neq \int_{-a}^{b} \alpha dx$

Therefore, the given function is not R-integrable on any interval.

**Ex. 5.3. Show that function $f(x) = x^3$ is integrable on any interval $[0,b]$.**

**Proof.** Let $P$ be any partiion of the interval $[0,b]$ obtained by dividing interval into $n$ −equal parts. i.e. $P = [\frac{0}{n} = 0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \ldots, \frac{nb}{n} = b]$

Let lower bounds of function in $\Delta x_i = \left(\frac{(i-1)k}{n}\right)^3$ and Upper bounds of function in $\Delta x_i = \left(\frac{ik}{n}\right)^3$

Therefore

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n$

$= 0.\frac{b}{n} + \left(\frac{b}{n}\right)^3 .\frac{b}{n} + \left(\frac{2b}{n}\right)^3 .\frac{b}{n} + \cdots + \left(\frac{b(n-1)}{n}\right)^3 .\frac{b}{n} = \frac{b^4}{n^4}[1^3 + 2^3 + \cdots + (n-1)^3]$

$= \frac{b^4(n-1)^2 n^2}{4n^4} = \frac{b^4}{4}\left(1 - \frac{1}{n}\right)^2$

Similarly

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n$

$= \left(\frac{b}{n}\right)^3 .\frac{b}{n} + \left(\frac{2b}{n}\right)^3 .\frac{b}{n} + \left(\frac{2b}{n}\right)^3 .\frac{b}{n} + \cdots + \left(\frac{bn}{n}\right)^3 .\frac{b}{n} = \frac{b^4}{n^4}[1^3 + 2^3 + \cdots + n^3]$

$= \frac{b^4 n^2 (n+1)^2}{4n^4} = \frac{b^4}{4}\left(1 + \frac{1}{n}\right)^2$

Therefore

$\int_{-0}^{b} \alpha dx = \sup L(P,f) = \frac{b^4}{4}$ and

$\int_{-0}^{b} \alpha dx = \inf U(P,f) = \frac{b^4}{4}$

$\Rightarrow \int_{-0}^{b} \alpha dx = \int_{-0}^{b} \alpha dx = \frac{b^4}{4}$

Therefore, the given function is R-integrable and $\int_0^b \alpha dx = \frac{b^4}{4}$.

## *5.4 INEQUALITIES FOR INTEGRALS*

We already prove that

$m(b-a) \leq \int_a^b f\, dx \leq M(b-a)\ when\ b \geq a$ …………………..(I)

If $b < a$, so that $a > b$ and

$m(a-b) \leq \int_b^a f\, dx \leq M(a-b)$

$\Rightarrow -m(a-b) \leq -\int_b^a f\, dx \leq -M(a-b)$

$\Rightarrow m(b-a) \leq \int_a^b f\, dx \leq M(b-a)$ when $b < a$………………(II)

**Deduction 1:** If $f$ is bounded and integrable on $[a,b]$, then there exists a number $k$ lying between bounds of $f$ such that $\int_b^a f\, dx = k(b-a)$

**Deduction 2:** If $f$ is continuous and integrable on $[a,b]$, then there exists a number $c$ lying between $a$ and $b$ such that $\int_b^a f\, dx = f(c)(b-a)$

**Deduction 3:** If $f$ is bounded and integrable on $[a,b]$, and $\alpha > 0$ is a number such that $|f(x)| \leq \alpha$ for all $x \in [a,b]$, then $\left|\int_b^a f\, dx\right| \leq \alpha|b-a|$.

**Proof.** Let $M$ and $m$ be the upper bounds and lower bounds of $f(x)$ respectively.

Let $\alpha > 0$ is a number such that $|f(x)| \leq \alpha$ for all $x \in [a,b]$

Hence for $b > a, -\alpha \leq f(x) \leq \alpha$

$\Rightarrow -\alpha \leq m \leq f(x) \leq M \leq \alpha$

$\Rightarrow -\alpha(b-a) \leq m(b-a) \leq \int_a^b f(x) \leq M(b-a) \leq \alpha(b-a)$

$\Rightarrow \left|\int_a^b f(x)\right| \leq \alpha(b-a)$

If $a > b$, we have

$\left|\int_a^b f(x)\right| \leq \alpha(a-b)$

Therefore $\left|\int_a^b f(x)\right| \leq \alpha|b-a|$.

The result is trivial for $a = b$.

**Deduction 4:** If $f$ is bounded and integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f\, dx \geq 0$ when $b \geq a$ and $\int_a^b f\, dx \leq 0$ when $b \leq a$

**Proof.** Because $f(x) \geq 0$ for all $x \in [a,b]$, then the lower bound of $f(x)\ i.e. m \geq 0$

From Inequality (I) and (II) , we get

$\int_a^b f\,dx \geq 0$ when $b \geq a$ and $\int_a^b f\,dx \leq 0$ when $b \leq a$

**Deduction 5 :** If $f$ and $g$ are bounded and integrable on $[a, b]$, such that $f(x) \geq g(x)$., for all $x \in [a, b]$.then

$\int_a^b f\,dx \geq \int_a^b g\,dx$ when $b \geq a$ and $\int_a^b f\,dx \leq \int_a^b f\,dx$ when $b \leq a$

**Proof.** It is given that $f \geq g$ then $f - g \geq 0$ for all $x \in [a, b]$.

Using deduction 4, we have

$\int_a^b (f - g)dx \geq 0$ $if$ $b \geq a$

$\Rightarrow \int_a^b f dx \geq \int_a^b g dx$ $if$ $b \geq a$

Similarly

$\int_a^b f dx \leq \int_a^b g dx$ $if$ $b \leq a$

**CHECK YOUR PROGRESS**

(CQ 1) $f$ is bounded and integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f\,dx \geq 0$ when $b \geq a$ and $\int_a^b f\,dx \leq 0$ when $b \leq a$. (T/F)

(CQ 2) $f(x) = x^3 + 1$ is not integrable on interval $[a, b]$. (T/F)

(CQ 3) $U(P, f) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i$. (T/F)

(CQ 4) If $f$ is integrable then________________________ .

## 5.5 REFINEMENT OF PARTITIONS AND TAGGED PARTITIONS

**Norm:** The norm (or mesh) of $P$ to be the number

$\mu(P) = \max\{x_1 - x_0,\ x_2 - x_1, \ldots, x_n - x_{n-1}\}$

**OR**

the norm of a partition is merely the length of the largest subinterval into which the partition divides $[a, b]$.

**Refinement:** A partition $P^*$ is said to be a refinement of $P$ if $P^* \supseteq P$ i.e. every point of P is a point of $P^*$.

Or we can say that $P^*$ refines $P$ or $P^*$ is finer than $P$.

If $P_1$ and $P_2$ are two partitions, then $P^* = P_1 \cup P_2$.

**Theorem 5.1. Suppose that $f : [a, b] \to R$ is bounded and $P$ and $P^*$ be partitions of [a, b] and refinement of $P$ respectively. Then**

(i) $L(P, f) \leq L(P^*, f)$

(ii) $U(P^*, f) \leq U(P, f)$

**Proof.** Let $P$ be partition of $[a,b]$ and $\boldsymbol{P}^*$ contains just one more point $'\alpha'$ than $P$.

Let $\alpha \in \Delta x_i$ $i.e$ $x_{i-1} < \alpha < x_i$.

It is given that the function $f$ is bounded over the interval $[a,b]$.

⇒ It is bounded in every subinterval $\Delta x_i$.

Let $\beta_1, \beta_2$ and $m_i$ be the infimum of f in the interval $[x_{i-1}, \alpha], [\alpha, x_i]$ and $[x_{i-1}, x_i]$ respectively.

Obviously $m_i \leq \beta_1$ and $m_i \leq \beta_2$.

Hence

$L(P^*, f) - L(P, f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + \beta_1(\alpha - x_{i-1}) + \beta_1(x_i - \alpha) + m_{i+1}\Delta x_{i+1} + \cdots + m_n\Delta x_n - (m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_i\Delta x_n + m_n\Delta x_n)$

$= \beta_1(\alpha - x_{i-1}) + \beta_2(x_i - \alpha) - m_i(x_i - x_{i-1})$

$= \beta_1\alpha - \beta_1 x_{i-1} + \beta_2 x_i - \beta_2\alpha - m_i x_i + m_i x_{i-1}$

$= \beta_1\alpha - \beta_1 x_{i-1} - m_i\alpha + m_i\,\alpha + \beta_2 x_i - \beta_2\alpha - m_i x_i + m_i x_{i-1}$

$= \alpha(\beta_1 - m_i) - x_{i-1}(\beta_1 - m_i) - m_i\,(x_i - \alpha) + \beta_2(x_i - \alpha)$

$= (\alpha - x_{i-1})(\beta_1 - m_i) + (\beta_2 - m_i)\,(x_i - \alpha)$

$x_i > \alpha > x_{i-1}$ and $\beta_1, \beta_2 \geq m_i$ ⇒$(\alpha - x_{i-1})$, $(x_i - \alpha), (\beta_1 - m_i)$ and $(\beta_2 - m_i)$ are positive.

Therefore, $L(P^*, f) - L(P, f) \geq 0$

If $P^*$ contains $p$ points more than $P$, we repeat the above reasoning $p$ times and conclude that

$L(P^*, f) \geq L(P, f)$

Similarly, we can prove that $U(P^*, f) \;\leq\; U(P, f)$

**Corollary If a refinement $\boldsymbol{P}^*$ of $\boldsymbol{P}$ contains k points more than $\boldsymbol{P}$ and $|\boldsymbol{f}(\boldsymbol{x})| \leq \boldsymbol{K}$, for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$, then**

**(i)** $L(P, f) \leq L(P^*, f) \leq L(P, f) + 2k\boldsymbol{K\mu}$

**(ii)** $U(P, f) \geq U(P^*, f) \geq U(P, f) - 2k\boldsymbol{K\mu}$

**Proof. .** Let $P$ be partition of $[a,b]$ and $\boldsymbol{P}^*$ contains just one more point $'\alpha'$ than $P$.

Let $\alpha \in \Delta x_i$ $i.e$ $x_{i-1} < \alpha < x_i$.

It is given that the function $f$ is bounded over the interval $[a,b]$.

⇒ It is bounded in every subinterval $\Delta x_i$.

Let $\beta_1, \beta_2$ and $m_i$ be the infimum of $f$ in the interval $[x_{i-1}, \alpha], [\alpha, x_i]$ and $[x_{i-1}, x_i]$ respectively.

Obviously $m_i \leq \beta_1$ and $m_i \leq \beta_2$.

Hence

$L(P^*, f) - L(P, f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + \beta_1(\alpha - x_{i-1}) + \beta_1(x_i - \alpha) + m_{i+1}\Delta x_{i+1} + \cdots + m_n\Delta x_n - (m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_i\Delta x_n + m_n\Delta x_n)$

$= \beta_1(\alpha - x_{i-1}) + \beta_2(x_i - \alpha) - m_i(x_i - x_{i-1})$

$= \beta_1\alpha - \beta_1 x_{i-1} \ + \beta_2 x_i - \beta_2\alpha - m_i x_i + m_i x_{i-1}$
$= \beta_1\alpha - \beta_1 x_{i-1} - m_i\alpha + m_i\,\alpha + \beta_2 x_i - \beta_2\alpha - m_i x_i + m_i x_{i-1}$
$= \alpha(\beta_1 - m_i) - x_{i-1}(\beta_1 - m_i) - m_i\,(x_i - \alpha) + \beta_2(x_i - \alpha)$
$= (\alpha - x_{i-1})(\beta_1 - m_i) + (\beta_2 - m_i)\,(x_i - \alpha)$
It us given that $|f(x)| \leq K$ for all $x \in [a,b]$, therefore
$-K \leq m_i \leq \beta_1 \leq K \ \Rightarrow \ K \geq -m_i$ and $K \geq \beta_1 \Rightarrow \ 2K \geq \beta_1 - m_i$ or $2K \geq \beta_1 - m_i \geq 0$
Similarly
$2K \geq \beta_2 - m_i \geq 0$
Therefore
$L(P^*, f) - L(P, f) \leq 2K(\alpha - x_{i-1}) + 2K\,(x_i - \alpha) = 2K(\alpha - x_{i-1} + x_i - \alpha) = 2K(x_i - x_{i-1})$
Therefore
$L(P^*, f) - L(P, f) \leq 2K\Delta x_i$
Let $\mu$ be the norm of $P$, hence
$L(P^*, f) - L(P, f) \leq 2K\mu$

**Let each additional point is introduced one by one, by repeating the above reasoning $k$ times, we get**
$L(P^*, f) - L(P, f) \leq 2Kk\mu \Rightarrow L(P^*, f) \leq L(P, f) + 2Kk\mu$
Also, $L(P, f) \leq L(P^*, f)$
Hence $L(P, f) \leq L(P, f) + 2Kk\mu$
Similarly, we can prove that $U(P, f) \geq U(P^*, f) \geq U(P, f) - 2kK\mu$

**<u>Darboux Theorem</u>**

**Theorem 5.2. If $f$ is bounded function on $[a, b]$ then to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that**
**(i) $U(P, f) < \int_a^{-b} f\,dx + \ \varepsilon$**
**(ii) $L(P, f) > \int_{-a}^{b} f\,dx - \ \varepsilon$**
**For every partition $P$ of $[a, b]$ with norm $\mu(P) < \delta$.**

**Proof.** It is given that $f$ is bounded on $[a, b]$. Hence there exists $\alpha > 0$ such that
$f(x) \leq \alpha \ \ for\ all\ x \in [a, b]$
Now
$\int_a^{-b} f\,dx = \inf U = \inf\,\{U(P, f): P \text{ is a partition of } [a, b]\}$
Hence for every $\varepsilon > 0$ there exists a partition $P' = \{x_0, x_1, x_3, \ldots, x_k\}$ of $[a, b]$ such that
$U(P_1, f) < \int_a^{-b} f\,dx + \frac{1}{2}\varepsilon$ ………………………..(1)
Also partition $P'$ contains $k - 1$ points other than $a$ and $b$.
Let $\delta$ be a positive number such that
$2(k-1)\alpha\delta = \frac{1}{2}\varepsilon$ ………………………………….(2).

Let $P$ be any partition such that $P = \{x_0, x_1, x_3, \dots, x_n\}$ with norm $\mu(P) < \delta$.
Assume $P^*$ be a refinement of $P$ and $P^{'}$ such that $P^* = P \cup P'$
$P^*$ be a refinement of $P \Rightarrow$ $P^*$ have $p-1$ more point than $P$ and also $f(x) \le \alpha$
Therefore
$U(P,f) \ge U(P^*,f) \ge U(P,f) - 2(p-1)\alpha\delta$ (Using previous corollary)

$$\Rightarrow U(P,f) - 2(p-1)\alpha\delta \le U(P^*,f)$$
$$\le U(P',f)$$
$$< \int_a^{-b} f\,dx + \frac{1}{2}\varepsilon \quad \text{(Using eq (1))}$$

Therefore
$$U(P,f) < \int_a^{-b} f\,dx + \frac{1}{2}\varepsilon + 2(p-1)\alpha\delta$$
Using equation (2), we get
$$U(P,f) < \int_a^{-b} f\,dx + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon < \int_a^{-b} f\,dx + \varepsilon$$
Similarly we can prove that $L(P,f) > \int_{-a}^{b} f\,dx - \varepsilon$

**NOTE**

- **Tags:** If a point $t_i$ has been selected from each subinterval $I_i = [x_{i-1}, x_i]$ for $i = 1,2,\dots,n$, then the points are called tags of the subintervals $I_i$.
- **Tagged Partition of $I$:** A set of ordered $P = \{([x_{i-1}, x_i], t_i); i = 1,2,\dots,n\}$ of subintervals and corresponding tags is called a tagged partition of $I$.

## *5.6 CONDITION OF INTEGRABILITY AND SOME PROPERTIES OF INTEGRABLE FUNCTIONS*

We already discussed that the bounded function is integrable if upper and lower integral are equal. Now we try to study the necessary and sufficient condition for integrability of a function.

**FIRST FORM**

**Theorem 5.3. The necessary and sufficient condition for integrability of a bounded function f is for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition $P$ of $[a,b]$with norm $\mu(P) < \delta$ and $U(P,f) - L(P,f) < \varepsilon$**
**Proof.** Necessary condition
Let $f$ be a bounded function and integrable over interval $[a,b]$,

Hence $\int_{-a}^{b} f\,dx = \int_{a}^{-b} f\,dx = \int_{a}^{b} f\,dx$
Let $\varepsilon > 0$ be any positive number.
By Darbaux's Theorem there exists a positive number $\delta$ such that foe every partition $P$ with norm $\mu(P) < \delta$
$U(P,f) < \int_{a}^{-b} f\,dx + \frac{1}{2}\varepsilon$ …………………..(1)
$L(P,f) > \int_{-a}^{b} f\,dx - \frac{1}{2}\,\varepsilon$ ……………………(2)
$\Rightarrow -L(P,f) < -\int_{-a}^{b} f\,dx + \frac{1}{2}\,\varepsilon$………………………………..(3)
By adding inequality (1) and (3), we get
$U(P,f) - L(P,f) < \int_{a}^{-b} f\,dx + \frac{1}{2}\varepsilon - \int_{-a}^{b} f\,dx + \frac{1}{2}\,\varepsilon = \varepsilon$
Hence for every partition $P$ of $[a,b]$with norm $\mu(P) < \delta$
$U(P,f) - L(P,f) < \varepsilon$
Sufficient Condition
Assume for every partition $P$ of $[a,b]$with norm $\mu(P) < \delta$ and
$U(P,f) - L(P,f) < \varepsilon$…………………………………………..(4)
for any partition $P$ of $[a,b]$, we have
$U(P,f) \geq \int_{a}^{-b} f\,dx \Rightarrow \int_{a}^{-b} f\,dx \leq U(P,f)$…………………………..(5)
$L(P,f) \leq \int_{-a}^{b} f\,dx \Rightarrow -\int_{-a}^{b} f\,dx \leq -L(P,f)$………………………..(6)
Adding inequality (5) and (6), we get
$\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx \leq U(P,f) - L(P,f)$
Using inequality (4), we get
$\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon$
Because $\varepsilon$ is any arbitrary positive number and also we know that a non negative number is less than every positive number.
Therefore it should be equal to 0.
i.e. $\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon = 0$
Therefore $\int_{a}^{-b} f\,dx = \int_{-a}^{b} f\,dx$ which implies that $f$ is integrable over interval $[a,b]$.

**SECOND FORM**

**Theorem 5.4. A bounded function $f$ is integrable on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition $P$ of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$**
**Proof.** Necessary condition
Let $f$ be a bounded function and integrable over interval $[a,b]$,
Hence $\int_{-a}^{b} f\,dx = \int_{a}^{-b} f\,dx = \int_{a}^{b} f\,dx$
Let $\varepsilon > 0$ be any positive number.
As we know that the

$\int_{-a}^{b} f\,dx$ =supremum of lower sums and $\int_{a}^{-b} f\,dx =$ infimum of upper sums

Hence there exists a partition $P'$ and $P''$ such that

$U(P', f) < \int_{a}^{-b} f\,dx + \frac{1}{2}\varepsilon$

$\Rightarrow U(P', f) < \int_{a}^{b} f\,dx + \frac{1}{2}\varepsilon$ ………………………..(1)

$L(P'', f) > \int_{-a}^{b} f\,dx - \frac{1}{2}\,\varepsilon$

$\Rightarrow L(P'', f) > \int_{a}^{b} f\,dx - \frac{1}{2}\,\varepsilon$

$\Rightarrow \int_{a}^{b} f\,dx < L(P'', f) + \frac{1}{2}\,\varepsilon$ …………………………(2)

Assume $P$ be the commom refinement of partitions $P'$ and $P''$ i.e. $P = P' \cup P''$

Therefore

$U(P, f) \le U(P', f) < \int_{a}^{b} f\,dx + \frac{1}{2}\varepsilon$ (using inequality (1))

$\Rightarrow U(P, f) < L(P'', f) + \frac{1}{2}\,\varepsilon + \frac{1}{2}\,\varepsilon = L(P'', f) + \varepsilon$

Therefore $U(P, f) - L(P, f) < \varepsilon$ for a partition $P$.

Sufficient Condition

Assume $\varepsilon < 0$ be any positive number. Consider $P$ be a partitions such that

$U(P, f) - L(P, f) < \varepsilon$……………………………………………..(3)

Now for any partition $P$ of $[a, b]$, we have

$U(P, f) \ge \int_{a}^{-b} f\,dx \Rightarrow \int_{a}^{-b} f\,dx \le U(P, f)$…………………………..(4)

$L(P, f) \le \int_{-a}^{b} f\,dx \Rightarrow -\int_{-a}^{b} f\,dx \le -L(P, f)$………………………..(5)

Adding inequality (4) and (5), we get

$\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx \le U(P, f) - L(P, f)$

Using inequality (4), we get

$\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon$

Because $\varepsilon$ is any arbitrary positive number and also we know that a non negative number is less than every positive number.

Therefore it should be equal to 0.

i.e. $\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon = 0$

Therefore $\int_{a}^{-b} f\,dx = \int_{-a}^{b} f\,dx$ which implies that $f$ is integrable over interval $[a, b]$.

## Integrability of the sum and difference of Integrable functions

**Theorem 5.5. Let $f_1$ and $f_2$ are two bounded and integrable function on $[a, b]$ then $f = f_1 + f_2$ is also integrable on $[a, b]$ and $\int_{a}^{b} f\,dx = \int_{a}^{b} f_1\,dx + \int_{a}^{b} f_2\,dx$**

**Proof.** Let $f_1$ and $f_2$ are two bounded $\Rightarrow f = f_1 + f_2$ is bounded on $[a, b]$.

Let $P$ be any partition $P$ of $[a,b]$ such that $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$.
Let $M_i'$ and $m_i'$ are the upper and lower bound of $f_1$ respectively and $M_i''$ and $m_i''$ are the upper and lower bound of $f_2$ respectively in $\Delta x_i$.
Assume $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively in $\Delta x_i$.
Therefore
$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$……………………………………………(1)
Multiplying inequality (1) by $\Delta x_i$, we get
$(m_i' + m_i'')\Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le (M_i' + M_i'')\Delta x_i$
Adding all these inequalities for $i = 1,2,3,\dots,n$, we get
$\sum_{i=1}^{n}(m_i' + m_i'')\Delta x_i \le \sum_{1=1}^{n} m_i \Delta x_i \le \sum_{1=1}^{n} M_i \Delta x_i \le \sum_{1=1}^{n}(M_i' + M_i'')\Delta x_i$
$\Rightarrow L(P,f_1) + L(P,f_2) \le L(P,f) \le U(P,f) \le U(P,f_1) + U(P,f_2)$
$U(P,f) \le U(P,f_1) + U(P,f_2)$ ………………………………….(2)
$L(P,f_1) + L(P,f_2) \le L(P,f)$
$-L(P,f) \le -(L(P,f_1) + L(P,f_2))$ …………………………………….(3)
Let $\varepsilon > 0$ be any positive number.
It is given that $f_1$ and $f_2$ are integrable. Hence for any partition $P$ there exists $\delta > 0$ such that the norm $\mu(P) < \delta$, we have
$U(P,f_1) - L(P,f_1) < \frac{1}{2}\varepsilon$…………………………(4)
$U(P,f_2) - L(P,f_2) < \frac{1}{2}\varepsilon$…………………………...(5)
From (2),(3),(4) and (5), we get
$U(P,f) - L(P,f) \le U(P,f_1) + U(P,f_2) - \big(L(P,f_1) + L(P,f_2)\big)$
$$= U(P,f_1) - L(P,f_1) + U(P,f_2) - L(P,f_2) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$
Therefore
$U(P,f) - L(P,f) < \varepsilon$ .
Hence the function $f$ is integrable.
$f_1$ and $f_2$ are integrable and $\varepsilon > 0$ is any positive number.
Using Darboux's theorem, there exists $\delta > 0$ such that for all partitions $P$ whose norm $\mu(P) < \delta$, we have
$U(P,f_1) < \int_a^b f_1\,dx + \frac{1}{2}\varepsilon$ ……………(6)
And
$U(P,f_2) < \int_a^b f_2\,dx + \frac{1}{2}\varepsilon$ ……………(7)
Using inequality (2), we get
$\int_a^b f\,dx \le U(P,f) \le U(P,f_1) + U(P,f_2)$
Using inequalities (6) and (7), we get
$\int_a^b f\,dx < \int_a^b f_1\,dx + \frac{1}{2}\varepsilon + \int_a^b f_2\,dx + \frac{1}{2}\varepsilon = \int_a^b f_1\,dx + \int_a^b f_2\,dx + \varepsilon$
As we know $\varepsilon$ is arbitrary, therefore
$\int_a^b f\,dx \le \int_a^b f_1\,dx + \int_a^b f_2\,dx$ …………………………..(8)
Now replacing $f_1$ and $f_2$ with $(-f_1)$ and $(-f_2)$ respectively, we get

$\int_a^b (-f)\,dx \leq \int_a^b (-f_1)\,dx + \int_a^b (-f_2)\,dx$
i.e. $\int_a^b f\,dx \geq \int_a^b f_1\,dx + \int_a^b f_2\,dx$……………………………..(9)
From inequality (8) and (9), we get
$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$

**Theorem 5.6. Let $f_1$ and $f_2$ are two bounded and integrable function on $[a,b]$ then $f = f_1 - f_2$ is also integrable on $[a,b]$ and $\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx$**

**Proof.** Let $f_1$ and $f_2$ are two bounded $\Rightarrow f = f_1 + (-f_2)$ is bounded on $[a,b]$.
Let $P$ be any partition $P$ of $[a,b]$ such that $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$.
Let $M_i'$ and $m_i'$ are the upper and lower bound of $f_1$ respectively and $M_i''$ and $m_i''$ are the upper and lower bound of $f_2$ respectively in $\Delta x_i$.
$\Rightarrow -M_i''$ and $-m_i''$ are the upper and lower bound of $(-f_2)$ respectively in $\Delta x_i$.
Assume $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively in $\Delta x_i$.
Therefore
$m_i' + (-m_i'') \leq m_i \leq M_i \leq M_i' + (-M_i'')$
$\Rightarrow$ $m_i' - M_i' \leq m_i \leq M_i \leq M_i'' - m_i''$……………………………………………(1)
Multiplying inequality (1) by $\Delta x_i$, we get
$(m_i' - M_i'')\Delta x_i \leq m_i \Delta x_i \leq M_i \Delta x_i \leq (M_i' - m_i'')\Delta x_i$
Adding all these inequalities for $i = 1,2,3,\dots,n$, we get
$\sum_{i=1}^{n}(m_i' - M_i')\Delta x_i \leq \sum_{1=1}^{n} m_i \Delta x_i \leq \sum_{1=1}^{n} M_i \Delta x_i \leq \sum_{1=1}^{n}(M_i' - m_i'')\Delta x_i$
$\Rightarrow L(P,f_1) - U(P,f_2) \leq L(P,f) \leq U(P,f) \leq U(P,f_1) - L(P,f_2)$
$U(P,f) \leq U(P,f_1) - L(P,f_2)$ ………………………………….(2)
$L(P,f_1) - U(P,f_2) \leq L(P,f)$
$-L(P,f) \leq U(P,f_2) - L(P,f_1)$ …………………………………….(3)
Let $\varepsilon > 0$ be any positive number.
It is given that $f_1$ and $f_2$ are integrable. Hence for any partition $P$ there exists $\delta > 0$ such that the norm $\mu(P) < \delta$, we have
$U(P,f_1) - L(P,f_1) < \frac{1}{2}\varepsilon$…………………………(4)
$U(P,f_2) - L(P,f_2) < \frac{1}{2}\varepsilon$…………………………...(5)
From (2),(3),(4) and (5), we get
$U(P,f) - L(P,f) \leq U(P,f_1) - L(P,f_2) + U(P,f_2) - L(P,f_1)$
$= U(P,f_1) - L(P,f_1) + U(P,f_2) - L(P,f_2) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$
Therefore
$U(P,f) - L(P,f) < \varepsilon$ .

Hence the function $f$ is integrable.
$f_1$ and $f_2$ are integrable and $\varepsilon > 0$ is any positive number.
Using Darboux's theorem, there exists $\delta > 0$ such that for all partitions $P$ whose norm $\mu(P) < \delta$, we have

$U(P, f_1) < \int_a^b f_1\,dx + \frac{1}{2}\varepsilon$ ……………(6)

And

$L(P, f_2) > \int_a^b f_2\,dx + \frac{1}{2}\varepsilon$

$\Rightarrow -L(P, f_2) < -\int_a^b f_2\,dx + \frac{1}{2}\varepsilon$ ……………(7)

Using inequality (2), we get

$\int_a^b f\,dx \le U(P, f) \le U(P, f_1) - L(P, f_2)$

Using inequalities (6) and (7), we get

$\int_a^b f\,dx < \int_a^b f_1\,dx + \frac{1}{2}\varepsilon - \int_a^b f_2\,dx + \frac{1}{2}\varepsilon = \int_a^b f_1\,dx - \int_a^b f_2\,dx + \varepsilon$

As we know $\varepsilon$ is arbitrary, therefore

$\int_a^b f\,dx \le \int_a^b f_1\,dx - \int_a^b f_2\,dx$ …………………………..(8)

Now replacing $f_1$ and $f_2$ with $(-f_1)$ and $(-f_2)$ respectively, we get

$\int_a^b (-f)\,dx \le \int_a^b (-f_1)\,dx - \int_a^b (-f_2)\,dx$

i.e. $\int_a^b f\,dx \ge \int_a^b f_1\,dx - \int_a^b f_2\,dx$…………………………..(9)

From inequality (8) and (9), we get

$\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx$ .

**Oscillation:** The oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$ of numbers.
Let $M$ and $m$ be the upper and lower bounds of $f$ on $[a, b]$ respectively.
$\Rightarrow m \le f(x_1) \le M$ and $m \le f(x_2) \le M$ for all $x_1, x_2 \in [a, b]$
$\Rightarrow |f(x_1) - f(x_2)| \le M - m$ for all $x_1, x_2 \in [a, b]$...........................(1)
$\Rightarrow M - m$ is an upper bound of $\{f(x_1) - f(x_2)$ , for all $x_1, x_2 \in [a, b]\}$
Let $\varepsilon > 0$ be any positive number, because $M$ is supremum of $f$.
Therefore there exists $y \in [a, b]$ such that

$f(y) > M - \frac{1}{2}\varepsilon$ …………………………………………(2)

Similarly there exists $z \in [a, b]$ such that

$f(z) > m + \frac{1}{2}\varepsilon$ …………………………………………(3)

From inequalities (2) and (3), we conclude that there exist $x, y \in [a, b]$ such that

$f(y) - f(z) > M - \frac{1}{2}\varepsilon - m - \frac{1}{2}\varepsilon = M - m - \varepsilon$

Or $|f(y) - f(z)| > M - m - \varepsilon$…………………………(4)

From inequalities (1) and (4), we conclude that
$M - m$ is an upper bound and also number less than $M - m$ cannot be upper bound of given set.
Hence $M - m = \sup\{\,|f(y) - f(z)| : y, z \in [a, b]\}$......................(A)

**Theorem 5.7. If $f$ and $g$ are two bounded and integrable functions on $[a,b]$ then the product $fg$ is also bounded and integrable on $[a,b]$.**

**Proof.** It is given that $f$ and $g$ are two bounded therefore there exists $\alpha$ such tha

$|f(x)| \leq \alpha$ and $|g(x)| \leq \alpha$ for all $x \in [a,b]$

$\Rightarrow |fg(x)| = |f(x)||g(x)| \leq \alpha.\alpha \leq \alpha^2$

It implies that $fg$ is bounded on $[a,b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a,b]$.

Let $M_i'$ and $m_i'$ are the upper and lower bound of $f$ respectively and $M_i''$ and $m_i''$ are the upper and lower bound of $g$ respectively in $\Delta x_i$.

Assume $M_i$ and $m_i$ are the upper and lower bound of $fg$ respectively in $\Delta x_i$.

Now for all $x, x' \in \Delta x_i$,

$(fg)(x') - (fg)(x) = f(x')g(x') - f(x)g(x)$

$= f(x')g(x') - f(x)g(x') + f(x)g(x') - f(x)g(x) = g(x')(f(x') - f(x)) + f(x)(g(x') - g(x))$

It implies that

$|(fg)(x') - (fg)(x)| = |g(x')\big(f(x') - f(x)\big) + f(x)\big(g(x') - g(x)\big)|$

$\leq |g(x')||f(x') - f(x)| + |f(x)||\,g(x') - g(x)|$

Hence, From inequality (A), we get

$M - m \ \leq \alpha(M' - m') + \alpha(M'' - m'')$ ……………………(1)

Let $\varepsilon > 0$ be given number and it is given that $f$ and $g$ integrable on interval $[a,b]$.

Therefore there exists a positive number $\delta > 0$ such that for any partition $P$ with norm $\mu(P) < \delta$

$U(P,f) - L(P,f) \leq \frac{\varepsilon}{2\alpha}$ …………………………………..(2) and

$U(P,g) - L(P,g) \leq \frac{\varepsilon}{2\alpha}$ …………………………………..(3)

Now multiply inequality (1) with $\Delta x_i$, we get

$(M - m)\Delta x_i \ \leq \alpha(M' - m')\Delta x_i + \alpha(M'' - m'')\Delta x_i$

Adding all these inequalities for $i = 1,2,3, \ldots, n$, we get

$\sum_{1=1}^{n}(M - m)\Delta x_i \ \leq \sum_{1=1}^{n}\alpha(M' - m')\Delta x_i + \sum_{1=1}^{n}\alpha(M'' - m'')\Delta x_i$

$\Rightarrow \sum_{1=1}^{n} M\Delta x_i - \sum_{1=1}^{n} m\Delta x_i \leq \alpha(\sum_{1=1}^{n} M'\Delta x_i - \sum_{1=1}^{n} m'\Delta x_i) \ + \alpha(\sum_{1=1}^{n} M''\Delta x_i - \sum_{1=1}^{n} m''\Delta x_i)$

$\Rightarrow U(P,fg) - L(P,fg) \leq \alpha\big(U(P,f) - L(P,f)\big) \ + \alpha\big(U(P,g) - L(P,g)\big)$

$\leq \alpha\frac{\varepsilon}{2\alpha} + \alpha\frac{\varepsilon}{2\alpha}$

Therefore $U(P,fg) - L(P,fg) \leq \varepsilon$

Hence we conclude that $fg$ is integrable on $[a,b]$.

**Theorem 5.8. If $f$ and $g$ are two bounded and integrable functions on $[a,b]$ and there exists a positive number $k$ such that $|g| \geq k\ for\ all\ x \in [a,b]$ then the $f/g$ is also bounded and integrable on $[a,b]$.**

**Proof.** It is given that $f$ and $g$ are two bounded therefore there exists $\alpha$ such that

$|f(x)| \leq \alpha$ and $k \leq |g(x)| \leq \alpha \Rightarrow \frac{1}{k} \geq \frac{1}{|g(x)|} \geq \frac{1}{\alpha}$ for all $x \in [a,b]$

$\Rightarrow |(f/g)(x)| = |f(x)|/|g(x)| \leq \alpha.\frac{1}{k} \leq \frac{\alpha}{k}$

It implies that $fg$ is bounded on $[a,b]$.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be any partition of $[a,b]$.

Let $M_i'$ and $m_i'$ are the upper and lower bound of $f$ respectively and $M_i''$ and $m_i''$ are the upper and lower bound of $g$ respectively in $\Delta x_i$.

Assume $M_i$ and $m_i$ are the upper and lower bound of $f/g$ respectively in $\Delta x_i$.

Now for all $x, x' \in \Delta x_i$,

$$\left|\left(\frac{f}{g}\right)(x') - \left(\frac{f}{g}\right)(x)\right| = \left|\frac{f(x')}{g(x')} - \frac{f(x)}{g(x)}\right| = \left|\frac{f(x')g(x) - f(x)g(x')}{g(x)g(x')}\right|$$

$$= \left|\frac{f(x')g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x')}{g(x)g(x')}\right|$$

$$= \left|\frac{g(x)\big(f(x') - f(x)\big) - f(x)\big(g(x') - g(x)\big)}{g(x)g(x')}\right|$$

$$\leq \alpha \frac{|f(x') - f(x)|}{|g(x)g(x')|} + \alpha \frac{|g(x') - g(x)|}{|g(x)g(x')|}$$

Hence, From inequality (A), we get

$M - m \leq \alpha.(M' - m').\frac{1}{k^2} + \alpha.(M'' - m'').\frac{1}{k^2}$

Hence

$M - m \leq \frac{\alpha}{k^2}(M' - m') + \frac{\alpha}{k^2}(M'' - m'')$ ……………………(1)

Let $\varepsilon > 0$ be given number and it is given that $f$ and $g$ integrable on interval $[a,b]$.

Therefore there exists a positive number $\delta > 0$ such that for any partition $P$ with norm $\mu(P) < \delta$

$U(P,f) - L(P,f) \leq \frac{\varepsilon k^2}{2\alpha}$ …………………………………..(2) and

$U(P,g) - L(P,g) \leq \frac{\varepsilon k^2}{2\alpha}$ …………………………………..(3)

Now multiply inequality (1) with $\Delta x_i$, we get

$(M - m)\Delta x_i \leq \frac{\alpha}{k^2}(M' - m')\Delta x_i + \frac{\alpha}{k^2}(M'' - m'')\Delta x_i$

Adding all these inequalities for $i = 1,2,3, \dots, n$, we get

$\sum_{1=1}^{n}(M - m)\Delta x_i \leq \sum_{1=1}^{n}\frac{\alpha}{k^2}(M' - m')\Delta x_i + \sum_{1=1}^{n}\alpha(M'' - m'')\Delta x_i$

$\Rightarrow \sum_{1=1}^{n} M\Delta x_i - \sum_{1=1}^{n} m\Delta x_i \leq \frac{\alpha}{k^2}(\sum_{1=1}^{n} M'\Delta x_i - \sum_{1=1}^{n} m'\Delta x_i) + \frac{\alpha}{k^2}(\sum_{1=1}^{n} M''\Delta x_i - \sum_{1=1}^{n} m''\Delta x_i)$

$$\Rightarrow U(P,fg) - L(P,fg) \leq \frac{\alpha}{k^2}\big(U(P,f) - L(P,f)\big) + \frac{\alpha}{k^2}\big(U(P,g) - L(P,g)\big)$$

$$\leq \frac{\alpha}{k^2}\frac{\varepsilon k^2}{2\alpha} + \frac{\alpha}{k^2}\frac{\varepsilon k^2}{2\alpha}$$

Therefore $U(P,fg) - L(P,fg) \leq \varepsilon$

Hence we conclude that $f/g$ is integrable on $[a,b]$.

**Theorem 5.9. If $f$ ís bounded and integrable functions on $[a,b]$ then $|f|$ is also bounded and integrable on $[a,b]$ and also $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$.**

**Proof.** It is given that $f$ is bounded therefore there exists $\alpha$ such that
$|f(x)| \le \alpha$ for all $x \in [a,b]$
It implies that the function $|f|$ is bounded.
Since $f$ is integrable, for a given positive number $\varepsilon > 0$ there exists a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a,b]$ and such that
$U(P,f) - L(P,f) < \varepsilon$ …………………..(1)
Let $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively and $M_i'$ and $m_i'$ are the upper and lower bound of $g$ respectively in $\Delta x_i$.
Now for all $x, x' \in \Delta x_i$,
$\big||f|(x) - |f|(y)\big| = \big||f(x)| - |f(y)|\big| \le |f(x) - f(y)|$
$\Rightarrow M_i' - m_i' \le M - m$..........................(2)
Now multiply inequality (2) with $\Delta x_i$, we get
$(M_i' - m_i')\Delta x_i \le (M_i - m_i)\Delta x_i$
Adding all these inequalities for $i = 1,2,3, \dots, n$, we get
$\sum_{1=1}^{n}(M_i' - m_i')\Delta x_i \le \sum_{1=1}^{n}(M_i - m_i)\Delta x_i$
$\Rightarrow \sum_{1=1}^{n} M_i'\Delta x_i - \sum_{1=1}^{n} m_i'\Delta x_i \le \sum_{1=1}^{n} M_i\ \Delta x_i - \sum_{1=1}^{n} m_i\Delta x_i$
$\Rightarrow U(P,|f|) - L(P,|f|) \le U(P,f) - L(P,f)$
Using inequality (1), we get
$U(P,|f|) - L(P,|f|) < \varepsilon$ . Hence $|f|$ is integrable on $[a,b]$.
We Know that if $f$ and $g$ are bounded and integrable on $[a,b]$ such that $f \ge g$ then
$\int_a^b f\,dx \le \int_a^b g\,dx$ when $b \le a$
Hence $\int_a^b f\,dx \le \int_a^b |f|\,dx$
and $-\int_a^b f\,dx = \int_a^b (-f)\,dx \le \int_a^b |f|\,dx$
$\Rightarrow \left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$

**NOTE:** The Converse of the above theorem is not true. For example, the function

$$f(x) = \begin{cases} 1, & when\ x\ is\ rational \\ -1, & when\ x\ is\ irrational \end{cases}$$

Here $\int_a^{-b} f\,dx = b - a$ but $\int_a^{-b} f\,dx = a - b$
It implies that $f$ is not integrable.
But $|f(x)| = 1$ $for\ all\ x$, therefore $\int_a^b |f|\,dx$ exists and equal to $b - a$.
Here we observe that $f|$ is integrable.

**CHECK YOUR PROGRESS**

(CQ 5) If $f$ and $g$ are two bounded and integrable functions on $[a, b]$ then the product $fg$ is also bounded and but not integrable on $[a, b]$. (T/F)
(CQ 6) The oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)|: x_1, x_2 \in [a, b]\}$ of numbers. (T/F)
(CQ 7) Upper Darbaux sum=_________
(CQ 8) Lower darbaux sum=_______________

## *5.7 RIEMANN SUM*

**Riemann Sum:** Let $P'$ is the tagged partition then the Riemann sum of a function $f : [a, b] \to \mathbb{R}$ corresponding to $P'$ can be defined as
$S(f, P') = \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1})$
If the function $f$ is positive on $[a, b]$, then the Riemann Sum is the sum of the areas of $n$ rectangles whose bases are the subintervlas $I_1 = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$. See Fig 5.2.

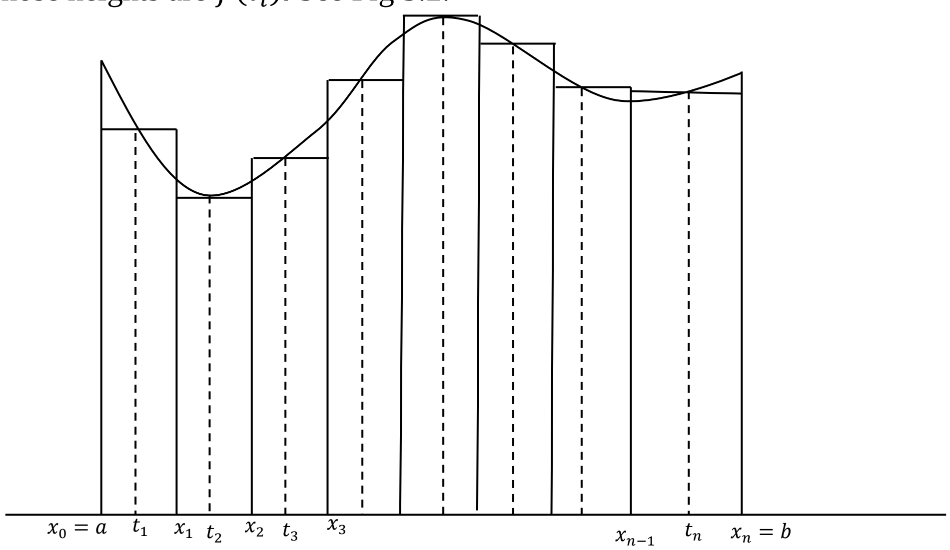


***Fig 5.3. A Riemann Sum***

**Theorem 5.12. If $f: [a, b] \to \mathbb{R}$ is continuous, then $f$ is Riemann integrable.**
**Proof.** Let $\varepsilon > 0$ be given.
Now $f$ is continuous on $[a, b] \Rightarrow$ It is also uniformly continuous.
Therefore there exists a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta$.
For any large integer $N$ we assume an equally spaced partition $x_k = a + kh$, with $h = \frac{b-a}{N}$ and $k = 0,1, \dots, N$. We choose $N$ so large that $\frac{b-a}{N} < \delta$.

Now function $f$ is continuous on any of the intervals $[x_{k-1}, x_k]$,
Hence there must exist points $c_k, d_k \in [x_{k-1}, x_k]$ where $f$ attains its minimum and maximum, respectively, i.e.
$f(c_k) \le f(x) \le f(d_k)$ $for\ all\ x \in [x_{k-1}, x_k]$.
Let $s, t: [a, b] \to \mathbb{R}$ are two step functions such that on each interval $[x_{k-1}, x_k)$
$s(x) = f(c_k)$ and $t(x) = f(d_k)$.
Therefore, we conclude that $s(x) \le f(x) \le t(x)$ for some $x \in [x_{k-1}, x_k)$
Since $|c_k - d_k| \le \frac{b-a}{N} < \delta$ then for any $x \in [x_{k-1}, x_k)$
$t(x) - s(x) = f(d_k) - f(c_k) < \frac{\varepsilon}{b-a}$.
This also holds for each interval $[x_{k-1}, x_k)$ $(k = 1,2, \dots, N)$
Hence we shown that $0 \le t(x) - s(x) < \frac{\varepsilon}{b-a}$ for all $x \in [a, b]$
Now compare the integrals of $t$ and $s$ and since $t \le s + \frac{\varepsilon}{b-a}$
Then $\int_a^b t(x)dx \le \int_a^b \left(s(x) + \frac{\varepsilon}{b-a}\right)dx = \int_a^b s dx + \varepsilon$

## *5.8 INTEGRATION AND DIFFERENTIATION*

**Theorem 5.13. If a function $f$ is bounded and integrable on $[a, b]$ then the function $F$ defined as $F(x) = \int_a^b f(x)dx$ is continuous on $[a, b]$ and also if $f$ is continuous at a point $c$ of $(a, b)$ then $F$ is derivable at $c$ and $F'(c) = f(c)$.**

**Proof.** It is given that $f$ is bounded, therefore there exists a positive number $\alpha > 0$, such that
$|f(x)| \le \alpha\ for\ all\ x \in [a, b]$
If $x'$ and $x''$ are two points of $[a, b]$ such that $a \le x' \le x'' \le b$, then
$|F(x'') - F(x')| = \left|\int_{x'}^{x''} f(x)dx\right| \le \alpha(x'' - x')$ (From Deduction 3)
Hence for a given $\varepsilon > 0$,
$|F(x'') - F(x')| < \varepsilon$ if $|x'' - x'| < \frac{\varepsilon}{\alpha}$
Therefore, $F$ is continuous on $[a, b]$.
Let $f$ be continuous at a point $c$ of $(a, b)$, therefore for any positive number $\varepsilon > 0$ there exists $\delta$ such that
$|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$
Assume $c - \delta < k_1 \le k_2 \le t < c + \delta$
Therefore

$$\left|\frac{F(k_1)-F(k_2)}{k_1-k_2}-f(c)\right| \le \left|\frac{1}{k_2-k_1}\int_{k_1}^{k_2}\big(f(x)-f(c)\big)dx\right|$$
$$\le \frac{1}{|k_2-k_1|}\int_{k_1}^{k_2}|f(x)-f(c)|dx < \varepsilon$$

Hence $F'(c) = f(c)$.
This theorem sometimes known as the First Fundamental Theorem of Integral Calculus.

**NOTE:**
Continuity of $f$ at any point of $[a, b]$ implies derivability of $F$ at that point.

**Fundamental Theorem of Calculus**

**Theorem 5.14. A function $f$ is bounded and integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$, then**
$\int_a^b f(x)dx = F(b) - F(a)$

**Proof.** It is given that $F' = f$ is bounded and integrable on $[a, b]$.
Therefore for every given $\varepsilon > 0$ there exists a positive number $\delta$ such that for every partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$, with norm $\mu(P) < \delta$.
$\left|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f(x)dx\right| < \varepsilon$ ……………………….(1)
For every choice of points $t_i$ in $\Delta x_i$.
Because we have freedom in the selection of points $t_i$ in $\Delta x_i$., we choose them in a particular way as follows:
By Lagrange Mean value theorem, we have
$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i \qquad (i = 1,2, \dots, n)$
Hence $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$
It implies that $\sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a)$.
From inequality (1), we get
$\int_a^b f(x)dx = F(b) - F(a)$
This theorem is also known as the Second Fundamental theorem of Integral Calculus.

**CHECK YOUR PROGRESS**

(CQ 9) If $f: [a, b] \to \mathbb{R}$ is continuous, then $f$ is Riemann integrable. (T/F)
(CQ 10) Fundamental theorem of calculus states that________________.

## *5.9 MEAN VALUE THEOREM*

**First Mean Value theorem**

**Theorem 5.15. A function $\boldsymbol{f}$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$, then there exists a number $\boldsymbol{k}$ in $[\boldsymbol{a}, \boldsymbol{b}]$ uch that $\int_a^b \boldsymbol{f}\, \boldsymbol{dx} = \boldsymbol{f}(\boldsymbol{k})(\boldsymbol{b} - \boldsymbol{a})$**

**Proof.** It is given that $f$ is continuous on $[a, b]$, therefore f is Riemann Integrable on $[a, b]$.
Let $M$ and $m$ are the upper and lower bound of $f$ on $[a, b]$ respectively.
As we know that
$m(b - a) \leq \int_a^b f\, dx \leq M(b - a)$
Hence there exists a real number $\gamma \in [m, M]$ such that
$\int_a^b f\, dx = \gamma(b - a)$
Because $f$ is continuous on $[a, b]$, it attains every value between m and M.
Hence, there exists a number $k \in [a, b]$ such that $f(k) = \gamma$.
Therefore, $\int_a^b f\, dx = f(k)(b - a)$

## *5.10 SUMMARY*

In this unit, we discussed about Riemann integral and its properties. Also we proved some important theorem related to Riemann integral.

## *5.11 GLOSSARY*

1. Set- a well defined collection of elements
2. Continuity- curve can be drawn without picking up the pencil
3. Derivative- the rate of change of a function with respect to a variable
4. Integral- a function of which a given function is the derivative.

## *5.13 REFERENCES*

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## *5.12 SUGGESTED READINGS*


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2. W. Rudin (2019) Principles of Mathematical Analysis, McGraw-Hill Publishing, 1964.
3. Tom M. Apostol (1996). Mathematical Analysis (2nd edition), Narosa Book Distributors Pvt Ltd-New Delhi.
4. Pawan K. Jain and Khalil Ahmad (2005). Metric spaces, 2nd Edition, Narosa.


## *5.14 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) If f is continuous and positive on [a,b] then show that $\int_a^b f\, dx$ is also positive.

(TQ 2) Prove that A function $f$ is continuous on $[a, b]$, then there exists a number $k$ in $[a, b]$ uch that $\int_a^b f\, dx = f(k)(b - a)$

(TQ 3) Show that $\lim I_n$, where $I_n = \int_a^b \frac{\sin nx}{x}\, dx$, n $\in \mathbb{N}$ exista and equal to $\frac{\pi}{2}$

(TQ 4) State and Prove the First Fundamental Theorem of Integral Calculus.

(TQ 5) Explain Riemann Sums.

.**Fill in the blanks**

(TQ 6) $\int_a^b \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) dx$ is ______________________.

(TQ 7) $f(x) = \begin{cases} 0, \text{when } x \text{ is rational} \\ 1, \text{ when } x\ is \text{ irrational} \end{cases}$ then $f(x)$ is _______________.

## *5.15 ANSWERS*

| | | |
|---|---|---|
| (CQ 1) T | (CQ 2) F | (CQ 3) T |
| (CQ 4) $U(P, f) = L(P, f)$ | (CQ 5) F | (CQ 6) T |
| (CQ 7) $\sum M_i\, x_i$ | (CQ 8) $\sum m_i\, x_i$ | (CQ 9) T |
| (TQ 6) sin 1. | (TQ 7) not integrable | |

# UNIT 6: RIEMANN-STIELTJES INTEGRAL

**CONTENTS**



## *6.1 INTRODUCTION*

The learners will recall from elementary calculus that to find the area of the region under the graph of a pose function $f$ defined on $[a, b]$, we subdivide the interval $[a, b]$ into a finite number of subintervals, say *n*, the $i^{th}$ subinterval having length $\Delta x_i$, and we consider sums of the form $\sum_{i=1}^{n} f(u_i)\Delta x_i$ where $u_i$ is some point in the $i^{th}$ subinterval. Such a sum is an approximation to the area by means of rectangles.

If $f$ is sufficiently continuous in $[a, b]$. The two concepts, derivative and integral, arise in entirely different ways and it is a remarkable fact indeed that the two are intimately connected. If we consider the definite integral of a continuous function $f$ as a function of its upper limit, say we write $\int_{i=1}^{n} f(u_i)\Delta x_i$.. Then $F$ has a derivative and $F'(x) = f(x)$. This important result shows that differentiation and integration are, in a sense, inverse operations.

In this unit we study the process of integration in some detail. Actually we consider a more general concept than that of Riemann namely Riemann-Stieltjes integral, which involves two functions $f$ and $h$. The symbol for such an integral $\int_a^b f(x)d\left(h(x)\right)$ and the usual

Riemann integral occurs as the special case in which $h(x) = x$. When$h$ has a continuous derivative, the definition is such that the Stieltjes integral becomes the Riemann integral. However, the Stieltjes integral still makes sense when $h$ is not differentiable or even when $h$ is discontinuous.

Problems in physics which involve mass distributions that are partly discrete and partly continuous can also be treated by using Stieltjes integrals. In the mathematical theory of probability this integral is a very useful tool that makes possible the simultaneous treatment of continuous and discrete random variables.

In previous unit we studied about Riemann integral. In this unit we will study Riemann -Stieltjes integral.

## *6.2 OBJECTIVES*

In this Unit, we will discussed about

1. Basics of Riemann-stieltjes integral
2. Important theorems of Riemann-stieltjes integral
3. Relationship between Riemann-stieltjes integral and Riemann integral

## *6.3 RIEMANN -STIELTJES INTEGRAL*

Now we will discuss the definition of Riemann- Stieltjes integral of a function $f$ on an interval $[a, b]$.

Let $f : [a, b] \to \mathbb{R}$ be bounded function and $\alpha$ be a monotonically increasing function on $[a, b]$.

Let $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ such that $a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n$ be any Partition of $[a, b]$ . then

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \quad i = 1,2,3, \dots, n$

According to the definition of monotone function $\alpha(a)$ and $\alpha(b)$ are finite

therefore $h$ is bounded on $[a, b]$,

Because $h$ is monotonically increasing function then clearly $\Delta\alpha_i \geq 0, i = 1, 2, 3, \dots, n$.

Let $M_i$ and $m_i$ be upper bound and lower bound of $f(x)$ in interval $[x_{i-1}, x_i]$.

i.e. $M_i = \sup f(x)$ , $m_i = \inf f(x)$ where $x \in [x_{i-1}, x_i]$) for each $P \in \mathcal{P}([a, b])$.

Now we define
$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \, \Delta\alpha_i$ and $L(P, f, h) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i$
and $U(P, f, \alpha)$ and $L(P, f, \alpha)$ are called the Upper and Lower Riemann Stieltjes sums respectively.
Also
$\int_{a}^{-b} f d(h(\alpha)) = \inf U(P, f, \alpha)$ and $\int_{-a}^{b} f d(\alpha(x)) = \sup L(P, f, \alpha)$
where the inf and the sup are taken over all partitions $P$ of $[a, b]$, are called the Upper and Lower Riemann Stieltjes integrals of $f$ over $[a, b]$ respectively.
$\int_{a}^{-b} f(x) d(\alpha(x)) = \int_{-a}^{b} f(x) d(\alpha(x))$ then $f$ is said to be Riemann Stieltjes integrable integral of $f$ with respect to $\alpha$ over $[a, b]$ and we write $f \in R(\alpha)$.
It is also denoted by $\int_{a}^{b} f(x) d(\alpha(x))$
If we put $\alpha(x) = x$ then we conclude that the Riemann integral is the special condition of the Riemann- Stietjes integral.

**Theorem 6.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\alpha$ be a monotonically increasing function on $[a, b]$. Consider $P$ be any Partition of $[a, b]$. Then $U(P, f, \alpha)$ and $L(P, f, h)$are bounded.**

**Proof.** It is given that $f$ is bounded, therefore there exist two real numbers $m$ and $M$ such that $m \leq f(x) \leq M$ where $a \leq x \leq b$
Therefore for every partition $P$ of $[a, b]$ we have
$m \leq m_i \leq M_i \leq M$
Multiplying inequality (1) by $\Delta h_i$, we get
$m\Delta\alpha_i \leq m_i \, \Delta\alpha_i \leq M_i\Delta\alpha_i \leq M\Delta\alpha_i$
Adding all these inequalities for $i = 1,2,3, \dots, n$, we get
$\sum_{i=1}^{n} m\Delta\alpha_i \leq \sum_{i=1}^{n} m_i \, \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \, \Delta\alpha_i \leq \sum_{i=1}^{n} M\Delta\alpha_i$
$\Rightarrow m\big(\alpha(b) - \alpha(a)\big) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M\big(\alpha(b) - \alpha(a)\big)$
Hence $L(P, f, \alpha)$ and $U(P, f, \alpha)$ form a bounded set.
**NOTE**
By the definition of lower and upper Riemann-Stietjes integrals we conclude that from above theorem that the upper and lower integrals are defined for every bounded function $f$ are bounded also.

**Theorem 6.2. Let $P^*$ be a refinement of the partition P of $[a, b]$, then $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.**

**Proof.** Let $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ such that $a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n$ be any Partition of $[a,b]$ .and $P^*$ the refinement of $P$ contains only one point $x'$ more than $P$ such that $x_{i-1} < x^* < x_i$ where $x_{i-1}$ and $x_{i-1}$ are two consecutive points of P.

Let $m_i', m_i''$ and $m$ are the infimum of $f(x)$ in $[x_{i-1}, x^*]$, $[x^*, x_i]$ and $[x_{i-1}, x_i]$, respectively Clearly we can see that

$m_i \leq m_i'$ and $m_i \leq m_i''$.

Therefore

$L(P^*, f, \alpha) - L(P, f, \alpha)$

$= m_i'[\alpha(x^*) - \alpha(x_{i-1})] + m_i''[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})]$

$= m_i'[\alpha(x^*) - \alpha(x_{i-1})] + m_i''[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})]$

$= m_i'[\alpha(x^*) - \alpha(x_{i-1})] + m_i''[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x^*)] - m_i[+m_i\ [\alpha(x^*) - \alpha(x_{i-1})]$

$= m_i'[\alpha(x^*) - \alpha(x_{i-1})] + m_i''[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x^*)] - m_i\ [\alpha(x^*) - \alpha(x_{i-1})]$

$= (m_i' - m_i)\big(\alpha(x^*) - \alpha(x_{i-1})\big) + (m_i'' - m_i)\big(\alpha(x_i) - \alpha(x^*)\big) \geq 0$

Therefore $L(P^*, f, \alpha) \leq L(P, f, \alpha)$

Let each additional point is introduced one by one, by repeating the above reasoning $k$ times, we get same result.

Similarly, we can prove that $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

**Theorem 6.3. If $P_1$ and $P_2$ are any two partitions of [a, b] then $\boldsymbol{L(P_1, f, \alpha) \leq U(P_2, f, \alpha)}$ and $\boldsymbol{L(P_2, f, \alpha) \leq U(P_1, f, \alpha)}$.**

**Proof.** Consider $P_1$ and $P_2$ be any two partitions of $[a,b]$. Let $= P_1 \cup P_2$, then $P$ is the common refinement of $P_1$ and $P_2$. Then

$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_1, f, \alpha)$……………………………(1)

and

$L(P_2, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$ …………….(2)

Comparing inequalities (1) and ((2), we get

$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$ and $L(P_2, f, \alpha) \leq U(P_1, f, \alpha)$.

**NOTE:**

$\int_{-a}^{b} f(x) d(\alpha(x)) \leq \int_{a}^{-b} f(x) d(\alpha(x))$

**Theorem 6.4. A function $f$ is a Riemann Stieltjes integrable with respect to $h$ iff for every $\varepsilon > 0$ there exists a partition $P$ such that $\boldsymbol{U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon}$**

**Proof.** Let P be any partition of $[a, b]$ such that. $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$

Then $L(P, f, \alpha) \leq \int_{-a}^{b} f(x) d(\alpha(x)) \leq \int_{a}^{-b} f(x) d(\alpha(x)) \leq U(P, f, \alpha)$

It implies that

$U(P, f, \alpha) \leq \int_{a}^{-b} f(x) d(\alpha(x))$…………………..(1)

$L(P, f, \alpha) \geq \int_{-a}^{b} f\, f(x) d(\alpha(x))$……………………(2)

$\Rightarrow -L(P, f, \alpha) \leq -\int_{-a}^{b} f(x) d(\alpha(x))$…………………………………..(3)

By adding inequality (1) and (3), we get

$U(P, f, \alpha) - L(P, f, \alpha) \leq \int_{a}^{-b} f(x) d(\alpha(x)) - \int_{-a}^{b} f(x) d(\alpha(x))$

Now It is given that for any positive real number $\varepsilon$

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Hence $\int_{a}^{-b} f(x) d(\alpha(x)) - \int_{-a}^{b} f(x) d(\alpha(x)) < \varepsilon$

Because $\varepsilon$ is any arbitrary positive number and also we know that a non negative number is less than every positive number.

Therefore it should be equal to 0.i.e. $\int_{a}^{-b} f(x) d(\alpha(x)) - \int_{-a}^{b} f(x) d(\alpha(x)) = 0$

Hence $\int_{a}^{-b} f(x) d(\alpha(x)) = \int_{-a}^{b} f(x) d(\alpha(x))$ which implies $f$ is a Riemann Stieltjes integrable with respect to $\alpha$.

Converse

Let $f$ is a Riemann Stieltjes integrable with respect to $h$.

Hence $\int_{a}^{-b} f(x) d(\alpha(x)) = \int_{-a}^{b} f(x) d(\alpha(x)) = \int_{a}^{b} f(x) d(\alpha(x))$

Let $\varepsilon > 0$ be any positive number.

By Definition of Riemann Stieltjes integrale there exists two partitions $P_1$ and $P_2$ such that

$U(P_1, f, \alpha) < \int_{a}^{-b} f(x) d(\alpha(x)) + \frac{1}{2}\varepsilon$ …………………..(1)

$L(P_2, f, \alpha) > \int_{-a}^{b} f(x) d(\alpha(x)) - \frac{1}{2}\varepsilon$ ……………………(2)

Consider a partition $P$ such that $P = P_1 \cup P_1$ be the common refinement of $P_1$ and $P_2$.

Then $U(P, f, \alpha) \leq U(P_1, f, \alpha)$

Using inequality (1), we get

$U(P, f, \alpha) < \int_{a}^{-b} f(x) d(\alpha(x)) + \frac{1}{2}\varepsilon$ ……………………………(3)

Similarly $L(P, f, \alpha) \geq L(P_1, f, \alpha)$

$L(P, f, \alpha) > \int_{-a}^{b} f(x) d(\alpha(x)) - \frac{1}{2}\varepsilon$

$\Rightarrow -L(P,f,\alpha) < -\int_{-a}^{b} f(x)d\big(\alpha(x)\big) + \frac{1}{2}\,\varepsilon$……………………………(4)

Adding inequality (3) and (4), we get

$U(P,f,\alpha) - L(P,f,\alpha)$

$$< \int_{a}^{-b} f(x)d\big(\alpha(x)\big) + \frac{1}{2}\varepsilon - \int_{-a}^{b} f(x)d\big(\alpha(x)\big) + \frac{1}{2}\,\varepsilon$$

$$= \varepsilon$$

Hence $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

**Theorem 6.5. If $\boldsymbol{f} \in \boldsymbol{R}(\alpha)$ and if $\boldsymbol{t_i}$ is arbitrary point in $[\boldsymbol{x_{i-1}}, \boldsymbol{x_i}]$, then** $\left|\sum_{i=1}^{n} \boldsymbol{f}(\boldsymbol{t_i})\Delta\boldsymbol{\alpha_i} - \int_{a}^{-b} f(x)d\big(\alpha(x)\big)\right| < \boldsymbol{\varepsilon}$

**Proof.** It is given that $f \in R(\alpha)$, therefore for any given $\varepsilon > 0$

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ ………………………..(1)

Let $P$ be any partition $P$ of $[a,b]$ such that $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$.

Let $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively and $t_i$ is arbitrary point in $[x_{i-1}, x_i]$,

$L(P,f,\alpha) \le \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \le U(P,f,\alpha)$

$\Rightarrow \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \le U(P,f,\alpha)$…………………..(2)

and $L(P,f,\alpha) \le \int_{a}^{b} f(x)d\big(\alpha(x)\big) \le U(P,f,\alpha)$

$\Rightarrow L(P,f,\alpha) \le \int_{a}^{b} f(x)d\big(\alpha(x)\big)$

$\Rightarrow -\int_{a}^{b} f(x)d\big(\alpha(x)\big) \le -L(P,f,\alpha)$…………………(3)

Adding inequalities (2) and (3), we get

$\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_{a}^{b} f(x)d\big(\alpha(x)\big) \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ (from inequality (1))

Therefore

$\left|\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_{a}^{b} f(x)d\big(\alpha(x)\big)\right| < \varepsilon$

**Theorem 6.6. Let $\boldsymbol{f}$ be a continuous function on [a, b] and function $\alpha$ be monotonic increasing on $[\boldsymbol{a}, \boldsymbol{b}]$ then $\boldsymbol{f}$ is a Riemann Stieltjes integrable with respect to $\alpha$ on $[\boldsymbol{a}, \boldsymbol{b}]$.**

**Proof.** Let $\varepsilon > 0$ be any positive number.

Now we choose $\eta > 0$ such that $\alpha(b) - \alpha(a) < \frac{\varepsilon}{\eta}$.

It is given that $f$ is continuous in theinterval $[a,b]$ therefore it is uniformly continuous on [a, b], So there exists a $\delta > 0$ such that

$|f(x) - f(y)| < \eta$ whenever $|x - y| < \delta \; for\ all\ x, y \in [a,b]$.

………(1)

Let $P$ be any partition of $[a,b]$ such that. $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ such that

$M_i = \sup f(x)$ and $m_i = \inf f(x)$ $(x_{i-1} \le x \le x_i)$

then $\Delta x_i < \delta$ for all $i$, therefore

From inequality (1), we get

$M_i - m_i < \eta \quad (i = 1, \dots, n)$

Hence

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i$$
$$< \eta\big(\alpha(a) - \alpha(b)\big)$$
$$< \eta\frac{\varepsilon}{\eta} < \varepsilon$$

Therefore $f$ is a Riemann Stieltjes integrable with respect to $\alpha$ on $[a,b]$.

**CHECK YOUR PROGRESS**

(CQ 1) Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $\alpha$ be a monotonically increasing function on $[a,b]$. Consider $P$ be any Partition of $[a,b]$. Then $U(P,f,\alpha)$ and $L(P,f,h)$ are bounded. (T/F)

(CQ 2) If $f \in R(\alpha)$ and if $t_i$ is arbitrary point in $[x_{i-1}, x_i]$, then $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^{-b} f(x)d\big(\alpha(x)\big)\right| > \varepsilon$. (T/F)

(CQ 3) If $P_1$ and $P_2$ are any two partitions of [a, b] then $L(P_1,f,\alpha) \le U(P_2,f,\alpha)$ and $L(P_2,f,\alpha) \le U(P_1,f,\alpha)$. (T/F)

## *6.4 SOME IMPORTANT THEOREMS*

**Theorem 6.7. a)If $f_1, f_2 \in R(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in R(\alpha)$ and**

$$\int_a^b (f_1 + f_2)(x)\, d\big(\alpha(x)\big) = \int_a^b f_1(x)\, d\big(\alpha(x)\big) + \int_a^b f_2(x)\, d\big(\alpha(x)\big)$$

**b) If $f_1, f_2 \in R(\alpha)$ on $[a,b]$ and $f_1(x) \le f_2(x)$ *for all* $x \in [a,b]$ then**

$$\int_a^b f_1(x)\, d\big(\alpha(x)\big) \le \int_a^b f_2(x)\, d\big(\alpha(x)\big)$$

**c) If $f \in R(\alpha)$ on $[a,b]$ and $k$ is a constant then $cf \in R(\alpha)$ and**

$$\int_a^b (cf)(x)\, d\big(\alpha(x)\big) = c\int_a^b f(x)\, d\big(\alpha(x)\big)$$

**d) If $f \in R(\alpha)$ on $[a,b]$ and if $a < k < b$ then $f \in R(\alpha)$ over $[a,k]$ and $[k,b]$ and**

$$\int_a^k f(x)\, d\big(\alpha(x)\big) + \int_k^b f(x)\, d\big(\alpha(x)\big) = \int_a^b f(x)\, d\big(\alpha(x)\big)$$

**Proof. (a)** Let $f_1, f_2 \in R(\alpha)$ on $[a,b]$ and $f = f_1 + f_2$

Clearly $f = f_1 + f_2$ is bounded on $[a,b]$.

Let $P$ be any partition $P$ of $[a,b]$ such that $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$.

Let $M_i'$ and $m_i'$ are the upper and lower bound of $f_1$ respectively and $M_i''$ and $m_i''$ are the upper and lower bound of $f_2$ respectively in $\Delta x_i$.

Assume $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively in $\Delta x_i$.

Therefore $m_i' \leq m_i''$ and $M_i' \leq M_i''$

Multiplying inequality (1) by $\Delta\alpha_i$, we get

$(m_i' + m_i'')\Delta\alpha_i \leq m_i\Delta\alpha_i \leq M_i\Delta\alpha_i \leq (M_i' + M_i'')\Delta\alpha_i$

Adding all these inequalities for $i = 1,2,3, \dots, n$, we get

$\sum_{i=1}^{n}(m_i' + m_i'')\Delta\alpha_i \leq \sum_{1=1}^{n} m_i\Delta\alpha_i \leq \sum_{1=1}^{n} M_i\Delta\alpha_i \leq \sum_{1=1}^{n}(M_i' + M_i'')\Delta\alpha_i$

$\Rightarrow L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$

$U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$ ……………………………….(2)

$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f, \alpha)$

$-L(P, f, \alpha) \leq -(L(P, f_1, \alpha) + L(P, f_2, \alpha))$ …………………………….(3)

Let $\varepsilon > 0$ be any positive number.

It is given that $f_1, f_2 \in R(\alpha)$. Hence there exists partitions $P_1$ and $P_2$ such that

$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{1}{2}\varepsilon$ and

$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{1}{2}\varepsilon$

Let $P = P_1 \cup P_2$ then

$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{1}{2}\varepsilon$,………………………….(4)

$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{1}{2}\varepsilon$…………………………...(5)

From (2),(3),(4) and (5), we get

$U(P, f, \alpha) - L(P, f, \alpha)$

$\leq U(P, f_1, \alpha) + U(P, f_2, \alpha) - \big(L(P, f_1, \alpha) + L(P, f_2, \alpha)\big)$

$= U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha)$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$

Therefore, $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ .

Hence the function $f \in R(\alpha)$.

Since the upper integral is the infimum of upper sums, hence there exists partitions $P_1$ and $P_2$ such that

$U(P_1, f_1, \alpha) < \int_a^b f_1\, d\alpha + \frac{1}{2}\varepsilon$ and $U(P_2, f_2, \alpha) < \int_a^b f_2\, d\alpha + \frac{1}{2}\varepsilon$

If P be a partition such that $P = P_1 \cup P_2$, then

$U(P, f_1, \alpha) < \int_a^b f_1\, d\alpha + \frac{1}{2}\varepsilon$ ……………(6)

and

$U(P, f_2, \alpha) < \int_a^b f_2\, d\alpha + \frac{1}{2}\varepsilon$ ……………(7)

Using inequality (2), we get

$\int_a^b f d\alpha \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$

Using inequalities (6) and (7), we get

$\int_a^b f\, d\alpha < \int_a^b f_1\, d\alpha + \frac{1}{2}\varepsilon + \int_a^b f_2\, d\alpha + \frac{1}{2}\varepsilon = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + \varepsilon$

As we know $\varepsilon$ is arbitrary, therefore

$\int_a^b f\, d\alpha \leq \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$ …………………………..(8)

Now replacing $f_1$ and $f_2$ with $(-f_1)$ and $(-f_2)$ respectively, we get

$\int_a^b (-f)\, d\alpha \leq \int_a^b (-f_1) d\alpha + \int_a^b (-f_2)\, d\alpha$

i.e. $\int_a^b f\, d\alpha \geq \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$…………………………..(9)

From inequality (8) and (9), we get

$\int_a^b f\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2 d\alpha$

(b) Similarly we can prove that $\int_a^b f\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2 d\alpha$

Let $P$ be any partition of $[a, b]$ and $f(x) = f_2(x) - f_1(x) \geq 0\ for\ all\ x \in [a, b]$ and $\alpha$ monotonically increasing function on $[a, b]$ then

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0.$

Also if $m_i = \inf f(x)\ where\ a \leq x_{i-1} \leq x \leq x_i \leq b$, then $L(P, f, \alpha) \geq 0$. Therefore $\int_{-a}^b f\, d\alpha = \sup L(P, f, \alpha) \geq 0$

Therefore $\int_{-a}^b f\, d\alpha = \int_a^b f_2\, d\alpha - \int_a^b f_1 d\alpha \geq 0.$

Hence $\int_a^b f_2\, d\alpha \geq \int_a^b f_1 d\alpha.$

(c) Let $f \in R(\alpha)$ on $[a, b]$ and $k$ be a constant.

If $k = 0$ then the proof is trivial, but if $k \neq 0$ then say $k > 0$

$f \in R(\alpha)$ then for given $\varepsilon > 0$ there is a partition $P$ of $[a, b]$ i.e. $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon/k$ ……………………………(10)

Now $(kf)(x) = kf(x)$, therefore $\sup(cf)(x) = c \sup f(x)$ and $\inf(cf)(x) = c \inf f(x)$ of $[a, b]$ which implies

$U(P, kf, \alpha) = k\, U(P, f, \alpha)$ and $L(P, kf, \alpha) = kL(P, f, \alpha)$

Therefore

$U(P,kf,\alpha) - L(P,kf,\alpha) = k\,U(P,f,\alpha) - kL(P,f,\alpha) =$
$k\big(\,U(P,f,\alpha) - L(P,f,\alpha)\big)$
From inequality (10), we get
$U(P,kf,\alpha) - L(P,kf,\alpha) < k\,\frac{\varepsilon}{k} < \varepsilon$
Therefore $kf \in R(\alpha)$
Similarly when $k < 0$ then $kf \in R(\alpha)$.
(d) It is given that $f \in R(\alpha)$ for given $\varepsilon > 0$ there is a partition $P$ of $[a,b]$ i.e. $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ such that
, $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ ……………………………(11)
Where $U(P,f,\alpha) = \sum_{1=1}^{n} M_i \Delta\alpha_i \quad and\ M_i\ is\ upper\ bounds\ in\ \Delta x_i$ and
$L(P,f,\alpha) = \sum_{1=1}^{n} m_i \Delta\alpha_i \quad and\ m_i\ is\ lower\ bounds\ in\ \Delta x_i$
Let $x_r = k$ , then $U(P,f,\alpha) = \sum_{1=1}^{r} M_i \Delta\alpha_i + \sum_{1=r}^{n} M_i \Delta\alpha_i$ and
$L(P,f,\alpha) = \sum_{1=1}^{r} m_i \Delta\alpha_i + \sum_{1=r}^{n} m_i \Delta\alpha_i$.
Therefore
$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{1=1}^{r} (M_i - m_i)\Delta\alpha_i + \sum_{1=r}^{n} (M_i - m_i)\Delta\alpha_i$
From inequality (1) we get that
$\sum_{1=1}^{r} (M_i - m_i)\Delta\alpha_i + \sum_{1=r}^{n} (M_i - m_i)\Delta\alpha_i < \varepsilon$
It implies that $\sum_{1=1}^{r} (M_i - m_i)\Delta\alpha_i < \varepsilon$ and $\sum_{1=r}^{n} (M_i - m_i)\Delta\alpha_i < \varepsilon$
Which implies $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ on $[a,k]$ and $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ on $[k,b]$.
Thus $f \in R(\alpha)$ on $[a,k]$ and $[k,b]$
We already prove that if $t_i$ is arbitrary point in $[x_{i-1}, x_i]$ of $[a,b]$then
$\left|\sum_{i=1}^{n} \boldsymbol{f(t_i)\Delta\alpha_i} - \int_a^b f(x)d\big(\alpha(x)\big)\right| < \boldsymbol{\varepsilon}$
Now $f \in R(\alpha)$ on $[a,b]$, $[a,k]$ and $[k,b]$, therefore for $\varepsilon > 0$, we have
$\left|\sum_{i=1\,\{[a,b]\}}^{n} \boldsymbol{f(t_i)\Delta\alpha_i} - \int_a^b f(x)d\big(\alpha(x)\big)\right| < \frac{\boldsymbol{\varepsilon}}{\mathbf{3}}$ on $[a,b]$,……………(12)
$\left|\sum_{i=1\,\{[a,k]\}}^{n} \boldsymbol{f(t_i)\Delta\alpha_i} - \int_a^k f(x)d\big(\alpha(x)\big)\right| < \frac{\boldsymbol{\varepsilon}}{\mathbf{3}}$ on $[a,k]$ ,…………(13)
$\left|\sum_{i=1\,\{[k,b]\}}^{n} \boldsymbol{f(t_i)\Delta\alpha_i} - \int_k^b f(x)d\big(\alpha(x)\big)\right| < \frac{\boldsymbol{\varepsilon}}{\mathbf{3}}$ on $[k,b]$,……………(14)
Therefore
$\left|\int_a^b f(x)d\big(\alpha(x)\big) - \left(\int_a^k f(x)d\big(\alpha(x)\big) + \int_k^b f(x)d\big(\alpha(x)\big)\right)\right|$
$=\Big|\int_a^b f(x)d\big(\alpha(x)\big) - \left(\int_a^k f(x)d\big(\alpha(x)\big) + \int_k^b f(x)d\big(\alpha(x)\big)\right) +$
$\sum_{i=1\,\{[a,k]\}}^{n} f(t_i)\Delta\alpha_i + \sum_{i=1\,\{[k,b]\}}^{n} f(t_i)\Delta\alpha_i - \sum_{i=1\,\{[a,b]\}}^{n} f(t_i)\Delta\alpha_i\Big|$
$=\Big|\left(\int_a^b f(x)d\big(\alpha(x)\big) - \sum_{i=1\,\{[a,b]\}}^{n} f(t_i)\Delta\alpha_i\right) + \Big(\sum_{i=1\,\{[a,k]\}}^{n} f(t_i)\Delta\alpha_i -$
$\int_a^k f(x)d\big(\alpha(x)\big) + \sum_{i=1\,\{[k,b]\}}^{n} f(t_i)\Delta\alpha_i - \int_k^b f(x)d\big(\alpha(x)\big)\Big)\Big|$
$\le \left|\sum_{i=1\,\{[a,b]\}}^{n} f(t_i)\Delta\alpha_i - \int_a^b f(x)d\big(\alpha(x)\big)\right| + \Big|\sum_{i=1\,\{[a,k]\}}^{n} f(t_i)\Delta\alpha_i -$

$\int_a^k f(x)d\big(\alpha(x)\big)\Big| + \Big|\sum_{i=1\,\{[k,b]\}}^n f(t_i)\Delta\alpha_i - \int_k^b f(x)d\big(\alpha(x)\big)\Big|$

From inequalities (12), (13) and (14), we gert

$\Big|\int_a^b f(x)d\big(\alpha(x)\big) - \Big(\int_a^k f(x)d\big(\alpha(x)\big) + \int_k^b f(x)d\big(\alpha(x)\big)\Big)\Big| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$

As we know $\varepsilon$ is arbitrary then letting $\varepsilon \to 0$, we get

$\Big|\int_a^b f(x)d\big(\alpha(x)\big) - \Big(\int_a^k f(x)d\big(\alpha(x)\big) + \int_k^b f(x)d\big(\alpha(x)\big)\Big)\Big| = 0$

$\Rightarrow \int_a^b f(x)d\big(\alpha(x)\big) - \Big(\int_a^k f(x)d\big(\alpha(x)\big) + \int_k^b f(x)d\big(\alpha(x)\big)\Big) = 0$

$\Rightarrow \int_a^b f(x)d\big(\alpha(x)\big) = \int_a^k f(x)d\big(\alpha(x)\big) + \int_k^b f(x)d\big(\alpha(x)\big)$

**Theorem 6.8. a) lf $\boldsymbol{f} \in \boldsymbol{R}(\alpha)$ on $[\boldsymbol{a}, \boldsymbol{b}]$ then $|\boldsymbol{f}| \in \boldsymbol{R}(\alpha)$ and**

$\Big|\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{d}\big(\alpha(\boldsymbol{x})\big)\Big| \le \int_{\boldsymbol{a}}^{\boldsymbol{b}} |\boldsymbol{f}(\boldsymbol{x})|\, \boldsymbol{d}\big(\alpha(\boldsymbol{x})\big)$

**b) lf $\boldsymbol{f} \in \boldsymbol{R}(\alpha_1)$ and $\boldsymbol{f} \in \boldsymbol{R}(\alpha_2)$, then $\boldsymbol{f} \in \boldsymbol{R}(\alpha_1 + \alpha_2)$ then**

$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{d}\big((\alpha_1 + \alpha_2)\,(\boldsymbol{x})\big) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{d}\big(\alpha_1\,(\boldsymbol{x})\big) + \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{d}\big(\alpha_2\,(\boldsymbol{x})\big)$

**c) lf $\boldsymbol{f} \in \boldsymbol{R}(\alpha)$ on $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{k}$ is a constant then $\boldsymbol{f} \in \boldsymbol{R}(\boldsymbol{k}\alpha)$ and**

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{d}\big((k\alpha)(\boldsymbol{x})\big) = \boldsymbol{k}\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{d}\big(\alpha(\boldsymbol{x})\big)$$

**Proof. a)** It is given that $f$ is bounded therefore there exists $k$ such that

$|f(x)| \le k$ for all $x \in [a, b]$

It implies that the function $|f|$ is bounded.

Since $f \in R(\alpha)$ on $[a, b]$ and $\alpha$ be a monotonically increasing function on $[a, b]$.

Then for a given positive number $\varepsilon > 0$ there exists a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ and such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ …………………..(1)

Let $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively and $M_i'$ and $m_i'$ are the upper and lower bound of $g$ respectively in $\Delta x_i$.

Now for all $x, x' \in \Delta x_i$,

$\big||f|(x) - |f|(y)\big| = \big||f(x)| - |f(y)|\big| \le |f(x) - f(y)|$

$\Rightarrow M_i' - m_i' \le M - m$……………………….(2)

Now multiply inequality (2) with $\Delta\alpha_i$, we get

$(M_i' - m_i')\Delta\Delta\alpha_i \le (M_i - m_i)\Delta\alpha_i$

Adding all these inequalities for $i = 1,2,3, \dots, n$, we get

$\sum_{1=1}^n (M_i' - m_i')\Delta\alpha_i \le \sum_{1=1}^n (M_i - m_i)\Delta\alpha_i$

$\Rightarrow \sum_{1=1}^n M_i'\Delta\alpha_i - \sum_{1=1}^n m_i'\Delta\alpha_i \le \sum_{1=1}^n M_i\ \Delta\alpha_i - \sum_{1=1}^n m_i\Delta\alpha_i$

$\Rightarrow U(P, |f|, \alpha) - L(P, |f|, \alpha) \le U(P, f, \alpha) - L(P, f, \alpha)$

Using inequality (1), we get

$U(P,|f|,\alpha) - L(P,|f|,\alpha) < \varepsilon$ . Hence $|f| \in R(\alpha)$ on $[a,b]$.

We Know that if $f\ g \in R(\alpha)$ on $[a,b]$ such that $f \le g$ then

$\int_a^b f\,dx \le \int_a^b g\,dx$ when $a \le b$

Hence $\int_a^b f\,dx \le \int_a^b |f|\,dx$

and $-\int_a^b f\,dx = \int_a^b (-f)\,dx \le \int_a^b |f|\,dx$

$\Rightarrow \left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$

**b)** It is given that $f \in R(\alpha_1)$ and $f \in R(\alpha_1)$ then for given $\varepsilon > 0$ there exists partitions $P_1$

and $P_2$ of [a, b] such that

$U(P_1,f,\alpha_1) - L(P_1,f,\alpha_1) < \frac{\varepsilon}{2}$ and $U(P_2,f,\alpha_2) - L(P_2,f,\alpha_2) < \frac{\varepsilon}{2}$

If $P = P_1 \cup P_2$ then

$U(P,f,\alpha_1) - L(P,f,\alpha_1) < \frac{\varepsilon}{2}$……………..(3) and

$U(P,f,\alpha_2) - L(P,f,\alpha_2) < \frac{\varepsilon}{2}$ ……………………..(4)

Let $\alpha = \alpha_1 + \alpha_2$ and $M_i$ be upper bound of $f$ in $\Delta x_i$, thereore

$U(P,f,\alpha) = \sum_{i=1}^n M_i[\alpha(x_i) - \alpha(x_{i-1})] = \sum_{i=1}^n M_i[(\alpha_1 + \alpha_2)(x_i) - (\alpha_1 + \alpha_2)(x_{i-1})]$

$= \sum_{i=1}^n M_i[\alpha_1(x_i) - \alpha_1(x_{i-1})] + \sum_{i=1}^n M_i[\alpha_2(x_i) - \alpha_2(x_{i-1})]$

Therefore, $U(P,f,\alpha) = U(P,f,\alpha_1) + U(P,f,\alpha_2)$,

Similarly $L(P,f,\alpha) = L(P,f,\alpha_1) + L(P,f,\alpha_2)$,

So, $U(P,f,\alpha) - L(P,f,\alpha) = [U(P,f,\alpha_1) + U(P,f,\alpha_2))] - [L(P,f,\alpha_1) + L(P,f,\alpha_2)]$

From equation (3) and (4), we get

$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

Therefore, $f \in R(\alpha)$ $i.e. f \in R(\alpha_1 + \alpha_2)$

$$\int_a^b f\,d\alpha = inf\ U(P,f,\alpha) = \inf\{U(P,f,\alpha_1) + U(P,f,\alpha_2)\}$$
$$\ge inf\ U(P,f,\alpha_1) + inf\ U(P,f,\alpha_2)$$

$\int_a^b f\,d\alpha \ge \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$ ……………………………..(5)

Similarly

$\int_a^b f\,d\alpha = \sup L(P,f,\alpha_1) \le \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$ ……………(6)

From inequalities (5) and (6), we get

$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$

(c) Similarly we can prove (c) part.

## *6.5 INTEGRAL AS A LIMIT OF SUM*

**Integral as a limit of sum:** Let $P = \{x_0, x_1, \dots, x_n\}$ such that $a = x_0 \le x_1 \le \dots \le x_n \le b$ be any Partition of $[a, b]$ and $t_i \in [x_{i-1}, x_i]$. Assume the sum
$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i$
$S(P, f, \alpha)$ converges to $X$ as $\mu(P) \to 0, i.e. \lim_{\mu(P)\to 0} S(P, f, \alpha) = X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|S(P, f, \alpha) - X| < \varepsilon$, for every partition $P$ with mesh $\mu(P) < \delta$ and every choice of $t_i$ in $\Delta x_i$.

**Theorem 6.9. If $\lim_{\mu(P)\to 0} S(P, f, \alpha)$ exists then $f \in R(\alpha)$ and $\lim_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f\, d\alpha$.**

**Proof.** Let $\lim_{\mu(P)\to 0} S(P, f, \alpha)$ exists and is equal to $X$.
Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ for every partition $P = \{x_0, x_1, \dots, x_n\}$ with mesh $\mu(P) < \delta$ and every choice of $t_i$ in $\Delta x_i$ such that
$|S(P, f, \alpha) - X| < \frac{\varepsilon}{2}$ i.e. $\frac{-\varepsilon}{2} < S(P, f, \alpha) - X < \frac{\varepsilon}{2}$
Or $\frac{-\varepsilon}{2} + X < S(P, f, \alpha) < \frac{\varepsilon}{2} + X$……………………………………..(1)
Assume $M_i$ and $m_i$ are the upper and lower bound of $f$ respectively in $\Delta x_i$ and let the points $t_i \in \Delta x_i$ therefore
$m_i \le f(t_i) \le M_i$
Multiplying inequality (1) by $\Delta\alpha_i$, we get
$m_i\Delta\alpha_i \le f(t_i)\Delta\alpha_i \le M_i\Delta\alpha_i$
Adding all these inequalities for $i = 1,2,3, \dots, n$, we get
$\sum_{i=1}^{n} m_i\Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i\Delta\alpha_i$
$\Rightarrow L(P, f, \alpha) \le S((P, f, \alpha) \le U(P, f, \alpha)$ ……………………..(2)
From inequalities (1) and (2), we get
$\frac{-\varepsilon}{2} + X < L(P, f, \alpha) \le U(P, f, \alpha) < \frac{\varepsilon}{2} + X$
i.e. $\frac{-\varepsilon}{2} + X < L(P, f, \alpha)$
$-L(P, f, \alpha) < \frac{\varepsilon}{2} - X$…………….(3)
and $U(P, f, \alpha) < \frac{\varepsilon}{2} + X$ ……………..(4)
Adding inequalities (3) and (4), we get
$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon}{2} + X + \frac{\varepsilon}{2} - X < \varepsilon$
It implies that $f \in R(\alpha)$ over $[a, b]$.

Since $S((P, f, \alpha)$ and $\int_a^b f \, d\alpha$ lies between $U(P, f, \alpha)$and $L(P, f, \alpha)$. Therefore

$S((P, f, \alpha) \leq U(P, f, \alpha)$ ………………………….(5) and

$L(P, f, \alpha) \leq \int_a^b f \, d\alpha$ i.e. $-\int_a^b f \, d\alpha \leq -L(P, f, \alpha)$……………………(6)

Adding inequalities (5) and (6), we get

$S((P, f, \alpha) - \int_a^b f \, d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

It implies that $\lim_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f \, d\alpha$

**Ex. 6.1. A function $\boldsymbol{\alpha}$ increases on $[\boldsymbol{a}, \boldsymbol{b}]$ and it is continuous at $\boldsymbol{t}$ where $\boldsymbol{a} \leq \boldsymbol{t} \leq \boldsymbol{b}$. Function $\boldsymbol{f}$ is defined such that $\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} \mathbf{1}, & \boldsymbol{when\ x = t} \\ \mathbf{0}, & \boldsymbol{when\ x \neq t} \end{cases}$. Then prove that $\boldsymbol{f} \in \boldsymbol{R}(\boldsymbol{\alpha})$ over $[\boldsymbol{a}, \boldsymbol{b}]$ and $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f} \, \boldsymbol{d\alpha} = \mathbf{0}$.**

**Sol.** Let $P$ be any partition of $[a, b]$ such that $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ and let $t$ in $\Delta x_i$.

It is given that function $\alpha$ increases on $[a, b]$ and it is continuous at $t$.

Therefore for $\varepsilon > 0$ we can choose $\delta > 0$ such that

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) < \varepsilon$ whenever $|x_i - x_{i-1}| < \delta$

Let $P$ be any partition with $\mu(P) < \delta$, Now

$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i = \Delta\alpha_i$ and $L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i = 0$

Therefore over all the partitions P with $\mu(P) < \delta$

$\int_a^{-b} f \, d\alpha = \inf \ U(P, f, \alpha) \ = 0 = \ \sup \ L(P, f, \alpha) = \int_{-a}^b f \, d\alpha$

which implies $f \in R(\alpha)$ and $\int_a^b f \, d\alpha \ = 0$

## *6.6 RELATION BETWEEN RIEMANN INTEGRAL AND RIEMANN-STIELGES INTEGRAL*

**Theorem 6.10. Consider $\boldsymbol{\alpha}$ increases monotonically and $\boldsymbol{\alpha}' \in \boldsymbol{R}[\boldsymbol{a}, \boldsymbol{b}]$. Let $\boldsymbol{f}$ be a bounded real function on $[\boldsymbol{a}, \boldsymbol{b}]$. Then $\boldsymbol{f} \in \boldsymbol{R}(\boldsymbol{\alpha})$ if and only if $\boldsymbol{f\alpha}' \in \boldsymbol{R}[\boldsymbol{a}, \boldsymbol{b}]$. Also $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f} \, \boldsymbol{d\alpha} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{\alpha}'(\boldsymbol{x}) \, \boldsymbol{dx}$**

**Proof.** It is given that $\alpha' \in R[a, b]$

Because $f$ is bounded, there exists $M > 0$, such that

$|f(x)| \leq M$, for all $x \in [a,b]$……………………………….(1)

As $f, \alpha' \in R[a,b]$, therefore $f\alpha' \in R[a,b]$. Hence for $\mu(P) < \delta_1$ and $t_i \in \Delta x_i$ there exists a positive number $\delta_1 > 0$ such that

$\sum_{i=1}^{n} f(t_i)\alpha(t_i)\Delta x_i - \int_a^b f\alpha'\, dx < \frac{\varepsilon}{2}$ ………………………………(2)

and

for $\mu(P) < \delta_2$ and $t_i \in \Delta x_i$ there exists a positive number $\delta_2 > 0$ such that

$\sum_{i=1}^{n} \alpha'(t_i)\Delta x_i - \int_a^b \alpha'\, dx < \frac{\varepsilon}{4M}$ ………………………………(3)

Now for $\mu(P) < \delta_2$ and $t_i, s_i \in \Delta x_i$ , from inequality (3), we get

$\sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)|\Delta x_i < 2.\frac{\varepsilon}{4M} = \frac{\varepsilon}{2M}$ ……………………..(4)

Let $\delta = min\,(\delta_1, \delta_2)$ and $P$ be any partition of $[a,b]$ with $\mu(P) < \delta$.

According to Mean value theorem for real valued function for all $t_i \in [x_{i-1}, x_i]$ there are points $s_i \in [x_{i-1}, x_i]$ such that

$\alpha\,(x_i) - \alpha\,(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})$

$\Rightarrow \Delta\alpha_i = \alpha\,'(s_i)\ \Delta x_i$ , for $i = 1, \dots, n.$ ……………….(5)

Thus

$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\alpha'\, dx\right| = \left|\sum_{i=1}^{n} f(t_i)\alpha\,'(s_i)\ \Delta x_i - \int_a^b f\alpha'\, dx\right|$

(Using equation (5))

$= \left|\sum_{i=1}^{n} f(t_i)\alpha\,'(s_i)\ \Delta x_i - \int_a^b f\alpha'\, dx + \sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\ \Delta x_i - \right.$

$\left.\sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\ \Delta x_i\right|$

$= \left|\sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\ \Delta x_i - \int_a^b f\alpha'\, dx + \sum_{i=1}^{n} f(t_i)\alpha\,'(s_i)\ \Delta x_i - \right.$

$\left.\sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\ \Delta x_i\right|$

$= \left|\sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\ \Delta x_i - \int_a^b f\alpha'\, dx + \sum_{i=1}^{n} f(t_i)\ (\alpha\,'(s_i) - \right.$

$\left.\alpha\,'(t_i))\ \Delta x_i\right|$

$\leq \left|\sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\,\Delta x_i - \int_a^b f\alpha'\, dx\right| + |\sum_{i=1}^{n} f(t_i)\ (\alpha\,'(s_i) -$

$\alpha\,'(t_i))\ \Delta x_i|$

$= \left|\sum_{i=1}^{n} f(t_i)\alpha\,'(t_i)\,\Delta x_i - \int_a^b f\alpha'\, dx\right| + |\sum_{i=1}^{n} |f(t_i)||(\alpha\,'(s_i) -$

$\alpha\,'(t_i))\ \Delta x_i|$

From inequalities (1), (2) and (4), we get

$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\alpha'\, dx\right| < \frac{\varepsilon}{2} + M\frac{\varepsilon}{2M} < \varepsilon$

Therefore for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all the partitions with $\mu(P) < \delta$,

$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\alpha'\,dx\right| < \varepsilon$

Therefore $\sum_{i=1}^{n} f(t_i)\Delta\alpha_i$ exists and $\sum_{i=1}^{n} f(t_i)\Delta\alpha_i = \int_a^b f\alpha'\,dx$

## *6.7 SUMMARY*

In this Unit, we discussed about the theory of Riemann-stieltjes integration. Some Basic theorems are also constructed.

## *6.8 GLOSSARY*

1. Set- a well defined collection of elements
2. Derivative- the rate of change of a function with respect to a variable
3. Integral- the area under the curve of a graph of the function.

## *6.11 REFERENCES*

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## *6.9 SUGGESTED READINGS*

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## *6.12 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\alpha$ be a monotonically increasing functionon $[a, b]$. Consider $P$ be any Partition of $[a, b]$. Prove that $U(P, f, \alpha)$ and $L(P, f, h)$are bounded.

(TQ 2) Explain Riemann -stieltjes integral.

(TQ 3) Prove that if $P_1$ and $P_2$ are any two partitions of [a, b] then $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$ and $L(P_2, f, \alpha) \leq U(P_1, f, \alpha)$.

(TQ 4) Derive relation between Riemann integral and Riemann-stielges integral.

**Fill in the blanks**

(TQ 5) Riemann-stielges integral is generalization of __________________.

(TQ 6) The maximum of the length of the components is defined as the ____ of the partition

## *6.13 ANSWERS*

| | | |
|---|---|---|
| (CQ 1) T | (CQ 2) F | (CQ 3) T |
| (TQ 6) Riemann integral | (TQ 7) norm | |

# UNIT 7: IMPROPER INTEGRAL

**CONTENTS**



## *7.1 INTRODUCTION*

The concept of Riemann integral which discussed in previous unit requires the range of integration is finite and the integrand remains bounded in that domain. If either (or both) of these assumptions is not satisfied it is necessary to attach a new interpretation to the integral.

In case the integrand $f$ becomes infinite in the interval $a \leq x \leq b$, i.e. $f$ has points of infinite discontinuity (singular points) in $[a, b]$ or the limits of integration $a$ or $b$ (or both) become infinite, the symbol $\int_a^b f dx$ is called an improper (or infinite or generalised) integral. Thus dx dx

$\int_1^\infty \frac{1}{x}\, dx, \int_{-\infty}^0 e^x\, dx$ are examples of improper integrals.

For the sake of distinction, the integrals which are not improper are called proper integrals.

Thus $\int_0^1 \frac{\cos x}{x}\, dx$ is a proper integral.

It will be assumed throughout that the number of singular points in any interval is finite and, therefore, when the range of integration is infinite, that all the singular points can be included in a finite interval. The restriction on the number is not necessary for the existence of the improper integral, but consideration of the discussion is beyond our limits. Further, it is assumed once for all that in a finite interval which encloses no point of infinite discontinuity (singular point) the integrand is hounded and integrable.

## *7.2 OBJECTIVES*

In this Unit, we will

- analyze about Improper Integral
- Study some theorems based on Improper Integral
- Analyze types of improper Integral

## *7.3 INTEGRATION OF UNBOUNDED FUNCTIONS WITH FINITE LIMITS OF INTEGRATION*

**Convergence at the left end:** Let $f$ be a functuion defined on an interval $[a, b]$ except some finite number of points. Let $a$ be the only point of infinite discontinuity of $f$ so that the integral $\int_{a+k}^{b} f(x)\,dx$ exists for every $k$, $0 < k < b - a$. The improper integral $\int_{a}^{b} f(x)\,dx$ is defined as the limit of $\int_{a+k}^{b} f(x)\,dx \ \ when\, k \to 0^+$ such that $\int_{a}^{b} f(x)\,dx = \lim_{n\to\infty} \int_{a+k}^{b} f(x)\,dx$ . If this limit exists and finite then the improper integral $\int_{a}^{b} f(x)\,dx$ is said to be **converge at $\boldsymbol{a}$ otherwise it is said to be divergent.**

**Convergence at the right end:** Let $f$ be a functuion defined on an interval $[a, b]$ except some finite number of points. Let $b$ be the only point of infinite discontinuity of $f$ so that the integral $\int_{a}^{b-k} f(x)\,dx$ exists for every $k$, $0 < k < b - a$. The improper integral $\int_{a}^{b} f(x)\,dx$ is defined as the limit of $\int_{a}^{b-k} f(x)\,dx \ \ when\, k \to 0^+$ such that $\int_{a}^{b} f(x)\,dx = \lim_{n\to\infty} \int_{a}^{b-k} f(x)\,dx$ . If this limit exists and finite then the improper integral $\int_a^b f(x)\,dx$ is said to be **converge at $b$ otherwise it is said to be divergent.**

**Converges at both the end points:** If the end points $a$ and $b$ are the only points of infinite discontinuity of $f$, then for any point c within the interval $[a, b]$, the imprioper integral $\int_a^b f(x)\,dx$ is can be $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. If both the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ then the improper integral converges otherwise it is divergent. It is also defined as $\int_a^b f(x)\,dx = \lim_{\substack{k\to 0^+ \\ r\to 0^+}} \int_{a-k}^{b-r} f(x)\,dx$ **.** It the improper integral exists if the limit exists.

Convergence of interior points: If an interior point $c$, $a < c < b$, is the only point of infinite discontinuity of $f$, then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

The improper integral $\int_a^b f(x)\,dx$ is convergent if both the integrals on the RHS exist in accordance with the definitions given above.

Similarly if the function has a finite number of points of discontinuity, $c_1, c_2, c_3, \dots, c_n$ within $[a, b]$ where $a \le c_1 \le c_2 \le c_3 \le \dots \le c_n \le b$ the improper integral $\int_a^b f(x)\,dx$ defined as

$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \dots + \int_{c_n}^b f(x)\,dx$ is said to be convergent if all the integrals on the R.H.S are convergent otherwise it is divergent.

**Ex. 7.1. Examine the Convergence of $\int_0^1 \frac{dx}{x^3}$**

**Sol.** The only point of discontinuity is 0.

Therefore

$\int_0^1 \frac{dx}{x^3} = \lim_{k\to 0} \int_k^1 \frac{dx}{x^3}$ where $0 < k < 1$

Hence, $\int_0^1 \frac{dx}{x^3} = \lim_{k\to 0}(1 - \frac{1}{2k^2}) = \infty$

It implies that $\int_0^1 \frac{dx}{x^3}$ is divergent.

**Ex. 7.2. Examine the Convergence of $\int_0^\pi \frac{dx}{\sin x}$**

**Sol.** The points of discontinuity are 0 and $\pi$.

Therefore

$\int_0^{\pi} \frac{dx}{\sin x} = \lim_{\substack{k\to 0 \\ r\to 0}} \int_k^{\pi-r} \frac{dx}{\sin x}$ where $0 < k < \pi$ and $0 < r < \pi$

Now $\int_k^{\pi-r} \frac{dx}{\sin x} = \int_k^{\pi-r} \operatorname{cosec} x$

$$= \int_k^{\pi-r} \frac{\operatorname{cosec} x\,(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\,dx$$

$= \int_k^{\pi-r} \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x\,\cot x}{(\operatorname{cosec} x - \cot x)}\,dx$

$= \int_k^{\pi-r} \frac{\mathrm{d}(\operatorname{cosec} \mathrm{x}) - \mathrm{d}(\cot x)}{(\operatorname{cosec} x - \cot x)}\,dx$

$= \int_k^{\pi-r} \frac{\mathrm{d}(\operatorname{cosec} \mathrm{x} - \cot x)}{(\operatorname{cosec} x - \cot x)}\,dx$

$= \log|\{\operatorname{cosec}(\pi - r) - \cot\,(\pi - r)\}| - \log|\operatorname{cosec} k - \cot k\}|$

$\log \frac{|\operatorname{cosec}(\pi-r) - \cot\,(\pi-r)\}|}{|\operatorname{cosec} k - \cot k|}$

Therefore

$\lim_{\substack{k\to 0 \\ r\to 0}} \int_k^{\pi-r} \frac{dx}{\sin x} = \lim_{\substack{k\to 0 \\ r\to 0}} \log \frac{|\operatorname{cosec}(\pi-r) - \cot\,(\pi-r)\}|}{|\operatorname{cosec} k - \cot k|} = \infty$

It implies that $\int_0^{\pi} \frac{dx}{\sin x}$ is divergent.

## *7.4 CONVERGENCE TEST*

Assume $a$ be the left end of the interval and the only point of infinite discontinuity of $f$ in $[a, b]$. When the integrand keeps the same sign, positive or negative, in a small neighbourhood of $a$, we may suppose that $f \geq 0$ and if $f \leq 0$, it can be replaced by $(-f)$, to test the convergence of $\int_a^b f\,dx$.

**Theorem 7.1. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f\,dx$ at $a$, where $f \geq 0$ $in$ $[a, b]$, is there exists a positive number $N$, independent of $k$ such that $\int_{a+k}^b f\,dx < N$, $0 < k < b - a$.**

**Proof.** By definition improper integral $\int_a^b f\,dx$ at $a$ if for $0 < k < b - a$, $\int_{a+k}^b f\,dx \to finite\ as \lim k \to 0^+$.

Now it is given that $f \geq 0\ in\ [a, b]$, $\int_{a+k}^b f\,dx$ is monotone increasing as $k$ decreases i.e. $\int_{a+k}^b f\,dx \to finite\ as\ k \to 0$ iff it is bounded above.

That is there exists a $N > 0$ independent of $k$ such that $\int_{a+k}^{b} f\, dx < N$, $0 < k < b - a$.

**Comparison test I**

**Theorem 7.2. Let $f$ band $g$ be two functions such that $f(x) \le g(x)$, $for\ all\ x\ in\ [a,b]$, then**
**(i) $\int_a^b f\, dx$ converge if $\int_a^b g\, dx$ converges and (ii) $\int_a^b g\, dx$ diverges if $\int_a^b f\, dx$ diverges**

**Proof.** Assume functions $f$ and $g$ such that both are bounded and integrable in $[a + k, b]$,
$0 < k < b - a$, and $a$ is the only point of infinite discontinuity in $[a, b]$.
Since $f$ and $g$ are positive and $f(x) \le g(x)\ for\ all\ x\ in\ [a, b]$.
Therefore,
$\int_{a+k}^{b} f\, dx \le \int_{a+k}^{b} g\, dx$ ……………………………………….(1)
(i) Let $\int_a^b g\, dx$ be convergent, hence there exists a positive number $N > 0$ such that
$\int_{a+k}^{b} g\, dx < N,\ for\ 0 < k < b - a$
From inequality (1), we get
$\int_{a+k}^{b} f\, dx < N,\ for\ 0 < k < b - a$
It implies that $\int_a^b f\, dx$ converges at $a$
(ii) It is given that $\int_a^b f\, dx$ diverges at $a$,
then $\int_{a+k}^{b} f\, dx$ is not bounded above.
From inequality (1), we get
$\int_{a+k}^{b} g\, dx$ is also not bounded above.
Therefore $\int_a^b g\, dx$ diverges at $a$.

**Comparison test II**

**Theorem 7.3. Let $f$ band $g$ be two positive functions defined on $[a, b]$ such that $\lim_{n\to\infty} \frac{f(x)}{g(x)} = l$, where $l$ is positive finite number, then the two integrals $\int_a^b f\, dx$ and $\int_a^b g\, dx$ converge and diverge together at $a$.**

**Proof.** It is given that $l$ is positive finite number i.e. $l > 0$.
Let $\varepsilon$ be a positive number such that $l - \varepsilon > 0$.
As $\lim_{n\to\infty} \frac{f(x)}{g(x)} = l$, hence there exists a neighbourhood $(a, c)$ and $a < c < b$ such that
$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon$ when $x \in (a, c)$
$\Rightarrow -\varepsilon < \frac{f(x)}{g(x)} - l < \varepsilon \Rightarrow l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon$
Therefore
$g(x)(l - \varepsilon) < f(x) < g(x)(l + \varepsilon)$ when $x \in (a, c)$…………..(1)
Now, if $\int_a^b f\,dx$ converges at $a$ then $\int_a^c f\,dx$ also converges
By comparision test I, we get
$\int_a^c g(x)(l - \varepsilon)\,dx$ converges at $a$.
It implies that $\int_a^b g(x)dx$ converges at $a$.
From inequality (1), we get
$f(x) < g(x)(l + \varepsilon)$ when $x \in (a, c)$
If $\int_a^b f(x)dx$ diverges at $a$ then
$\int_a^c g(x)(l + \varepsilon)\,dx$ diverges at $a$.
It implies that $\int_a^b g(x)dx$ diverges at $a$.
Similarly we can prove that $\int_a^b f(x)dx$ converges and diverges with $\int_a^b g(x)dx$.

**Theorem 7.4. The improper integral $\int_a^b \frac{dx}{(x-a)^n}\,dx$ converges if and only if $n < 1$.**

**Proof.** As we can see that $\int_a^b \frac{dx}{(x-a)^n}\,dx$ is proper integral if $n \leq 1$ and improper for other values of $n$.
It is observed that the only point of infinite discontinuity is at $a$.
Now for $n \neq 1$,

$$\int_a^b \frac{dx}{(x-a)^n}\,dx = \lim_{\lambda\to 0^+} \frac{dx}{(x-a)^n}, \quad 0 < \lambda < b - a$$

$$= \lim_{\lambda\to 0^+} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\lambda^{n-1}}\right]$$

$$= \begin{cases} \frac{1}{1-n}\,(b-a)^{n-1}, & if\ n < 1 \\ \infty, & if\ n > 1 \end{cases}$$

For n=1

$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a} = \lim_{\lambda\to 0^+} \int_{a+\lambda}^b \frac{dx}{x-a}$

$= \lim_{\lambda\to 0^+}[\log(b-a) - \log\lambda] = \infty$

Hence the improper integral $\int_a^b \frac{dx}{(x-a)^n}\ dx$ converges if and only if $n < 1$.

**Ex. 7.3. Test the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$**

**Proof.** Let $f(x) = \frac{1}{\sqrt{1-x^4}} = \frac{1}{\sqrt{1-x^2}\sqrt{1+x^2}}$

We can clearly see that $\frac{1}{\sqrt{1+x^2}}$ is bounded function on [0,1] and let $M$ be its upper bound.

Therefore $f(x) \le \frac{M}{\sqrt{1-x^2}}$

Also $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent.

Therefore by comparision test $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ is convergent.

**Cauchy's test**

**Theorem 7.5. The improper integral $\int_a^b f\ dx$ converges at a iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\int_{a+\lambda_1}^{a+\lambda_2} f\ dx\right| < \varepsilon$ where $\lambda_1$ *and* $\lambda_2$ tends to 0.**

**Proof.** As we know the improper integral $\int_a^b f\ dx$ is said to exist when $\lim_{\lambda\to 0^+}\int_{a+\lambda}^b f\ dx$ exists finitely.

Let $F(\lambda) = \int_{a+\lambda}^b f\ dx$ i.e. $F(\lambda)$is a function of $\lambda$.

According to Cauchy's criterion for finite limits (already studied in graduation)

$F(\lambda) \to finite\ limit\ \ as\ \lambda \to 0$ iff for iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all positives $\lambda_1, \lambda_2 < \delta$,

$|\boldsymbol{F}(\lambda_1) - F(\lambda_2)| < \varepsilon \Rightarrow \left|\int_{a+\lambda_1}^b f\ dx\ - \int_{a+\lambda_2}^b f\ dx\right| < \varepsilon$

$\left|\int_{a+\lambda_1}^b f\ dx\ + \int_b^{a+\lambda_2} f\ dx\right| < \varepsilon \Rightarrow \left|\int_{a+\lambda_1}^{a+\lambda_2} f\ dx\right| < \varepsilon$

**CHECK YOUR PROGRESS**

(CQ 1) Test for convergence

(i) $\int_0^1 \frac{x^n}{1-x} dx$

Answer…………………………………………………………………………………………

(ii) $\int_0^{\pi/2} \sin^{p-1} x \cos^{q-1} x \ dx$

Answer……………………………………………………………………………………………

## 7.5 *ABSOLUTE CONVERGENCE AND INFINITE RANGE OF INTEGRATION*

**Absolute Convergence: The improper integral** $\int_a^b f\, dx$ is said to be absolutely convergent if $\int_a^b |f|\, dx$ is convergent.

**Theorem 7.6. Every absolutely convergent integral is convergent.**

**Proof:** Let $\int_a^b |f|\, dx$ exists. Now we prove that $\int_a^b f\, dx$ exists.

$\int_a^b |f|\, dx$ exists, hence by Cauchy's test for integral, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all positives

$\left| \int_{a+\lambda_1}^{a+\lambda_2} |f|\, dx \right| < \varepsilon$ …………………………………….(1)

As we proved in previous unit that

$\left| \int_{a+\lambda_1}^{a+\lambda_2} f\, dx \right| \leq \int_{a+\lambda_1}^{a+\lambda_2} |f|\, dx$ …………………………………….(2)

Therefore, from equation (1) and (2), we get

$\left| \int_{a+\lambda_1}^{a+\lambda_2} f\, dx \right| < \varepsilon \Rightarrow \int_a^b f\, dx$ exists. $\int_a^b f\, dx$

**Ex. 7.4. Prove that $\int_0^1 \frac{\sin\frac{1}{x}}{x^p} dx$, $p > 0$, converges absolutely for $p < 1$.**

**Proof.** Consider $f(x) = \frac{\sin\frac{1}{x}}{x^p}, p > 0.$

Here we can see that 0 is the only point of infinite discontinuity and f does not keep the same sign in any neighbourhood of 0.

Now in interval [0,1]

$|f(x)| = \left|\frac{\sin\frac{1}{x}}{x^p}\right| = \frac{\left|\sin\frac{1}{x}\right|}{x^p} < \frac{1}{x^p}$

Also $\int_0^1 \frac{dx}{x^p}\ dx$ converges iff p<1.

Hence by comparision test

$\int_0^1 \left|\frac{\sin\frac{1}{x}}{x^p}\right| dx$ converges and hence $\int_0^1 \frac{\sin\frac{1}{x}}{x^p}\, dx$ converges absolutely for $p < 1$.

**Convergence at ∞:** $\int_a^\infty f\ dx,\ x \geq a$ is defined as the limit of $when$ k tends to ∞ such that $\int_a^\infty f\ dx\ = lim_{k\to\infty} \int_a^k f\ dx$. When limit exists and is finite then the improper integral $\int_a^\infty f\ dx$ is said to be convergent, otherwise it is divergent.

**Convergence at $-\infty$:** $\int_{-\infty}^b f\ dx,\ x \leq b$ is defined as the limit of $\int_{-\infty}^b f\ dx,$ when l tends to ∞ such that $\int_{-\infty}^b f\ dx\ = lim_{l\to-\infty} \int_l^b f\ dx$. When limit exists and is finite then the improper integral $\int_{-\infty}^b f\ dx$ is said to be convergent, otherwise it is divergent.

**Convergence at both ends:** $\int_{-\infty}^\infty f\ dx,\ for\ all\ x\ \ \int_{-\infty}^b f\ dx + \int_b^\infty f\ dx,$ where $b$ is any real. If both the improper integral exists then the given improper integral converges otherwise is divergent.

$$\int_{-\infty}^{\infty} f\ dx\ = \lim_{\substack{k\to\infty \\ l\to-\infty}} \int_l^k f\ dx$$

**Integrals of unbounded functions with infinite limits of integration:** When the infinite range of integration includes a finite number of points of infinite discontinuity.

$\int_{-\infty}^\infty f\ dx\ = \int_{-\infty}^b f\ dx + \int_b^\infty f\ dx + \int_a^b f\ dx.$

When all the integrals exists then $\int_{-\infty}^\infty f\ dx$ converges otherwise is divergent.

**Ex. 7.5. Examine the convergence of $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$**

**Sol.** $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}\ = \lim_{\substack{k\to\infty \\ l\to-\infty}} \int_l^k \frac{dx}{1+x^2}$

$$= \lim_{\substack{k\to\infty \\ l\to-\infty}} (\tan^{-1} k - \tan^{-1} l)$$
$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

**Ex. 7.6. Examine the convergence of $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$**

**Sol.** $\int_0^\infty x^3 e^{-x^2} dx \ = \lim_{k\to\infty} \int_0^k x^3 e^{-x^2} dx \ = \lim_{k\to\infty} f(k)$ where
$f(k) = \int_0^k x^3 e^{-x^2} dx$
Let $x^2 = t \Rightarrow 2xdx = dt$
Therefore
$f(k) = \frac{1}{2}\int_0^{k^2} te^{-t}dt \ = \frac{1}{2}\left\{t\int_0^{k^2} e^{-t}dt \ - \int_0^{k^2}\left\{\frac{d}{dt}(t)\int_0^{k^2} te^{-t}dt\right\} dt\right\}$
$= \frac{1}{2}\left\{|-te^{-t}|_0^{k^2} + \int_0^{k^2} e^{-t}\, dt\right\} = \frac{1}{2}\left[-k^2e^{-k^2} + |-e^{-t}|_0^{k^2}\right] =$
$\frac{1}{2}\left[-k^2e^{-k^2} - e^{-k^2} + 1\right] = \frac{1}{2}\left[1 - (1+k^2)e^{-k^2}\right]$
Hence $\lim_{k\to\infty} f(k) = \lim_{k\to\infty} \frac{1}{2}\left[1 - (1+k^2)e^{-k^2}\right] = \frac{1}{2}$
Therefore $\int_0^\infty x^3 e^{-x^2} dx \ = \frac{1}{2}$, converges

## *7.6 SOME COMPARISION TESTS FOR CONVERGENCE AT ∞*

**Theorem 7.7. A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f\, dx$ at $a$, where $f$ is positive in $[a, k]$ there exists a positive number $N$, independent of $k$ such that $\int_a^b f\, dx < N, for\ every\ k \geq a.$**

**Proof.** The integral $\int_a^\infty f\, dx$ is said to be convergent if $\int_a^k f\, dx$ tends to a finite limit as $k$ tends to $\infty$.
Since $f$ is positive in $[a, k]$, the positive function of $k$, $\int_a^k f\, dx$ is monotone increasing as k increases and hence $\int_a^k f\, dx \to finite$ iff there exists a positive number $N$, independent of $k$ such that $\int_a^b f\, dx < N, for\ every\ k \geq a$.

**Theorem 7.8. Let $f$ band $g$ be two positive functions such that $f(x) \leq g(x),\ for\ all\ x\ in\ [a, b]$, then**
**(i) $\int_a^{\infty} f\, dx$ converge if $\int_a^{\infty} g\, dx$ converges and (ii) $\int_a^{\infty} g\, dx$ diverges if $\int_a^{\infty} f\, dx$ diverges**

**Proof.** Assume functions $f$ and $g$ such that both are bounded and integrable in $[a, k],\ k \geq a$
Since $f$ and $g$ are positive and $f(x) \leq g(x)\ for\ all\ x\ in\ [a, k]$.
Therefore,
$\int_a^k f\, dx \leq \int_a^k g\, dx$ ……………………………………….(1)
(i) Let $\int_a^{\infty} g\, dx$ be convergent, hence there exists a positive number $N > 0$ such that
$\int_a^k g\, dx < N,\ for\ k \geq a$
From inequality (1), we get
$\int_a^k f\, dx < N,\ for\ k \geq a$
It implies that $\int_a^b f\, dx$ converges
(ii) It is given that $\int_a^{\infty} f\, dx$ divergent then the positive function $\int_a^k f\, dx$ is not bounded above.
From inequality (1), we get
$\int_a^k g\, dx$ is also not bounded above.
Hence $\int_a^k g\, dx$ diverges.

**Theorem 7.9. (i) Let $f$ band $g$ be two positive functions defined on $[a, x]$ such that $\lim_{n\to\infty} \frac{f(x)}{g(x)} = l$, where $l$ is positive finite number, then the two integrals $\int_a^{\infty} f\, dx$ and $\int_a^{\infty} g\, dx$ converge and diverge together at $a$.**
**(ii) If $\frac{f}{g} \to 0$ and $\int_a^{\infty} g\, dx$ converges then $\int_a^{\infty} f\, dx$ converges and if $\frac{f}{g} \to \infty$ and $\int_a^{\infty} g\, dx$ diverges then $\int_a^{\infty} f\ \ dx$ diverges.**

**Proof.** It is given that $l$ is positive finite number i.e. $l > 0$.
Let $\varepsilon$ be a positive number such that $l - \varepsilon > 0$.

As $\lim_{n\to\infty}\frac{f(x)}{g(x)} = l$, hence there exists a number $k(> a)$, however large, such that for al $x > k$,

$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon \;\; \Rightarrow -\varepsilon < \frac{f(x)}{g(x)} - l < \varepsilon \Rightarrow l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon$

Therefore

$g(x)(l-\varepsilon) < f(x) < g(x)(l+\varepsilon)$ …………………………..(1)

Now, if $\int_a^\infty f\,dx$ converges at $a$ then $\int_k^\infty f\,dx$ also converges

From previous theorem, we get

$\int_k^\infty g(x)\,dx$ converges at $\infty$.

It implies that $\int_a^b g(x)dx$ converges at $a$.

From inequality (1), we get

$f(x) < g(x)(l+\varepsilon)$ when $x > k > a$

If $\int_a^\infty f(x)dx$ diverges at then by previous theorem $\int_a^\infty g(x)\,dx$ diverges at $\infty$.

It implies that $\int_a^b g(x)dx$ diverges at $\infty$.

Similarly we can prove that $\int_a^b f(x)dx$ converges and diverges with $\int_a^b g(x)dx$.

(ii) When $\frac{f}{g} \to 0$, we can find $k$ so that $\frac{f(x)}{g(x)} < \varepsilon \;\; for\ all \;\; x > k$

$\Rightarrow \frac{f(x)}{g(x)} > N \Rightarrow f(x) > Ng(x) \;\; for\ all\ x \geq k$

Hence, if $\int_a^\infty g\,dx$ converges then $\int_a^\infty f\,dx$ converges

When $\frac{f}{g} \to \infty$, we can find $k$ and $N$ such that $\frac{f(x)}{g(x)} > N \;\; or\ f(x) > ng(x) \;\;\; for\ all\ x \geq k$.

Hence, if $\int_a^\infty g\,dx$ diverges then $\int_a^\infty f\,dx$ diverges.

**Ex. 7.7. Show that the improper integral $\int_a^\infty \frac{K}{x^n}dx, a > 0$, where $K$ is a positive constant, converges iff $n > 1$.**

**Proof.** Now

$$\int_0^\alpha \frac{K}{x^n}dx = \begin{cases} K\,log\,\frac{\alpha}{a}, & n = 1 \\ \frac{1}{1-n}\left[\frac{1}{\alpha^{n-1}} - \frac{1}{a^{n-1}}\right], & n \neq 1 \end{cases}$$

Therefore

$\int_0^\infty \frac{K}{x^n} dx = \lim_{n\to\infty} \int_0^\infty \frac{K}{x^n} dx = \begin{cases} +\infty, & n \le 1 \\ \frac{K}{(n-1)a^{n-1}}, & n > 1 \end{cases}$

Hence $\int_0^\infty \frac{K}{x^n} dx$ converges iff $n > 1$.

**Ex. 7.8. Test the convergence of integral**

$$\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$$

**Proof:** Assume

$f(x) = \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} = \frac{x \tan^{-1} x}{x^{\frac{4}{3}}(1+x^{-4})^{1/3}} = \frac{\tan^{-1} x}{x^{\frac{1}{3}}(1-x^{-4})^{\frac{1}{3}}}$ and $g(x) = \frac{1}{x^{\frac{1}{3}}}$

Hence

$\frac{f(x)}{g(x)} = \frac{\frac{\tan^{-1} x}{x^{\frac{1}{3}}(1-x^{-4})^{\frac{1}{3}}}}{\frac{1}{x^{\frac{1}{3}}}} = \frac{\tan^{-1} x}{(1+x^{-4})^{1/3}}$

Taking lim n tends to infinity we get

$\frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{\tan^{-1} x}{(1+x^{-4})^{1/3}} = \tan^{-1} \infty = \frac{\pi}{2}$

Therefore,

$\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ behaves alike .

As $\int_0^\infty g(x)dx = \int_0^\infty \frac{1}{x^{\frac{1}{3}}} dx$ $(p < 1)$ diverges, hence $\int_0^\infty f(x)dx = \int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$ diverges.

**Ex. 7.9. The integral $\int_0^\infty e^{-x}x^{m-1}dx$ is convergent iff $m > 0$.**

**Proof.** Assume $f(x) = e^{-x}x^{m-1}$

It clearly observe that integrand $f$ has infinite discontinuity at 0 if $m < 1$.

Therefore we will check convergence at 0 and $\infty$.

Now

$\int_0^\infty e^{-x}x^{m-1}dx = \int_0^1 e^{-x}x^{m-1}dx + \int_1^\infty e^{-x}x^{m-1}dx = I_1 + I_2$

Now we will check convergence of Integral $I_1$ at 0.

Let $g(x) = \frac{1}{x^{1-m}}$ such that $\frac{f(x)}{g(x)} = \frac{e^{-x}x^{m-1}}{x^{m-1}} = e^{-x}$.

Taking limit $x$ tends to 0, we get

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} e^{-x} = 1$

$\int_0^1 f\,dx$ and $\int_0^1 g\,dx$ converge and diverge together .

$\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^{1-m}}dx$ converges iff $1-m<1$ $i.e. m>0$ .

Therefore $\int_0^1 f(x)dx$ converges iff $m>0$ i.e. $I_1=\int_0^1 e^{-x}x^{m-1}dx$ converges iff $m>0$

Now we will check convergence of Integral $I_2$ at $\infty$.

Let $h(x)=1/x^2$ such that $\frac{f(x)}{h(x)}=\frac{e^{-x}x^{m-1}}{x^{-2}}=e^{-x}x^{m+1}$

Taking limit $x$ tends to $\infty$, we get

$\lim_{x\to\infty}\frac{f(x)}{h(x)}=\lim_{x\to\infty} e^{-x}x^{m+1}=0$

$\int_1^\infty f\,dx$ and $\int_1^\infty g\,dx$ converge and diverge together .

$\int_1^\infty g(x)dx=\int_1^\infty \frac{1}{x^2}dx$ converges if.

Therefore $\int_1^\infty f(x)dx$ converges i.e. $I_2=\int_1^\infty e^{-x}x^{m-1}dx$ converges for all $m$.

Hence $\int_0^\infty e^{-x}x^{m-1}dx$ is convergent iff $m>0$.

## *7.7 CAUCHY'S TEST FOR CONVERGENCE AT ∞ AND ABSOLUTE CONVERGENCE*

**Cauchy's test for convergence at ∞**

**Theorem 7.10. The improper integral $\int_a^\infty f\,dx$ converges at a iff for every $\varepsilon>0$ there exists $K_0>0$ such that $\left|\int_{K_1}^{K_2} f\,dx\right|<\varepsilon$ for all $K_1,K_2>K_0$**

**Proof.** As we know the improper integral $\int_a^\infty f\,dx$ is said to exist when $\lim_{\lambda\to\infty}\int_a^\lambda f\,dx$ exists finitely.

Let $F(\lambda)=\int_a^\lambda f\,dx$ i.e. $F(\lambda)$is a function of $\lambda$.

According to Cauchy's criterion for finite limits (already studied in graduation)

$F(\lambda)\to finite\ limit\ \ as\ \lambda\to\infty$ iff for every $\varepsilon>0$ there exists $K_0>0$ such that for all $K_1,K_2>K_0$,

$|\boldsymbol{F}(\lambda_1)-F(\lambda_2)|<\varepsilon \Rightarrow \left|\int_a^{\lambda_1} f\,dx - \int_a^{\lambda_2} f\,dx\right|<\varepsilon$

$\left|\int_{\lambda_1}^{a} f\,dx + \int_a^{\lambda_2} f\,dx\right|<\varepsilon \Rightarrow \left|\int_{\lambda_1}^{\lambda_2} f\,dx\right|<\varepsilon.$

**Ex. 7.10. Show that improper integral $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.**

**Proof.** Here we can see that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. i.e. 0 is not point of discontinuity.

Let $\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^\infty \frac{\sin x}{x} dx$

Now we will test the convergence of $\int_1^\infty \frac{\sin x}{x} dx$ at $\infty$.

For any $\varepsilon > 0$, let $K_1, K_2$ be two numbers greater than $2/\varepsilon$.

Now

$$\int_{K_1}^{K_2} \frac{\sin x}{x} dx = \left| -\frac{\cos x}{x} \right|_{K_1}^{K_2} - \int_{K_1}^{K_2} \frac{\cos x}{x^2} dx$$

$$\Rightarrow \int_{K_1}^{K_2} \frac{\sin x}{x} dx = \frac{\cos K_1}{K_1} - \frac{\cos K_2}{K_2} - \int_{K_1}^{K_2} \frac{\cos x}{x^2} dx$$

Taking modulus on both side we get

$$\left| \int_{K_1}^{K_2} \frac{\sin x}{x} dx \right| = \left| \frac{\cos K_1}{K_1} - \frac{\cos K_2}{K_2} - \int_{K_1}^{K_2} \frac{\cos x}{x^2} dx \right|$$

$$\Rightarrow \left| \int_{K_1}^{K_2} \frac{\sin x}{x} dx \right| \le \left| \frac{\cos K_1}{K_1} \right| + \left| \frac{\cos K_2}{K_2} \right| + \left| \int_{K_1}^{K_2} \frac{\cos x}{x^2} dx \right|$$

$$\le \left| \frac{\cos K_1}{K_1} \right| + \left| \frac{\cos K_2}{K_2} \right| + \int_{K_1}^{K_2} \left| \frac{\cos x}{x^2} \right| dx$$

$$\le \frac{1}{|K_1|} + \frac{1}{|K_2|} + \left| \int_{K_1}^{K_2} \frac{1}{x^2} dx dx \right| \qquad (\, because\ |\cos x| \le 1)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Thus by Cauchy's test $\int_1^\infty \frac{\sin x}{x} dx$ is convergent.

Thus $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

**Absolute Convergence: The improper integral** $\int_a^\infty f\,dx$ is said to be absolutely convergent if $\int_a^\infty |f|\,dx$ is convergent.

**Theorem 7.11.** $\boldsymbol{\int_a^\infty f\,dx}$ **exists if** $\boldsymbol{\int_a^\infty |f|\,dx}$ **exists.**

**Proof:** Let $\int_a^\infty |f|\,dx$ exists. Now we prove that $\int_a^\infty f\,dx$ exists.
$\int_a^\infty |f|\,dx$ exists, hence by Cauchy's test for integral, for every $\varepsilon > 0$ $K_0 > 0$ such that for all $K_1, K_2 > K_0$,
$\left|\int_{K_1}^{K_2} |f|\,dx\right| < \varepsilon$ ……………………………………(1)
As we proved in previous unit that
$\left|\int_{K_1}^{K_2} f\,dx\right| \le \int_{K_1}^{K_2} |f| dx$ … … … … … … … … … … … … … …(2)
Therefore, from equation (1) and (2), we get
$\left|\int_{K_1}^{K_2} f\,dx\right| < \varepsilon \Rightarrow \int_a^\infty f\,dx$ exists.

**Ex. 7.11. Show that** $\boldsymbol{\int_1^\infty \frac{\sin x}{x^p}\,dx}$ **converges absolutely if** $\boldsymbol{p > 1}$.

**Proof.** Now
$\left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} \le \frac{1}{x^p} \quad for\ all\ x \ge 1$
Also $\int_1^\infty \frac{1}{x^p}\,dx$ converges only if $p > 1$.
Therefore, $\int_1^\infty \left|\frac{\sin x}{x^p}\right| dx$ converges only if $p > 1$.
Hence $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges absolutely if $p > 1$.
**Note:** $\int_1^\infty \frac{\sin x}{x^p}\,dx$ not converges absolutely if $p - 1$.

**CHECK YOUR PROGRESS**

( CQ 2) Test for convergence
(i) $\int_0^\infty \frac{x}{x^3+1}\,dx$
Answer……………………………………………………………………
(ii) $\int_0^\infty x^{2n+1}\,e^{-x^2}\,dx$
Answer……………………………………………………………………

# 7.8 SOME TESTS FOR ABSOLUTE CONVERGENCE AND CONVERGENCE AT $\infty$

**Test for Absolute convergence**

**Theorem 7.12. If a function $\varphi$ is bounded on interval $[a,\infty)$ and integrable in $[a,K]$ where $K \geq a$ and $\int_a^\infty f\,dx$ absolutely convergent at $\infty$, then $\int_a^\infty f\varphi\,dx$ is absolutely convergent at $\infty$.**

**Proof.** It is given that $\varphi$ is bounded on interval $[a,\infty)$.
So there exists a positive $\lambda$ such that
$f(x) \leq \lambda$ for all $x$ in $[a,\infty)$……………………………………..(1)
Since $|f| > 0$ and $\varphi$ $\int_a^\infty f\,dx$ absolutely convergent at $\infty, i.e.$ $\int_a^\infty |f|\,dx$ convergent.
So there exists a real number $N$ such that
$\int_a^\infty |f|\,dx \leq N$ for all $x$ in $[a,\infty)$……………………………………(2)
By using equation (1) and equation (2), we get
$|f\varphi(x)| = |f(x)\varphi(x)| \leq |f(x)||\varphi(x)| \leq N|f|$
i.e $|f\varphi(x)| \leq \lambda|f|$ for all $x$ in $[a,\infty)$
$\Rightarrow \int_a^K |f\varphi(x)|dx \leq \lambda \int_a^K |f|\,dx \leq \lambda N$ for all $x$ in $[a,\infty)$
Therefore, the positive function $\int_a^K |f\varphi(x)|dx$ is bounded above by $\lambda N$, for $K \geq a$.
Hence $\int_a^\infty f\varphi\,dx$ is absolutely convergent at $\infty$.

**Abel's test**

**Theorem 7.13. If $\varphi$ is bounded and monotonic in $[a,\infty)$ and $\int_a^\infty f\,dx$ is convergent at $\infty$, then $\int_a^\infty f\varphi\,dx$ is convergent at $\infty$.**

**Proof:** It is given that $\varphi$ is bounded and monotonic in $[a,\infty)$ and it is integrable in $[a,\infty)$.
Therefore it is integrable in $[a,K)$ where $K$ is a real number.
Let $K_1$ and $K_2$ be any two real number such that $K_1, K_2 \geq a$ and $\lambda$ is lie between $K_1$ and $K_2$.
Then By Second Mean value theorem, we get

$\int_{K_1}^{K_2} f\varphi\, dx \ = \varphi(K_1) \int_{K_1}^{\lambda} f\ dx \ + \varphi(K_2) \int_{\lambda}^{K_2} f dx$ ………………….(1)

Let $\varepsilon > 0$ be arbitrary. As $\varphi$ is bounded in $[a, \infty)$, there exists a positive number $N$ exists such that

$for\ al |\varphi(x)| \leq N \qquad x \geq a$

Therefore, we can say that

$|\varphi(K_1)| \leq N$ and $|\varphi(K_2)| \leq N$ …………………………(2)

As $\int_a^{\infty} f\ dx$ is convergent at ∞, there exists a number $K_0$ exists such that

$\left|\int_{K_1}^{K_2} f dx\right| < \frac{\varepsilon}{2N} \qquad for\ all\ K_1, K_2 \geq K_0$ ……………………………(3)

Let the numbers $K_1$ and $K_2$ in (1) be greater than $K_0$ so that the number λ is also greater than $K_0$.

Hence using equation (3), we get

$\left|\int_{K_1}^{\lambda} f\ dx\right| < \frac{\varepsilon}{2N}$ and $\left|\int_{\lambda}^{K_2} f\ dx\right| < \frac{\varepsilon}{2N}$ ……………………………….(4)

Now taking modulus on both side of equation (1), we get

$\left|\int_{K_1}^{K_2} f\varphi\ dx\right| = \left|\varphi(K_1) \int_{K_1}^{\lambda} f\ dx \ + \varphi(K_2) \int_{\lambda}^{K_2} f dx\right|$

$\Rightarrow \left|\int_{K_1}^{K_2} f\varphi\ dx\right| \leq |\varphi(K_1)| \left|\int_{K_1}^{\lambda} f\ dx\right| + |\varphi(K_2)| \left|\int_{\lambda}^{K_2} f dx\right|$

Using equations (2) and (4), we get

$$\left|\int_{K_1}^{K_2} f\varphi\ dx\right| \leq |\varphi(K_1)| \left|\int_{K_1}^{\lambda} f\ dx\right| + |\varphi(K_2)| \left|\int_{\lambda}^{K_2} f dx\right|$$
$$< N.\frac{\varepsilon}{2N} + N.\frac{\varepsilon}{2N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

By Cauchy's test, $\int_a^{\infty} f\varphi\ dx$ is convergent at ∞.

**Dirichlet's test**

**Theorem 7.14. If $\boldsymbol{\varphi}$ is bounded and monotonic in $[\boldsymbol{a}, \boldsymbol{K})$ and tends to 0 as $\boldsymbol{x} \to \infty$ and $\int_{\boldsymbol{a}}^{\boldsymbol{K}} \boldsymbol{f\ dx}$ is bounded at $\boldsymbol{for\ K \geq a}$, then $\int_{\boldsymbol{a}}^{\infty} \boldsymbol{f\varphi\ dx}$ is convergent at ∞.**

**Proof.** It is given that $\varphi$ is bounded and monotonic in $[a, \infty)$.

Hence it is integrable in $[a, K)$ for all $K\ in\ [a, \infty)$.

As $f$ is integrable in $[a, K)$.

Let $K_1$ and $K_2$ be any two real number such that $K_1, K_2 \geq a$ and $\lambda$ is lie between $K_1$ and $K_2$.

Then By Second Mean value theorem, we get

$\int_{K_1}^{K_2} f\varphi\ dx \ = \varphi(K_1) \int_{K_1}^{\lambda} f\ dx \ + \varphi(K_2) \int_{\lambda}^{K_2} f dx$ …………….(1)

As $\int_a^{K} f\ dx$ is bounded when $K \geq a$.

Therefore there exists a number $l > 0$ such that

$\left|\int_a^K f\,dx\right| \le l \quad for\ all\ x\ in\ [a, \infty).$

Therefore

$$\left|\int_{K_1}^{\lambda} f\,dx\right| = \left|\int_a^{\lambda} f\,dx - \int_{K_1}^{\lambda} f\,dx\right|$$

$$\le \left|\int_a^{\lambda} f\,dx\right| + \left|\int_{K_1}^{\lambda} f\,dx\right| \le l + l = 2l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Similarly

$$\left|\int_{K_2}^{\lambda} f\,dx\right| = 2l \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Let $\varepsilon > 0$ be arbitrary.

As $\varphi$ tends to 0 as $x \to \infty$, therefore there exists $K_0 > 0$ such that

$\varphi(K_1) < \frac{\varepsilon}{4l}$ and $\varphi(K_2) < \frac{\varepsilon}{4l}$ where $K_1, K_2 \ge K_0$……………(4)

Now taking modulus on both side of equation (1), we get

$$\left|\int_{K_1}^{K_2} f\varphi\,dx\right| = \left|\varphi(K_1)\int_{K_1}^{\lambda} f\,dx + \varphi(K_2)\int_{\lambda}^{K_2} f dx\right|$$

$$\Rightarrow \left|\int_{K_1}^{K_2} f\varphi\,dx\right| \le |\varphi(K_1)| \left|\int_{K_1}^{\lambda} f\,dx\right| + |\varphi(K_2)| \left|\int_{\lambda}^{K_2} f dx\right|$$

Using equations (2), (3) and (4), we get

$$\left|\int_{K_1}^{K_2} f\varphi\,dx\right| < \frac{\varepsilon}{4l}.2l + \frac{\varepsilon}{4l}.2l < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Hence, by cauchy's test, $\int_a^{\infty} f\varphi\,dx$ is convergent at $\infty$.

**Ex. 7.12. The integral $\int_1^{\infty} \frac{\log x \sin x}{x} dx$**

**Proof.** Let $f(x) = \sin x$ and $\varphi(x) = \frac{\log x}{x}$.

Now $\left|\int_0^{\infty} \sin x\ dx\right|$ is bounded above and $\varphi$ is monotonic decreasing to 0 as $x \to \infty$.

Hence, the given integral converges by diritchlet's test.

## *7.9 SUMMARY*

In this unit we discussed about improper integral. We also discussed about the absolute convergence and some tests to check the convergence of improper integral at $\infty$.

## *7.10 GLOSSARY*

1. Set- a well defined collection of elements
2. Derivative- the rate of change of a function with respect to a variable
3. Set- a well defined collection of elements
4. Derivative- the rate of change of a function with respect to a variable
5. Integral- continuous analog of a sum, used to calculate areas, volumes.
6. Absolute convergence- converge even when you take absolute value of each term

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## *7.13 TERMINAL QUESTION*

**<u>Long Answer Questions</u>**

(TQ 1) Test the convergence of the improper integral $\int_1^2 \frac{xdx}{\sqrt{x-1}}$

(TQ 2) Test the convergence of the improper integral $\int_1^{-1} \frac{(x-1)dx}{x^{5/3}}$

(TQ 3) Test the convergence of the improper integral $\int_1^{-1} \frac{x^n \log x dx}{(1+x)^\wedge 2}$

(TQ 4) Show that the integral $\int_0^{\pi/2} \sin x \log \sin x \ dx$ converges to the value $\log 2 - 1$.

(TQ 5) Prove Diritchlet test of improper integral.

.

**<u>Fill in the blanks</u>**

(TQ 6) $\int_a^b x \log \sin x \ dx$ is ____________________.

(TQ 7) $\int_1^\infty \frac{\sqrt{x}}{(1+x^2)^2} \ dx$ ______________.

(TQ 7) $\int_1^\infty \frac{\tan^{-1} x}{x^2} \ dx$ ______________.

(TQ 8) $\int_0^\infty \frac{x \log x}{(1+x^2)^2} \ dx$ converges to ______________

(TQ 9) $\int_0^\infty \frac{x^{p-1}}{1+x} \ dx$ is ______________

## *7.14 ANSWERS*

| | | |
|---|---|---|
| (CQ 1) (i) div. | (CQ 1) (ii) conv. $p > 0, q > 0$ | (CQ 2) (i) cov. |
| (CQ 2) (ii) conv. | | |
| (TQ 1) C to 8/3 | (TQ 2) Divergent | (TQ 3) Conv. for $n > -1$ |
| (TQ 6) $-\frac{\pi^2}{2} \log 2$ | (TQ-7) $\frac{1}{2} + \frac{\pi}{4}$ | (TQ-8) $\frac{\pi}{4} + \frac{1}{2} \log 2$ |
| (TQ-8) 0 | (TQ-9) converges | |

# BLOCK III: UNIFORM CONVERGENCE AND LEBEGUE INTEGRAL

# UNIT–8: POINTWISE CONVERGENCE OF SEQUENCE OF FUNCTIONS

**CONTENTS:**



## *8.1 INTRODUCTION*

In previous chapter we learned about improper integral. In the undergraduate level, we have learnt and studied sequences of real numbers. For the convenient of readers, we would like to summarize the following about the sequences of real numbers

**Sequence of Numbers:** A sequence of real numbers is a real-valued function defined on a set of natural numbers. We usually denote a sequence by $\{a_n\}$ where, $a_n$ (a real number) is the nth term of the sequence. A sequence $\{a_n\}$ is called bounded if there exists a real number k such that $|a_n| \le k$ for all n.

**Limit of Sequence of Numbers**: A sequence $\{a_n\}$ converges to a limit $a$ if for every $\varepsilon > 0$ there exists a natural number m such that $|a_n - a| < \varepsilon$ for all $n \ge m$.

**Cauchy Sequence**: A sequence $\{a_n\}$ of real numbers is called a Cauchy sequence if $\varepsilon > 0$ there exists a natural number k such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq k$.

In this unit we apply the above mentioned and more concepts of sequences of numbers to study the sequence of functions.

## *8.2 OBJECTIVES*

After studying this unit, you should be able to -

- Understand the difference between the sequence of numbers and sequence of functions.
- Understand through solved example that the properties of boundedness, continuity, differentiability and integrations are not preserved under the pointwise convergence of the sequence of functions.
- Prepare a background for the further study of uniform convergence of the sequence of functions.

## *8.3 SEQUENCE OF FUNCTIONS*

Let $A \subseteq R$ be given and suppose that for each $n \in N$ there is a function $f_n : A \to R$ i.e. for each $n$ there is a function of real numbers, we shall say that $\langle f_n \rangle$ is a sequence of functions on $A$ to $R$. It clearly indicate that for each $x \in A$, there is a sequence $\langle f_n(x) \rangle$ of real numbers. For example $f_n : R \to R$ is defined by $f_n(x) = \frac{x}{n}$, for $x = \frac{1}{2}$ and $x = 1$ there are two sequences of real numbers $\left\langle \frac{1}{2n} \right\rangle$ and $\left\langle \frac{1}{n} \right\rangle$ respectively. Similarly reader can find number of sequences by choosing different values of $x$.

## *8.4 POINTWISE CONVERGENCE OF SEQUENCE OF FUNCTIONS*

For certain values of $x \in A$ the sequence $\langle f_n(x) \rangle$ may converge, and for other values of $x \in A$ this sequence may not converge. For each $x \in A$ for which the sequence converges, there is a uniquely determined real number

$\lim f_n(x)$. In general, the value of this limit, when it exists, will depend on the choice of the point $x \in A$. Thus there arises in this way a function whose domain consists of all numbers $x \in A$ for which the sequence $\langle f_n(x) \rangle$ converges.

***Definition:*** Suppose that $\{f_n\}$ is a sequence of functions $f_n : A \to R$ and $f : A \to R$. Then $f_n \to f$ pointwise on $A$ if for every $x \in A$ the sequence $\{f_n(x)\}$ of numbers converges to a number $f(x)$. In this case $f$ is called the limit on $A$ of the sequence $\{f_n\}$. When such a function $f$ exists , we say that the sequence $\{f_n\}$ converges pointwise on $A$.

**Example11.1:** suppose that $f_n : R \to R$ is defined by

$$f_n(x) = \frac{x}{n}$$

Then, for $x = 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n} = 0$

Now, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x.\lim_{n\to\infty} \frac{1}{n} = 0$ for all $x \in R$

Hence, $f_n \to f$ pointwise on $R$, where $f : R \to R$ such that $f(x) = 0$ for all $x \in R$.

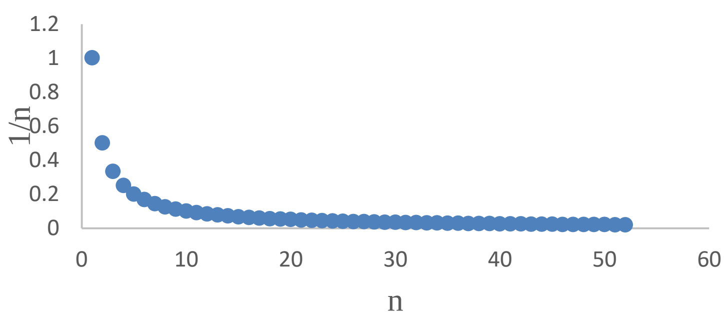


fig.8.1 Convergence of sequence 1/n

**Lemma 11.1:** A sequence $\{f_n\}$ of functions $f_n : A \to R$ converges to a function $f : A \to R$ if and only if for each $\varepsilon > 0$ and for each $x \in A$ there is a natural number $m$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$.

**Proof:** Let the sequence $\{f_n\}$ converges to a function $f : A \to R$. Then the sequence of numbers $\{f_n(x)\}$ converges to a number $f(x)$ for all $x \in A$. Hence it follows from the definition of convergence of the sequence of numbers that for each $x \in A$ and for each $\varepsilon > 0$ there is a natural number $m$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$.

Let $\{f_n\}$ is a sequence of functions $f_n : A \to R$ and $f : A \to R$. Suppose for each $\varepsilon > 0$ and for each $x \in A$ there is a natural number $m$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$. It shows that each sequence of numbers $\{f_n(x)\}$ converges to a number $f(x)$ for all $x \in A$.

Hence, the lemma.

**Note:** Here, it is worth noting that $m$ depends on both $\varepsilon$ and $x \in A$.

**Lemma11.2:** A sequence $\{f_n\}$ of functions $f_n : A \to R$ converges pointwise to a function $f : A \to R$ then $f$ is unique.

**Proof:** Suppose that sequence $\{f_n\}$ converges to a function $f : A \to R$. Let us take an arbitrary number $x \in A$ Then $\lim_{n\to\infty} f_n(x) \to f(x)$. We know that limit of a sequence of numbers is unique. Hence $f$ is unique.

## 8.5 POINTWISE CONVERGENCE AND BOUNDEDNESS

**Example11.2:** Suppose that $f_{n:} : (0,1) \to \mathrm{R}$ is defined by

$$f_n(x) = \frac{n}{nx+1}$$

Then, since $x \neq 0$ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{nx+1} = \lim_{n\to\infty} \frac{1}{x + \frac{1}{n}} = \frac{1}{x}$, So, $f_n \to f$ pointwise where $f:(0,1) \to \mathrm{R}$ is given by

$$f(x) = \frac{1}{x}$$

We have $|f_n(x)| = \left|\frac{n}{nx+1}\right| < n$ for all $x \in (0,1)$, so each $f_n$ is bounded in $(0,1)$, but their pointwise limit $f$ is not. Thus, a sequence of bounded functions may converge pointwise to an unbounded function.

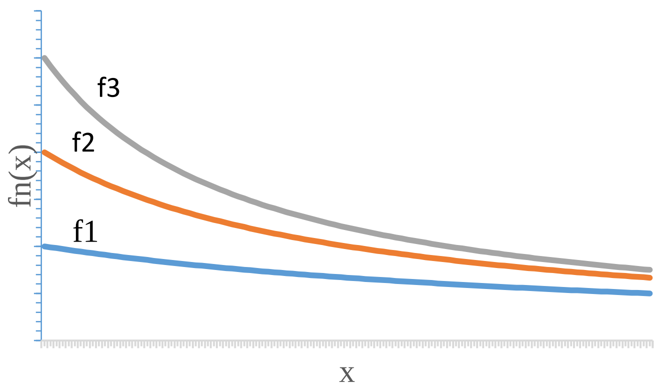


fig.8.2

**Example11.3:** Suppose that $f_{n:} : [0,1] \to \mathrm{R}$ is defined by

$$f_n(\mathrm{x}) = \begin{cases} 2\mathrm{n}^2 x & \text{if } 0 \le \mathrm{x} \le \dfrac{1}{2\mathrm{n}} \\ 2n^2(1/n - \mathrm{x}) & \text{if } \dfrac{1}{2\mathrm{n}} \le \mathrm{x} \le \dfrac{1}{\mathrm{n}} \\ 0 & \text{if } \dfrac{1}{\mathrm{n}} \le \mathrm{x} \le 1 \end{cases}$$

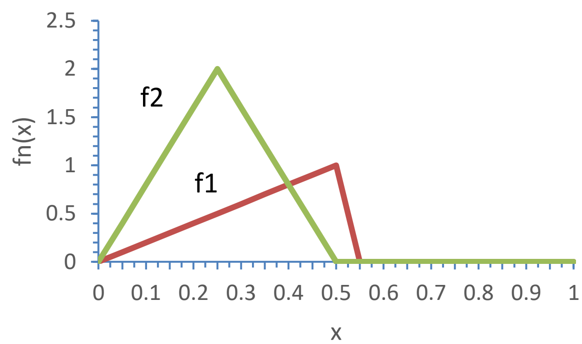


fig. 8.3

Here, we would like to show two members of the sequence so that reader can easily understand the problem

$$f_1(\mathrm{x}) = \begin{cases} 2x & \text{if } 0 \le \mathrm{x} \le \dfrac{1}{2} \\ 0 & \text{if } \dfrac{1}{2} < \mathrm{x} \le 1 \end{cases}$$

$$f_2(\mathrm{x}) = \begin{cases} 8x & \text{if } 0 \le \mathrm{x} \le \dfrac{1}{4} \\ 8(1/2 - \mathrm{x}) & \text{if } \dfrac{1}{4} < \mathrm{x} < \dfrac{1}{2} \\ 0 & \text{if } \dfrac{1}{2} \le \mathrm{x} \le 1 \end{cases}$$

One can easily conclude that $f_n(x) = 0$ for $0 < x \le 1$ and for all $n \ge 1/x$ and if $x = 0$, then $f_n(x) = 0$ for all $n$. Hence $f_n \to 0$ pointwise on $[0, 1]$.

Here, max $f_1 = 1$, max $f_2 = 2$ . . . max $f_n = n \to \infty$ as $n \to \infty$. Thus, a pointwise convergent sequence of functions need not be bounded, even if it converges to zero.

## *8.6 CONTINUITY AND POINTWISE CONVERGENCE*

**Example11.4:** Suppose that $f_{n:} : [0,1] \to \mathrm{R}$ is defined by

$f_n(x) = x^n$. If $0 \le x < 1$, then

$f_n(x) = x^n \to 0$ as $n \to \infty$,

if $x = 1$ then $f_n(x) = x^n = 1$ for all $n$.

So $f_n \to f$ pointwise where

$$f(\mathrm{x}) = \begin{cases} 0 & \text{if } 0 \le \mathrm{x} < 1 \\ 1 & \text{if } \mathrm{x} = 1 \end{cases}$$

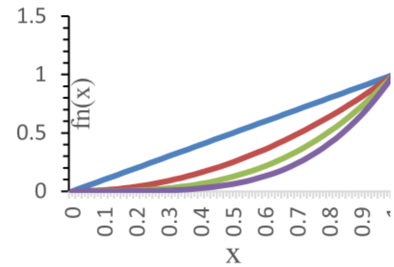

fig. 11.4

Although each $f_n$ is continuous on $[0, 1]$, their pointwise limit $f$ is not. Thus, a sequence of continuous functions can converge pointwise to a discontinuous function.

**Note:** In present example each $f_n$ is differentiable on $[0, 1]$, their pointwise limit $f$ is not. Hence, a sequence of differential functions can converge pointwise to a non-differential function.

## *8.7 SOLVED EXAMPLES*

**Ex. 8.1.** Suppose that $f_{n:} : [-2, 2] \to \mathrm{R}$ is defined by

$$f_n(x) = 2 - \frac{x^2}{n}$$

for $x \in [-2, 2]$ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2 - \frac{x^2}{n} = 2$

Hence, $\{f_n\}$ converges to $f : [-2, 2] \to \mathrm{R}$ defined by

$f(x) = 2$.

**Ex. 8. 2:** Suppose that $f_{n:} : [-1, 1] \to \mathrm{R}$ is defined by

$$f_n(x) = \left(1 - \frac{nx^2}{n+1}\right)^{n/2}$$

for $x \in [-1, 0) \cup (0,1]$ $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\left(1 - \dfrac{nx^2}{n+1}\right)^{n/2} = 0$ and for $x = 0$ $\lim\limits_{n\to\infty} f_n(x) = 1$ therefore, $\{f_n\}$ converges to $f : [-1,1] \to \mathrm{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1, 0) \cup (0,1] \\ 1 & \text{if x} = 0 \end{cases}.$$

**Ex. 8.3:** Let $A_{n:}$ be the set of all numbers of the form $x = \dfrac{p}{q} \in [0,\ 1]$, where, $p$ and $q$ are integers with no common factors and $1 \le q \le n$ Suppose that $f_{n:} : [0,1] \to \mathrm{R}$ is defined by

$$f_n(x) = \begin{cases} 1 \text{ if x} \in \mathrm{A_n} \\ 0 \text{ if x} \notin \mathrm{A_n} \end{cases}$$

For an irrational number, $x \notin A_n$ $\lim\limits_{n\to\infty} f_n(x) = 0$ and for a rational number $x$ there exists a natural number $m$ such that $x \in A_n$ for all $n \ge m$ then $\lim\limits_{n\to\infty} f_n(x) = 1$. Hence, $\{f_n\}$ converges to $f : [0,1] \to \mathrm{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if x is irrational} \end{cases}.$$

**Note:** it is illustrated that each $f_{n:} : [0,1] \to \mathrm{R}$ is integrable, while their limit $f : [0,1] \to \mathrm{R}$ is not. Thus, a sequence of integrable functions may converge to a not integrable function.

**Ex. 8.4:** Suppose that $f_{n:} : [0,\ \infty[ \to [0,\ \infty[$ is defined by

$$f_n(x) = x^n e^{-nx}$$

Here, $|f_n(x)| = \left|x^n e^{-nx}\right| \le e^{-n}$ for all $x \in [0,\ \infty[$. Hence $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0,\ \infty[$.

**Ex. 8.5.** Show that the sequence $\{f_n\}$, where $f_n(x) = nx(1-x^2)^n$ and the sequence of integrals $\left\{\int_0^1 f_n dx\right\}$ are pointwise convergent for $0 \le x \le 1$but $\lim_{n\to\infty}\left\{\int_0^1 f_n dx\right\} \ne \int_0^1 \lim_{n\to\infty}\{f_n\}dx$.

## *8.8 SUMMARY*

In this unit, we have learned about the difference between the sequences of numbers and sequences of functions. We have learned that the pointwise convergence is not sufficient to preserve the properties of boundedness, continuity, integrability and differentiability. Solved examples and similar exercises is given to understand the facts easily.

## *8.9 GLOSSARY*

1. Sequence – a function from set of natural number to a set.
2. Boundedness – absolute value of each member of a sequence less than or equal to a real number.
3. Continuity – No break in a curve.
4. Preserve – to keep something safe.

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## *8.12 TERMINAL QUESTIONS*

**Q 1.** Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is pointwise convergent for all real values of $x$ .

**Q 2.** Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$ is pointwise convergent for all $x \geq 0$ .

**Q 3.** Give an example to show that a sequence of continuous function can converge pointwise to a discontinuous function.

**Q 4.** Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ is pointwise convergent on for all real values of $x$ but the sequence of differential $\{f_n'\}$ is not .

# UNIT 9: UNIFORM CONVERGENCE AND CONTINUITY

**CONTENTS:**



## *9.1 INTRODUCTION*

In the previous unit, we show that the properties of boundedness, continuity, integrations and differentiations are not preserved under pointwise convergence of the sequence of functions. So we need a more general concept, uniform convergence to handle the above properties. Uniform convergence preserves these properties in the sense that if each term of uniform convergent sequence (series) of functions possesses these properties then the limit function also possesses these properties. In this unit we will study the concept of uniform convergence for sequences and series of functions.

## 9.2 OBJECTIVES

After studying this unit, you should be able to -

- Understand the difference between the pointwise convergence and uniform convergence.
- Understand through theorems and solved examples that the properties of boundedness, continuity, differentiability and integrations are preserved under the uniform convergence of the sequence of functions with some additional hypotheses.
- Understand the conditions for term by term Integration and differentiation in series of functions.

## 9.3 SEQUENCE OF FUNCTIONS

Let $A \subseteq R$ be given and suppose that for each $n \in N$ there is a function $f_n : A \to R$ i.e. for each $n$ there is a function of real numbers, we shall say that $\langle f_n \rangle$ is a sequence of functions on $A$ to $R$. It clearly indicate that for each $x \in A$, there is a sequence $\langle f_n(x) \rangle$ of real numbers. For example $f_n : R \to R$ is defined by $f_n(x) = \frac{x}{n}$, for $x = \frac{1}{2}$ and $x = 1$ there are two sequences of real numbers $\left\langle \frac{1}{2n} \right\rangle$ and $\left\langle \frac{1}{\text{n}} \right\rangle$ respectively. Similarly reader can find number of sequences by choosing different values of $x$.

## 9.4 UNIFORM CONVERGENCE OF SEQUENCE OF FUNCTIONS

For certain values of $x \in A$ the sequence $\langle f_n(x) \rangle$ may converge, and for other values of $x \in A$ this sequence may not converge. For each $x \in A$ for which the sequence converges, there is a uniquely determined real number $\lim f_n(x)$. In general, the value of this limit, when it exists, will depend on the choice of the point $x \in A$. Thus there arises in this way a function whose domain consists of all numbers $x \in A$ for which the sequence $\langle f_n(x) \rangle$ converges.

***Definition12.1:*** Suppose that $\{f_n\}$ is a sequence of functions $f_n : A \to R$ and $f : A \to R$. Then $f_n \to f$ uniformly on $A$ if for $\varepsilon > 0$ and for every $x \in A$ there exists a natural number $m$ (independent on $x$ but dependent on $\varepsilon$ ) such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$.

**Example12.1:** suppose that $f_n : [0,1] \to R$ is defined by

$f_n(x) = \dfrac{1}{x+n}$, then $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$ i.e. the sequence $\{f_n\}$ is pointwise convergent to $f : [0,1] \to \mathrm{R}$ where, $f(x) = 0$.

Now, for any $\varepsilon > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$$

Let us take a natural number $m \geq 1/\varepsilon$ , then for $\varepsilon > 0$, there exists a natural number $m$ such that

$|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$.

Hence the sequence is uniformly convergent in $[0,1]$.

**Note:** Every uniformly convergent sequence $\{f_n\}$ is pointwise convergent (the uniform limit function is same as the pointwise limit function) but converse is not true.

**Example12.2** :- suppose that $f_n : [0,1] \to R$ is defined by

$f_n(x) = x^n$ then we know that sequence $\{f_n\}$ is pointwise convergent to $g : [0,1] \to \mathrm{R}$ where, $g(x) = 0, \text{if } x \in [0,1)$ and $g(x) = 1, \text{if } x = 1$.

Now, let $\varepsilon = 1/3$, for a natural number $n$, $x = (1/2)^{1/n} \in [0,1)$ then $|f_n(x) - f(x)| = \left| \left((1/2)^{1/n}\right)^n - 0 \right| = 1/2 > \varepsilon$ it follows that $\{f_n\}$ is not uniformly convergent.

## 9.5 CAUCHY CRITERION FOR UNIFORM CONVERGENCE

**Theorem12.1:** A sequence $\{f_n\}$ of functions $f_n : A \to R$ converges uniformly to a function $f : A \to R$ if and only if for every $\varepsilon > 0$ and for all $x \in A$ there exists a natural number $N$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq N$.

**Proof:** Let the sequence $\{f_n\}$ converges uniformly to a function $f : A \to R$. Then for each $x \in A$ and for each $\varepsilon > 0$ there is a natural number $N$ such that $|f_k(x) - f(x)| < \varepsilon/2$ for all $k \geq N$. Let $x \in A$, $n, m \geq N$. Then $|f_n(x) - f_m(x)| \leq |f_n x) - f(x)| + |f_m x) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Suppose that for every $\varepsilon > 0$ and for all $x \in A$ there exists a natural number $N$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq N$. Let $x \in A$ then condition shows that $\{f_n(x)\}$ is a Cauchy sequence of real numbers and therefore, $\{f_n(x)\}$ is convergent for all $x \in A$. Thus the sequence $\{f_n\}$ is converges pointwise to a function $f : A \to R$. For a fixed $m \geq N$, $x \in A$ and $n \to \infty$ we have $\lim_{n\to\infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| < \varepsilon$. Thus $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in A$. Hence $\{f_n\}$ converges uniformly.

**Theorem12.2:** A sequence $\{f_n\}$ of functions $f_n : A \to R$ converges pointwise to a function $f : A \to R$ and let $M_n = \sup_{x \in A} |f_n(x) - f(x)|$. Then

$f_n \to f$ uniformly on $A$ if and only if $M_n \to 0$ as $n \to \infty$.

**Proof:** Suppose that sequence $\{f_n\}$ converges uniformly to a function $f : A \to R$, so that for every $\varepsilon > 0$ there exists an integer m such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$ and for all $x \in A$ this implies that $M_n = \sup_{x \in A} |f_n(x) - f(x)| \leq \varepsilon$, for all $n \geq m$. Thus $M_n \to 0$ as $n \to \infty$.

Let $M_n \to 0$ as $n \to \infty$, so that for every $\varepsilon > 0$ there exists an integer m such that $M_n < \varepsilon$ for all $n \geq m$ i.e. $\underset{x \in A}{Sup}\left|f_n(x) - f(x)\right| < \varepsilon$ for all $n \geq m$ or

$|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$, for all $x \in A$. Hence, $f_n \to f$ uniformly on $A$.

**Example12.3:** Suppose that $f_{n:} : (0,1) \to$ R is defined by

$$f_n(x) = \frac{x}{nx^2 + 1}$$

So, $f_n \to f$ pointwise where $f : [0,1] \to$ R is given by

$f(x) = 0$

$$M_n = \underset{x \in [0,1]}{Sup}\left|f_n(x) - f(x)\right| = \underset{x \in [0,1]}{Sup}\left|\frac{x}{nx^2 + 1}\right| = \frac{1}{2\sqrt{n}}$$

$M_n \to 0$ as $n \to \infty$.

Hence $\{f_n\}$ converges uniformly on [0, 1].

## *9.6 UNIFORM CONVERGENCE AND BOUNDEDNESS*

**Theorem12.3:** A sequence $\{f_n\}$ of bounded functions $f_n : A \to R$ converges uniformly to a function $f : A \to R$ then $f : A \to R$ is bounded.

**Proof:** Since $f_n : A \to R$ converges uniformly to a function $f : A \to R$ then for any $\varepsilon > 0$ there exists an integer m such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq m$ and for all $x \in A$ or $|f(x)| < |f_n(x)| + \varepsilon$ for a natural number $n \geq m$ and for all $x \in A$. Since $f_n(x)$ is bounded therefore, there exists $M > 0$ such that $|f_n(x)| < M$ for all $x \in A$ therefore $|f(x)| < M + \varepsilon$ for all $x \in A$. Hence $f : A \to R$ is bounded.

**Remark1:** the converse of the theorem is not true i.e. sequences of bounded functions exist which have a bounded limit but are not uniformly convergent.

**Example12.4:** suppose that $f_n : [0,1] \to R$ is defined by

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \text{ for all } x \in [0,1].$$

Now, $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\frac{nx}{1+n^2x^2}\right| = 1/2$

$M_n$ does not tend to 0 as $n \to \infty$ therefore the sequence is not uniform

Here, $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is bounded for all n and the limit function $f(x) = 0$ is bounded but the sequence $\{f_n\}$ is not uniformly convergent.

## 9.7 UNIFORM CONVERGENCE AND CONTINUITY

**Theorem12.4:** A sequence $\{f_n\}$ of continuous functions $f_n : A \to R$ converges uniformly to a function $f : A \to R$ then $f : A \to R$ is continuous.

**Proof:** Let $f_n : A \to R$ is continuous at $x = a$. Since $f_n : A \to R$ converges uniformly to a function $f : A \to R$ then for any $\varepsilon > 0$ there exists an integer m such that

$|f_n(x) - f(x)| < \varepsilon/3$ for all $n \ge m$ and for all $x \in A$

For a particular $n \ge m$ and $x = a$

$|f_n(a) - f(a)| < \varepsilon/3$

Since $f_n(x)$ is continuous at $x = a$ therefore,

$|f_n(x) - f_n(a)| < \varepsilon/3$ whenever, $|x-a| < \delta$

Now, for $|x-a| < \delta$

$$|f(x) - f(a)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(x) - f(a)|$$
$$= \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Hence $f : A \to R$ is continuous.

**Remark1:** the converse of the theorem is not true i.e. sequences of continuous functions exist which have a continuous limit but are not uniformly convergent.

**Example12.5:** suppose that $f_n : [0,1] \to R$ is defined by

$f_n(x) = nxe^{-nx^2}$

$f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

Now, $M_n = \sup_{x\in[0,1]} \left| f_n(x) - f(x) \right| = \sup_{x\in[0,1]} nxe^{-nx^2} = \sqrt{n/2e}$

$M_n \to \infty$ as $n \to \infty$ therefore the sequence is not uniform

Here, $f_n(x) = nxe^{-nx^2}$ is continuous for all n and the limit function $f(x) = 0$ is continuous but the sequence $\{f_n\}$ is not uniformly convergent.

## 9.8 UNIFORM CONVERGENCE AND INTEGRABILITY

**Theorem12.5:** A sequence $\{f_n\}$ of integrable functions $f_n : [a,\mathrm{b}] \to R$ converges uniformly to a function $f : [a,\mathrm{b}] \to R$ then $f$ is integrable on $[a,\mathrm{b}]$.

**Proof:** Since the sequence $\{f_n\}$ is uniformly convergent then for any $\varepsilon > 0$ there exists an integer m such that

$\left| f_n(x) - f(x) \right| < \varepsilon / 3(b-a)$ for all $n \ge m$ and for all $x \in [a,\mathrm{b}]$

For a particular $n = m$

$\left| f_m(x) - f(x) \right| < \varepsilon / 3$

(1)

For this fixed m , since $f_m$ is integrable, therefore one can choose a partition P of $[a,\mathrm{b}]$, such that

$\mathrm{U}(\mathrm{P}, f_m) - \mathrm{L}(\mathrm{P}, f_m) < \varepsilon / 3$

From (1)

$f_m(x) - \varepsilon / 3(b-a) < f(x) < f_m(x) + \varepsilon / 3(b-a)$

$U(P, f) < U(P, f_m) + \varepsilon / 3$

$L(P, f) > L(P, f_m) - \varepsilon / 3$

Therefore, $\mathrm{U}(\mathrm{P}, f) - \mathrm{L}(\mathrm{P}, f) < \mathrm{U}(\mathrm{P}, f_m) - \mathrm{L}(\mathrm{P}, f_m) + 2\varepsilon / 3 < \varepsilon$.

Hence $f$ is integrable on $[a,\mathrm{b}]$.

**Note:** In the present case $\lim_{n\to\infty} \int_a^x f_n \, dt = \int_a^x f \, dt$.

## *9.9 UNIFORM CONVERGENCE AND DIFFERENTIABILITY*

**Theorem12.6:** A sequence $\{f_n\}$ of differentiable functions $f_n:[a,\mathrm{b}]\to R$ converges at least at one point $c\in[a,\mathrm{b}]$. If the sequence of differentials $\{f_n'\}$ converges uniformly then the given sequence $\{f_n\}$ converges uniformly to a function $f:[a,\mathrm{b}]\to R$.

**Proof:** Since $\{f_n(c)\}$ is convergent then there exists a natural number $N_1$ such that

$$|f_n(c)-f_m(c)|<\varepsilon/2 \text{ for all } n,m\geq N_1$$

Again Since $\{f_n'\}$ is uniformly convergent then there exists a natural number $N_2$ such that

$$|f_n'(x)-f_m'(x)|<\varepsilon/2(b-a) \text{ for all } n,m\geq N_2 \text{ and for all } x\in[a,\mathrm{b}].$$

We know that $(f_n-f_m)$ is a differential function on $[a,\mathrm{b}]$, therefore by Lagrange's mean value theorem for any two points $x,y\in[a,\mathrm{b}]$, we get for $x<\mathrm{z}<\mathrm{y}$, for all $n,m\geq N_2$

$$|\{f_n(x)-f_m(x)\}-\{f_n(y)-f_m(y)\}|=|x-y|\,|f_n'(z)-f_m'(z)|<|x-y|\varepsilon/2(b-a)<\varepsilon/2$$

Now, for $n,m\geq N=\max\{N_{1,}N_2\}$, we have

$$|f_n(x)-f_m(x)|\leq|\{f_n(x)-f_m(x)\}-\{f_n(c)-f_m(c)\}|+|f_n(c)-f_m(c)|<\varepsilon/2+\varepsilon/2=\varepsilon$$

Hence the sequence $\{f_n\}$ is uniformly convergent on $[a,\mathrm{b}]$.

**Note:** Here one can easily find that $f'(x)=F(x)=\lim_{n\to\infty}f_n'(x)$.

## *9.10 SERIES OF FUNCTIONS*

We know that the sum of an infinite series of real numbers as the limit of the sequence of partial sums. Similarly, the series of functions can be defined, as follows

**Definition:** Let $A \subseteq R$ be a non-empty set. A series of functions $A \to R$ is a sum

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \ldots,$$

Where, $\{f_n\}$ is a sequence of functions $A \to R$. Each function $f_n$, $n = 1,\ 2,\ 3 \ldots$, is called a term of the series $\sum_{n=1}^{\infty} f_n$.

The partial sums of series $\sum_{n=1}^{\infty} f_n$ are defined by

$$s_n = \sum_{i=1}^{n} f_i\ ,\ n = 1, 2, 3, \ldots$$

The series of functions $\sum_{n=1}^{\infty} f_n$ is pointwise (Uniform) convergent if the sequence of partial sums $\{s_n\}$ is pointwise (Uniform) convergent. If $\{s_n\}$ converges pointwise (uniform) to a function $f : A \to R$, we say $\sum_{n=1}^{\infty} f_n$ converges pointwise (uniform) to the function $f : A \to R$.

## 9.11 DIVERGENCE TEST FOR SERIES OF FUNCTIONS

Let $A \subseteq R$ be a non-empty set, and let $\sum_{n=1}^{\infty} f_n$ be a series of functions $A \to R$. If $\{f_n\}$ does not converge pointwise (Uniform) to the zero function, then $\sum_{n=1}^{\infty} f_n$ is not pointwise (uniform) convergent.

## 9.12 CAUCHY CRITERION FOR UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

**Theorem 12.7:** A series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function $f : A \to R$ if and only if for every $\varepsilon > 0$ and for all $x \in A$ there exists a natural number $N$ such that $\left| f_{m+1}(x) + f_{m+2}(x) + f_{m+3}(x) + \ldots + f_n(x) \right| < \varepsilon$ for all $n, m \geq N$.

**Proof:** Let the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function $f : A \to R$. Then the sequence of partial sums $\{s_n\}$converges uniformly to the function $f : A \to R$. Therefore by Cauchy criterion for uniform converge of sequence, for every $\varepsilon > 0$and for all $x \in A$there exists a natural number $N$ such that

$|s_n(x) - s_m(x)| < \varepsilon$ for all $n, m \geq N$ .

Or $\left| \sum_{i=1}^{n} f_n - \sum_{i=1}^{m} f_m \right| < \varepsilon$

without loss of generality, assume that $n > m$.

Hence $|f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x)| < \varepsilon$ for all $n, m \geq N$

Suppose $|f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x)| < \varepsilon$ for all $n, m \geq N$ , $x \in A$for all implies

$\left| \sum_{i=1}^{n} f_n - \sum_{i=1}^{m} f_m \right| < \varepsilon$or $|s_n(x) - s_m(x)| < \varepsilon$ for all $n, m \geq N$ , $x \in A$.

The sequence of partial sums $\{s_n\}$. Hence the series $\sum_{n=1}^{\infty} f_n$ converges uniformly.

## *9.13. WEIERSTRASS'M-TEST*

**Theorem 12.8:** Let $A \subseteq R$ be a non-empty set, and let $\sum_{n=1}^{\infty} f_n$ be a series of functions . suppose that for each natural number $n$ there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive number such that $|f_n(x)| \leq M_n$ for all $n$ and for all $x \in A$ then the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

**Proof:** Since $\sum_{n=1}^{\infty} M_n$ is convergent, therefore for any $\varepsilon > 0$there exists a natural number $N$ such that $|M_{m+1}(x) + M_{m+1}(x) + \ldots + M_n(x)| < \varepsilon$ for all $n, m \geq N$ , $x \in A$hence for all $x \in A$and for all $n, m \geq N$ , we have

$$\left| f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x) \right| \le \left| f_{m+1}(x) \right| + \left| f_{m+2}(x) \right| + \ldots + \left| f_n(x) \right|$$
$$\le \mathrm{M}_{m+1} + M_{m+2} + \ldots + \mathrm{M}_n$$
$$< \varepsilon$$

Hence $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

**Example 12.6:** Let us consider a series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ for all real $x$.

$\left| \frac{\sin nx}{n^2} \right| \le \frac{1}{n^2}$ for all n.

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

Hence $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is uniformly convergent.

**Note:** Weierstrass's M-test is applicable to the series which are absolutely convergent. When this is not the case, we can use Abel's test and Dirichlet's test

The next three theorems show that uniform convergence series of functions preserve the continuity and the series subjected to the appropriate hypotheses can be integrated and differentiated term by term. The proof of these theorems are similar to the proof of corresponding theorems for sequences of functions.

**Theorem 12.9**: If a series $\sum_{n=1}^{\infty} f_n$ of continuous terms $f_n : A \to R$, is uniformly convergent to a function $f : A \to R$ then $f : A \to R$ is continuous.

**Theorem 12.10**: If a series $\sum_{n=1}^{\infty} f_n$ of integrable terms $f_n : A \to R$, is uniformly convergent to a function $f : A \to R$ then $f : A \to R$ is integrable and $\sum_{n=1}^{\infty} \int_a^x f_n \,\mathrm{dt} = \int_a^x f \, dt$.

**Theorem 12.11:** A series $\sum_{n=1}^{\infty} f_n$ of differentiable functions $f_n : [a, \mathrm{b}] \to R$ converges at least at one point $c \in [a, \mathrm{b}]$. If the series of differentials

$\sum_{n=1}^{\infty} f_n'$ converges uniformly then the given series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function $f:[a,b]\to R$ and $f'(x)=F(x)=\sum_{n=1}^{\infty} f_n'(x)$.

## *9.14 SOLVED EXAMPLES*

**EXAMPLE 1:** Suppose that $f_{n:}: \mathrm{R}\to\mathrm{R}$ is defined by

$$f_n(x)=\frac{\sin nx}{n}$$

Then $f_{n:}\to 0$ pointwise on R.

Now, $|f_n(x)|=\left|\frac{\sin nx}{n}\right|\le\frac{1}{n}$,

Therefore $|f_n(x)-0|<\varepsilon$ for all $x\in R$ if $n>1/\varepsilon$ .

Hence, $f_n(x)=\frac{\sin nx}{n}$ is uniformly convergence.

**EXAMPLE 2:** Suppose that $f_{n:}:[0,1/2]\to\mathrm{R}$ is defined by

$f_n(x)=x^n$, converges pointwise to the limit 0.

Now, for $\varepsilon\ge 1$, $|f_n(x)-0|=|x^n|<\varepsilon$ for all n and for all $x\in[0,1/2]$

And for $0<\varepsilon<1$, $|f_n(x)-0|=|x^n|<\varepsilon$ for all $n\ge\frac{\log 1/\varepsilon}{\log 2}$ and for all $x\in[0,1/2]$.

Hence $f_n(x)=x^n$ is uniformly convergent in $[0,1/2]$

**Note:** $f_n(x)=x^n$ is not uniformly convergent in $[0,1]$ since for $0<\varepsilon<1$

$$|f_n(x)-0|=|x^n|<\varepsilon$$

which is not possible for $x=1$.

**EXAMPLE 3**: Suppose that $f_{n:}:[0,\ \infty[\to[0,\ \infty[$ is defined by

$$f_n(x)=x^n e^{-nx}$$

$$f(x)=\lim_{n\to\infty} f_n(x)=0 \text{ for all } x\in[0,\ \infty[.$$

Now, $M_n=\sup_{x\in[0,\infty[}|f_n(x)-0|=|x^n e^{-nx}|=e^{-n}$

$M_n\to 0$ as $n\to\infty$ therefore the sequence is uniformly convergent.

**EXAMPLE 4**: Let us consider a series $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{n^p(1+x^{2n})}$ for all real $x$.

$$\left|\frac{(-1)^n x^{2n}}{n^p(1+x^{2n})}\right| \le \frac{1}{n^p} \text{ for all n.}$$

We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent for $p>1$

Hence $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{n^p(1+x^{2n})}$ is uniformly convergent for $p>1$.

**EXAMPLE 5:** Let us consider a series

$$\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^5}{1+x^8}+\ldots \quad \text{for all } |x|\le 1/2.$$

$$f_n(x)=\frac{2^n x^{2n-1}}{1+x^{2n}}$$

$$|f_n(x)| = \left|\frac{2^n x^{2n-1}}{1+x^{2n}}\right| \le 2^n\frac{1}{2^{2n-1}} = \frac{1}{2^{n-1}} \text{ for all n.}$$

We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ is convergent.

Hence $\sum_{n=1}^{\infty}\frac{2^n x^{2n-1}}{1+x^{2n}}$ is uniformly convergent.

## *9.15 SUMMARY*

In this unit, we have learned about the uniform convergence of sequences and series of functions. Uniform convergence is a more general concept than pointwise convergence. Uniform convergence preserves the properties of continuity, differentiability and integrability. Solved examples on each property are given and similar exercises left to the reader for detail understanding of the concepts.

## *9.16 GLOSSARY*

1. Series– sum of infinite terms of numbers or functions.
2. Partial sum – sum of n terms of a series.
3. Finite Interval – initial and end values are finite numbers.
4. Differentials – derivative of functions.

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## *9.19 TERMINAL QUESTIONS*

**Q 1.** Test the following sequences for uniform convergence.

**i.** $\{x^n\}$ on $[0,\ 1]$ ii. $\dfrac{\sin nx}{\sqrt{n}}$ on $[0,\ 2\pi]$

**Q 2.** Show that the sequence $\{f_n\}$, where $f_n(x) = \tan^{-1} nx$ is uniformly convergent in any interval $[a, \mathrm{b}]$, $a > 0$.

**Q 3.** Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{x}{n+x}$ is uniformly convergent in any interval $[0, \mathrm{b}]\; b < \infty$.

**Q 4.** Show that the sequence $\{f_n\}$, where $f_n(x) = e^{-nx}$ is uniformly convergent in any interval $[a, \mathrm{b}]$where $a$ and $b$ are positive numbers.

**Q 5.** Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{n^2 x}{1+n^3x^2}$ is not uniformly convergent on $[0,1]$.

**Q 6.** Show that the series $\sum_{n=1}^{\infty} \dfrac{x}{1+nx^2}$ is uniformly convergent for all real $x$.

**Q 7.** Show that the series $(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + \ldots$ is not uniformly convergent on $[0,1]$

**Q 8.** Test the following series for uniform convergence.

**i.** $\sum_{n=1}^{\infty} \dfrac{\sin(x^2+n^2x)}{n(n+1)}$ for all real numbers. ii. $\sum_{n=1}^{\infty} \dfrac{x^4}{(1+x^4)^n}$ on $[0,1]$.

iii. $\sum_{n=1}^{\infty} a^n \cos nx$, $0 < a < 1$for all real numbers iv. $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^p}$ for all real numbers, $p > 1$

**Q 9.** Show that the series $\sum_{n=1}^{\infty} \dfrac{x}{n^p + n^q x^2}$ is uniformly convergent over any finite interval $[a, \mathrm{b}]$, for $p > 1$, $q \geq 0$ .

**Q 10.** Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{x}{1+nx^2}$ is uniformly convergent to a function $f$ on $[0,1]$and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is true if $x \neq 0$and false if $x = 0$.

**Q 11.** Show that the sequence $\{f_n\}$, where

$$f_n(x) = \begin{cases} n^2 x & 0 \le x < 1/n \\ -n^2 x + 2n & 1/n \le x < 2/n \\ 0 & 2/n \le x \le 1 \end{cases}$$ is not uniformly convergent on $[0,1]$

# UNIT 10: LEBEGUE INTEGRAL

**Contents**



## *10.1 INTRODUCTION*

In previous unit we studied Uniform convergence. In this unit we will study lebesgue integral.

We begin by reminding ourselves that two sets of real numbers are said to be equivalent if a bijective mapping exists between them. The idea of "cardinality" of sets of real numbers, which actually counts the number of elements in sets, is created as a result. Any two open intervals (a,b) and (c,d) are equivalent, as is widely known. Do they appear the same when plotted on the expanded real line, though? The answer is "NO" because the interval's length is a very noticeable property when we draw intervals.

Now we define $l(I)$ as the length of a bounded interval I (open, closed, or half open) with end points $a$ and $b$ where $a < b$.

The length of an unbounded interval $I$, or we can say that an interval whose left end point is $-\infty$or the right end point is$+\infty$, is defined as infinity and is usually expressed as $l(I) = \infty$.

Is it conceivable to apply the concept of "length" of intervals to arbitrary sets of real numbers, which is the obvious question that now arises? How long should an open set or a closed set be? Be aware that it is quite reasonable to define the length of an open set as the sum of the lengths of the intervals that make up the open set. While it is simple to define the length of a bounded closed set $F$ as $b - a - l(K)$ where $F \subset (a, b)$ and $O = (a, b) - F$.

However, the class of open sets and bounded closed sets is too narrow for our needs, so we'd need to know the sizes of more complicated sets. For instance, how big should the set of irrational integers in (5, 8) be? But more crucially, when is it possible to determine the "length" of any given set of real numbers? One of the most logical ways to respond to these kinds of issues is through the theory of Lebesgue measure, which addresses these challenges.

However, the class of open sets and bounded closed sets is too narrow for our needs, so we'd need to know the sizes of more complicated sets. For instance, how big should the set of irrational integers in (2, 4) be? But more crucially, when is it possible to determine the "length" of any given set of real numbers? One of the most logical ways to respond to these kinds of issues is through the theory of Lebesgue measure, which addresses these challenges. In this chapter we will discussed about lebesgue integral.

## *10.2 OBJECTIVES*

In this Unit, we will

1. analyze about lebesgue measure
2. Studied some basic definition of lebesue integration
3. construct important theorems based on lebesgue measure

## *10.3 LEBESGUE OUTER MEASURE*

As we know that a countable number of open intervals can cover any set $X$ of real numbers. This number can be viewed as a approximate measure of the set $X$ because the length of an open interval is a positive number and the sum of the lengths of these open intervals is uniquely defined regardless of the order of the terms. Naturally, we are interested in finding the best possible approximation, which gives rise to the idea of Lebesgue outer measure, which is defined below.

**Lebegue outer measure:** Let $X$ be any subset of $\mathbb{R}$, then the Lebesgue outer measure of $X$ is defined by

$$m^*(X) = inf\{\sum_n l(I_n) : \{I_n\}_{n\in\mathbb{N}}\}$$

where $\{I_n\}$ is a countable collection of open intervals such that $X \subseteq \bigcup_{n=1}^{\infty} I_n$.

**NOTE:** Let $X$ be any subset of $\mathbb{R}$ and $m^*(X)$ be Lebegue outer measure. Then

(i) $0 \leq m^*(X) \leq \infty$ for any $X \subset \mathbb{R}$.

(ii) $m^*(X) = 0$ if $X$ is an empty set.

**Theorem 10.1. Let $X$ be any subset of $\mathbb{R}$ and $m^*(X)$ be Lebegue outer measure. Then**

**(i) $Y \subset \mathbb{R}$ such that $X \subset Y$ implies $m^*(X) \leq m^*(Y)$**

**(ii) If $x \in \mathbb{R}$ we have $m^*(A + x) = m^*(A)$**

**(iii) The Lebesgue outer measure of an interval is equal to its length.**

**Proof:** (i) As we know that any cover of $Y$ by open intervals is also a cover of $X$.

Therefore $m^*(X) \leq m^*(Y)$

(ii) Let $\varepsilon > 0$ be given.

we may try to find a countable collection of open intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $A \subset \bigcup_n I_n$ In and $\sum_n l(I_n) < m^*(X) + \varepsilon$.

Obviously $A + x \subset \bigcup_n l(I_n + x)$

Therefore, $m^*(A + x) \leq \sum_n l(I_n + x) = \sum_n l(I_n) < m^*(X) + \varepsilon$.

As $\varepsilon > 0$ is arbitrary hence it follows that

$m^*(X + x) \leq m^*(X)$. ……………………………….(1)

Now $X = (X + x) - x$

$\Rightarrow m^*(X) \leq m^*(X + x)$. …………………………..(2)

From inequality (1) and (2), we get

$m^*(X) = m^*(X + x)$

(vi) Let $I = [a, b]$ be a closed bounded interval. Now we will prove that $m^*(I) = b - a$.

Let ε > 0 be given such that $I = [a, b] \subset (a - \varepsilon, b + \varepsilon)$

By definition of outer measure, we get

$m^*(I) \leq b - \varepsilon - (a + \varepsilon) \leq b - a + 2\,\varepsilon$

$\varepsilon$ is arbitrary, therefore

$m^*(I) \leq b - a$ ………………………………………………(3)

Now we prove that μ ∗ (I) ≥ b − a.

Let $\{I_n\}_{n\in N}$ any countable collection of open intervals.

As we know that $[a, b]$ is compact.

Therefore Every open cover of $[a, b]$ has a finite subcover.

Also the sum of the lengths of intervals from a sub-collection of $\{I_n\}_{n\in N}$ can not exceed $l(I_n)$ and so it is sufficient to prove $\sum_n l(I_n) \geq b - a$ for a finite collection $\{I_n\}$ of open intervals covering $[a, b]$.

Since $a \in [a, b]$, so there exists a member of $\{I_n\}_{n\in N}$, be $(a_1, b_1)$, such that $a_1 < a < b1$.

If $b_1 > b$ then $[a, b] \subset (a_1, b_1)$ and the proof is complete.

If not, then $b_1 \in [a, b]$. Then there exists another member $(a_2, b_2)$ (say) with $a_2 < b_1 < b_2$. If $b_2 > b$ then the proof is finished.

If not, then we proceed as above. In this way we get members $(a_1, b_1), (a_2, b_2), \ldots\ldots.$ from $\{I_n\}_{n\in N}$ such that $a_{i+1} < b < b_{i+1}$.

The collection $\{I_n\}$ is finite, so this process must stop after finite number of steps.

But note that if it stops after $k$ steps then we must have $a_k < b < b_k$.

Now clearly

$\sum_n l(I_n) \geq \sum_{i=1}^{n} (b_i - a_i)$

$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots .(b_1 - a_1)$

$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots (a_2 - b_1) - a_1 \geq b_k - a_1$

$\geq b - a$

Thus we have $m^*(I) \geq b - a$.

Therefore $m^*(I) = (b - a)$.

Next let $I$ be any finite interval.

Let $\varepsilon > 0$ be given.

Choose a closed interval $K \subset I$ such that $l(K) > l(I) - \varepsilon$.

Then $l(I) - \varepsilon < l(K) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = l(\bar{I}) = l(I)$.

Again since this is true for any ε > 0 hence we get $l(I) = m^*(I)$.

Finally let $I$ be an infinite interval. For any $H > 0$, we can find a closed interval $\beta \subset I$ with $l(\beta) > H$.

Then $m^*(I) \geq m^*(\beta) = l(\beta) > H$.

Since this is true for any $H > 0$ so we must have $m^*(I) = \infty = l(I)$.

This completes the proof.

**Theorem 10.2. Outer measure is countably subadditive, i.e, if $\{\boldsymbol{X_k}\}_{\boldsymbol{k=1}}^{\infty}$ be any countable collection of sets, then**

$$\boldsymbol{m}^*(\cup_{\boldsymbol{k=1}}^{\infty} \boldsymbol{X_k}) \leq \sum_{\boldsymbol{k=1}}^{\infty} \boldsymbol{m}^*(\boldsymbol{X_k})$$

**Proof.** Let one of the $X_k$ has infinite outer measure, the inequality holds trivially.

Assume each $X_k, k = 1,2,3, \ldots$..has finite outer measure.

Let $\varepsilon > 0$. For each $k \in \mathbb{N}$, there is a countable collection $\{I_k\}_{i=1}^{\infty}$ of open, bounded intervals such that

$X_k \subseteq \cup_{k=1}^{\infty} I_{k_i}$ and $\sum_{k=1}^{\infty} l\left(I_{k_i}\right) < m^*(X_k) + \frac{\varepsilon}{2^k}$

Now $\left\{I_{k_i}\right\}_{1\leq k,i\leq\infty}$ is a countable collection of open, bounded intervals that covers $\cup_{k=1}^{\infty} X_k$.

It is countable (countable collection of set is also countable)

By the definition of outer measure, we get

$$\begin{aligned} m^*(\cup_{k=1}^{\infty} X_k) &\leq \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} l\left(I_{k_i}\right)\right) \\ &< \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} l\left(I_{k_i}\right)\right) \\ &< \sum_{k=1}^{\infty}\left(m^*(X_k) + \frac{\varepsilon}{2^k}\right) \\ &= \sum_{k=1}^{\infty} m^*(X_k) + +\varepsilon \end{aligned}$$

Because this true for each $\boldsymbol{\varepsilon} > 0$, it also true for $\boldsymbol{\varepsilon} = 0$. Therefore

$m^*(\cup_{k=1}^{\infty} X_k) \leq \sum_{k=1}^{\infty} m^*(X_k)$

**NOTE:** if $\{X_k\}_{k=1}^{n}$ be any Finite collection of sets, then

$m^*(\cup_{k=1}^{n} X_k) \leq \sum_{k=1}^{n} m^*(X_k)$

**It is called finite subadditivity property.**

**Theorem 10.3. If $X$ is countable then $m^*(X) = 0$.**

**Proof:** Consider $A$ be a countable set defined as $A = \{a_n\}_{n=1}^{\infty}$.

Suppose $\varepsilon > 0$.

For each $n \in \mathbb{N}$, define $I_n = (a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}})$.

Now the countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covers $X$.

Hence

$0 \leq m^*(X) \leq \sum_{i=1}^{\infty} l(I_n) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

As $\varepsilon > 0$ is arbitrary.

Therefore $m^*(X) = 0$.

**NOTE:**

$H \subset \mathbb{R}$ is called a $G_\delta$ set if it can be expressed as the intersection of a countable number of open sets.

Similarly a set $S \subset \mathbb{R}$ is called a $S_\sigma$ set if it can be expressed as the union of a countable number of closed sets.

**CHECK YOUR PROGRESS**

(CQ 1) Define Lebegue outer measure

ANSWER___________________________________

(CQ 2) Let $X$ be any subset of $\mathbb{R}$ and $m^*(X)$ be Lebegue outer measure. Then prove that $Y \subset \mathbb{R}$ such that $X \subset Y$ implies $m^*(X) \leq m^*(Y)$

ANSWER____________________________________

## *10.4 THE $\sigma$ −ALGEBRA OF.LEBESGUE MEASURABLE SETS*

The four benefits of outer measure are that

(i) it is defined for all sets of real numbers,

(ii) an interval's outer measure is its length,

(iii) it is countably subadditive, and

(iv) it is translation invariant.

Outer measure, however, is not countably additive. Infact, it is not even finitely additive there are disjoint sets X and Y for which $m^*(X \cup Y) < m^*(X) + m^*(Y)$.

**Measurable:** Let $A$ be any set then a set $X$ is said to be measurable if $m^*(A) = m^*(A \cap X) + m^*(A \cap X^c)$

**NOTE:**

If $X$ is measurable and $Y$ is any set disjoint from $X$, then

$m^*(X \cup Y) = m * ((X \cup Y) \cap X)) + m^* ((X \cup Y) \cap X^c) = m^*(A) + m^*(B)$

$X$ is measurable if and only if for each set $S$ we have $m^*(S) \geq m^*(S \cap X) + m * (S \cap X^c)$.

**Theorem 10.4. Any countable set is measurable.**

**Proof** Consider the set $X$ have outer measure zero.

Assume $S$ be any set.

As we know that

$S \cap X \subseteq X$ and $S \cap X^c \subseteq S$

Also

$m^*(S \cap X) < m^*(X) = 0$ and $m^*(S \cap X^c) < m^*(S)$.

Therefore

$$m^*(S) > m^*(S \cap X^c) = 0 + m^*(S \cap X^c)$$

$= m^*(S \cap X) + m^*(S \cap X^c),$

Hence $X$ is measurable.

**Theorem 10.5. The union of a finite collection of measurable sets is measurable**

**Proof.** Fist we will try to prove that the union of two measurable sets $X_1$ and $X_2$ is measurable.

Assume $S$ be any set.

As $X_1$ is measurable, so we have

$m^*(S) = m^*(S \cap X_1) + m^*(S \cap {X_1}^c)$ ……………………..(1)

$X_2$ is measurable, hence

$m^*(S \cap {X_1}^c) = m^*\big((S \cap {X_1}^c) \cap X_2\big) + m^*\big((S \cap {X_1}^c) \cap X_2^c\big)$ ……….(2)

Using equation (2) in equation (1), we get

$m^*(S) = m^*(S \cap X_1) + m^*\big((S \cap {X_1}^c) \cap X_2\big) + m^*\big((S \cap {X_1}^c) \cap X_2^c\big)$ ..(3)

As we know

$(S \cap {X_1}^c) \cap X_2^c = S \cap (X_1 \cap X_2)^c$ ……………. (4)

Using equation (4) in equation (3), we get

$m^*(S) = m^*(S \cap X_1) + m^*\big((S \cap {X_1}^c) \cap X_2\big) + m^*(S \cap (X_1 \cap X_2)^c)$

We also know

$(S \cap X_1) \cup (S \cap X_1^c \cap X_2) = S \cap (X_1 \cup X_2)$

Therefore

$(S \cap X_1) \cup (S \cap X_1^c \cap X_2) = S \cap (X_1 \cup X_2)$

Using finite subadditivity of outer measure, we get

$m^*(S \cap X_1) + m^*(S \cap X_1^c \cap X_2) \geq m^*\big(S \cap (X_1 \cup X_2)\big)$

Therefore

$m^*(S) \geq m^*\big(S \cap (X_1 \cup X_2)\big) + m^*(S \cap (X_1 \cap X_2)^c)$

Hence $X_1 \cup X_2$ is measurable.

let $\{X_k\}_{k=1}$ be any finite collection of measurable sets. We try to justify that the union of finite measurable set is also measurable i.e. $\cup_{k=1}^{n} X_k$ is measurable.

For general $n$, by induction. This is trivial for n = 1.

Suppose it is true for $n - 1$ i.e. $\cup_{k=1}^{n-1} X_k$ is measurable.

For $n$

$\cup_{k=1}^{n} X_k = \cup_{k=1}^{n-1} X_k \cup X_n$

It is given that $X_n$ is measurable which implies $\cup_{k=1}^{n-1} X_k \cup X_n$ is measurable.

Therefore, $\cup_{k=1}^{n} X_k$ is measurable.

## *10.5 LEBESGUE MEASURE AND COUNTABLE ADDITIVITY*

**Lebesgue measure:** The restriction of the set function outer measure to the class of measurable sets is called **Lebesgue measure**. It is denoted by $m$, so that if $X$ is a measurable set, its Lebesgue measure, $m(X)$, is defined by

$m(X) = m^*(X)$.

**Theorem 10.6. Lebesgue measure is countably additive, i.e., if $\{X_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets, then its union $\cup_{k=1}^{\infty} X_k$ also is measurable and**

$$\boldsymbol{m(\cup_{k=1}^{\infty} X_k) \leq \sum_{k=1}^{\infty} m(X_k)}$$

**Proof**. $\{X_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets

It implies that $\cup_{k=1}^{\infty} X_k$ is measurable sets. (because union of measurable sets is measurable)

Outermeasure is countably subadditive.

Therefore

$m(\cup_{k=1}^{\infty} X_k) \leq \sum_{k=1}^{\infty} m(X_k)$ ……………………………………………(1)

As we know that $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint then

$m^*\ ([\cup_{k=1}^{n} E_k])\ = \sum_{k=1}^{n} m^*(E_k\ ).$

Therefore for each natural number $n$, $m^*\ ([\cup_{k=1}^{\infty} X_k])\ = \sum_{k=1}^{n} m^*(X_k\ ).$

Since $\cup_{k=1}^{\infty} X_k$ contains $\cup_{k=1}^{n} X_k$, by the monotonicity of outer measure and the preceding equality.

$m(\cup_{k=1}^{\infty} X_k) \geq \sum_{k=1}^{n} m^*(X_k\ )\ \ for\ each\ n$

The left-hand side of this inequality is independent of $n$.

Therefore

$m(\cup_{k=1}^{\infty} X_k) \geq \sum_{k=1}^{\infty} m^*(X_k\ )$ …………………………………………..(2)

From the inequalities (1) and (2), we get

$m(\cup_{k=1}^{\infty} X_k) = \sum_{k=1}^{\infty} m^*(X_k\ )$

**Cantor set:**

Let $I\ = [0, 1]$ be the closed and bounded interval.

The first subdivide $I$ into three intervals of equal length $1/3$ and remove the interior of the middle interval, i.e., remove the interval $(1/3, 2/3)$ from the interval $[0, 1]$ to obtain the closed set $I_1$, which is the union of two disjoint closed intervals, each of length $\frac{1}{3}$

$I_1\ = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

We now repeat this on each of the two intervals in $I_1$ to obtain a closed set $I_2$, which is the union of $2^2$ closed intervals, each of length $\frac{1}{3^2}$

$I_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$

Again repeat this step on each of the four intervals in $I_2$ to obtain a closed set $I_3$, which is the union of $2^3$ closed intervals, each of length $\frac{1}{3^3}$.

We continue this above steps countably many times to obtain the countable collection of sets $\{I_n\}_{n=1}^{\infty}$.

Now we define the Cantor set $C$ by

$C = \cap_{n=1}^{\infty} I_n$

The collection $\{I_n\}_{n=1}^{\infty}$ have the following two properties:

(i) $\{I_n\}_{n=1}^{\infty}$ is a descending sequence of closed sets.

(ii) For each $n$, $I_n$ is the disjoint union of $2^n$ closed intervals, each of length $\frac{1}{3^n}$.

**Theorem 10.7. The Cantor set $C$ is a closed, uncountable set of measure zero.**

**Proof.** The intersection of any collection of closed sets is closed.

Therefore $C$ is closed.

Each closed set is measurable hence each $I_n$ and $C$ itself is measurable.

Now each $I_n$ is the disjoint union of $2^n$ intervals, each of length $\frac{1}{3^n}$,

Using finite additivity of Lebesgue measure, we get

$m(I_n) = \left(\frac{2}{3}\right)^n$.

By the monotonicity of measure, we get

$m(C) \leq m(I_n) = \left(\frac{2}{3}\right)^n, for\ all\ k, m(C) = 0.$

Now we will prove that $C$ is uncountable.

Assume $C$ is countable.

Let $\{c_n\}_{n=1}^{\infty}$ be an enumeration of $C$.

Hence we have two disjoint Cantor intervals whose union is $I_1$ and does not contain the point $c_1$, let it be $A_1$.

Similarly one of the two disjoint Cantor intervals in $I_2$ whose union is $A_2$ fails to contain the point $c_2$; denote it by $A_2$.

Continuing in this way, we construct a countable collection of sets $F_{n=1}^{\infty}$ which, for each n,

having the following properties:

(i) $A_n$ is closed and $A_{n+1} \subseteq A_n$

(ii) $A_n \subseteq C_n$ and

(iii) $c_n \notin A_n$ .

From (i) and the Nested Set Theorem we get

The intersection $\cap_{k=1}^{\infty} I_n$ is nonempty.

Let the point $x$ belong to this intersection.

By property (ii), $\cap_{k=1}^{\infty} A_k \subseteq \cap_{k=1}^{\infty} I_k = C$ and therefore the point $x$ belongs to $C$.

However, $(c_n\}_{n=1}^{\infty}$ is an enumeration of $C$. Therefore $x = c_m$ for some index $m$.

Hence $c_m = x \in \cap_{k=1}^{\infty} A_n \subseteq A_m$ It contradicts property (iii).

Thus $C$ must be uncountable.

**CHECK YOUR PROGRESS**

(CQ 1) Define Cantor set

ANSWER_________________

(CQ 2) Prove that measure of cantor set is 0.

ANSWER____________________________________

## *10.6 POINTWISE AND UNIFORM CONVERGENCE*

Let $\{f_n\}$ be sequence of functions with common domain $X$, if $f$ a function on $X$ and a subset $Y$ of $X$, then

a) The sequence $\{f_n\}$ converges to $f$ pointwise on $Y$ if

$\lim_{n\to\infty} f_n(x) = f(x) for\ all\ x \in Y$

b) The sequence $\{f_n\}$ converges to $f$ pointwise $a.e.$ on $Y$ if $\{f_n\}$ converges to $f$ pointwise on $Y \sim Z$ , where $m\,(Z) = 0$.

c) The sequence $\{f_n\}$converges to $f$ uniformly on $Y$ if for each $\varepsilon > 0$, there is a positive natural number $n_0$ such that $|f - f_n| < \varepsilon \; on \; Y \; for \; all \; n \geq n_0$

**Egoroff s Theorem**

**Theorem 10.8. Let $X$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $X$ that converges pointwise on $X$ to the real-valued function $f$. Then for each $\varepsilon > 0$, there is a closed set $S$ contained in $X$ such that $\{f_n\} \to f$ uniformly on $A$ and $m\,(X - A) < \varepsilon$.**

**Proof.** First we prove that for each $\delta_1 > 0$ and $\delta_2 > 0$, there is a measurable subset $Y$ of $X$ and a positive natural number $n_0$ such that

$|f_n - f| < \delta_1 \; on \; Y \; forall \; n > n_0 \; and \; m(X - Y) < \delta_2$.

As $f$ is real valued function therefore for each $k$, the function $|f - f_k|$ is properly defined.

$f$ is measurable $\Rightarrow$ the set $\{x \in X| \; |f(x) - f_k(x)| < \delta_1\}$ is measurable.

The intersection of acountable collection of measurable sets is measurable.

Hence

$X_n = \{x \in X| \; if \; |f(x) - f_k(x)| < \delta_1 for \; all \; k \geq n$ is a measurable set.

Let $\{X_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets.

$\{f_n\}$ converges pointwise to $f$ on $X \Rightarrow X = \cup_{n=1}^{\infty} X_n$

Now continuity of measure implies that

$m(X) = \lim_{n\to\infty} m(X_n)$

Since $m\,(X) < \infty$, we choose a positive natural number $n_0$ such that $m\,(X_{n_0}) > m\,(X) - \varepsilon$.

Let $A = X_n$ and by the excision property of measure, $m\,(X - A) = m\,(X) - m\left(X_{n_0}\right) < \varepsilon$ Now for each $n \in \mathbb{N}$, let $A_n$ be a measurable subset of $A$ and $n_0(n)$ be any positive natural number such that if $\delta_1 = \frac{\varepsilon}{2^{n+1}}$ and $\delta_2 = \frac{1}{n}$ then

$m(X - A_n) < \frac{\varepsilon}{2^{n+1}}$……………………..(1) and

$|f_k - f| < \frac{1}{n}$ $on\ X_n\ \ for\ all\ k \geq n_0(n)$ ………………(2)

Consider $A = \cap_{n=1}^{\infty} A_n$

By De Morgan's Identities, the countably subadditivity of measure and (1),

$m(X - A) = m\left(\cup_{n=1}^{\infty} (X - A_n)\right) \leq \sum_{n=1}^{\infty} m(X - A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}$.

Now we try to prove that that $\{f_n\}$ converges to $f$ uniformly on $X$.

Also let $\varepsilon >\ 0$. We choose a natural number $m$ such that $\frac{1}{m} <\ \varepsilon$.

From inequality (2), we get

$|f_k - f| < \varepsilon\ on\ A\ for\ k \geq n_0(m)$

Therefore $\{f_n\}$ converges to $f$ uniformly on $A$ and $m\ (X\ -\ A) < \frac{\varepsilon}{2}$.

Let a closed set $F$ contained in $A$ for which $m\ (A\ -\ S) < \frac{\varepsilon}{2}$.

Therefore $m\ (X -\ S)\ <\ \varepsilon$ and $\{f\} \rightarrow f$ uniformly on $S$.

## *10.7 LEBESGUE INTEGRATION*

A real-valued function $f$ defined on $[a, b]$ is called a step function if there is a partition $P\ =\ \{x_0, x_1, \ldots, x_n\ \}$ of $[a, b]$ and numbers $k_1, \ldots, k_n$, such that for $1\ <\ i\ <\ n$,

$\omega(x)\ =\ c_i\ if\ x_{i-1}\ <\ x\ <\ x_i$.

We know that

$L(\omega, P) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = U(\omega, P)$

From above and the definition of the upper and lower Riemann integrals, we coclude that a step function $\omega$ is Riemann integrable and

$\int_a^b \omega = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$

Therefore, we may reconstruct the definition of the lower and upper Riemann integrals as follows:

$\int_a^b \omega = \sup\left\{\int_X \varphi \mid \varphi \ is\ step\ function\ and\ \varphi \leq f\ on\ X\ \right\}$ and

$\int_a^b \omega = \inf\left\{\int_X \varphi \mid \varphi \ is\ step\ function\ and\ \varphi \geq f\ on\ X\ \right\}$

**NOTE:**

A step function takes only a finite number of values and each interval is measurable.

Thus a step function is simple. Since the measure of a singleton set is zero and the measureof an interval is its length, we infer from the linearity of Lebesgue integration for simplefunctions defined on sets of finite measure that the Riemann integral over a closed, boundedinterval of a step function agrees with the Lebesgue integral.

Let $f$ be a bounded real-valued function defined on a set of finite measure $X$.

We define the **lower and upper Lebesgue integral**, respectively, of $f$ over $X$ to be

$\sup\left\{\int_X \varphi \mid \varphi\ simple\ and\ \varphi \leq f\ on\ X\ \right\}$

and

$\inf\left\{\int_X \varphi \mid \varphi\ simple\ and\ \varphi \geq f\ on\ X\ \right\}$

As we know that $f$ is bounded, by the monotonicity property of the integral for simple functions, the lower and upper integrals are finite and the upper integral is always at least as large as the lower integral.

**Lebesgue integrable:** A bounded function $f$ on a domain $X$ of finite measure is said to be Lebesgue integrable over $X$ provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the Lebesgue integral, or simply the integral, of $f$ over $X$ and is denoted by $\int_X f$.

**Theorem 10.9. Let $f$ be a bounded function defined on the closed, bounded interval $[a,b]$. If $f$ is Riemann integrable over $[a,b]$, then it is Lebesgue integrable over $[a,b]$ and the two integrals are equal**.

**Proof** Let $f$ is Riemann integrable on interval $I = [a,b]$,

$$sup\left\{\int_I \varphi \middle|\ \varphi \text{ } is\ a\ step\ function, \varphi \leq f\right\}$$

$$= inf\left\{\int_I \varphi \middle|\ \varphi \text{ } is\ a\ step\ function, \varphi \geq f\right\}$$

To prove that f is Lebesgue integrable we will try prove that

$$sup\left\{\int_I \varphi \middle|\ \varphi \text{ } is\ a\ simple\ function, \varphi \leq f\right\}$$

$$= inf\left\{\int_I \varphi \middle|\ \varphi \text{ } is\ a\ simple\ function, \varphi \geq f\right\}$$

Although, Every step function is a simple function and, as we have already established, for a step function, the Riemann integral and the Lebesgue integral are the equal.

Hence the first inequality $\Rightarrow$ the second inequality

Therefore

$$sup\left\{\int_I \varphi \middle|\ \varphi \text{ } is\ a\ simple\ function, \varphi \leq f\right\}$$

$$= inf\left\{\int_I \varphi \middle|\ \varphi \text{ } is\ a\ simple\ function, \varphi \geq f\right\}$$

**Theorem 10.10. Let $f_1$and $f_2$ be bounded measurable functions on a set of finite measure $X$. Then for any $a$ and $b$ then $\int_X (af_1 + bf_2) = a\int_X f_1 + b\int_X f_2$ . Also if $f \leq g$ $on$ $X, then$ $\int_X f_1 \leq \int_X f_2$ .**

**Proof** Let $f_1$and $f_2$ be bounded measurable functions on a set of finite measure $X$.

We Know that linear combination of measurable bounded functions is measurable and bounded.

Therefore $af_1 + bf_2$ is integrable over $X$.

If $\varphi$ is a simple function then and $a\varphi$ is also simple function

Let $a > 0$.

As Lebesgue integral is equal to the upper lebesgue integral, hence

$$\int_X af_1 = inf_{\varphi \geq af} \int_X \varphi = a\, inf_{\left[\frac{\varphi}{a}\right] \geq f} \int_X \left[\frac{\varphi}{a}\right] = a\int_X f$$

Let $a < 0$.

As Lebesgue integral is equal to the upper lebesgue integral and lower lebesgue integral, hence

$$\int_X af_1 = inf_{\varphi \leq af} \int_X \varphi = a\, sup_{\left[\frac{\varphi}{a}\right] \leq f} \int_X \left[\frac{\varphi}{a}\right] = a\int_X f$$

If $a = b = 1$.

Let $\varphi_1$ and $\varphi_2$ be simple functions for which $f_1 \leq \varphi_1$and $f_2 \leq \varphi_2$ on $X$.

Then $\varphi_1 + \varphi_2$ is a simple function and $f_1 + f_2 \leq \varphi_1 + \varphi_2$on $X$.

Because $\int_X (f + g)$ is equal to the upper Lebesgue integral of $f_1 + f_2$ over $X$, therefore

$\int_X (f_1 + f_2) \leq \int_X (\varphi_1 + \varphi_2) = \int_X \varphi_1 + \int_X \varphi_2$ ( linearity of integration for simple functions)

Because $\varphi_1$ and $\varphi_2$ vary among simple functions such that $f \leq \varphi_1$ and $f \leq \varphi_2$.

Therefore $g.l.b.(\int_X (f_1 + f_2)) = \int_X f_1 + \int_X f_2$

From above inequalities we conclude that $\int_X (f_1 + f_2)$ is a lower bound for these same sums. Therefore,

$\int_X (f_1 + f_2) = \int_X f_1 + \int_X f_2$

Let $\Phi_1$ and $\Phi_2$be simple functions for which $\Phi_1 \leq f_1$ and $\Phi_2 \leq f_2$ on $X$.

Then $\Phi_1 + \Phi_2 \leq f_1 + f_2$ on $X$ and $\Phi_1 + \Phi_2$ is simple.

Because $\int_X (f_1 + f_2)$ is equal to the lower Lebesgue integral of $f_1 + f_2$ over $X,$

$\int_X (f_1 + f_2) \geq \int_X (\Phi_1 + \Phi_2) = \int_X \Phi_1 + \int_X \Phi_2$ (by the linearity of integration for simple functions)

Because $\Phi_1$ and $\Phi_2$ vary among simple functions such that $f \leq \Phi_1$ and $f \leq \Phi_2$.

Therefore $l.u.b( \int_X (f_1 + f_2)) = \int_X f_1 + \int_X f_2$

From above inequalities we conclude that $\int_X (f_1 + f_2)$ is a upper bound for these same sums. Therefore,

$\int_X (f_1 + f_2) \geq \int_X f_1 + \int_X f_2$

Therefore,

$\int_X (f_1 + f_2) \geq \int_X f_1 + \int_X f_2$

This completes the proof of linearity of integration.

Let $f_1 \leq f_2$ on $X$. Assume $f = f_1 - f_2$ on $X$.

Therefore $\int_X f_2 - \int_X f_1 = \int_X (f_2 - f_1) = \int_X f$.

The function $h$ is nonnegative.

Hence $h > 0$ on $X,$ where $\varphi = 0$ on $X$.

Since the integralof $h$ equals its lower integral, $\int_X f \geq \int_X \varphi = 0$.

Therefore, $\int_X f_1 \leq \int_X f_2$.

**Theorem 10.11. Let $f$ be a bounded measurable function on a set of finite measure $X$. Then $\left|\int_X f\right| \le \int_X |f|$**

**Proof** It is given that function $|f|$ is measurable and bounded.

As we know that

$-|f| \le f \le |f|$ on $X$.

Therefore, by linearity and monotonicity of integration

$-\int_X |f| \le \int_X f \le \int_X |f|$ i.e. $\left|\int_X f\right| \le \int_X |f|$.

**Theorem 10.12. Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure $X$. If $\{f_n\} \to f$ uniformly on $X$, then $\lim_{n\to\infty} \int_X f_n = \int_X f$**

Proof It is given that $f_n$ is bounded measurable function on X, for all $n \in \mathbb{N}$ and $\{f_n\} \to f$ uniformly on $X$

Therefore $f$ is bounded.

As $f$ is the pointwise limit of a sequence $\{f_n\}$.

Therefore $f$ is measurable.

Let $\varepsilon > 0$ be any positive number. We Choose $n_0 \in \mathbb{N}$ such that

$|f - f_n| < \frac{\varepsilon}{m(X)}$ $on\ X\ for\ all\ n \ge n_0$ ……………………………(1)

By the linearity and monotonicity of integration $for\ all\ n \ge n_0$, we get

$\left|\int_X f - \int_X f_n\right| \le \left|\int_X (f - f_n)\right|$

$\le \int_X |f - f_n|$

Using inequality (1), we get

$\left|\int_X f - \int_X f_n\right| < \int_X \frac{\varepsilon}{m(X)} \le \frac{\varepsilon}{m(X)}.m(X) = \varepsilon$

Hence $\lim_{n\to\infty} \int_X f_n = \int_X f$

**The Bounded Convergence Theorem**

**Theorem 10.13. Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure $X$. Consider $\{f_n\}$ be uniformly pointwise bounded on $X$, i.e., there exists a number $K > 0$ such that $|f_n| \leq K \; on \; X \; for \; all \; n$. If $\{f_n\} \to f$ pointwise on $X$, then $\lim_{n\to\infty} \int_X f_n = \int_X f$**

**Proof.** It is given that $\{ f_n\}$ uniformly pointwise bounded on $X$,

Hence $f$ is measurable .

Also $|f_n| \leq \; K \; on \; X \; for \; all \; n \Rightarrow |f| \leq K \; on \; X$.

Let $Y$ be any measurable subset of $X$ and $n$ be any natural number, then

$\int_X f - \int_X f_n = \int_X (f - f_n) = \int_A (f - f_n) + \int_{X-A} (f - f_n)$

Hence

$\int_X f - \int_X f_n = \int_A (f - f_n) + \int_{X-A} f - \int_{X-A} f_n$

$$\Rightarrow \left|\int_X f - \int_X f_n\right| \leq \left|\int_A (f - f_n)\right| + \left|\int_{X-A} f\right| + \left|\int_{X-A} f_n\right|$$

$$\leq \int_A |f - f_n| + \int_{X-A} |f| + \int_{X-A} |f_n|$$

$$\leq \int_A |f - f_n| + K.m(X - A) + K.m(X - A)$$

Therefore

$\left|\int_X f - \int_X f_n\right| \leq \int_A |f - f_n| + 2K.m(X - A)$ ……………………………(1)

Let $\varepsilon > 0$ be any number.

As $m(X) < \infty$ and $f$ is real-valued.

According to Egoroff's Theorem, there is a measurable subset $A$ of $X$ for which $\{f_n\} \to f$ uniformly on $A$ and $m\,(X - \; A) < \frac{\varepsilon}{4K}$.

By uniform convergence, there is a $n_o \in \mathbb{N}$ for which

$|f_n - f| < \frac{\varepsilon}{2\,m(X)} \; on \; A \; for \; all \; n \geq n_o$

Hence, for $n > n_0$ From inequality (1) and the monotonicity of integration, we get

$\left|\int_X f_n \; - \int_X f\;\right| \leq \frac{\varepsilon}{2\,m(X)}.m(X) + 2K.m(X - A) < \varepsilon$

Therefore the sequence of integrals $\left\{\int_X f_n\right\}$ converges to $\int_X f$ .

## *10.8 SUMMARY*

In this unit we discussed about lebesgue measure and lebesgue integration. We discussed some important proof on this unit

## *10.9 GLOSSARY*

1. Set- a well defined collection of elements
2. Derivative- the rate of change of a function with respect to a variable
3. Integral- continuous analog of a sum, used to calculate areas, volumes.
4. Absolute convergence- converge even when you take absolute value of each term.

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## *10.10 SUGGESTED READINGS*


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## *10.12 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Define lebesgue measure

(TQ 2) Define lebesgue integration

(TQ 3) State and Prove Egoroff theorem.

(TQ 4) State and prove boundedness convergence theorem.

(TQ 5) Define outer measure.

.**Fill in the blanks**

(TQ 6) Cantor set is ______________________.

(TQ 7) Cantor set has measure ______________.

## *10.13 ANSWERS*

| | | |
|---|---|---|
| (TQ 6) uncountable | (TQ 7) 0 | |

# BLOCK IV: METRIC SPACE

# UNIT 11: METRIC SPACES

**CONTENTS**



## *11.1 INTRODUCTION*

In previous unit, we studied Lebesgue integral. In this unit we analyze about metric space and its properties.

A metric space in mathematics is a collection with a concept of distance between its components, which are typically referred to as points. A metric or distance function is used to calculate the distance.[1] The most common environment for exploring many of the ideas of mathematical analysis and geometry is in metric spaces.

There are not many criteria for the distance concept given by the metric space axioms. Metric spaces have a lot of versatility because of their generality. The idea is also potent enough to represent a number of intuitive facts about what distance means. As a result, generic conclusions regarding metric spaces can be used in a wide range of situations.

With the use of examples, we will learn the fundamentals of metric space in this unit.

René Maurice Fréchet was a French mathematician who lived from 2 September 1878 to 4 June 1973. He was the first to define metric spaces and made significant advances to general topology.
His remarkable 1906 PhD thesis, "Some Points of the Functional Calculus," was his first significant contribution to the field. The term "metric space" was first used by Fréchet, even though Hausdorff is responsible for the nomenclature. The degree of abstraction used by Fréchet is comparable to that of group theory; he proves theorems inside a carefully selected axiomatic system that can then be applied to a wide range of specific instances.

***Fig. 11.1. René Maurice Fréchet (Source:https://en.wikipedia.org/wiki/Ren%C3%A9_Maurice_Fr%C3%A9chet#/media/File:Frechet.jpeg)***

## *11.2 OBJECTIVES*

In this Unit, we will

1. Analyze basic definition of metric space
2. Illustrate same examples of metric space
3. Studied some theoorems based on metric space
4. Discussed open set , closed set, limit point etc.

## *11.3 METRIC SPACE*

Let $X \neq \emptyset$ be a set. A metric on the set $X$ is essentially just a rule for calculating the distance between any two elements of $X$.

**Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set $X$ is defined as a function $d: X \times X \to [0, \infty)$ such that the following conditions are satisfied

$(M1)$ $d(x, y) \geq 0$ $\forall x, y \in X$ (self distance)

$(M2)$ $d(x,y) = 0$ if and only if $x = y$ $\forall x, y \in X$ (Positivity)
$(M3)$ $d(x,y) = d(y,x); \forall x, y \in X$ (Symmetry property)
$(M4)$ $d(x,y) \leq d(x,z) + d(z,y); \forall x, y, z \in X$ (Triangle inequality)

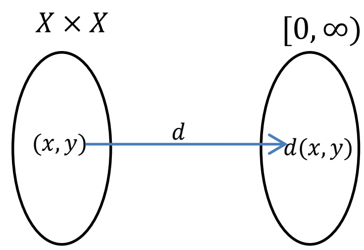


***Fig.11. 2. Metric Space***

A metric space is an ordered pair $(X, d)$ where $X$ is a nonempty set and $d$ is a metric on $X$.

**Pseudo-metric:** Let $X \neq \emptyset$ be a set then the pseudo-metric on the set $X$ is defined as a function $d: X \times X \to [0, \infty)$ such that it satisfies axioms $(M1), (M3)$ and $(M4)$ of metric space and the axiom
$(M^*2) d(x,x) = 0 \; for \; all \; x.$
Every Metric is pseudo-metric but pseudo-metric need not to be metric.
**NOTE:**
Metric $d$ is also known as distance function.
For a Pseudo-metric $x = y \Rightarrow d(x,y) = 0$ but converse may not be true.

**Examples**

- Let $X$ be any set and define the function $d : X \times X \to \mathbb{R}$ by $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$
  Then $d$ is a metric on $X$ and called the discrete metric.
- The set $C[0,1]$ consisting of all real valued continuous functions defined on $[0,1]$ with function d defined by $d(f,g) = \int_0^1 |f(x) - g(x)| \, dx \;\; for \; all \; f, g \in C[0,1]$. I a metric space.

**Ex. 11.1. Prove that with $d(x,y) = |x - y|$, the absolute value of the difference $x - y$, for each $x, y \in \mathbb{R}$, $(\mathbb{R}, d)$ is a metric space.**
**Proof.** It is given that $\boldsymbol{d(x,y) = |x - y|}$
Clearly we see that $\boldsymbol{d(x,y)}$ satisfied $(M1)$, $(M2)$ and $(M3)$ conditions
Now for all $x, y, z \in \mathbb{R}$
$d(x,y) = |x - y|$

$= |(x - z) + (z - y)|$
$\leq |x - z| + |z - y|$
$= d(x,z) + d(z,y).$

$d(x,y)$ satisfied $(M4)$ conditions.
Hence $(\mathbb{R}, d)$ is a metric space.

**Ex 11.2. Let $\boldsymbol{X}$ be the set of all continuous functions $\boldsymbol{f} : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$. Then $\boldsymbol{d(f,g) = sup\{|f(x) - g(x)| : x \in [a,b]\}}$ is a metric on $\boldsymbol{X}$.**

**Proof.** It is given that $\boldsymbol{d(x,y) = |x - y|}$
Clearly we can see that $d(x,y)$ satisfied $(M1)$ and $(M3)$ conditions
$(M2)$ $d(f,g) = 0 \Leftrightarrow sup\{|f(x) - g(x)|\} = 0 \Leftrightarrow |f(x) - g(x)| = 0 \Leftrightarrow f = g$
$\Leftrightarrow d(x,y)$ satisfied $(M2)$ conditions.
$(M4)$ $d(f,g) = sup\{|f(x) - g(x)|\}$
$= sup\{|f(x)| - h(x) + h(x) - g(x)|\}$
$\leq sup\{|f(x) - h(x)|\} + sup\{|h(x) - g(x)|\}$
$\leq d(f,h) + d(h,g)$
Hence $d(f,g)$ is a metric on $X$.

**Ex.11.3. If $\mathbb{R}^{\boldsymbol{n}}$ be the set of all ordered n-tuples with function $\boldsymbol{d}$ defined by**

**$\boldsymbol{d(x,y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}}$ $\boldsymbol{for\ all\ x = (x_1, x_2, x_3, \ldots, x_n), y = (y_1, y_2, y_3, \ldots, y_n) \in \mathbb{R}^n}$ is a metric space**

**Proof.** It is given that $d(x,y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$.
Clearly we can see that $d(x,y)$ satisfied $(M1), (M2)$ and $(M3)$ conditions.
To prove $(M4)$ condition we will use the Cauchy's Schwarz inequality that is

$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$ .

There is nothing to prove if $b_i = 0\ for\ all\ 1 \leq i \leq n$.
Let $b_i \neq 0\ for\ some\ i$, then $\sum_{i=1}^{n} b_i^2 > 0$.
Let $x$ be any real number, then
$\sum_{i=1}^{n}(a_i - xb_i)^2 \geq 0 \Rightarrow \sum_{i=1}^{n}(a_i^2 - 2xa_ib_i + x^2b_i^2) \geq 0$
$\Rightarrow \sum_{i=1}^{n} a_i^2 - 2x\sum_{i=1}^{n} a_ib_i + x^2\sum_{i=1}^{n} b_i^2 \geq 0$
As $x$ is arbitrary, so it is true for all $\in \mathbb{R}$ . Now $\sum_{i=1}^{n} b_i^2 > 0$. So

$x^2 \sum_{i=1}^{n} b_i^2 - 2x \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} a_i^2 = 0$

which is a quadratic equation in $x$ and we can observe that the discriminant is non negative if

$(\sum_{i=1}^{n} a_i b_i)^2 \geq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$

Now

$\left( (\sum_{i=1}^{n} (x_i - z_i)^2 )^{\frac{1}{2}} + (\sum_{i=1}^{n} (z_i - y_i)^2 )^{\frac{1}{2}} \right)^2$

$= \sum_{i=1}^{n} (x_i - z_i)^2 + \sum_{i=1}^{n} (z_i - y_i)^2 + 2(\sum_{i=1}^{n} (x_i - z_i)^2 )^{\frac{1}{2}} (\sum_{i=1}^{n} (z_i - y_i)^2 )^{\frac{1}{2}}$

$\geq \sum_{i=1}^{n} (x_i - z_i)^2 + \sum_{i=1}^{n} (z_i - y_i)^2 + 2 \sum_{i=1}^{n} (x_i - z_i) \sum_{i=1}^{n} (z_i - y_i)$

(By Schwarz inequality)

$= \sum_{i=1}^{n} \left( (x_i - z_i) + (z_i - y_i) \right)^2 = \sum_{i=1}^{n} (x_i - y_i)^2$

Hence

$\left( (\sum_{i=1}^{n} (x_i - z_i)^2 )^{\frac{1}{2}} + (\sum_{i=1}^{n} (z_i - y_i)^2 )^{\frac{1}{2}} \right)^2 \geq \sum_{i=1}^{n} (x_i - y_i)^2$

$\Rightarrow (\sum_{i=1}^{n} (x_i - z_i)^2 )^{\frac{1}{2}} + (\sum_{i=1}^{n} (z_i - y_i)^2 )^{\frac{1}{2}} \geq (\sum_{i=1}^{n} (x_i - y_i)^2 )^{\frac{1}{2}}$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \ for\ all\ x, y, z \in \mathbb{R}^n$

$(\mathbb{R}^n, d)$ is a metric space.

**Ex 11.4. Let $C[a,b]$ be the set of all continuous functions $f : [a,b] \to \mathbb{R}$. Then $d(f,g) = \left( \int_a^b (f(x) - gx)^2 dx \right)^{\frac{1}{2}}$ is a metric on $C[a,b]$.**

**Proof.** It is given that $d(f,g) = \left( \int_a^b (f(x) - gx)^2 dx \right)^{\frac{1}{2}}$

Clearly we can see that $d(x,y)$ satisfied $(M1)$ and $(M3)$ conditions

$(M2)$ $d(f,g) = 0 \Leftrightarrow \left( \int_a^b (f(x) - g(x))^2 dx \right)^{\frac{1}{2}} = 0 \Leftrightarrow f(x) - g(x) = 0 \Leftrightarrow f = g$

$d(x,y)$ satisfied $(M2)$ conditions.

$(M4)$ Let $\varphi$ be a function such that for $\beta \in [a,b]$

$\varphi(\beta) = \int_a^b \left( \beta f(x) + g(x) \right)^2 dx$

$= \int_a^b \left( \beta^2 f^2(x) + g^2(x) + 2\beta f(x) g(x) \right) dx$

$= \beta^2 \int_a^b f^2(x) dx + \int_a^b g^2(x) dx + 2\beta \int_a^b f(x) g(x) dx$

Because $\varphi(\beta) \geq 0, for\ all\ \beta \in [a,b],$ hence the discriminant of the quadratic in $\beta$ is non positive if

$\left(\int_a^b f(x)g(x)dx\right)^2 \geq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$

$\Rightarrow \int_a^b f(x)g(x)dx \geq$

$\left(\int_a^b f^2(x)dx \int_a^b g^2(x)dx\right)^{\frac{1}{2}}$..............................................(1)

Now

$\left(\left(\int_a^b \big(f(x)-h(x)\big)^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b \big(h(x)-g(x)\big)^2 dx\right)^{\frac{1}{2}}\right)^2$

$= \int_a^b \big(f(x)-h(x)\big)^2 dx + \int_a^b \big(h(x)-g(x)\big)^2 dx + 2\left(\int_a^b \big(f(x)-h(x)\big)^2 dx\right)^{\frac{1}{2}} \left(\int_a^b \big(h(x)-g(x)\big)^2 dx\right)^{\frac{1}{2}}$

From inequality (1), we get

$\left(\left(\int_a^b \big(f(x)-h(x)\big)^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b \big(h(x)-g(x)\big)^2 dx\right)^{\frac{1}{2}}\right)^2 \geq \int_a^b \big(f(x)-h(x)\big)^2 dx + \int_a^b \big(h(x)-g(x)\big)^2 dx + 2\int_a^b \big(f(x)-h(x)\big)dx \int_a^b \big(h(x)-g(x)\big)dx$

$= \int_a^b ((f(x)-h(x))^2 + (h(x)-g(x))^2 + 2(f(x)-h(x))(h(x)-g(x))dx$

$= \int_a^b ((f(x)-h(x)+h(x)-g(x))^2 dx$

$= \int_a^b ((f(x)-g(x))^2 dx$

Hence

$\left(\int_a^b \big(f(x)-h(x)\big)^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b \big(h(x)-g(x)\big)^2 dx\right)^{\frac{1}{2}} \geq \left(\int_a^b ((f(x)-g(x))^2 dx\right)^{\frac{1}{2}}$

i.e. $d(f,g) \leq d(f,h) + d(h,g)$

$d$ a metric on $C[a,b]$.

**Ex 11.5. Let $X$ be the set of all sequences of complex numbers. Then function $d$ which is defined as $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{(1+|x_n-y_n|)}$ $for\ all\ x = \{x_n\},\ y = \{y_n\} \in X$ is a metric on $X$.**

**Proof.** It is given that $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{(1+|x_n-y_n|)}$

Clearly we can see that $d(x,y)$ satisfied $(M1), (M2)$ and $(M3)$ conditions

$(M4)$ To prove the triangle inequality we first establish following inequality

Let $0 \leq \mu \leq \delta$, then

$\mu + \delta\mu \leq \delta + \delta\mu \quad i.e. \quad \mu(1+\delta) \leq \delta(1+\mu)$

Now dividing both side by $(1+\mu)(1+\delta)$, we get

$\frac{\mu}{(1+\mu)} \leq \frac{\delta}{(1+\delta)}$ …………………………..(1)

Now for any $x = \{x_n\},\ y = \{y_n\}\ and\ z = \{z_n\} \in X$, we have

$0 \leq |x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$

From inequality (1), we conclude

$$\frac{|x_n-y_n|}{1+|x_n-y_n|} \leq \frac{|x_n-z_n|+|z_n-y_n|}{1+|x_n-z_n|+|z_n-y_n|}$$

$$\leq \frac{|x_n-z_n|}{1+|x_n-z_n|+|z_n-y_n|} + \frac{|z_n-y_n|}{1+|x_n-z_n|+|z_n-y_n|}$$

Hence

$$\frac{|x_n-y_n|}{1+|x_n-y_n|} \leq \frac{|x_n-z_n|}{1+|x_n-z_n|} + \frac{|z_n-y_n|}{1+|z_n-y_n|}$$

Now multiplying $\frac{1}{2^n}$ and taking summation w.r.t $n$, we get

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n-y_n|}{(1+|x_n-y_n|)} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n-y_n|}{(1+|x_n-y_n|)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n-y_n|}{(1+|x_n-y_n|)}$$

$\Rightarrow d(x,y) \leq d(x,z) + d(y,z)$

Hence $d$ is metric on $X$.

$(X,d)$ is said to be metric space. It is also known as **Frechet space.**

**Theorem 11.1. Let $(X,d)$ be a metric space and let $x,y$ and $z$ be any points of $X$. Then $d(x,y) \geq |d(x,z) - d(z,y)|$**

**Proof.** It is given that $(X,d)$ is a metric space.

Let $x,y,z \in \mathbb{R}$, Using Triangle inequality in metric, we get

$d(x,z) \leq d(x,y) + d(y,z)$

$= d(x,y) + d(z,y)$ (By $(M2)$)

Subtracting both side with $d(z,y)$, we get

$d(x,z) - d(z,y) \leq d(x,y)$………………………………..(1)

Again Using Triangle inequality in metric, we get

$d(z,y) \leq d(z,x) + d(x,y)$

$= d(x,z) + d(x,y)$ (By $(M2)$)

Subtracting both side with $d(x,z)$, we get

$d(z,y) - d(x,z) \leq d(x,y)$ …………………………………...(2)

From inequality (1) and (2), we get

$d(x,y) \geq |d(z,y) - d(x,z)|$

**Some Postulates for a metric**

**Theorem 11.2. Let $X \neq \emptyset$. Then a mapping of $d: X \times X \to \mathbb{R}$ is metric iff the following conditions are satisfied:**
$(M'1)$ $d(x, y) = 0 \; iff \; x = y \; for \; all \; x, y \in X$
$(M'2)$ $d(x, y) \leq d(x, z) + d(y, z); \forall x, y, z \in X$ **(Triangle inequality)**
**Proof.** Let $d$ be a metric space. Then it satisfied all four axioms of metric.
From $(M4)$ condition; $\forall x, y, z \in X$, we get
$d(x, y) \leq d(x, z) + d(z, y)$
From $(M2)$ condition i.e. $d(y, z) = d(z, y)$, we get
$d(x, y) \leq d(x, z) + d(y, z)$ which is $(M'2)$ condition
Clearly we can see that $(M2)$condition is as same as $(M'1)$.
Let condition holds $(M'1)$ and $(M'2)$.
Assume $x$ and y be any two points in $X$.
Using $(M'2)$ conditions for $x, x$, and $y$, we get
$d(x, x) \leq d(x, y) + d(x, y)$
$\Rightarrow d(x, x) \leq 2d(x, y)$ ……………………………….(1)
From $(M'1)$, we have $d(x, y) = 0$
Hence inequality (1) become
$2d(x, y) \geq 0 \Rightarrow d(x, y) \geq 0$ i.e. $(M1)$ condition of metric space.
Clearly see that $(M'1)$ condition is same as $(M2)$ condition of metric space.
Again using $(M'2)$ conditions for $x, y$, and $x$, we get
$d(x, y) \leq d(x, x) + d(y, x)$
$\Rightarrow d(x, y) \leq 0 + d(y, x)$ (From $(M'1)$ condition)
$\Rightarrow d(x, y) \leq d(y, x)$……………………………(2)
Using $(M'2)$ conditions for $y, x$, and $y$, we get
$d(y, x) \leq d(y, y) + d(x, y)$
$\Rightarrow d(y, x) \leq 0 + d(x, y)$ (From $(M'1)$ condition)
$\Rightarrow d(y, x) \leq d(x, y)$……………………………(3)
From inequality (2) and (3), we get
$d(x, y) = d(y, x)$, $(M3)$ condition of metric space. ……………………………(4)
From $(M'2)$ condition, we get
$d(x, y) \leq d(x, z) + d(y, z)$
$d(x, y) \leq d(x, z) + d(z, x)$ (using equation (6))
It is $(M4)$ condition of metric space

## *11.4 DIAMETER AND BOUNDED AND UNBOUNDED METRIC SPACE*

**Diameter:** Let $(X, d)$ be a metric space and let $Y$ be a non empty subset of $X$. Then the diameter of $Y$, denoted by $\delta(Y)$ be defined as
$\delta(Y) = sup\,\{d(x, y): x, y \in Y\}$ i.e. diameter is the supremum of the set of all distance between point of $Y$.
**Distance between point and set:** let $Y$ be a non empty subset of $X$ and $p \in X$ then distance between point $p$ and $Y$ is defined as
$d(p, Y) = inf\,\{d(p, x): x\ in\ Y\}$ .
If $p \in Y$ then $d(p, Y) = 0$
**Distance between two set:** let $Y_1$ and $Y_2$ be a non empty subset of $X$ then distance between $Y_1$ and $Y_2$ is defined as
$d(Y_1, Y_2) = inf\,\{d(x, y): x\ in\ Y_1\ \ and\ y\ in\ Y_1\}$

**NOTE:**
$d(Y_1, Y_2) \geq 0$ and $d(Y_1, Y_2) \geq 0\ if\ and\ only\ if\ Y_1 \cap Y_2 \neq \emptyset$
$d(Y, \emptyset) = \infty$ where $\emptyset$ is an empty set.

**Theorem 11.3.** let $Y_1$ and $Y_2$ be a non empty subset of $(X, d)$. Then
$\delta(Y_1 \cup Y_2) \leq \delta(Y_1) + \delta(Y_2) + d(Y_1, \boldsymbol{Y_2})$
**Proof.** et $Y_1$ and $Y_2$ be a non empty subset of $(X, d)$ and $x$ and $y$ be any two points such that $x, y \in Y_1 \cup Y_2$.
Hence, Following condition arises
(i) $x, y \in Y_1 \Rightarrow d(x, y) \leq \delta(Y_1)$
(ii) $x, y \in Y_2 \Rightarrow d(x, y) \leq \delta(Y_2)$
(iii) $x \in Y_1, y \in Y_2$
Let $a \in Y_1$, $b \in Y_2$, using triangle inequaity we get
$d(x, y) \leq d(x, a) + d(a, y)$
$\leq d(x, a) + d(a, b) + d(b, y)$ (triangle inequality)
(iv) $y \in Y_1, x \in Y_2$
Let $a \in Y_2$, $b \in Y_1$, using triangle inequaity we get
$d(x, y) \leq d(x, b) + d(b, y)$
$\leq d(x, b) + d(b, a) + d(a, y)$ (triangle inequality)
Hence

$\delta(Y_1 \cup Y_2) \leq \delta(Y_1) + \delta(Y_2) + d(a,b)$
As $a$ and $b$ are arbitrary, hence
$\delta(Y_1 \cup Y_2) \leq \delta(Y_1) + \delta(Y_2) + d(Y_1, Y_2)$

**Bounded Metric spaces:** Let $(X,d)$ be a metric space. Then $X$ is said to be bounded if there exists $K \in \mathbb{R}^+$ such that $d(x,y) \leq K \; for \; all \; x,y \in X$.
**Bounded Metric spaces:** Let $(X,d)$ be a metric space. Then $X$ is said to be unbounded if it is not bounded.

Example: A discrete metric space $(X,d)$ where $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ is bounded because $\delta(X) = 1$.

**Theorem 11.4. Let $(X,d)$ be a metric space and $K \in \mathbb{R}^+$, then there exists a metric $d_1$ on $X$ such that the metric space $(X,d_1)$ is bounded with $\delta(X) = K$.**

**Proof.** Let we define $d_1$ such that $d_1(x,y) = \frac{Kd(x,y)}{\mathbf{1+d(x,y)}} \; for \; all \; x,y \in X$.
Now we will prove that $d_1(x,y)$ is a metric space.
$(M1)$ $d(x,y) \geq 0 \;\; \forall x,y \in X$ and $k > 0$

$$\Rightarrow \frac{Kd(x,y)}{\mathbf{1+d(x,y)}} \geq 0 \Rightarrow d_1(x,y) \geq 0 \;\; \forall x,y \in X$$

$(M2)$ $d_1(x,y) = 0 \Leftrightarrow \frac{Kd(x,y)}{\mathbf{1+d(x,y)}} = 0 \Leftrightarrow Kd(x,y) = 0$
$\Leftrightarrow d(x,y) = 0 \; as \; k > 0 \;\Leftrightarrow x = y$ (Positivity property of metric $d$)

$(M3)$ $d_1(x,y) = \frac{Kd(x,y)}{\mathbf{1+d(x,y)}}$

$$= \frac{Kd(y,x)}{\mathbf{1+d(y,x)}} \text{ (Symmetry property of metric } d)$$

$$= d_1(y,x)$$

$(M4)$ $d_1(x,y) = \frac{Kd(x,y)}{\mathbf{1+d(x,y)}} = \frac{Kd(x,y)+K-K}{\mathbf{1+d(x,y)}}$

$$= K - \frac{K}{\mathbf{1+d(x,y)}}$$

From triangle inequality of metric $d$, we have

$d(x,y) \leq d(x,z) + d(y,z) \Rightarrow \frac{1}{1+d(x,y)} \geq \frac{1}{1+d(x,z)+d(y,z)}$

$\Rightarrow \frac{-K}{d(x,y)} \leq \frac{-K}{1+d(x,z)+d(y,z)}$

Adding K on both side we get

$\Rightarrow K - \frac{K}{1+d(x,y)} \leq K - \frac{K}{1+d(x,z)+d(y,z)}$

Hence from equation (1), we get

$d_1(x,y) \leq K - \frac{K}{1+d(x,z)+d(y,z)}$

$= \frac{K(1+d(x,z)+d(y,z)-1)}{1+d(x,z)+d(y,z)} = \frac{K\left(d(x,z)+d(y,z)\right)}{1+d(x,z)+d(y,z)}$

$= \frac{Kd(x,z)}{1+d(x,z)+d(y,z)} + \frac{Kd(y,z)}{1+d(x,z)+d(y,z)} = d_1(x,z) + d_1(z,y)$

Therefore, $d_1(x,y) \leq d_1(x,z) + d_1(z,y)$

Hence $(X, d_1)$ is a metric space.

Here we can see that $d_1(x,y) = \frac{Kd(x,y)}{\mathbf{1+d(x,y)}} = \frac{K}{\frac{1}{d(x,y)}+1} \leq K \; for \; all \; point \; x, y \in X.$

Hence $(X, d_1)$ is a bounded metric space.

**CHECK YOUR PROGRESS**

(CQ 1) Prove that $d_1(x,y) = \frac{d(x,y)}{\mathbf{1+d(x,y)}}$ is a metric on $X$.

ANSWER__________________________________

(CQ 2) Prove that if $d_1$ and $d_2$ be two metric spaces then $d'(x,y) = d_1(x,y) + d_2(x,y)$is metric space.

ANSWER______________________________________

## *11.5 OPEN SET*

**Open Sphere:** Let $(X, d)$ be a metric space and let $x_0 \in X$. If $r$ be any real number then the set $x \in X: d(x, x_0) < r$ is said to be open sphere or open ball.

Here $x_0$ is said to be centre of the open sphere and $r$ is called the radius of the open sphere.

Open sphere of centre $x_0$ and radius $r$ is denoted by $S(x_0, r)$.

Therefore mathematically $S(x_0, r) = \{x \in X: d(x, x_0) < r\}$

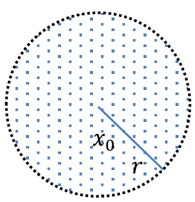


***Fig 11. 3. Open***

**Closed Sphere:** Let $(X, d)$ be a metric space and let $x_0 \in X$. If $r$ be any real number then the set $S[x_0, r) = \{x \in X: d(x, x_0) \leq r\}$ is said to be closed sphere or closed ball.

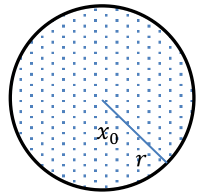


***Fig 11.4. Closed Sphere***

**NOTE:**

- Sphere or open sphere or open ball or open cell or open disc are same.
- In the usual metric space $\mathbb{R}^n$, the open sphere $S(r, x_0)$ is circular disc $|x - x_0| < r$ and $x_0 \in \mathbb{R}^n$ and $r > 0$

**Ex 11.6. Let $(X, d)$ *be* the usual metric space such that $d(x, y) = |x - y|$ *for* $[0, 1]$ . Then find $S\left(\frac{1}{3}, 1\right)$ and $S\left[\frac{1}{4}, \frac{1}{4}\right]$.**

**Proof**. $S\left(\frac{1}{3}, 1\right) = \{x \in [0,1]: \left|x - \frac{1}{3}\right| < 1$

$$= \{x \in [0,1]: -1 < x - \frac{1}{3} < 1$$

$$= \{x \in [0,1]: -1 + \frac{1}{3} < x < 1 + \frac{1}{3}$$

$$= \{x \in [0,1]: -\frac{2}{3} < x < \frac{4}{3}$$

$$= [0,1]$$

$$S\left[\frac{1}{5}, \frac{3}{5}\right] = \{x \in [0,1]: \left|x - \frac{1}{5}\right| \le \frac{3}{5}$$

$$= \{x \in [0,1]: -\frac{3}{5} \le x - \frac{1}{5} \le \frac{3}{5}$$

$$= \{x \in [0,1]: -\frac{3}{5} + \frac{1}{5} \le x \le \frac{3}{5} + \frac{1}{5}$$

$$= \{x \in [0,1]: -\frac{2}{5} \le x \le \frac{4}{5}$$

$$= \left[0, \frac{4}{5}\right]$$

**Theorem 11.5. Let $(X, d)$ be a metric space and let $p \notin S(x_0, r)$ where $x_0 \in X$ and $r > 0$. Then**

$$\boldsymbol{d(p, S(x_0, r)) \ge d(x_0, p) - r}$$

**Proof** Assume $x$ be any point of $(x_0, r)$ .

Using Triangle inequality $(M4)$ in $x_0, p, x$, we get

$d(x_0, p) \le d(x_0, x) + d(x, p)$

Subtracting both sides $d(x_0, x)$, we get

$d(x_0, p) - d(x_0, x) \le d(x_0, x) + d(x, p) - d(x_0, x)$

$\Rightarrow d(x_0, p) - d(x_0, x) \leq d(x, p)$ ……………………..(1)
But $x \in S(x_0, r) \Rightarrow d(x, x_0) < r$
$\Rightarrow -d(x, x_0) > -r$ ……………………..(2)
From inequality (1) and (2), we get
$d(x_0, p) - r \leq d(x, p)$
$\Rightarrow d(x_0, p) - r \leq d(p, x)$ (By $(M3)$ ……………(3)
Here we can see that the point $x$ is arbitrary, therefore inequality true for all $x \in S(x_0, r)$.
Hence
$d(x_0, p) - r \leq d\big(p, S(x_0, r)\big)$ or $d\big(p, S(x_0, r)\big) \geq d(x_0, p) - r$

**Neighbourhood of a point in metric space:** Let $(X, d)$ be a metric space and $x_0 \in X$. A subset $Y$ of $X$ is said to be neighbourhood of a point $x_0$ there exists $r > 0$ such that $S(x_0, r) \subseteq Y$.

**Open sets in metric space**
Let $(X, d)$ be a metric space. A subset $Y$ of $X$ is said to be open or $d$ −open in $X$ if $Y$ is neighbourhood of each of it points.
**OR**
Let $(X, d)$ be a metric space. A subset $Y$ of $X$ is said to be open or $d$ −open in $X$ iff for each $x \in Y$, there exists $r > 0$ such that $S(x, r) \subseteq Y$.

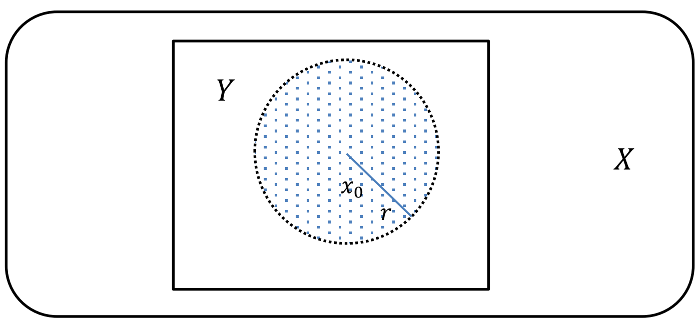

***Fig 11.5. Open Set***

**Example:**
$[0,2]$ is open in $X = [0,2]$ because $d$ and $X$ is open set in $X$.
The Cantor set $C$ is not open set.

**Ex 11.7. Every open interval is an open set in real line.**
**Proof.** Let $I$ be an opent interval on $\mathbb{R}$ and let $x \in I$.

Assume $\varepsilon = min\,\{x-a, b-x\}$. Then $I_x = (x-\varepsilon, x+\varepsilon)$ is an open sphere such that $x$ is centre and $I_x \subset I$.
Therefore $I$ is open.

**Theorem 11.6. Every open sphere is open set in metric space.**
**Proof.** Let $(X,d)$ be a metric space. Let $S(x_o, r)$ be an open sphere in $X$.
Now we will try to prove that for each point $x \in S(x_0, r)$ there exists an open sphere centred at $x$ and contained in $S(x_0, r)$ i.e. $S(x, r') \subseteq S(x_0, r)\ for\ all\ x \in S(x_0, r)$.
It is given that $x \in S(x_0, r)$
$\Rightarrow d(x, x_0) < r \Rightarrow r - d(x, x_0) > 0$.
Let $r' = r - d(x, x_0) > 0$.
Now we show that $S(x, r') \subseteq S(x_0, r)$.
Let $x' \in S(x, r')$, then $d(x', x) < r'$……………………………….(1)
Now using triangle inequality $(M4)$ in $x', x_0$ and $x$
$d(x', x_0) \leq d(x', x) + d(x, x_0)$
$\qquad < r' + d(x, x_0)$ (by inequality (1))
$\qquad = r - d(x, x_0) + d(x, x_0)$
Hence $d(x', x_0) < r$ i.e. $x' \in S(x_0, r)$
Hence $S(x, r') \subseteq S(x_0, r)$.

**Theorem 11.7. Union of arbitrary collection of open set is open in metric space.**
**Proof.** Let $(X,d)$ be a metric space.
Assume $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of open subsets of $X$ and $G = \cup_{\lambda \in \Lambda} G_\lambda$.
Now we prove that $G$ is open.
Let $x \in G$. As G is union of arbitrary collection of open set $G_\lambda$.
Hence $x \in G_\lambda$ for some $\lambda \in \Lambda$.
It is given that $G_\lambda$ is open set.
there exists an open sphere centred at $x$ and contained in $G_\lambda$ i.e.
$S(x, r) \subseteq G_\lambda$ where $r > 0$.
$G_\lambda \subseteq G \Rightarrow S(x, r) \subseteq G$
Thus we conclude that for each $x \in G$ there exists an open sphere $S(x, r)$ such that $S(x, r) \subseteq G$.
Therefore $G$ is an open set.

**Theorem 11.8. Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. Then $Y$ is open iff it is neighhbourhood of each of its points.**

**Proof.** It is given that $(X, d)$ is a metric space and $Y$ is a subset of $X$.

Let $Y$ is open and let $x \in Y$ be any point.

By the definition of open set in metric space

$x \in S(x,r) \subseteq Y, where\ r > 0$.

i.e. $Y$ is neighbourhood of $x$.

As $x$ is arbitrary , hence $Y$ neighhbourhood of each of its points.

Converse

Let $Y$ neighhbourhood of each of its points.

Now we will prove that $Y$ is open.

let $x_i \in Y$ be any point.

$Y$ is neighbourhood of $x \Rightarrow$ there exists an open ball $S(x_i, r_i)$ such that $x_i \in S(x_i, r_i) \subseteq Y, where\ r > 0$.

Let $S = \cup \{S(x_i, r_i) : x_i \in Y\ and\ r_i > 0\}$

Now we will prove that $S = Y$.

If $x \in Y$ then $x \in S(x,r) \Rightarrow x \in \cup \{S(x_i, r_i): x_i \in Y\ and\ r_i > 0\} = S$

$\Rightarrow Y \subseteq S$……………………………………………………………….(1)

Again if $y \in S \Rightarrow y \in S(x,r) for\ some\ x \in Y$.

But $S(x,r) \subseteq Y \Rightarrow y \in Y$

$\Rightarrow S \subseteq Y$ ……………………………………………………………….(1)

From (1) and (2), we get

$S = Y$

We know that the union of a collection of open set is open.

Hence $S$ is open $\Rightarrow Y$ is open

**Theorem 11.9. Intersection of finite number of open set is open in metric space.**

**Proof.** Let $(X, d)$ be a metric space.

Assume $\{G_i: 1 \le i \le n\}$ be a finite collection of open subsets of $X$ and $H = \cap_{1 \le i \le n} G_i$.

Now we prove that $H$ is open.

Let $x \in H$. As $H$ is **intersection of finite number of open set** $G_i$.

Hence $x \in G_i$ for all $i = 1,2,3, . . n$.

It is given that each $G_i$ is open set.

Therefore there exist open spheres centred at $x$ and contained in $G_i$ i.e.

$S(x, r_i) \subseteq G_i\ for\ all\ i = 1,2, … . n$ where $r_i > 0$.

Assume $r = \min\{r_i, r_2, … , r_n\}$

Then $S(x,r) \subseteq S(x,r_i) for\ all\ i = 1,2, \dots . n$
$\Rightarrow S(x,r) \subseteq G_i for\ all\ i = 1,2, \dots . n$
$\Rightarrow S(x,r) \subseteq \cap_{1 \le i \le n} G_i \Rightarrow \Rightarrow S(x,r) \subseteq H$
Thus we conclude that for each $x \in H$ there exists an open sphere $S(x,r)$ such that $S(x,r) \subseteq H$.
Therefore $H$ is an open set.

**Ex 11.8. The intersection of infinite collection of open set is not necessary an open set.**
**Sol.** Let $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right), for\ all\ n \in \mathbb{N}$.
Then each $G_n$ is open set as every open interval $I$ an open set.
Here we can see that $\cap_{n=1}^{\infty} G_n = \cap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$
But $\{0\}$ is not open set because there does not exist any $\varepsilon > 0$ such that $S(0,\varepsilon) \subseteq \{0\}$.
Hence the intersection of infinite collection of open set is not necessary an open set.

**Theorem 11.10. A subset of a metric space is open iff it is the union of family of open sphere.**
**Proof.** Let $(X,d)$ be a metric space and $Y \subseteq X$ such that $Y$ the union of family of open sphere.
If the family is empty then union of empty family spheres is empty$\Rightarrow Y$ is open.
Let $Y$ be union of non empty family $F$ of open sphere and $x$ be any point of $Y$.
Therefore $x \in S(x_0,r)$ for some $S(x_0,r) \in F$.
If $x \neq x_0$, i.e. $x$ is not centre of sphere $S(x_0,r)$.
Then there exists a sphere $S(x,r')$ such that $x$ is centere of given sphere.
$S(x,r') \subseteq S(x_0,r) \subseteq Y$.
Therefore for each $x \in Y$ here exists a open sphere such that $x$ is centere of given sphere and contained in $Y$.
Hence $Y$ is open
*Converse*
let $Y$ be open.
If $Y$ is empty then it is the union of empty family spheres.
Let $Y$ is non empty and let $x$ be any point of $Y$.
$Y$ is open $\Rightarrow$ there exist an open sphere centred at $x$ s.t. contained in $Y$. i.e.

$S(x, r) \subseteq Y$ where $r_i > 0$.
Therefore
$Y \subseteq \cup \{S(x, r): for\ all\ x\ in\ Y\} \subseteq Y$.
Hence $Y = \cup \{S(x, r): for\ all\ x\ in\ Y\}$. i.e. $Y$ the union of family of open sphere.

**Theorem 11.11. Every non empty open set on the real line is the union of a countable collection of pairwise disjoint open intervals.**
**Proof.** Let $Y \neq \emptyset$ be a subset of $\mathbb{R}$.
It is given that $Y$ is open $\Rightarrow$ For each $x \in Y$, there exists an open sphere $S(x, r)\ where\ r > 0$ such that $S(x, r) \subseteq Y$.
Let $I_x$ be the union of all open interval which contain $x$ are contained in $Y$. i.e.
$I_x = \cup \{I_i: x \in I_i\ and\ I_i \subseteq Y\}$
Hence we can easily conclude that

(i) $I_x$ is open interval such that $x \in I_x\ and\ I_x \subseteq Y$
(ii) If $p$ is another point in $I_x$, then $I_x = I_p$.
(iii) If $p$ and $x$ two distinct points of $Y$ then either $I_p \cap I_x = \emptyset\ or\ I_p \cap I_x \neq \emptyset$ .
Let $q \in I_p \cap I_x$ then $q \in I_p$ and $q \in I_x$
Hence $I_q = I_p$ and $I_q = I_x \Rightarrow I_p = I_x$

Let $I$ be the collection of all distinct $I_x$ for points $x \in Y$.
$I$ is a disjoint collection of open intervals and $Y$ is the union of such collection.
Now we will prove that $I$ is countable.
Let $Y_r = set\ of\ all\ rational\ numbers\ in\ Y \Rightarrow Y_r$ is non empty.
Now we define a mapping $f: Y_z \to I$ such that
$f(z)$ be unique interval in $I$ to which $z$ belong for each $z \in Y_z$.
Clearly, $Y_r$ is countable as it is subset of set of rational number $\mathbb{Q}$.
Hence $I$ is countable.

**Equivalent Metrics**
Let $d$ and $d'$ are two metrics on the same set $X$. Then $d$ and $d'$ are equivalent iff every $d$ −open set is $d'$ −open and every $d'$ −open is $d$ −open set .
**Ex 11.9. Let $(X, d)$ be a metric space and then mapping $d': X \times X \to \mathbb{R}$ such that**

$\boldsymbol{d'(x,y) = \frac{Kd(x,y)}{1+d(x,y)} > 0}$ **is also a metric $\boldsymbol{X}$. Also show that $\boldsymbol{d}$ and $\boldsymbol{d'}$ are equivalent.**

**Proof.** In previous example we already prove that $d'$ is metric for $X$.
Now we will prove that $d$ and $d'$ are equivalent.
Let $S(x,r)\ where\ r > 0$, be $d$ −open sphere centred at $x \in X$ and $S'(x,r')\ where\ r' = \frac{Mr}{1+r}$, be $d'$ −open sphere centred at $x \in X$.
Now we will try to prove that $S'(x,r') \subseteq S(x,r)$.
Let $y \in S'(x,r') \Rightarrow d'(x,y) < r' \Rightarrow \frac{\boldsymbol{Kd(x,y)}}{\boldsymbol{1+d(x,y)}} < \frac{Kr}{1+r}$
$\Rightarrow Kd(x,y) + rKd(x,y) < Kr + rKd(x,y) \Rightarrow d(x,y) < r$
Therefore $y \in S(x,r)$
As $y$ is arbitrary so we can say that $S'(x,r') \subseteq S(x,r)$.
Now let $S'(x,r')\ where\ r' > 0$, be $d'$ −open sphere centred at $x \in X$.
Because $d'(x,y) \le K\ for\ every\ x\ and\ y\ in\ X$.
Therefore $0 < r' \le K$.
Let $S(x,r)\ where\ r = \frac{r'}{K-r'}$, be $d$ −open sphere centred at $x \in X$.
Now we will prove that $S(x,r) \subseteq S'(x,r')$.
Now $y \in S(x,r) \Rightarrow d(x,y) < r$…………….(1)
It is given $d'(x,y) = \frac{Kd(x,y)}{1+d(x,y)} \Rightarrow d(x,y) = \frac{d'(x,y)}{K-d'(x,y)}$
Therefore Using inequality (1), we get
$\frac{d'(x,y)}{K-d'(x,y)} < \frac{r'}{K-r'} \Rightarrow Kd'(x,y) - rd'(x,y) < Kr' - r'd'(x,y) \Rightarrow$
$d'(x,y) < r'$
As $y$ is arbitrary so we can say that which implies $S(x,r) \subseteq S'(x,r')$.
Hence every $d$ −open set is $d'$ −open and every $d'$ −open is $d$ −open set .
Therefore, $d$ and $d'$ are equivalent.

## *11.6 INTERIOR, EXTERIOR, FRONTIER AND BOUNDARY OF A SET*

**Interior point:** Let $(X,d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called an interior point of $Y$ if there exists an open ball with centre $x$ contained in $Y$, i.e.,

$$x \in S(x,r) \subseteq Y\ for\ some\ r > 0$$

**Interior of Set:** The set of all interior points of $Y$ is called the interior of $Y$ and is denoted by $Int\ (Y)$ or ° .

$$Int\ (Y) = \{x \in Y\ such\ that\ \in\ S(x,r)\ \subseteq Y\ for\ some\ r\ >\ 0\}$$

**Theorem 11.12. Let $Y$ be a subset of a metric space $(X,d)$. Then**
**(i) $Int(Y)$ is an open subset of $Y$ that contains every open subset of $Y$**
**(ii) $Y$ is open if and only if $Int(Y) = Y$**

**Proof. .** (i) Let $x \in Int(Y\ )$ be arbitrary.
Then there exists an open ball $S(x,r)$ such that $S(x,r)\ \subseteq\ Y$.
As we know every open sphere is open set
Hence $S(x,r)$ is an open set
$\Rightarrow$ each point of $S(x,r)$ is the centre of some open ball contained in $S(x,r)$ and thus contained in $Y$.
Hence each point of $S(x,r)$ is an interior point of $Y$, i.e., $S(x,r) \subseteq Int(Y)$.
Therefore, $x$ is the centre of an open ball contained in $Int\ (A)$.
As $x \in Int(Y\ )$ is arbitrary, Hence each $x \in Int(Y\ )$ is the centre of an open ball contained in $Int(Y\ )$.
Hence, $Int(Y\ )$ is open.
Now we will show that $Int(Y\ )$ contains every open subset $G\ \subseteq\ A$.
Let $x \in G$ Since $G$ is open
$\Rightarrow$there exists an open ball $S(x,r)\ \subseteq\ G\ \subseteq\ A \Rightarrow x \in Int(Y)$.
Therefore, $x \in G\ \Rightarrow x \in Int(Y)$.
Thus $G \subseteq Int(Y)$.
(ii) is immediate from (i).

**Theorem 11.13. Let $(X,d)$ be a metric space and $Y_1, Y_2$ be subsets of $X$. Then**
(i) $Y_1 \subseteq Y_2 \Rightarrow\ Int(Y_1) \subseteq Int(Y_2)$
(ii) $Int\ (Y_1 \cap Y_2) = Int(Y_1) \cap Int(Y_2)$
(iii) $Int(Y_1 \cup Y_2) \subseteq Int(Y_1) \cup Int(Y_2)$
Proof. (i) Let $x \in Int(Y_1)$ then there exists an $r\ >\ 0$ such that $S(x,r)\ \subseteq Y_1$.
It is given that $Y_1 \subseteq Y_2$
$\Rightarrow S(x,r)\ \subseteq Y_2$
Hence $x \in int(Y_2)$. Thus $Int(Y_1) \subseteq Int(Y_2)$
(ii) As we know that
$Y_1 \cap Y_2 \subseteq Y_1$ and $Y_1 \cap Y_2 \subseteq Y_2$

From property (i) we get,
$Y_1 \cap Y_2 \subseteq Y_1 \Rightarrow Int(Y_1 \cap Y_2) \subseteq Int(Y_1)$
Similarly
$Y_1 \cap Y_2 \subseteq Y_2 \Rightarrow Int(Y_1 \cap Y_2) \subseteq Int(Y_2)$
Hence $Int(Y_1 \cap Y_2) \subseteq Int(Y_1) \cap Int(Y_2)$
Let $x \in Int(Y_1) \cap Int(Y_2) \Rightarrow x \in Int(Y_1)$ and $x \in Int(Y_2)$
Hence, there exists $\rho_1$ and $\rho_2$ such that $S(x, \rho_1) \subseteq Y_1$ and $S(x, \rho_2) \subseteq Y_2$.
Let $\rho = \min\{\rho_1, \rho_2\} \Rightarrow \rho > 0$ and $S(x, \rho) \subseteq Y_1 \cap Y_2$.
Therefore $x \in Int(Y_1 \cap Y_2)$.
$\Rightarrow Int(Y_1) \cap Int(Y_2) \subseteq Int(Y_1 \cap Y_2)$.
Hence$Int(Y_1) \cap Int(Y_2) = Int(Y_1 \cap Y_2)$.

(iii) As we know that
$Y_1 \subseteq Y_1 \cup Y_2$ and $Y_2 \subseteq Y_1 \cap Y_2$
From property (i) we get,
$Y_1 \subseteq Y_1 \cup Y_2 \Rightarrow Int(Y_1) \subseteq Int(Y_1 \cup Y_2)$
Similarly
$Y_2 \subseteq Y_1 \cup Y_2 \Rightarrow Int(Y_2) \subseteq Int(Y_1 \cup Y_2)$
Hence $Int(Y_1 \cup Y_2) \subseteq Int(Y_1) \cup Int(Y_2)$

## SOME IMPORTANT TERMS

**Exterior points:** Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called a exterior point of $Y$ if it is an interior point of the complement of $Y$ i.e. $Y^c$.
**Exterior of Set:** The set of all exterior points of $Y$ is called the exterior of $Y$ and is denoted by $ext\ (Y)$ or $Y^e$. i.e. $ext(A) = int(A^c)$
**Frontier points:** Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called a frontier point of $Y$ if it is neither interior or nor exterior point of $Y$.
**Frontier of Set:** The set of all frontier points of $Y$ is called the frontier of $Y$ and is denoted by $Fr\ (Y)$.
**Boundary point:** Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called a boundary point of $Y$ if it is frontier point of $Y$ and belong to $Y$.
**Boundary of Set:** The set of all boundary points of $Y$ is called the boundary of $Y$ and is denoted by $b\ (Y)$.

**Dense set:** Let $(X, d)$ be a metric space and let $Y_1$ and $Y_2$ be subsets of $X$. Then $Y_1$ is said to be dense in $Y_1$ if $Y_2 \subseteq \overline{Y_1}$.
**Everywhere Dense:** Let $(X, d)$ be a metric space and let $Y_1$ be a subset of $X$. Then $Y_1$ is said to be dense in $X$ or everywhere dense if $\overline{Y_1} = X$.
**Nowhere Dense:** Let $(X, d)$ be a metric space and let $Y_1$ be a subset of $X$. Then $Y_1$ is said to be nowhere dense in $X$ if interior of the closure of $Y$ is empty.

## *11.7 CLOSED SET*

**Limit Point:** Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called a limit point (an accumulation point) if every neighbourhood of $x$ contains a point of $Y$ distinct from $x$.
**Derived Set:** The set of all limit points of $Y$ is called the derived set of $Y$ and denoted by $D(Y)$.
**Adherent Point:** Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called an adherent point of $Y$ if every neighbourhood of $x$ contains a point of $Y$ (not necessarily distinct from $x$).
**Adherence of Set:** The set of all adherent points of $Y$ is called the adherence of $Y$. It is denoted by $Adh(Y)$.
**Isolated points:** Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is called a islolated point of $Y$ if $x \in X$ but not limit point of $Y$.

**Theorem 11.14. Let $(X, d)$ be a metric space and let $Y$ be a subset of $X$. A point $x \in X$ is limit point if every open sphere $S(x, r)$, centred at $x$ and $r > 0$ contains infinitely many points of $Y$.**
**Proof.** Let the sphere $S(x_0, r)$ contains only a finite number of points of $Y$. Let $a_1, a_2, \dots, a_n$ are the points distinct from $x_0$ and belong $to\ S(x_0, r) \cap F$.
Let $\delta = \min\{d(a_1, x_0), d(a_2, x_0), \dots, d(a_n, x_0)\}$.
Then the ball $S(x_0, d)$ contains no point of $Y$ distinct from $x_0$, which contradict the our assumption that $x_0$ is a limit point of $Y$.
Hence every open sphere $S(x, r)$, centred at $x$ and $r > 0$ contains infinitely many points of $Y$.

**Closed Sets:** Let $(X, d)$ be a metric space. A subset $Y$ of $X$ is said to be closed or $d$-closed if the compliment of $Y$ is open.

**OR**

A subset $Y$ of the metric space $(X, d)$ is said to be closed if it contains each of its limit points, i.e., $D(Y) \subseteq Y$.

**Example:**

- The set $\mathbb{Z}$ of integers is a closed subset of the real line.
- The set $Y = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots \ldots\}$ is not closed in $\mathbb{R}$ as $D(Y) = \{0\} \nsubseteq Y$.
- Every subset of discrete metric space is closed.

**Perfect set:** If closed set have no isolated points then it is called Perfect set.

**Theorem 11.15. Let $\boldsymbol{Y}$ be a subset of the metric space $(\boldsymbol{X}, \boldsymbol{d})$ and $\boldsymbol{D}(\boldsymbol{Y})$ be the derived set of Y then $\boldsymbol{D}(\boldsymbol{Y})$ is a closed subset of $(\boldsymbol{X}, \boldsymbol{d})$.**

**Proof. Let $\boldsymbol{Y}$ be a subset and $\boldsymbol{D}(\boldsymbol{Y})$ be its derived set.**

If $D(\boldsymbol{Y}) = \emptyset \Rightarrow D(D(\boldsymbol{Y})) = \emptyset$ hence there is nothing to prove.

If $D(\boldsymbol{Y}) \neq \emptyset$ and let $x \in D(D(\boldsymbol{Y}))$ .

Consider an arbitrary open ball $S(x, r)$ with centre $x$ and radius $r$.

Using the definition of limit point, we get

there exists a point $y \in S(x, r)$ such that $y \in D(Y)$.

If $\rho = r - d(y, x)$, then $S(y, \rho)$ contains infinitely many points of $Y$. (From previous theorem}.

We know that each open ball is open set.

Hence $S(y, \rho) \subseteq S(x, r)$

$\Rightarrow$ infinitely many points of $Y$ lie in $S(x, r)$.

Thus, $x$ is a limit point of $Y$, $x \in D(\boldsymbol{Y})$ .

Hence, $D(\boldsymbol{Y})$ contains all its limit points and therefore $D(\boldsymbol{Y})$ is closed.

**Theorem 11.16. Let $(\boldsymbol{X}, \boldsymbol{d})$ be a metric space and let $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ be subsets of $\boldsymbol{X}$.**

**(i) If $\boldsymbol{Y_1} \subseteq \boldsymbol{Y_2}$ then $\boldsymbol{D}(\boldsymbol{Y_1}) \subseteq \boldsymbol{D}(\boldsymbol{Y_2})$**

**(ii) $\boldsymbol{D}(\boldsymbol{Y_1} \cup \boldsymbol{Y_2}) = \boldsymbol{D}(\boldsymbol{Y_1}) \cup \boldsymbol{D}(\boldsymbol{Y_2})$**

**(iii) $\boldsymbol{D}(\boldsymbol{Y_1} \cap \boldsymbol{Y_2}) \subseteq \boldsymbol{D}(\boldsymbol{Y_1}) \cap \boldsymbol{D}(\boldsymbol{Y_2})$**

**Proof. (i)** let $Y_1$ and $Y_2$ be subsets of $X$ such that $Y_1 \subseteq Y_2$.

Let $x \in D(Y_1)$ then $x$ is a limit point.

By definition of limit point

every neighbourhood of $x$ contains a point of $Y_1$ other than $x$.

$\Rightarrow$ every neighbourhood of $x$ contains a point of $Y_2$ other than $x$. ( because $\boldsymbol{Y_1} \subseteq \boldsymbol{Y_2}$)

i.e. $x \in D(Y_1) \Rightarrow D(Y_1) \subseteq D(Y_2)$

(ii) As we know that $Y_1 \subseteq Y_1 \cup Y_2$ and $Y_2 \subseteq Y_1 \cup Y_2$
Using (i) property, we get
$D(Y_1) \subseteq D(Y_1 \cup Y_2)$ and $D(Y_2) \subseteq D(Y_1 \cup Y_2)$
Hence
$D(Y_1) \cup D(Y_2) \subseteq D(Y_1 \cup Y_2)$ ……………………(1)
Now we try to prove that $D(Y_1 \cup Y_2) \subseteq D(Y_1) \cup D(Y_2)$
Let $x \notin D(Y_1) \cup D(Y_2)$
$\Rightarrow x \notin D(Y_1) \cup D(Y_2)$ (From (i))
$\Rightarrow x \notin D(Y_1)$ and $x \notin D(Y_2)$
$\Rightarrow$ there exists neighbourhood $M_1$ and $M_2$ of x such that
$Y_1 \cap (M_1 \cap M_2 - \{x\}) = \emptyset$ and $Y_2 \cap (M_1 \cap M_2 - \{x\}) = \emptyset$
(because $M_1 \cap M_2 \subseteq M_1$ and $M_1 \cap M_2 \subseteq M_2$)
Hence $(Y_1 \cup Y_2) \cup (M_1 \cap M_2 - \{x\}) = \emptyset$.
Therefore $x$ is not a limit point of $Y_1 \cup Y_2$.
(As neighbourhood of $x$ $i.e.$ $M_1 \cap M_2$ containing no point of $(Y_1 \cup Y_2)$
Therefore $D(Y_1 \cup Y_2) \subseteq D(Y_1) \cup D(Y_2)$………(1).
From (1) and (2), we get
$D(Y_1 \cup Y_2) = D(Y_1) \cup D(Y_2)$

(iii) As we know that $Y_1 \cap Y_2 \subseteq Y_1$ and $Y_1 \cap Y_2 \subseteq Y_2$
Using (i) property, we get
$D(Y_1 \cap Y_2) \subseteq D(Y_1)$ and $D(Y_1 \cap Y_2) \subseteq D(Y_2)$
Hence
$D(Y_1 \cap Y_2) \subseteq D(Y_1) \cap D(Y_2)$

**Theorem 11.17. Let $(X, d)$ be a metric space. Then $\emptyset$ and $X$ are closed.**
**Proof.** As we know the empty set has no limit points, then essential condition that a closed set
contain all its limit points is satisfied. Hence $\emptyset$ is closed.
Similarly $X$ contains all points $\Rightarrow$ contains all its limit points
Hence $X$ is closed

**Theorem 11.18. Let $(X, d)$ be a metric space and $Y$ be a subset of $X$. Then $Y$ is closed in $X$ iff $Y^c$ is open in $X$.**
**Proof.** Assume $Y$ is closed in $X$.

Now we will prove that $Y^c$ is open in $X$.
If $Y = \emptyset \Rightarrow Y^c = X,$ therefore $Y^c$ is open in $X$. ($\because$ $X$ is open in $X$).
Let $Y \neq \emptyset \Rightarrow Y^c \neq \emptyset$.
Let $x$ be any point in $Y^c$.
Now $Y$ is closed in $X \Rightarrow D(Y) \subseteq Y$ and $x \notin Y$
$\Rightarrow x \notin D(Y),$ hence $x$ cannot be a limit point of $Y$.
Therefore there exists an $r > 0$ such that $S(x,r) \subseteq Y^c$.
Hence, each point of $Y^c$ is contained in an open ball contained in $Y$.
Thus $Y^c$ is open.
**Converse**
Let $Y^c$ is open. Now we will prove that $Y$ is closed.
Let $x \in X$ be a limit point of $X$.
Assume $x \notin Y \Rightarrow x \in Y^c$
$Y^c$ is open. Therefore, there exists $r > 0$ such that $S(x,r) \subseteq Y^c$.
$i.e.,$ $S(x,r) \cap Y = \emptyset$
Hence, $x$ cannot be a limit point of $Y$, which is a contradiction.
Therefore, $x$ belongs to $Y \Rightarrow D(Y) \subseteq Y. i.e. Y$ is closed.

**Theorem 11.19. Let $(X,d)$ be a metric space. Then**
**(i) any intersection of closed sets is closed.**
**(ii) a finite union of closed sets is closed.**
**Proof.** (i) Let $\{Y_\alpha\}$ be a family of closed sets in $X$ and $Y = \cap_\alpha Y_\alpha$
In previous theorem we prove that $Y$ is closed in $X$ iff $Y^c$ is open in $X$.
Therefore we will try to prove that $Y^c$ is open in $X$.
$Y = \cap_\alpha Y_\alpha \Rightarrow Y^c = \cup_\alpha Y_\alpha$ (By De Morgan's law)
It is given that each $Y_\alpha$ is closed$\Rightarrow$ each ${Y_\alpha}^c$ is open.
As we prove earlier that the arbitrary union of open set is open.
Hence, $Y^c = \cup_\alpha Y_\alpha$ is open.
Therefore $Y$ is closed.
(ii) Let $\{Y_n\}$ be a family of finite closed sets in $X$ and $Y = \cup_n Y_\alpha$
Now we will try to prove that $Y^c$ is open in $X$.
$Y = \cup_n Y_\alpha \Rightarrow Y^c = \cap_n Y_\alpha$ (By De Morgan's law)
It is given that each $Y_n$ is closed$\Rightarrow$ each ${Y_n}^c$ is open.
As we prove earlier that the finite intersection of open set is open.
Hence, $Y^c = \cap_n Y_\alpha$ is open.
Therefore $Y$ is closed.

**NOTE:** An arbitrary union of closed sets need not be closed.

Let $\bar{S}\left(0,1-\frac{1}{n}\right), n \geq 2$ be a closed subset of the complex plane,

but $\cup_{n=2}^{\infty} \bar{S}\left(0,1-\frac{1}{n}\right) = S(0,1)$ is not closed.

As each point $z$ satisfying $|z| = 1$ is a limit point of $S(0,1)$ but is not contained in $S(0,1)$.

**Closure of set:** Let $Y$ be a subset of a metric space $(X, d)$. The set $Y \cup D(Y)$ is called the closure of $Y$ and is denoted by $\bar{Y}$.

**Theorem 11.20. The closure $\bar{Y}$ of $Y \subseteq X$, where $(X, d)$ is a metric space, is closed.**

**Proof.** As we know that

$\bar{Y} = Y \cup D(Y)$

$\Rightarrow D(\bar{Y}) = D\big(Y \cup D(Y)\big)$……………………………..(1)

In earlier we already prove that $D(Y \cup D(Y)) = D(Y) \cup D\big(D(Y)\big)$

Hence $D\big(Y \cup D(Y)\big) = D(Y) \cup D\big(D(Y)\big)$

$\subseteq D(Y) \cup D(Y) = D(Y) \subseteq \bar{Y}$ (because $D\big(D(Y)\big) \subseteq D(Y)$)

Therefore

$D(\bar{Y}) \subseteq \bar{Y}$

Hence, closure $\bar{Y}$ of $Y$ is closed.

NOTE:

If $Y_1 \subseteq Y_2$ then $\bar{Y}_1 \subseteq \bar{Y}_2$

## *11.8 SUMMARY*

In this unit we discussed about metric space with illustrative examples. We proved some important theorems on this unit.

## *11.9 GLOSSARY*

1. Set- a well defined collection of elements
2. metric- a notion of distance between its elements
3. pseudometric- distance between two distinct points can be zero

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## *11.12 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Define Metric soace.

(TQ 2) Define Open and Closed set

(TQ 3) Prove that the closure $\bar{Y}$ of $Y \subseteq X$, where $(X, d)$ is a metric space, is closed.

(TQ 4) Let $X$ be the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Then prove that $d(f, g) = sup\{|f(x) - g(x)| : x \in [a, b]\}$ is a metric on $X$.
(TQ 5) Define Fronterior set.

.**<u>Fill in the blanks</u>**

(TQ 6) Euclid space is ______________________.

(TQ 7) Empty set and $X$ is ______________.

## *11.13 ANSWERS*

| | | |
|---|---|---|
| (TQ 6) metric space | (TQ 7) Open and closed set | |

# UNIT 12: COMPLETENESS

**CONTENTS**



## *12.1 INTRODUCTION*

It makes intuitive sense that if there are no "points missing" from a space (either inside or at the boundary), it is complete. For instance, even though one can create a Cauchy sequence of rational numbers that converges to it, the set of rational numbers is not complete since, for instance, $\sqrt{2}$ is "missing" from it (see further instances below).

As will be discussed below, it is always possible to "fill all the holes," resulting in the completion of a particular space.

In earlier chapter we analyzed Metric space. In this chapter we will studied complete metric space.

Felix Hausdorff (November 8, 1868 – January 26, 1942), a German mathematician better known by his pen name Paul Mongré, is regarded as one of the pioneers of modern topology. He also made significant contributions to set theory, descriptive set theory, measure theory, and functional analysis.

A theory of topological and metric spaces was developed by Hausdorff through his work in topology.

***Fig. 12.1. Felix Hausdorff***
(Source:https://en.wikipedia.org/wiki/Felix_Hausdorff#/media/File:Hausdorff_1913-1921.jpg)

## *12.2 OBJECTIVES*

In this Unit, we will

1. analyze about bases
2. Understand about subspaces and product metric space
3. Prove some theorems based on bases and completeness
4. Illustrate completeness using some examples.

## *12.3 BASES*

**Base:** Let $(X, d)$ be a metric space and $x \in X$. Let $\{G_k\}_{k\in\Lambda}$ be a family of open sets, each containing $x$.

**Local base:** The family $\{G_k\}_{k\in\Lambda}$ is said to be a local base at $x$ if for every nonempty open set $G$ containing $x$, there exists a set $G_i$ in the family $\{G_k\}_{k\in\Lambda}$ such that $x \in G_i \subseteq G$.

**Example:**

In the metric space $\mathbb{R}^2$ with the Euclidean metric,

let $G_k = S(x,k) where\ x = (x_1, x_2)$ and $k > 0, k \in \mathbb{R}$.

The family $\{G_k : k > 0\ and\ k \in \mathbb{R}\}$ is a family of balls and is a local base at $x$.

**Theorem 12.1. In any metric space, there is a countable base at each point.**

**Proof.** Let $(X,d)$ be a metric space and $x \in X$.

The family of open balls centred at $x$ and having rational radii,

i.e. $\{S(x,r): r \in \mathbb{Q}^+\}$ is a countable base at $x$.

If $G$ is an open set and $x \in G$,

Hence, there exists an $\varepsilon > 0$ (e depending on $\varepsilon$) such $that\ x \in S(x,\varepsilon) \subseteq G$.

Let $r$ be a positive rational number less than $\varepsilon$.

Then $x \in S(x,r) \subseteq S(x,\varepsilon) \subseteq G$.

**Base for the open sets:** A family $\{G_k\}_{k\in\Lambda}$ of nonempty open sets is called a base for the open sets of $(X,d)$ if every open subset of $X$ is a union of a subfamily of the family $\{G_k\}_{k\in\Lambda}$..

**NOTE:**

If $G$ is an arbitrary nonempty open set and $x \in G$, then there exists a set $G_l$ in the family such that $x \in G_l \subseteq G$.

**Theorem 12.2.**. The collection $\{S(x,\varepsilon): x \in X,\ \varepsilon > 0\}$ of all open balls in $X$ is a base for the open sets of $X$.

**Proof.** Let $G$ be a nonempty open subset of $X$ and let $x \in G$.

Therefore there exists a positive $\varepsilon(x)$ such that $x \in S(x,\varepsilon) \subseteq G$.

Hence $\{S(x, \varepsilon): x \in X, \ \varepsilon > 0\}$ of all open balls in $X$ is a base for the open sets of $X$.

**Second Countable:** A metric space is said to be second countable if it has a countable base for its open sets.

**Example:**

Let $(\mathbb{R}, d)$ be the real line with the usual metric. The collection $\{(x, y) : x, y \, rational\}$ of all open intervals with rational endpoints form a countable base for the open sets of R.

**Open cover of set:** Let $(X, d)$ be a metric space and $G$ be a collection of open sets in $X$. If for each $x \in X$ there is a member $G_i \subseteq G$ such that $x \in G_i$, then $G$ is called an open cover of $X$.

**Subcover of set:** A subcollection of $G$ which is itself an open cover of $X$ is called a subcover (or subcovering).

**Example:**

Let $X$ be the discrete metric space consisting of the five elements $v, w, x, y, z$.

The union of the family of subsets $\{\{v\}, \{v, w\}, \{x, y\}, \{v, y, z\}\}$ is $X$ and all subsets are open. Therefore the family is an open cover. The family $\{\{w, x\}, \{x, y\}, \{v, y, z\}\}$ is a proper subcover

**Everywhere dense:** A subset $Y$ of a metric space $(X, d)$ is said to be everywhere dense or simply dense if $\bar{Y} = X$, i.e., if every point of $X$ is either a point or a limit point of $x_0$ or given any point $x$ of $X$, there exists a sequence of points of $Y$ that converges to $x$.

**Example:** The set of rationals $\mathbb{Q}$ is a dense subset of $\mathbb{R}$ (usual metric) and so is the set of irrationals.

**Seperable:** The metric space $X$ is said to be separable if there exists a countable, everywhere dense set in X.

In other words, $X$ is said to be separable if there exists in $X$ a sequence $\{x_1, x_2, x_3, \dots\}$ such that for every $x \in X$, some sequence in the range of $\{x_1, x_2, x_3, \dots\}$ converges to $x$

**Theorem 12.3. Let $(X, d)$ be a metric space and $Y \subseteq X$. If $X$ is separable, then $Y$ with the induced metric is separable.**

**Proof.** Let $A = \{x_i : i = 1, 2, \dots\}$ be a countable dense subset of $X$.

If $A$ is contained in $Y$, then no need to prove.

Let we establish a countable dense subset of $Y$ whose points are arbitrarily close to those of $X$. For $n, m \in$, let $S_{n,m} = S(x_n, \frac{1}{m})$ and choose $y_{n,m} \in S_{n,m} \cap Y$ whenever this set is nonempty. We show that the countable set $\{y_{n,m}: n, m \in \mathbb{N}^+\}$ of $Y$ is dense in $Y$.

For this purpose, let $y \in Y$ and $\varepsilon > 0$.

Let $m$ be so large that $\frac{1}{m} < \frac{\varepsilon}{2}$ and find $x_n \in S\left(y, \frac{1}{m}\right)$.

Then $y \in S_{n,m} \cap Y$ and

$$d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $y_{n,m} \in S(y, \varepsilon)$.

Because $y \in Y$ and $\varepsilon$ are arbitrary, the theorem is proved.

## *12.4 SUBSPACES AND PRODUCT OF TWO METRIC SPACE*

**Subspace:** Let $(X, d)$ is a metric space and $X_1$ is a nonempty subset of $X$. The restriction $d'$ of $d : X_1 \to X_1$ is a metric for $Y$, as it clearly satisfies the all the axioms of metric space so $(X_1, d')$ is a metric space. i.e. metric space is called a subspace of $X$ or of $(X, d)$ and the restriction $d'$ is known as the metric induced by $d$ on $Y$.

**Hereditary Property:** A property of a metric space is said to be hereditary iff every subspace of that space has that property

**Theorem 12.4. Let $(X, d)$ be a metric space and $(Y, d^*)$ be a subspace. A subset $A^*$ of $Y$ is $d^*$-open iff there exists a $d$ −open subset $A$ of $X$ such that $A^* = A \cap Y$.**

**Proof.** Let $d^*$-open sphere in $Y$ with centre $y_0$ and radius $r$ denoted by

$S^*(y_0, r) = \{y \in Y: d^*(y, y_0) < r\}$

We can see that

$S^*(y_0, r) = Y \cap S(y_0, r)$ where $S(y_0, r)$ is $d$-open sphere in $X$ with centre $y_0$ and radius $r$.

Let $A^*$ be any $d^*$-open of Y.

Then to each $y \in B^*$, there exists a $d^*$-open sphere $S^*(y_0, r(y)) \subseteq A^*$.

Now

$A^* = \cup \{ S^*(y_0, r(y)): y \in A^*\}$

$= \cup \{ Y \cap S(y_0, r(y)): y \in A^*\}$

Using distributive law, we get

$A^* = Y \cap (\cup \{ S(y_0, r(y)): y \in A^*\})$

$= Y \cap A$ where $A = \cup \{ S(y_0, r(y)): y \in A^*\}$

We can see that $A$ is $d$ −open set as it is union of $d$ −open spheres.

*Converse*

Let $A^* = A \cap Y$ where $A$ is $d$ −open set.

Let $y$ be any point of $A^*$.

As $A^* = A \cap Y$, hence $y \in A$.

It is given that $A$ is open set.

Hence there exists an open sphere $S\big(y, r(y)\big) \subseteq A$.

$\Rightarrow S^*\big(y, r(y)\big) = Y \cap S\big(y, r(y)\big) \subseteq Y \cap A = A^*$.

Hence we see that for each $y \in A^*$, there exists $d^*$ $-$open sphere centred at $y$ and contained in $A^*$.

Therefore $A^*$ of $Y$ is $d^*$-open.

**Product of two metric space:**

**Theorem 12.5. Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces. For any pair of points $x = (x_1, x_2) and\ y = (y_1, y_2)\ in\ X_1 \times X_2$ defined by $d(x, y) = \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)}$. Then $d$ is a metric for $X_1 \times X_2$.**

**Proof.** It is given that if $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$

$$d(x, y) = \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)}$$

Easily we can see that $d(x, y) \geq 0\ for\ all\ x, y \in X_1 \times X_2$.

Hence $(M1)$ condition is satisfied

$$d(x, y) = 0 \Leftrightarrow \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)} = 0$$

$$\Leftrightarrow d_1^2(x_1, y_1) + d_2^2(x_2, y_2) = 0$$

$\Leftrightarrow d_1^2(x_1, y_1) = 0$ and $d_2^2(x_2, y_2) = 0$

$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$

$$\Leftrightarrow x = y$$

Therefore $(M2)$ condition is satisfied

$$\begin{aligned} d(x, y) &= \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)} \\ &= \sqrt{d_1^2(y_1, x_1) + d_2^2(y_2, x_2)} \\ &= d(y, x) \end{aligned}$$

Therefore $(M3)$ condition is satisfied

Let $z \in X_1 \times X_2$ such that $z = (z_1, z_2)$

$$d(x,y) = \sqrt{d_1^2(x_1,y_1) + d_2^2(x_2,y_2)}$$

Using triangle inequality of metric $d_1$ and $d_2$, we get

$$d(x,y) \leq \sqrt{\left(d_1(x_1,z_1) + (z_1,y_1)\right)^2 + \left(d_2(x_2,z_2) + (z_2,y_2)\right)^2}$$

$$\leq \sqrt{d_1^2(x_1,z_1) + d_2^2(x_2,z_2)} + \sqrt{d_2^2(z_1,y_1) + d_2^2(z_2,y_2)}$$

(Using Minkowski's inequality)

Therefore

$d(x,y) \leq \; d(x,z) + d(z,y)$ , satisfied $(M4)$ condition.

Thus $d$ is a metric for $X_1 \times X_2$.

**CHECK YOUR PROGRESS**

(CQ 1) Define Subspace and bases
(CQ 2) Let $(X,d)$ be a metric space and $(Y,d^*)$ be a subspace. Prove that a subset $A^*$ of $Y$ is $d^*$-open iff there exists a $d$ −open subset $A$ of $X$ such that $A^* = A \cap Y$.

## *12.5 COMPLETENESS*

**Sequence:** Let $(X,d)$ be a metric space. A sequence of points in $X$ is a function $f$ from $\mathbb{N}$ into $\mathbb{N}$. It is denoted by $\{x_n\}$.

**Limit of a sequence:** Let $d$ be a metric on a set $X$ and $\{x_n\}$ be a sequence in the set $X$. An element $x \in X$ is said to be a limit of $\{x_n\}$ if, for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ whenever $n \geq m$ .

We also say that $\{x_n\}$ converges to $x$, and write it in symbols as $x_n \to x$.

If there is no such $x$, we say that the sequence diverges.

A sequence is said to be convergent if it converges to some limit point, and otherwise divergent.

**Example:** Let $X = \mathbb{R}$ with $d(x, y) = |x - y| for\ all\ \ x, y \ \in \mathbb{R}$.

Let $\{x_n\}$ be a sequence of real numbers. It converges to $x \ \in \mathbb{R}$ in $(X, d)$ iff $\lim_{n\to\infty} d(x_n, x) \ = \lim_{n\to\infty} |x_n - \ x| = 0$.

**Theorem 12.6. If $\boldsymbol{A}$ is a subset of the metric space $(\boldsymbol{X}, \boldsymbol{d})$, then $\boldsymbol{d}(\boldsymbol{A}) = \ \boldsymbol{d}(\overline{\boldsymbol{A}})$.**

**Proof.** If $x, y \ \in \ A$, then there exist sequences $\{x_n\}$ and $\{x_n\}$ in $A$ such that $d(x, x_n) < \frac{\varepsilon}{2}$ and $d(y, y_n) < \ \frac{\varepsilon}{2}$ for $n \geq m$, where $\varepsilon \ > \ 0$ is arbitrary.

Now for $n \geq m$, we have

$$d(x, y) \leq \ d(x, x_n) + \ d(x_n, y_n) + \ d(y_n, y)$$

$$< \ \frac{\varepsilon}{2} + \ d(x_n, y_n) + \ \frac{\varepsilon}{2}$$

$$< \ \varepsilon + \ d(A)$$

$\Rightarrow d(\bar{A}) \leq \ d(A)$……………………(1)

As $\varepsilon \ > \ 0$ is arbitrary.

Hence $d(\bar{A}) \geq \ d(A)$……………………(2)

From inequalities (1) and (2), we get

$$d(\bar{A}) = \ d(A)$$

**Theorem 12.7. Let $(\boldsymbol{X}, \boldsymbol{d})$ be a metric space and if $\boldsymbol{x}$ is a limit point of subset $\boldsymbol{Y}$ of $\boldsymbol{X}$, then there exists a sequence $\{\boldsymbol{x_n}\}$, contains all distincts point of $\boldsymbol{Y}$ from $\boldsymbol{x_0}$ converges to $\boldsymbol{x_0}$.**

**Proof.** It is given that $x$ is a limit point of subset $Y$ of $X$

Hence every sphere with centre x must contain atleast a point of Y distinct from x.

Assume any point $x_1 \neq x \in Y$ and $r_1$ such that $r_1 = min\,\{1, d(x, x_1).\}$

Now the sphere $S(x, r_1)$ contains a point $x_2 \neq x \in S(x, r_1)$.

Again let $r_2 = min\,\{\frac{1}{2}, d(x, x_2).\}$

Similarly the sphere $S(x, r_2)$ contains a point $x_3 \neq x \in S(x, r_2)$ and let $r_3 = min\,\{\frac{1}{2}, d(x, x_2)\}$

Continue above process indefinitely.

Hence we construct a sequence of $\{x_n\}$, distinct point from $x$ such that if $r_n = \left\{\frac{1}{n}, S(x, x_n)\right\}$, the sphere $S(x, x_n)$ the point $x_{n+1} \neq x \in Y$.

Thus $d(x, x_n) < r_{n-1} \leq \frac{1}{n-1}$.

It implies that $S(x, x_n) \to 0\ as\ n \to \infty$.

Therefore $\{x_n\}$ converges to $x$.

**Cauchy sequence: .** Let $d$ be a metric on a set $X$. A sequence $\{x_n\}$ in the set $X$ is said to

be a Cauchy sequence if, for every $\varepsilon >\ 0$, there exists $n_0\ \in \mathbb{N}$ such that

$d(x_n, x_m) <\ \varepsilon\ whenever\ n, m \geq\ n_0$

**Example** The sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, does not satisfy Cauchy's criterion of convergence.

Indeed,

$$|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\leq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$= \frac{n}{2n} = \frac{1}{2}$$

So, $|x_n - x_m|\ is\ not\ tends\ to\ 0$.

**Ex 12.1. Let $C[0,1]$ be a set of continuous function, the sequence $f_1, f_2, f_3, \ldots..$ given by $f_n(x) = \frac{nx}{n+x}$ and metric $d$ is defined as $d: C[0,1] \times C[0,1] \to \mathbb{R}$ such that $d(x,y) = sup\{|x-y|: x \in [0,1]\}$. Prove that the sequence $f_n$ is cauchy sequence.**

**Proof.** For $m \geq n$

$f_m(x) - f_n(x) = \frac{mx}{m+x} - \frac{nx}{n+x} = \frac{mnx+mx^2-mnx-nx^2}{(m+x)(n+x)} = \frac{(m-n)x^2}{(m+x)(n+x)}$, which is continuous in [0,1], let its maximum at some points $x_0 \in [0,1]$.

Therefore,

$d(f_m, f_n) = sup\ \{|f_m(x) - f_n(x)|: x \in [0,1]\} = \frac{(m-n){x_0}^2}{(m+x_0)(n+x_0)} \leq \frac{x_0^2}{n+x_0} \leq \frac{1}{n} \to 0$ for large $m$ and $n$.

Moreover, the sequence $\{f_n\}$ converges to some limit.

For $f(x) = x$, $|f_n(x) - f(x)| = \left|\frac{nx}{n+x} - x\right| = \frac{x^2}{n+x} \leq \frac{1}{n} \to 0 \quad as\ n \to \infty$

Therefore, $\{f_n\}$ converges to the limit f, where $f\ (x) = x$ for all $x \in [0,1]$.

**Theorem 12.8. A convergent sequence in a metric space is a Cauchy sequence.**

**Proof.** Let $\{x_n\}$ be a sequence in a set $X$ with metric $d$.

Let $x$ be an element of $X$ such that $\lim_{n\to\infty} x_n = x$.

Given any $\varepsilon > 0$, there exists some natural number $m$ such that

$d(x_n, x) < \frac{\varepsilon}{2} = 2\ whenever\ n \geq m.$

Assume any natural numbers $n$ and $n'$ such that $n \geq m$ and $n' \geq m$.

Then $d(x_n, x) < \frac{\varepsilon}{2}$ and $d(x_{n'}, x) < \frac{\varepsilon}{2}$.

Hence $d(x_n, x_{n'}) \leq d(x_n, x) + d(x_{n'}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

**Complete metric space:** A metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ is convergent.

**Example:**

$d(x, y) = |x - y|\ for\ x, y\ \in\ \mathbb{R};$ is complete metric space.

$d(z, w) = |z_1 - z_2|\ for\ z_1 - z_2\ \in\ \mathbb{C}$ is complete metric space.

$d(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$ and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is complete metric space.

**Subsequence:** Let $\{x_n\}$ be a given sequence in a metric space $(X, d)$ and let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$.

**Subsequential limit:** If $\{x_{n_k}\}$ converges, its limit is called a subsequential limit of $\{x_n\}$.

**NOTE:** A sequence $\{x_n\}$ in $X$ converges to x if and only if every subsequence of it converges to $x$.

**Theorem 12.9. If a Cauchy sequence of points in a metric space $(\boldsymbol{X}, \boldsymbol{d})$ contains a convergent subsequence, then the sequence converges to the same limit as the subsequence.**

**Proof.** Let $\{x_n\}$ be a Cauchy sequence in $(X, d)$.

Then for every positive number $\varepsilon$ there exists an integer $m(\varepsilon)$ such that

$d(x_n, x_{n'}) < \varepsilon\ whenever\ n, n' \geq m(\varepsilon))$.

Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ and its limit by $x$.

It implies that $d(x_{n_{n'}}, x_n) < \varepsilon\ whenever\ n, n' \geq m(\varepsilon))$.

As $\{n_k\}$ is a strictly increasing sequence of positive integers.

Now, $d(x, x_n) \leq d\left(x, x_{n_{n'}}\right) + d\left(x_{n_{n'}}, x_n\right)$

$$< d\left(x, x_{n_{n'}}\right) + \varepsilon\ whenever\ n, n' \geq m(\varepsilon)).$$

Taking $n' \to \infty$ we get

$d(x, x_n) < \varepsilon \; whenever \; n, n' \geq m(\varepsilon)$).

Therefore, the sequence $\{x_n\}$ converges to $x$.

**Ex. 12.2. Prove that a metric space $X = \mathbb{R}^n$ with the metric given by**

$$\boldsymbol{d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}, \qquad p \geq 1}$$

**where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$ is a complete metric space.**

**Proof.** Let $\{x^{(k)}\}, x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)$ be a Cauchy sequence in $(X, d)$,

i.e., $d_p(x^{(k)}, x^{(m)} \to 0 \; as \; k, m \to \infty$.

Then, for a given $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that

$(\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}} < \varepsilon \;\; for \; all \; k, m \geq n_0(\varepsilon)$
………………………………..(1)

Therefore, $\left|x_\lambda^{(k)} - x_\lambda^{(m)}\right| < \varepsilon \, for \; all \; k, m \geq n_0(\varepsilon) \; and \; k = 1, 2, \ldots, n$.

Using Cauchy's principle of convergence, we get

$x_\lambda^{(k)}$ converges to a limit $x_\lambda$ for each $k = 1, 2, \ldots, n$. i.e. $\lim_{n\to\infty} x_\lambda^{(k)} = x_\lambda$.

Let $x = (x_1, x_2, \ldots, x_n)$ and $k \geq n_0(\varepsilon)$

From (1), we conclude that

$\left(\sum_{i=1}^{n} \left|x_\lambda^{(k)} - x_\lambda^{(m)}{}_i\right|^p\right)^{\frac{1}{p}} < \varepsilon^p \; for \; all \; m \geq n_0(\varepsilon)$
……………………………….(2)

Taking $m \to \infty$ in inequality(2), we get

$\sum_{i=1}^{n} \left|x_\lambda^{(k)} - x_\lambda\right|^p < \varepsilon^p \; for \; all \; k \geq n_0(\varepsilon)$

Therefore $x^{(k)} \to x$ in $(X, d)$.

**Ex.12.3. Let $X = C[a, b]$ and $d(f, g) = \sup\{|f(x) - g(x)|: a \leq x \leq b\}$ be the metric. Then prove that $(X, d)$ is a complete metric space.**

**Proof.** Let $\{f_n\}$ be a Cauchy sequence in $C[a, b]$. \

Then for every $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that $m, n \geq n_0(\varepsilon)$

$\Rightarrow d(f_k, f_m) = \sup\{|f_k(x) - f_m(x)|\; such\; that\; a \leq x \leq b\} < \varepsilon$

Precisely, for every $x \in [a, b]$, the sequence $\{f_n\}$ is a Cauchy sequence of numbers.

Using Cauchy's principle of convergence, we have

$f_n(x) \to f(x) as\; n \to \infty$

Now we will prove that $f \in C[a, b]$ and that $\lim_{n\to\infty} d(f_n, f) = 0$.

As we know that

$|f_k(x) - f_m(x)| < \varepsilon\; for\; every\; x \in [a, b] such\; that\; m, n \geq n_0(\varepsilon)$

Taking $m \to \infty$, we get

$|f_k(x) - f(x)| < \varepsilon\; such\; that\; m, n \geq n_0(\varepsilon)\; and\; \; for\; all\; x \in [a, b]$.
…………..(1)

Let $\alpha \in [a, b]$ and $\rho > 0$, hence there exists an integer $k_0(\rho)$ such that

$|f_k(x) - f(x)| < \frac{\rho}{3}\quad for\; every\; x \in [a, b] and\; k \geq k_0(\rho)$

Choose $n \geq k_0(\rho)$, then

$|f_k(x) - f_n(x)| < \frac{\rho}{3}\quad for\; every\; x \in [a, b]$
……………………………………..(2)

It is given that $f_n$ is continuous, there exists $\delta > 0$ such that

$|f_k(x) - f_n(x_0)| < \frac{\rho}{3}\quad for\; every\; |x - x_0| < \delta$
…………………………………(3)

Now

$|f(x) - f(x_0)| = |f(x) - f_k(x) + f_k(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$

$\leq |f(x) - f_k(x)| + |f_k(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$

Using inequality (2) and (3), we get

$|f(x) - f(x_0)| < \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3} < \rho$ whenever $|x - x_0| < \delta$.

Hence, $f \in C[a, b]$. Also from inequality (1), we have

$\lim_{n\to\infty} d(f_n, f) = 0$

**Cantor's Intersection theorem**

**Theorem 12.10. Let $(X, d)$ be a metric space. Then $(X, d)$ is complete iff, for every nested sequence $\{A_n\}$ of nonempty closed subsets of $X$ $i.e. A_1 \supseteq A_2 \supseteq A_3 \ldots. \supseteq A_n \supseteq \cdots.$ *such that* $d(A_n) \to 0$ *as* $n \to \infty$ then the intersection $\cap_{n=1}^{\infty} F_n$ contains one and only one point.**

**Proof.** Let (X, d) be a complete metric space.
Assume for each positive integer $n$, $x_n$ be any point in $A_n$.
It is given that $A_n \supseteq A_{n+1} \supseteq A_{n+2}$ ....
Hence $x_n, x_{n+1}, x_{n+2}, \ldots \ldots \ldots$.. all lie in $A_n$.
As $\lim_{n\to\infty} d(A_n) = 0$
For any $\varepsilon > 0$, there exists some integer $m$ such that $d(A_m) < e$.
Now, $x_m, x_{m+1}, x_{m+2}, \ldots \ldots \ldots$.. all lie in $A_m$.
For $k, n \geq m$, we then have $d(x_k, x_n) \leq d(A_m) < \varepsilon$.
Hence the sequence $\{x_n\}$ is a Cauchy sequence in the complete metric space $X$.
⇒ It is convergent.
Let $x \in X$ be such that $\lim_{n\to\infty} x_n = x$.
Now for any given $n$, we have the sequence $x_n, x_{n+1}, \ldots \in A_n$.
Hence $x = \lim_{n\to\infty} x_n$ lie on $\overline{A_n}$. i.e $x \in A_n$ (because $A_n$ is closed)
Therefore $x \in \cap_{n=1}^{\infty} A_n$
Let $y \in X$ and $y \neq x$, then $d(y, x) = t > 0$.
There exists $n$ large enough so that $d(A_n) < t = d(y, x) \Rightarrow y \notin A_n$..
Hence, $y \notin \cap_{n=1}^{\infty} A_n$.
It implies that the intersection $\cap_{n=1}^{\infty} F_n$ contains one and only one point.

*Converse*

let $\{x_n\}$ be any Cauchy sequence in $X$.

For each $n \in \mathbb{N}$, let

$A_n = \overline{\{x_m : m \geq n\}}$:

Then $\{A_n\}$ is a nested sequence of closed sets

$\{x_n\}$ is a Cauchy sequence $\Rightarrow \lim_{n\to\infty} d(A_n) = 0$

Let $\cap_{n=1}^{\infty} A_n = \{x\}$

If $\varepsilon > 0$, then there exists a natural number $m$ such that $d(F_m) < \varepsilon$.

But $x \in A_m$ and hence $n \geq m$ implies $d(x_n, x) < \varepsilon$.

Therefore $(X, d)$ is a metric space.

**Theorem 12.11. If $Y$ is a nonempty subset of a metric space $(X, d)$ and Y is subspace of $X$. Then $(Y, d^*)$ is complete iff $Y$ is closed in $X$.**

**Proof.** Let $Y$ be a complete subspace of $X$.

Now we will prove that $Y$ is closed in $X$.

Let $c \in X$ be a cluster point of $Y$.

Then for every natural number $n$, the open sphere $S(c, \frac{1}{n})$ must contain a point $a_n$ of $Y$.

We can easily see that $\{a_n\}$ converges to $c$.

As we know every convergent sequence is a Cauchy sequence.

Therefore, $\{a_n\}$ is a Cauchy sequence in $Y$.

$Y$ is complete$\Rightarrow c \in Y$.

Therefore $Y$ contains all its limit points which implies $Y$ is closed.

Converse

Let $Y$ be closed

Now we will prove that $(Y, d^*)$ is complete.

Let $\{x_n\}$ be any Cauchy sequence in $X$.

It is given that $X$ is complete.

Therefore

$\{x_n\}$ must converges to a point $x$ which lie on $X$.

Now we try to prove that $x_0 \in Y$.

If the range set of x consists finite number of distinct points, then

$\{x_n\} = \{x_1, x_2, \ldots .x_n, x_0, x_0, \ldots .\} : n \; is \; finite$

If the range set of $x$ consists infinite number of distinct points, then

$x_0$ is limit point of range set of $x$.

(because in a metric space $(X, d)$ if $x$ is a limit point of subset $Y$, then there exits a sequence $\{x_n\}$, contains all distincts point of $Y$ from $x_0$ converges to $x_0$)

Hence $x_0$ is also limit point of of $Y$.

$Y$ is closed$\Rightarrow x_0 \in Y$.

Cauhy sequence $\{x_n\}$ converges to $x_0 \in Y$.

Hence $Y$ is complete metric space.

**CHECK YOUR PROGRESS**

(CQ 3) State and Prove Cantor's intersection theorem
(CQ 4) Give one example of incomplete metric space

## *12.6 BAIRE'S CATEGORY THEOREM*

**Nowhere dense:** Let $(X, d)$ be a metric space. A subset $Y$ of $X$ is said to be nowhere dense if $int(Y)$ is empty, i.e., $Y$ contains no interior point.

**Category I:** A subset $F$ of $X$ is said to be of category I if it is a countable union of nowhere dense subsets.
**Category II:** Subsets that are not of category I are said to be of category II.

## Baire's Category Theorem

**Theorem 12.12. Any complete metric space is of category II.**

**Proof.** Let $(X, d)$ be a complete metric space and

$X = \cup_{n=1}^{\infty} A_n$ where each of the $A_n$ is nowhere dense.

As each $A_n$ is nowhere dense, each $(\overline{A_n})^c$ is everywhere dense.

Hence there atleast one point in in each of these sets $(\overline{A_n})^c$ .

Let $x_1 \in (\overline{A_1})^c$ .

Let $(\overline{A_1})^c$ is open, there exists $r > 0$ such that $S(x_1, r) \subseteq (\overline{A_1})^c$ .

For $\varepsilon_1 < r$, we have

$$\bar{S}\,(x_1, \varepsilon_1) \subseteq S(x_1, r) \subseteq (\overline{A_1})^c \subseteq {A_1}^c$$

$\Rightarrow \bar{S}\,(x_1, \varepsilon_1) \cap A_1 = \emptyset$

Using induction hypothesis, we get

there exist balls $S\,(x_m, \varepsilon_m)$ for $m = 1,2, \ldots \ldots . n-1$ such that

$\bar{S}\,(x_m\,, \varepsilon_m\,) \cap A_m \, where \; x_m \in (\overline{A_m})^c$ and

$\varepsilon_m \le \frac{1}{2}\varepsilon_{m-1} \, for \; k = 1\,2, \ldots, n-1.$

Similarly, we can construct the $n^{th}$ ball with the above properties.

Now we choose

$$x_n \in S(x_{n-1},, \varepsilon_m - 1) \cap (\overline{A_1})^c$$

Given element must exist,otherwise $S(x_{n-1},, \varepsilon_m - 1) \subseteq \overline{A_1}$

$\Rightarrow x_n \in int\,(\overline{A_1})$ , contradic the fact that $int\,(\overline{A_1})$ is empty.

Because the intersection $S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{A_1})^c$is open, there exists $\varepsilon > 0$ such that

$S\,(x_n, \varepsilon) \subseteq S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{A_1})^c$

Now we choose a positive $\varepsilon_n < \min\left\{\varepsilon, \frac{1}{2}\varepsilon_{n-1}\right\}$. Then

$\bar{S}(x_n, \varepsilon_n) \subseteq S\,(x_n, \varepsilon) \subseteq S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{A_n})^c$

It implies $\bar{S}(x_n, \varepsilon_n) \cap A_n = \emptyset$.

Also

$\varepsilon_n \le \frac{1}{2}\varepsilon_{n-1}$

Hence we can construct $n^{th}$ ball with the required properties has been constructed.

As $\bar{S}(x_n, \varepsilon_n) \subseteq \bar{S}(x_{n-1}, \varepsilon_{n-1})$, the balls $\bar{S}(x_n, \varepsilon_n)$form a nested sequence of nonempty closed balls in a complete metric space with diameters tending to 0.

As we know if $(X, d)$ be a metric space. Then $(X, d)$ is complete iff, for every nested sequence $\{F_n\}$ of nonempty closed subsets of $X \ and \ d(F_n) \to 0 \ as \ n \to \infty$

Then the intersection $\cap_{n=1}^{\infty} F_n$ contains one and only one point.

Hence there exists $x_0 \in \cap_{n=1}^{\infty} \bar{S}(x_n, \varepsilon_n)$

Now $\bar{S}(x_n, \varepsilon_n) \cap A_n = \emptyset$

⇒for every $n$, we have $x_0 \in A_n$ for any $n$,

i.e., $x_0 \in A_n^c \ for \ all \ n.$

Although, $\cap_{n=1}^{\infty} A_n^c = \emptyset$., acontradiction .

Hence $X$ is not of category I.

## *12.7 SUMMARY*

In this unit we discussed about bases, Cauchy sequence and completeness. We illustrate some examples of complete metric space.

## *12.8 GLOSSARY*

1. Space- a set with some added structure.
2. metric- a notion of distance between its elements
3. Completeness- no "points missing" from it

## *12.9 REFERENCES*

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## *12.10 SUGGESTED READINGS*

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## *12.11 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Define Subspace of metric space.

(TQ 2) Define bases and subspaces

(TQ 3) Prove Baier's Category theorem.

(TQ 4) Define Cauchy sequence and completeness.

(TQ 5) Prove that If a Cauchy sequence of points in a metric space $(X, d)$ contains a convergent subsequence, then the sequence converges to the same limit as the subsequence..

.**Fill in the blanks**

(TQ 6) The space $\mathbb{Q}$ of rational numbers, with the standard metric given by the absolute value of the difference, is ____________________.

(TQ 7) Let $(X, d)$ be a complete metric space. If A ⊆ X is a closed set, then $A$ is also _____________.

## *12.12 ANSWERS*

(TQ 6) not complete (TQ 7) complete

# UNIT 13: CONTINUOUS FUNCTION AND COMPACTNESS

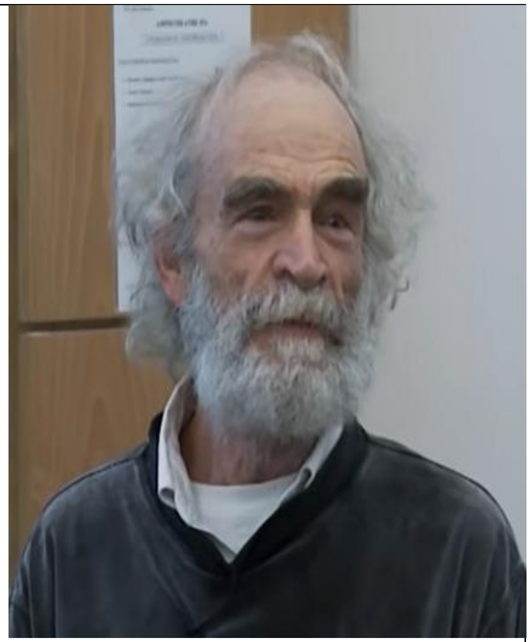
**CONTENTS**



## *13.1 INTRODUCTION*

The concept of compactness is an abstraction of a crucial quality held by some subsets of real numbers, which is of immense significance in the study of metric spaces or more generally in analysis. Every open cover of a closed and bounded subset of ℝ, according to the property in question, has a finite subcover and analysis is greatly affected by this.

In previous unit we studied completeness in metric space. In this unit we analyze continuous function and compactness.

<table>
<tr>
<td>Mikhael Leonidovich Gromov, a Russian-French mathematician best known for his work in geometry, analysis, and group theory, is also known by the names Mikhail Gromov, Michael Gromov, or Misha Gromov. He was born on December 23, 1943. He teaches mathematics at New York University and is a permanent member of France's Institut des Hautes Études Scientifiques.<br>The Gromov-Hausdorff metric, which was developed in 1981, gives the set of all metric spaces a metric space-like structure. In a broader sense, the choice of a point in each space can be used to determine the Gromov-Hausdorff distance between two metric spaces.</td>
<td><br>***Fig.13.1. Mikhael Leonidovich Gromov (Reference:https://en.wikipedia.org/wiki/Mikhael_Gromov_(mathematician)#/media/File:Mikhael_Gromov.jpg)***</td>
</tr>
</table>

## *13.2 OBJECTIVES*

In this Unit, we will

1. analyze about Compactness
2. Understand about Continuity
3. Prove some theorems based on compactness and continuity

## *13.3 CONNECTEDNESS*

**Seperated sets:** Let $(X, d)$ be a metric space. Two non empty subsets $X_1$ and $X_2$ are said to be **d-seperated if**

$$\overline{X_1} \cap X_2 = \emptyset \text{ and } X_1 \cap \overline{X_2} = \emptyset$$

**Example:**
Consider a metric space $(\mathbb{R}, d)$ and let $X_1 = (0,3), X_2 = (3,4)$ and $X_3 = [3,4]$. The sets $X_1$ and $X_2$ are separated because $\overline{X_1} = [0,3]$ and $\overline{X_2} = [3,4]$ and $\overline{X_1} \cap X_2 = \emptyset$ and $X_1 \cap \overline{X_2} = \emptyset$ but $X_1$ and $X_3$ are not separated.

$\boldsymbol{d}$ **−disconnected:** Let $(X, d)$ be a metric space. A subsets $Y$ of $X$ are said to be $d$ −disconnected if it is the union of two non empty d-seperated sets.
i.e. if there exists two set $W$ and $Z$ such that
$Y = W \cup Z$ and $\overline{W} \cap Z = \emptyset$ and $W \cap \bar{Z} = \emptyset$

$\boldsymbol{d}$ **−connected:** A set Y is said to be $d$ −connected if it is not $d$ −disconneted.

**Theorem 13.1. Let $(\boldsymbol{X}, \boldsymbol{d})$ be a metric space. Then the following statements are equivalent:**
**(i) $(\boldsymbol{X}, \boldsymbol{d})$ is disconnected;**
**(ii) there exist two nonempty disjoint subsets $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$, both open in $\boldsymbol{X}$, such that $\boldsymbol{X} = \boldsymbol{X_1} \cup \boldsymbol{X_2}$.**
**(iii) there exist two nonempty disjoint subsets $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ both closed in $\boldsymbol{X}$, such that $\boldsymbol{X} = \boldsymbol{X_1} \cup \boldsymbol{X_2}$**
**(iv) there exists a proper subset of $\boldsymbol{X}$ that is both open and closed in $\boldsymbol{X}$.**

**Proof.** (i)⇒(ii)
Let $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are nonempty .
$\overline{X_1} \cap X_2 = \emptyset$ and $X_1 \cap \overline{X_2} = \emptyset$.

Then $X_1 = X - \overline{X_2}$.
Although, $X_1 \subseteq X - \overline{X_2} \subseteq X - X_2 = X_1$.
Thus $X_1$ is open in X.
Similarly, $X_1$ is open in X.
Because $X_1$ and $\overline{X_2}$are disjoint,
⇒$X_1$ and $X_2$ are disjoint, which proves (ii) statement.
(ii) and (iii) are equivalent is trivial.

(iii)⇒(iv)

$X_1 = X - X_2$ is open. Hence $A$ is closed as well as open proper subset of $X$, hence (iv) condition is proved.

(iv)⇒(i)

Let $X_1$ be a proper subset of $X$ which is closed as well as open in X and let $X_2 = X - X_1$. Then $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$.

X_1 is closed implies $\overline{X_1} = X_1$

$\Rightarrow \overline{X_1} \cap X_2 = \emptyset$

Similarly, $X_1 \cap \overline{X_2} = \emptyset$.

hence (i) condition is proved.

**Theorem 13.2. Let $(X, d)$ be a metric space and let $X_1$ be a connected subset of $X$. If $X_2$ is a subset of $X$ such that $X_1 \subseteq X_2 \subseteq \overline{X_1}$, then $X_2$ is connected.**

**Proof.** We assume $X_2$ is disconnected.

Then there exists $Y_1, Y_2 \neq \emptyset$ such that

$Y_1 \cup Y_2 = X_2$ and $Y_1 \cap \overline{Y_2} = \emptyset$ and $\overline{Y_1} \cap Y_1 = \emptyset$.

It is given that $X_1 \subseteq X_2$, then $X_1 \subseteq Y_1 \cup Y_2$

We can see that connected set $X_1$ is contained in union of two separated set $Y_1$ and $Y_2$.

Also if $(X, d)$ be a metric space and $Y$ be connected subset of $X$ such that $Y \subseteq Y_1 \cup Y_2$ where $Y_1, Y_2$ is separated then either $Y \subseteq Y_1$or $Y \subseteq Y_2$.

Hence $X_1 \subseteq Y_1$or $X_1 \subseteq Y_2$.

Let $X_1 \subseteq Y_1$

$\Rightarrow \overline{X_1} \subseteq \overline{Y_1} \Rightarrow \overline{X_1} \cap Y_2 \subseteq \overline{Y_1} \cap Y_2 \Rightarrow \overline{X_1} \cap Y_2 \subseteq \emptyset$.

Since $\emptyset$ is a smallest subset

$\Rightarrow \overline{X_1} \cap Y_2 = \emptyset$..............................................(1)

Again $X_2 = Y_1 \cup Y_2$ and $X_2 = \overline{Y_1}$

$\Rightarrow Y_2 \subseteq X_2 \subseteq \overline{X_1}$

Therefore

$\overline{X_1} \cap Y_1 = Y_1$..............................................(2)

From (1) and (2), we get

$Y_2 = \emptyset$ , contradiction as $Y_2$ is non empty.

Thus $X_2$ is connected.

**Theorem 13.3. Closure of connected set is connected**

**Proof**. Let $(X, d)$ be a metric space and let $X_1$ be a connected subset of $X$.

Now we prove that $\overline{X_1}$ is connected.

In contrary, we assume $\overline{X_1}$ is disconnected.

Then there exists $Y_1, Y_2 \neq \emptyset$ such that

$Y_1 \cup Y_2 = \overline{X_1}$ and $Y_1 \cap \overline{Y_2} = \emptyset$ and $\overline{Y_1} \cap Y_1 = \emptyset$.

As we know that $X_1 \subseteq \overline{X_1}$

Thus $X_1 \subseteq Y_1 \cup Y_2$

if $(X, d)$ be a metric space and $Y$ be connected subset of $X$ such that $Y \subseteq Y_1 \cup Y_2$ where $Y_1, Y_2$ is separated then either $Y \subseteq Y_1$or $Y \subseteq Y_2$.

Hence $X_1 \subseteq Y_1$or $X_1 \subseteq Y_2$.

Let $X_1 \subseteq Y_1$

$\Rightarrow \overline{X_1} \subseteq \overline{Y_1} \Rightarrow \overline{X_1} \cap Y_2 \subseteq \overline{Y_1} \cap Y_2 \Rightarrow \overline{X_1} \cap Y_2 \subseteq \emptyset$.

Since $\emptyset$ is a smallest subset

$\Rightarrow \overline{X_1} \cap Y_2 = \emptyset$..............................................(1)

Again $\overline{X_1} = Y_1 \cup Y_2 \Rightarrow Y_2 \subseteq \overline{X_1}$

Therefore

$\overline{X_1} \cap Y_2 = Y_2$..............................................(2)

From (1) and (2), we get

$Y_2 = \emptyset$ , contradiction as $Y_2$ is non empty.

Thus $\overline{X_1}$ is connected.

**Locally connected at point:** Let $(X, d)$ be a metric space then it is locally connected at $x \in X$ iff every open neighbourhood of $x$ contains a connected open neighbourhood of $x$.
i.e. the collection of all connected open neighbourhood of $x$ forms a local base at $x$.

**Locally connected spaces: The metric space** $(X, d)$ is said to be locally connected if its locally connected at each of its points.

**Ex 13.1. Every discrete space $(X, d)$ is locally connected.**
**Sol.** Let $x$ be any point of $X$.
Therefore $\{x\}$ must be connected neighbourhood of $x$.
As we know every neighbourhood of $x$ must contain $\{x\}$.
Thus $(X, d)$ is locally conneceted.

**CHECK YOUR PROGRESS**

(CQ 1) Define disconnected set

ANSWER_____________________

(CQ 2) Give one example of separated set.

ANSWER______________________

## *13.4 COMPACT SET*

First we recall the definition of cover and subcover.

**Open cover of set:** Let $(X, d)$ be a metric space and $G$ be a collection of open sets in $X$. If for each $x \in X$ there is a member $G_i \subseteq G$ such that $x \in G_i$, then $G$ is called an open cover of $X$.

**Subcover of set:** A subcollection of $G$ which is itself an open cover of $X$ is called a subcover (or subcovering).

Now we define compact set as

**Compact Set:** A a metric space $(X, d)$ is said to be compact if every open covering $G$ of $X$ has a finite subcovering, i.e., there is a finite subcollection $\{G_1, G_2, \dots, G_n\} \in G$ such that $X = \cup_{i=1}^{\infty} G_i$.

**NOTE:**

- A nonempty subset $Y$ of $X$ is said to be compact if it is a compact metric space with the metric induced on it by $d$.
- A nonempty subset $Y$ is compact if every covering $G$ of $Y$ by relatively open sets of $Y$ has a finite subcovering.

**Example:**

- The interval $(0,1)$ in the metric space $(\mathbb{R}, d)$, where $d$ denotes the usual metric, is not compact. Now we will try to find an open covering such that given cover has no subcover. Consider the open covering $\left\{\left(\frac{1}{n}, 1\right) : n = 2, 3, \dots\right\}$ of (0,1) . We observed there is no subcover for open cover. Mathematically $\cup_{n=2}^{\infty} S\left(0,1 - \frac{1}{n}\right) \supseteq S(0,1)$. But no finite subcollection of $\left\{S\left(0,1 - \frac{1}{n}\right) : n = 2,3, \dots\dots\right\}$ covers open ball $S(0,1)$.
- Let $Y$ be a finite subset of a metric space $(X, d)$. Then $Y$ is compact.

**Theorem 13.4. Closed subsets of compact sets are compact.**

**Proof.** Let $Y$ be compact subset of metric space $X$.

Let $A \subseteq Y$ closed relative to Y and closed relative to $X$.

Now we will try to prove that $A$ is compact

Let $G = \{G_\lambda : \lambda \in \Lambda\}$ be an oper cover of A.

Then the collection

$M = \{G_\lambda : \lambda \in \Lambda\} \cup \{X - A\}$ forms an open cover of $Y$.

$Y$ is compact$\Rightarrow$there is a finite sub-collection $M^*$ of M which covers $Y$.

Therefore it also covers $A$.

If $X - A$ is a member of $M^*$, so we can remove it from $M^*$ and it still remain open cover of $A$.

Thus Finite subcollection of $G$ covers $A$.

Therefore $A$ is compact.

**Finite intersection property (F.I.P):** . A collection $F$ of sets in $X$ is said to have the finite intersection property if every finite subcollection of $F$ has a nonempty intersection.

**Theorem 13.5. Let $(\boldsymbol{Y}, \boldsymbol{d}^*)$ be a subspace of metric space $(\boldsymbol{X}, \boldsymbol{d})$. Prove that $\boldsymbol{Y}$ is compact w.r.t metric $\boldsymbol{d}^*$ iff $\boldsymbol{Y}$ is compact w.r.t metric $\boldsymbol{d}$ on $\boldsymbol{X}$.**

**Proof.** Let $F_\lambda$ is $d^* -$ open cover of $Y$.

$\Rightarrow Y \subseteq \cup_\lambda F_\lambda$.

Again $F_\lambda$ is $d^* -$ open cover

$\Rightarrow$there exists d-open $G_\lambda$ such that $F_\lambda = G_\lambda \cap Y \subseteq G_\lambda$

$\Rightarrow$there exists d-open $G_\lambda$ such that $\cup_\lambda F_\lambda \subseteq \cup_\lambda G_\lambda$

But $Y \subseteq \cup_i F_i$ and $Y \subseteq \cup_i G_i$

$\Rightarrow \{G_i\}$ is d-open cover of Y.

It is compact and therefore the cover $G_i$ must have finite reducible subcover.

Let $\left\{G_{\lambda_k}: k = 1,2,3, \ldots\right\}$ be subcover of $G_\lambda$.

$\Rightarrow Y \subseteq \cup_{\lambda=1}^{n} G_{\lambda_k}$

where

$Y \cap Y \subseteq Y \cap (\cap_{k=1}^{n} G_\lambda) = \cap_{k=1}^{n} \left(A \cap G_{\lambda_r}\right) = \cup F_{\lambda_k}$

$\Rightarrow Y \subseteq F_{\lambda_k}$ is a $d^* -$ open cover of A

$\Rightarrow$F is $d^*$ $-$compact.

*Converse*

Let $(Y, d^*)$ is a subspace of (X,d) and Y is $d^*$ $-$compact

Now we prove that Y is d-compact.

Let $G_\lambda$ is $d -$ open cover of $Y \Rightarrow Y \subseteq \cup_\lambda G_\lambda$.

Therefore $Y \cap Y \subseteq Y \cap (\cup_\lambda G_\lambda)$

It implies that $Y \subseteq \cup (Y \cap G_\lambda)$

Let $F_\lambda = G_\lambda \cap Y$ then $Y \subseteq \cup G_\lambda$

$\Rightarrow G_\lambda$ is $d$ $-$open $\Rightarrow G_\lambda = G_\lambda \cap Y$ is $d^*$ $-$open.

Therefore $F_\lambda$ is a $d^*$ $-$open cover of Y but $F_\lambda$ is $d^* -$ compact.

Hence given cover is reducible to finite subcover. i.e. $\{F_{\lambda_k}: 1 \leq k \leq n\}$
$\Rightarrow Y \subseteq \cup_{k=1}^{n} F_{\lambda_k} = \cup_{k=1}^{n} (F_{\lambda_k} \cap Y)$
$\Rightarrow Y \subseteq \cup_{k=1}^{n} \left(G_{\lambda_k} \cap Y\right) = \cup_{k=1}^{n} G_{\lambda_k}$
$\Rightarrow Y \subseteq \cup_{k=1}^{n} G_{\lambda}$
$\Rightarrow G_{\lambda}$ is finite subcover of the cover $G_{\lambda}$.
Therefore $Y$ is $d$ $-$compact.

**Bolzanno Weierstrass property (BWP):** A space $X$ is said to have Bolzanno weierstrass property(BWP) if every finite set in $X$ has a limit point.

**NOTE:**
A space with BWP is also said to be Frechet compact.

**Theorem 13.6. Let $(X, d)$ be a metric space and $Y$ a subset of $X$. If $Y$ is a compact subset of $(X, d)$, then $Y$ is closed and bounded.**
**Proof.** Let Y be a compact subset of $(X, d)$ and $y \in Y$, $x \in Y^c$. . For some positive real number $\varepsilon(y)$ such that $\varepsilon(y) < \frac{1}{2} d(x, y)$), there exist open balls $S\left(y, \varepsilon(y)\right)$ and $S(x, \varepsilon(y))$ with centres at $y$ and $x$, respectively, such that
$S\left(y, \varepsilon(y)\right) \cap S\left(x, \varepsilon(y)\right) = \emptyset$.
Also $Y \subseteq \cup_{y \in Y} S\left(y, \varepsilon(y)\right)$
It is given that $Y$ is compact,
Hence there exist $y_1, y_2, \dots, y_n$ such that
$Y \subseteq \cup_{y \in Y} S\left(y_i, \varepsilon(y_i)\right)$.
For each of the $y_i, i = 1, 2, \dots, n$, the open balls $S\left(x, \varepsilon(y_i)\right)$ satisfy
$S\left(y_i, \varepsilon(y_i)\right) \cap S\left(x, \varepsilon(y_i)\right) = \emptyset$.
Let $Z = \cap_{i=1}^{n} S(x, \varepsilon(y_i))$. T
Therefore $Z$ is an open subset of $X$ containing $\in Y^c$. .
Now we will try to prove that $Y \cap Z = \emptyset$.
Let $t \in Y \cap Z$, then $t \in S\left(y_k, \varepsilon(y_k)\right)$ for some $k$ in the set $\{1, 2, \dots, n\}$ and $t \in S\left(x, \varepsilon(y_k)\right)$.
Hence, $S\left(y_k, \varepsilon(y_k)\right) \cap S\left(x, \varepsilon(y_k)\right) = \emptyset$ andcontradicts the way we choose $\varepsilon(y_k)$.

Therefore, no point of $Y^c$ can be a limit point of $Y$.
Thus, all the limit points of $Y$ belong to $Y$, i.e., $Y$ is closed.
Now we will show that $Y$ is bounded.
Let $Y$ be not bounded, then there exist $x$ and $y$ in $Y$ such that for any for some positive real number $K, d(x,y) > K$.
Assume the open balls centred at the points of $Y$, each of radius 1.
Hence,
$Y \subseteq \cup_{y\in Y} \ S(y,1)$
$Y$ is compact⇒ there exist $y_1, y_2, \dots, y_n$ such that
$Y \subseteq \cup_{i=1}^{n} S(y_i,1)$
Let $k = max\,\{d(y_i, y_k): i,k = 1,2,\dots,n\}$.
There exist $x$ and $y$ in $Y$ such that
$d(x,y) > k+2$
As $x, y \in Y \Rightarrow$ there exist $y_i$ and $y_k$ such that
$x \in S(y_i,1)$ and $y \in S(y_k,1)$
Therefore,
$d(x,y) \leq d(x,y_i) + d(y_k,y_k) + d(y_k,y) < k + 2$, contradict our assumption for $x \ and \ y$.
Hence, $Y$ is bounded.

**Theorem 13.7. Let $(\mathbb{R}, d)$ be a metric space and $Y$ a subset of $\mathbb{R}$. Then $Y$ is a compact subset of $(X,d)$ $iff$ $Y$ is closed and bounded.**
**Proof.** Let $Y$ be a closed and boundedt subset of $(X,d)$.
$Y$ is a bounded⇒ there exist points $x, y \in \mathbb{R}$ such that $x < y$ and $Y \subseteq [x,y]$.
As we know that every closed and bounded interval on $\mathbb{R}$ is compact.
Hence $[x,y]$ is compact.
Again, $Y$ is closed in $\mathbb{R} \Rightarrow Y = Y \cap [x,y]$ is closed
Hence $Y$ is closed in $[x,y]$.
Now we know that closed subsets of compact sets are compact.
Hence $Y$ is compact.
*Converse*
Let $Y$ be compact. Now we will prove that $Y$ is closed and bounded.
Let $G = \{G_x : x \in Y\}\ where\ G_x = (x-1, x+1)$ be an open cover of Y.
$Y$ be compact
⇒there exists finite number of points $x_1, x_2, x_3, \dots. x_n \in Y$ such that

$Y \subseteq G_{x_1} \cup G_{x_2} \cup G_{x_3} \cup \dots . \cup G_{x_n}$
Let $K = max\{x_1, x_2, x_3, \dots . x_n\}$ and $k = min\ \{x_1, x_2, x_3, \dots . x_n\}$, then
$G_{x_1} \cup G_{x_2} \cup G_{x_3} \cup \dots . \cup G_{x_n} \subseteq [k-1, K+1] \Rightarrow Y \subseteq [k-1, K+1]$
Hence Y is bounded.
Again $Y$ be compact subset of $(\mathbb{R}, d)$ and we know that Every compact subset $Y$ of metric space $(X, d)$ is closed
$Y$ is closed.

**NOTE:**
The converse of the above theorem need not be true.
Let $X$ be an infinite set with the discrete metric d such that

$$d(x,y) = \begin{cases} 0\ if\ x = y \\ 1\ if\ x \neq y \end{cases}$$

We can easily see that the open ball $S(x, \frac{1}{2})$ is the set $\{x\}$ contain only $x$ and $d(x,y) \leq 1, \quad for\ all\ x, y\ in\ X \Rightarrow$each subset of $X$ is both closed and bounded. Therefore the open cover $\{\{x\}: x \in X\}$ has no finite subcover. $X$ is not compact.

**CHECK YOUR PROGRESS**

(CQ 3) Let $(\mathbb{R}, d)$ be a metric space and $Y$ a subset of $\mathbb{R}$. Then $Y$ is a compact subset of $(X, d) iff\ Y$ is closed and bounded. (T/F)
(CQ 4) Closed subsets of compact sets need not be compact. (T/F)
(CQ 5) A space $X$ is said to have _______ if every finite set in $X$ has a limit point.
(CQ 6) A subcollection of $G$ which is itself an open cover of $X$ is called a ________.

## *13.5 ε –NET AND TOTALLY BOUNDED*

**Countably compact spaces:** A metric space $X$ is said to be countably compact if every countable open cover of $X$ has a finite subcover.

**Sequentially compact spaces:** A metric space $(X, d)$ is known as sequentially compact if every sequence on X has a convergent subsequence.

**Theorem 13.8. A metric space $X$ is sequentially compact iff it has a B.W.P.**

**Proof.** Let $X$ be sequentially compact metric space.
Let $A$ be any infinite subset of $X$.
Let $\{x_n\}$ be any sequence distinct points of $A$.
As $X$ is sequentially compact
$\Rightarrow \{x_n\}$ contains convergent subsequence $\{x_{i_n}\}$ whose limit is $x$.
It implies that $x$ is a limit point of $A$.
Therefore $X$ has BWP.
Converse
Let $X$ has BWP.
Let $\{x_n\}$ be any sequence in $X$.
Let the range set be $A = \{x_1, x_2, \dots, x_n\}$ is finite.
$\Rightarrow$one of the point $x_{k_0}$ such that $x_i = x_{k_0}$ for infinitely many $i \in \mathbb{N}$.
It follows that $\{x\_k_0, x_{k_0}, \dots, x_{k_0}\}$ is a subsequence of $\{x_n\}$ which converges to $x_{k_0}$ in $X$.
Let the range set of $A$ is infinite.
As we know $X$ has a BWP $\Rightarrow A$ has a limit point say $x$.
Hence we can easy say that $\{x_n\}$ has a subsequence which converges to $x$.
Therefore $X$ is sequentially compact.

***ε* –net:** Let $(X, d)$ be a metric space and $\varepsilon$ be an arbitrary positive number. Then a subset $A \subseteq X$ is said to be an $\varepsilon-$net for $X$, for any given $x \in X$, there exists a point $y \in A$ such that $d(x, y) < \varepsilon$, i.e., $A$ is an $\varepsilon-$net for $X$ if $X = \cup \{S(y, \varepsilon): y \in A\}$.

**Finite *ε* –net:** A finite subset of X that is an e-net for X is called a finite $\varepsilon-$ net for $X$.

**Lebesgue number for covers:** Let $(X, d)$ be a metric space and let $G = \{G_\lambda: \lambda \in \Lambda\}$ be an open cover of $X$. A real number $l > 0$ is said to be lebesgue number for $G$ iff every subset of $X$ with diameter less than $l$ is contained in atleast one of $G_\lambda$.

## Lebesgue covering lemma

**Theorem 13.9. Every open cover of sequentially compact metric space has a lebesgue number.**

**Proof.** Let $(X, d)$ be a sequentially compact metric space.

Assume $G = \{G_\lambda : \lambda \in \Lambda\}$ be an oper cover of $X$.

Now we will try to prove that $G$ has a lebesgue number.

In contrary, let $G$ has no lebesgue number.

Hence for any natural number $n$, there exists a subset $L_n$ of $X$ such that

$0 < \delta(L_n) < \frac{1}{n}$ and $L_n \nsubseteq G_\lambda \quad for\ all\ \lambda \in \boldsymbol{\Lambda}$**…………………………..(1)**

Now we choose a point $l_n \in L_n$ for each $n \in \mathbb{N}$ and let a sequence $\{l_n\}$.

As X is sequentially compact, it should have a subsequence $\{l_{k_1}, l_{k_2}, \dots \dots \dots\}$ converges to a point $l \in X$.

In view of the fact that $G$ is cover of $X$, there exists an open set $G_{\lambda_0} \in \boldsymbol{G}$ such that $l \in G_{\lambda_0}$.

As $G_{\lambda_0}$ is open, there exists an open sphere $S(l, \varepsilon)$ with centre $l$ and radius $\varepsilon$ such that

$S(l, \varepsilon) \subseteq G_{\lambda_0}$ **…………………………..(2)**

Now $\{l_{k_n}\}$ converges to $l$

$\Rightarrow$ there exists a natural number $m$ such that $k_n \geq m$ .

$\Rightarrow l_{k_n} = S\left(l, \frac{\varepsilon}{2}\right)$**…………………………..(3)**

Now we choose a natural number $k_0 \geq m$ such that

$\frac{1}{k_0} < \frac{\varepsilon}{2}$**……………………………………..(4)**

From condition (3), we conclude that

$l_{k_0} \in S\left(l, \frac{\varepsilon}{2}\right)$

Also $l_{k_0} \in L_{k_0}$

Therefore $l_{k_0} \in S\left(l, \frac{\varepsilon}{2}\right) \cap L_{k_0}$……………………..(5)

From conditions (1) and (4), we get

$0 < \delta\left(L_{k_0}\right) < \frac{1}{k_0} < \frac{\varepsilon}{2}$ . ……………………..(6)

From Condition (5) and (60, we conclude that $L_{k_0}$ is a set of diameter which is less than $\frac{\varepsilon}{2}$.

Thus $L_{k_0} \subseteq S\left(c, \frac{\varepsilon}{2}\right)$

$\Rightarrow L_{k_0} \subseteq S(c, \varepsilon)$

From conditions (2) and (7), we see that $L_{k_0} \subseteq G_{\lambda_0}$, a contradiction.

Hence $G$ has a lebesgue number.

**Totally bounded:** The metric space $(X, d)$ is said to be totally bounded if, for any $\boldsymbol{\varepsilon} > 0$, there exists a finite $\boldsymbol{\varepsilon}$ −net for $(X, d)$.

A nonempty subset Y of $X$ is said to be totally bounded if the subspace $Y$ is totally bounded.

**Example:**

A bounded interval in $\mathbb{R}$ is a totally bounded metric space. Let the endpoints of the interval be $a$ and $b$ $(a < b)$ and $\varepsilon$ be an arbitrary positive number. Take an integer $n > \frac{b-a}{\varepsilon}$ and divide the interval into $n$ equal subintervals each of length $\frac{b-a}{n}$.

The points

$\left\{a + \frac{(k-1)(b-a)}{n} :\ k = 2, \ldots, n\right\}$ contain the required $\varepsilon$-net for the interval with endpoints $a$ and $b$.

Let $x$ be any point in the interval. Then $a \leq x \leq b$.

Then there exists an integer $\lambda \in \{1, 2, \ldots, n\}$ such that

$$a + \frac{(\lambda-1)(b-a)}{n} \leq x \leq a \leq a + \frac{\lambda(b-a)}{n}$$

Accordingly, the distance of $x$ from each of the endpoints of the interval

$\left[a + \frac{(\lambda-1)(b-a)}{n}, a + \frac{\lambda(b-a)}{n}\right]$

is less than or equal to $\frac{b-a}{n}$, which is strictly less than $\varepsilon$ in view of the way in which $n$ has been selected.

⇒any set containing at least one endpoint of each of the preceding subintervals, $k = 1, 2, \dots, n$, forms an $\varepsilon$ –net, the collection of points constitute the set.

**Theorem 13.10. Every sequentially compact metric space $(X, d)$ is totally bounded.**

**Proof.** Let $X$ be not totally bounded.

Then there exists $\epsilon > 0$ such that $X$ has no $\epsilon - net$.

Therefore if $x_1 \in X$ then there should be exists a point $x_2 \in X$ such that $d(x_1, x_2) \geq \epsilon$ (otherwise $\{x_1\}$ will be an $\epsilon - net$for $X$)

Again there must exists $x_3 \in X$ such that $d(x_2, x_3) \geq \epsilon$. (otherwise $\{x_1, x_2\}$ will be an $\epsilon - net$for $X$).

Continuing this process, we get sequence $\{x_1, x_2, x_3, \dots .\}$ such that $d(x_i, x_j) \geq \epsilon \ for\ i \neq j$.

Hence sequence $\{x_n\}$ cannot contain any convergent subsequence.

Therefore $X$ is not sequentially compact.

**Theorem 13.11. A metric space $X$ is compact if and only if it is sequentially compact.**

**Proof.** Let $X$ be compact.

As we know that Compact space has Bolzano weierstrass property (BWP).

Then $X$ has a BWP.

As we know that A metric space $X$ is sequentially compact if it has a B.W.P.

Hence $X$ is sequentially compact.

*Converse*

Let $X$ is sequentially compact then by Lebesgue covering lemma the cover $G$ has a lebesgue number $l > 0$.

As we know that every sequentially compact metric space $(X, d)$ is totally bounded.

$X$ is totally bounded.

By the definition of total boundness for $\epsilon = \frac{1}{3}$, there exists an $\epsilon$ −net

$A = \{x_i, x_2, \dots., x_n\}$ where $x_i \in \mathbb{N}\ for\ i = 1,2, \dots, n$ such that
$X = \cup \{S(x_i, \epsilon): x_i \in A\}$ ……………………………………………..(1)
By the definition of Lebesue number for each $i$ there exists an $A_{\lambda_i} \in G$ such that
$S(x_i, \epsilon) \subseteq A_{\lambda_i}$ ……………………………………………..(2)
From condition (1) and (2), we conclude that
$X = \cup \{A_{\lambda_i}: i = 1,2, \dots, n\}$ .
Therefore $X$ is compact.

**Theorem 13.12. Every compact metric space is complete.**
**Proof.** Let $X$ be a compact metric space and let $\{x_n\}$ be a Cauchy sequence in $X$.
As we know that compact metric space is sequentially compact
Hence $X$ is sequentially compact
Therefore sequence $\{x_n\}$ has a subsequence $\{x_{n_m}\}$ converges to some $x$ in $X$.
Now we will try to prove that sequence $\{x_n\}$ also converge to $x$.
Assume $\varepsilon > 0$ be given.
As $\{x_n\}$ is Cauchy sequence $\Rightarrow$ there exists $n_0 \in \mathbb{N}$ such that
$d(x_k, x_n) < \frac{\varepsilon}{2}\ when\ k, n \geq n_0$
Again since $\{x_{n_m}\}$ converges to , there exists $m_0 \in \mathbb{N}$ such that
$d\left(x_{n_m}, x\right) < \frac{\varepsilon}{2}\ when\ \ m \geq m_0$
Now we choose $m_1 \geq\ m_0$ such that $n_{m_1} \geq n$.
Then by condition (1) and (2), we get
$d\left(x_n, x_{n_{m_1}}\right) < \frac{\varepsilon}{2}\ \ where\ n \geq n_{m_1}$ and
$d(x_n, x)\ < \frac{\varepsilon}{2}$ where $n \geq n_{m_1}$
Using triangle inequality property, we get

$$d(x_n, x) <= d(x_n, x_{n_{m_1}}) + d(x_{n_{m_1}}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For every $n \geq n_{k_1}$, we get

$$lim_{n\to\infty} x_n = x.$$

Therefore $X$ is complete.

**Theorem 13.13. A metric space is compact if and only if it is complete and totally bounded.**

**Proof**. Let $(X, d)$ be a compact metric space.
As we know that every compact space is complete
Hence $(X, d)$ is complete.
Also every compact space is sequentially compact.
Thus $(X, d)$ is space is sequentially compact
As every sequentially compact is totally bounded
Hence $(X, d)$ is totally bounded.
i.e. $(X, d)$ is complete and totally bounded.

*Converse*

Let $(X, d)$ be complete and totally bounded.
Now we fist prove that $(X, d)$ is sequentially compact.
Consider an arbitrary sequence $s = \{x_1, x_2, x_3, \dots, x_n, ..\}$ in $X$.
$(X, d)$ is totally bounded.
$\Rightarrow$ there exists an $\varepsilon$ $-$net for $X$ such that $\varepsilon > 0$.
Assume $\varepsilon = 1$, then if $S(y_i, 1)$ be the open sphere of radius 1.
Then $X \subseteq \cup_{i=1}^{n} S(y_i, 1),\ y_i \in X$.
Now one of the sphere say $S(y_1, 1)$ must contain a subsequence
$s_1 = \{x_1^{(1)}, x_2^{(1)}, \dots \dots, x_n^{(1)}, \dots .\}$ of sequence $\{s\}$.
The distance between any two points of $s_1$ is less than 2.
Similarly, for $\varepsilon = \frac{1}{2}$, we obtain a subsequence
$s_2 = \{x_1^{(2)}, x_2^{(2)}, \dots \dots, x_n^{(2)}, \dots .\}$ of sequence $\{s\}$ such that $x_i^{(2)} \in S\left(y_1, \frac{1}{2}\right), y_2 \in X$.
Also the distance between any two points of $s_2$ is less than 1.
Proceeding by induction, for $\varepsilon = \frac{1}{k}$, we get a subsequence
$s_k = \{x_1^{(k)}, x_2^{(k)}, \dots \dots, x_n^{(k)}, \dots .\}$ such that $x_i^{(k)} \in S\left(y, \frac{1}{k}\right), y \in X\ and\ i = 1,2, ..n..$
The distance between any two points of $s_k$ is less than $\frac{2}{k}$.
Now we try to prove that the diagonal sequence
$s_d = \left\{x_1^{(1)}, x_2^{(2)}, \dots ., x_n^{n}, \dots \dots\right\}$ is a Cauchy subsequence of sequence $\{s\}$.
Now we see that

$$x_n^{(n)} \in S\left(y_n, \frac{1}{n}\right) \subseteq S\left(y_{n-1}, \frac{1}{n-1}\right) \subseteq \cdots \subseteq S(y_1, 1) \dots\dots\dots\dots\dots\dots\dots(1)$$

For each $\varepsilon > 0$, we choose $m \in \mathbb{N}$, such that $\frac{2}{m} > \varepsilon$.

Thus for $k, n \geq m$, from condition (1), we conclude that

$\Rightarrow x_n^{(n)}, x_k^{(k)} \in S\left(a_m, \frac{1}{m}\right)$

$\Rightarrow d\left(x_n^{(n)}, x_k^{(k)}\right) < \frac{2}{m}$

$\Rightarrow d\left(x_n^{(n)}, x_k^{(k)}\right) < \varepsilon.$

Hence $s_d$ is a cauchy subsequence of $s$.

As $X$ is complete, hence $s_d$ converges to a point in $X$.

Thus every sequence in X has a subsequence which converges to a point in X.

Therefore $X$ is sequentially compact.

Every sequentially compact is compact space.

**CHECK YOUR PROGRESS**

(CQ 7) Define compact set

ANSWER______________________

(CQ 8) What do you understand by FIP.

ANSWER______________________

## *13.6 CONTINUITY*

**Limit at a point:** Let $(X, d)$ and $(Y, d^*)$ be two metric spaces and let $f: X \rightarrow Y$ be a function of $X$ into $Y$. Assume $l \in X$ and $l^* \in Y$ Then $f(x)$ tends to limit $l^*$ as x tends l if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$d^*(f(x), l^*) < \varepsilon$ whenever $0 < d(x, l) < \delta$.

**Continuity at a point:** Let $(X, d)$ and $(Y, d^*)$ be two metric spaces and let $f: X \rightarrow Y$ be a function of $X$ into $Y$. F is continuous at a point $l \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$d^*(f(x), f(l)) < \varepsilon$ whenever $0 < d(x, l) < \delta$.

**Continuous map:** A function $f$ of a metric space $(X, d)$ into another metric space $(Y, d^*)$ is said to be continuous if it is continuous at every point of $X$.

**Theorem 13.14. Let $(X, d)$ and $(Y, d^*)$ be metric spaces and $A \subseteq X$. A function $f : A \to Y$ is continuous at $l \in A$ if and only if whenever a sequence $\{x_n\}$ in $A$ converges to $l$, the sequence $\{f(x_n)\}$ converges to $f(l)$.**

**Proof.** Assume the function $f : A \to Y$ is continuous at $l \in X$.

Let $\{x_n\}$ be a sequence in $A$ converging to$l$

Now we will try to prove that $\{f(x_n)\}$ converges to $f(l)$.

Let $\varepsilon$ be any positive real number.

$f$ is continuous at $l$

$\Rightarrow$ there exists some $\delta > 0$ such that $l \in X$ and

$d^*(f(x), f(l)) < \varepsilon$ whenever $0 < d(x, l) < \delta$.

Now

$\lim_{n\to\infty} x_n = l$, there exists some $\in \mathbb{N}$ such that

$d(x, l) < \delta$ when $m \geq n$

Therefore

$d^*(f(x), f(l)) < \varepsilon$ when $m \geq n$

Hence $\lim_{n\to\infty} f(x_n) = f(l)$.

*Converse*

Assume that every sequence $\{x_n\}$ in $A$ converging to $l$ has the property $\lim_{n\to\infty} f(x_n) = f(l)$.

Now we will try to prove that that $f$ is continuous at $l$.

In contrary, let $f$ is not continuous at $l$.

Then there must exists $\varepsilon > 0$ for which no $\delta > 0$ can satisfy the condition for $l \in X$ that

$d^*(f(x), f(l)) < \varepsilon$ whenever $0 < d(x, l) < \delta$.

i.e. for every $\delta > 0$ there must exits $x \in X$ such that

$0 < d(x, l) < \delta \Rightarrow d^*(f(x), f(l)) \geq \varepsilon$ .

Now for every natural number $n$, the number $\frac{1}{n} > 0$.

Thus there exists $x_n \in A$ such that

$d(x_n, x) < 1/n$ but $d^*(f(x_n), f(l)) \geq \varepsilon$ .

i.e. the sequence $\{x_n\}$ converges to l but sequence $\{f(x_n)\}$ not converges to $f(l)$.

But this contradict our assumption that every sequence $\{x_n\}$ converges to $l$ has condition that $\lim_{n\to\infty} f(x_n) = f(l)$.
Hence $f$ is continuous at $l$.

**Theorem 13.15. A function $\boldsymbol{f}$ of a metric space $(\boldsymbol{X}, \boldsymbol{d})$ into a metric space $(\boldsymbol{Y}, \boldsymbol{d}^*)$ is continuous at a point $\boldsymbol{l} \in \boldsymbol{X}$ if and only if for every $\boldsymbol{\varepsilon} > \mathbf{0}$, there exists $\boldsymbol{\delta} > \mathbf{0}$such that $\boldsymbol{S}(\boldsymbol{a}, \boldsymbol{\delta}) \subseteq \boldsymbol{f}^{-1}(\boldsymbol{S}(\boldsymbol{f}(\boldsymbol{l}), \boldsymbol{\varepsilon}))$, where $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{r})$ denotes the open ball of radius $\boldsymbol{r}$ with centre $\boldsymbol{x}$.**

**Proof.** The mapping $f : X \to Y$ is continuous at $l \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$

$d^*(f(x), f(l)) < \varepsilon$ whenever $0 < d(x, l) < \delta$,

i.e., $x \in S(l, \delta)) \Rightarrow f(x) \in S(f(l), \varepsilon)$

Therefore $f\left(S(l, \delta)\right) \subseteq S(f(l), \varepsilon)$.

Hence $S(l, \delta) \subseteq f^{-1}\left(S(f(l), \varepsilon)\right)$

**Theorem 13.16. Let $(\boldsymbol{X}, \boldsymbol{d})$ and $(\boldsymbol{Y}, \boldsymbol{d}^*)$ be metric spaces and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a function. Then $\boldsymbol{f}$ is continuous on $\boldsymbol{X}$ iff $\boldsymbol{f}^{-1}(\boldsymbol{G})$ is open in $\boldsymbol{X}$ for all open subsets $\boldsymbol{G} \subseteq \boldsymbol{Y}$.**

**Proof.** Let $f$ is continuous on $X$ and consider $G$ be an open subset of $Y$.
Now we prove that $f^{-1}(G)$ is open in $X$.
As $\emptyset$ and $X$ are open, we assume that $f^{-1}(G) \neq \emptyset$ and $f^{-1}(G) \neq X$
Let $x \in f^{-1}(G) \Rightarrow f(x) \in G$.
Now $G$ is open $\Rightarrow$ there exists $\varepsilon > 0$ such that $S(f(x), \varepsilon) \subseteq G$.
$f$ is continuous at $x$,
From previous theorem, we conclude that
For given $\varepsilon$ there exists $\delta > 0$ such that

$S(x, \delta) \subseteq f^{-1}\left(S(f(x), \varepsilon)\right) \subseteq f^{-1}(G)$.

Hence, every point $x$ of $f^{-1}(G)$ is interior point,
Therefore $f^{-1}(G)$ is open in $X$.

*Converse*

Let $f^{-1}(G)$ is open in $X$ for all open subsets $G$ of $Y$.
Let $x \in X$. As we know that eeach open ball is open set set in metric space.
Hence for each $\varepsilon > 0$, the set $S(f(x), \varepsilon)$ is open in $X$.
$\Rightarrow f^{-1}(S(f(x), \varepsilon))$ is open in $X$.
Now $x \in f^{-1}(S(f(x), \varepsilon))$,

⇒exists $\delta > 0$ such that $S(x,\delta) \subseteq f^{-1}(S(f(x),\varepsilon))$
From previous theorem, we conclude that
$f$ is continuous at $x$.

**Arbitrary close:** A point $x \in X$ is said to be arbitrary close to $X_1 \subseteq X$ iff $x$ is a limit point of $X_1$.

**Theorem 13.17. A function f from a metric space $(X,d)$ into another space $(Y,d^*)$ is continuous iff for every subset $X_1$ of $X$,**
$\boldsymbol{f[\overline{X_1}] \subseteq \overline{f[X_1]}}$
**$f: X \to Y$ is continuous iff for any $x \in X$ and any x arbitrary close to $\overline{X_1}$ implies $f(X)$ ai arbitrary close to $f[X_1]$.**

**Proof.** Let $f$ be continuous.
As we know that $f[\overline{X_1}]$ is closed in $Y$.
As we know that if $f$ is continuous function from a metric space $(X,d)$ into another space $(Y,d^*)$, then inverse image under $f$ of every $d^*$ −closed set is $d$ −closed.
$f^{-1}\{f[\overline{X_1}]\}$ is closed in $X$.
Hence
$\overline{f^{-1}\{f[\overline{X_1}]\}} = f^{-1}\{f[\overline{X_1}]\}$
Now $f(X_1) \subseteq \overline{\boldsymbol{f[X_1]}}$
$\Rightarrow X_1 \subseteq f^{-1}[f(X_1)] \subseteq f^{-1}\{\overline{\boldsymbol{f[X_1]}}\}$
$\Rightarrow \overline{X_1} \subseteq \overline{f^{-1}[f(X_1)]} = f^{-1}\{f[\overline{X_1}]\}$
Hence $\boldsymbol{f[\overline{X_1}] \subseteq \overline{f[X_1]}}$
*Converse*
*Let* $f[\overline{X_1}] \subseteq \overline{f[X_1]}$
*For every subset* $X_1$ *of* $X$, *let* $Y_1$ *be any closed subset in* $Y$ .
*Hence* $\overline{Y_1} \subseteq Y$
*Now* $f^{-1}[Y_1]$ *is a sunbset of* $X$ *implies*
$\overline{f[f^{-1}[Y_1]]} \subseteq \overline{Y_1}$
Hence $\overline{f^{-1}[Y_1]} \subseteq f^{-1}[\overline{Y_1}] = f^{-1}[Y_1]$
But $f^{-1}[Y_1] \subseteq \overline{f^{-1}[Y_1]}$
Therefore $f^{-1}[Y_1] = \overline{f^{-1}[Y_1]}$
$f^{-1}[Y_1]$ is closed in $X$.

If $f$ is function from a metric space $(X, d)$ into another space $(Y, d^*)$, such that inverse image under $f$ of every $d^*$ −closed set is $d$ −closed then $f$ is continuous.
Thus $f$ is continuous.

## *13.7 HOMEOMORPHISM*

**Open Mapping:** Let $(X, d)$ and $(Y, d^*)$ be two metric space. A mapping $f: X \to Y$ is said to be open mapping if $f[G]\ is\ d^*$-open whenever $G$ is $d$ −open.

**Closed Mapping:** Let $(X, d)$ and $(Y, d^*)$ be two metric space. A mapping $f: X \to Y$ is said to be closed mapping if $f[F]\ is\ d^*$-open whenever $F$ is $d$ −closed.

**Bicontinuous mapping:** Let $(X, d)$ and $(Y, d^*)$ be two metric space. A mapping $f: X \to Y$ is said to be bicontinuous mapping if $f$ is open and continuous.

**Homeomorphism:** Let $(X, d)$ and $(Y, d^*)$ be two metric space. A mapping $f: X \to Y$ is said to be homeomorphism if
(i) $f$ is one-one and onto i.e. $f$ is bijective.
(ii) $f$ is continuous
(iii) $f^{-1}$ is continuous.

**Homeomorphic spaces:** A metric space $(X, d)$ is said to be homeomorphic to another space $(Y, d^*)$ if there exists homeomorphism $f: X \to Y$. It is denoted by $X \approx Y$.
$Y$ is said to be homomorphic image of $X$.

**Example:**
In complex variable theory the mapping $w = \frac{z-a}{1+a\bar{z}}$ where $0 < |a| < 1$ of the closed disc $|z| \leq 1$ onto the closed disc $|w| \leq 1$ is a homeomorphism.

**Topological Property:** Let $(X, d)$ and $(Y, d^*)$ be two metric space. Then the metric $d$ and $d^*$ is equivalent if the identity mapping i.e.

$I: X \to X:\ I(x) = x\ for\ all\ x\ in\ X$ is homemorphism.

**Theorem 13.18. Homeomorphism is an equivalence relation in the collection of all metric spaces**.

**Proof.** *Reflexive:* Let $(X, d)$ be any metric space

Let $I$ be identity map defined as

$I: (X, d) \to (X, d) such\ that\ I(x) = x\ for\ all\ x \in X$.

then easily we can say that $I$ is homeomorphism.

Thus $(X, d) \approx (X, d)$.

*Symmetry:* Let $(X, d) \approx (Y, d^*)$.

There exists homeomorphism $f$ defined as

$f: (X, d) \to (Y, d^*)$ .

Thus $f$ is continuous and one-one onto and $f^{-1}$is contionuous..

Now we prove that $f^{-1}$ is homeomorphism which defined as $f^{-1}: (Y, d^*) \to (X, d)$

(i) $f$ is one-one and onto$\Rightarrow f^{-1}$ is one-one and onto.

(ii) $f^{-1}$ is continuous

(iii) $(f^{-1})^{-1} = f$ is continuous.

Hence $f^{-1}$ is homeomorphism.

Thus $(Y, d^*) \approx (X, d)$.

*Transitivity:* Let $(X, d) \approx (Y, d^*)$ and $(Y, d^*) \approx (Z, d')$.

Let $f$ and $g$ be two homeomorphism defined as

$f: (X, d) \to (Y, d^*)$ and $g: (Y, d^*) \to (Z, d')$.

Now we will show that $(X, d) \approx (Z, d')$

Now we define a composite mapping $gof: (X, d) \to (Z, d')$.

(i) $f$ and $g$ are one-one onto$\Rightarrow gof$ is one-one onto

(ii) $f$ and $g$ are continuous$\Rightarrow gof$ is continuous.

(iii) $f^{-1}$ and $g^{-1}$ are continuous

$\Rightarrow f^{-1} o\ g^{-1}$ are continuous

$\Rightarrow (gof)^{-1}$ are continuous.

Hence $gof$ is homeomorphism$\Rightarrow (X, d) \approx (Z, d')$.

## 13.8 CONTINUITY AND CONNECTEDNESS AND COMPACTNESS

**Theorem 13.19. A continuous image of a connected space is connected.**

**Proof.** Let $(X, d)$ and $(X, d^*)$ be two metric spaces and $f$ be a continuous mapping of $X$ onto $Y$.
Let $X$ is connected, we prove that $Y$ is connected.
In contrary, assume that Y is disconnected.
Then there exists subset $Y_1 \neq \emptyset$ of $Y$ such that $Y_1$ is open and closed.
It is given that $f$ is open mapping and $Y_1$ is proper subset of $Y$.
$\Rightarrow f^{-1}[Y_1]$ is proper subset of $X$.
$f$ is continuous $\Rightarrow f^{-1}[Y_1]$ is open and closed.
$\Rightarrow (X, d)$ is discoonected, a contradiction.
Thus $Y$ is connected.

**NOTE:**
Connectedness is a topological property.

**Theorem 13.20. A continuous image of compact space is compact.**
**Proof.** Let $(X, d)$ and $(X, d^*)$ be two metric spaces and $f$ be a continuous mapping of $X$ onto $Y$.
We prove that $f[X]$ is compact i.e. closed and bounded.
Let $\{G_\lambda : \lambda \in \Lambda\}$ **be an open cover of** $f[X]$.
$f$ is continuous $\Rightarrow$ $\{f^{-1}(G_\lambda) : \lambda \in \Lambda\}$ **is an open cover of** $f[X]$.
$X$ is compact
$\Rightarrow$ there exist finitely many indices $\lambda_1, \lambda_2, \dots ., \lambda_n$ such that
$X = f^{-1}\left(G_{\lambda_1}\right) \cup f^{-1}(G_2) \dots \cup f^{-1}\left(G_{\lambda_n}\right)$
$\Rightarrow X = f^{-1}\left(G_{\lambda_1} \cup G_2 \dots \dots \cup G_{\lambda_n}\right)$
Hence
$f[X] = f\left[f^{-1}\left(G_{\lambda_1} \cup G_2 \dots \dots \cup G_{\lambda_n}\right)\right]$.
Thus $f[X]$ is compact or $f[X]$ is closed and bounded.

**NOTE:**
Compactness is a topological property.

## 13.9 SUMMARY

In this unit we discussed about compactness, connectedness and continuity. We illustrate some examples and proved some important theorems.

## 13.10 GLOSSARY

1. Space- a set with some added structure.
2. metric- a notion of distance between its elements
3. Completeness- no "points missing" from it
4. Compact- open covering has a finite subcovering
5. Connectedness- cannot be represented as the union of two or more disjoint

## 13.11 REFERENCES

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## *13.13 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Define Compact metric space

(TQ 2) Define locally connected

(TQ 3) Prove a continuous image of compact space is compact.
(TQ 4) Define disconnected metric space.
(TQ 5) Prove that closure of connected set is connected

.**Fill in the blanks**

(TQ 6) A continuous image of a connected space is________________..

(TQ 7) Every discrete space $(X, d)$ is ______________.

## *13.14 ANSWERS*

| | |
|---|---|
| (CQ 3) T | (CQ 4) F |
| (CQ 5) BWP | (CQ 6) subcover |
| (TQ 6) connected | (TQ 7) locally connected |

# UNIT 14: FIXED POINT THEORY

**CONTENTS**



## *14.1 INTRODUCTION*

A fixed point of a function is a point that the function mappings to itself and maintains as fixed. Let $f: X \rightarrow X$ be a single-valued mapping and $X$ be a non-empty set. When $y$ remains unchanged under the mapping $f$, a point is said to be a fixed point of $T$ if $fy = y$. A fixed point is, in terms of graphics, the intersection of the graph of the curve $y = fx$ with the line $y = x$.

For instance, if $f(x) = x^2 - 5x + 9$ defines a function $f$ on the real numbers, then 3 is a fixed point of $f$.

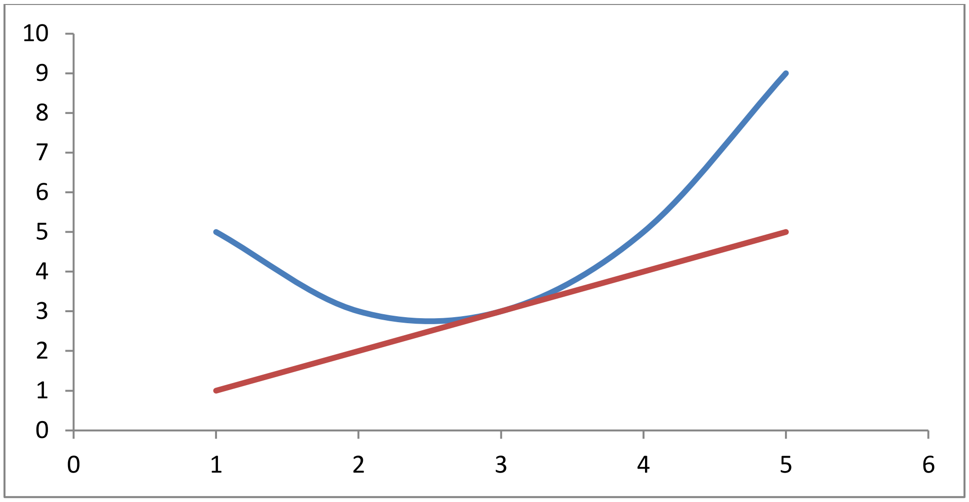


***Fig 14.1***

A mapping may have a single fixed point, many fixed points, an unlimited number of fixed points, or even no fixed points at all. The study of sufficient conditions on $X$ or $f$ that ensure that $f$ always has at least one fixed point is included in fixed point theory. Mappings (single or multi-valued) have solutions when certain conditions are met, according to fixed point theorems.

The Topological Fixed Point Theory and the Metric Fixed Point Theory are two subcategories of fixed point theory. However, because of the proving strategies used, the two classes are not truly mutually exclusive. While the latter mainly entails the study of fixed points depending on the mapping conditions and the spaces under examination, the former primarily involves the research of spaces with the fixed point property.

In previous unit we analyze compactness and continuity in metric space. In this unit we define fixed point in metric space and proved some important theorems based on fixed point theory.

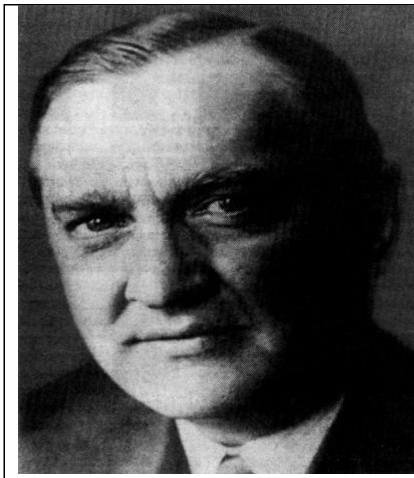

***Fig 14.2. Stephan Banach (Source: https://the-genius-of-autism.fandom.com/wiki/Stefan_Banach?file=Stefan-banach.jpg)***

Stefan Banach was a Polish mathematician who lived from 30 March 1892 to 31 August 1945. He is regarded as one of the most significant and influential mathematicians of the 20th century. He was a founding member of the Lwów School of Mathematics and the creator of contemporary functional analysis.

The idea of a complete normed vector space was formally axiomatized by Banach in his dissertation, which was finished in 1920 and published in 1922. It also lay the groundwork for the field of functional analysis.

His dissertation contained Banach's fixed point theorem, which was later expanded by his students and was based on earlier techniques created by Charles Émile Picard.

## *14.2 OBJECTIVES*

In this Unit, we will

1. analyze about fixed point
2. Prove some important fixed point theorems.

## *14.3 BANACH CONTRACTION PRINCIPLE*

The fundamental Banach Contraction Theorem (1922), developed by Polish mathematician Stephan Banach (1882–1945), is the simplest and most often used method in nonlinear analysis.

The fixed point of a self-mapping is guaranteed to exist and be unique by this theorem, which is the first fixed point result in metric fixed point theory. It also provides a method to find the fixed points of these mappings.

First we defined some important terms.

**Lipschitzian mapping:** A mapping $f$ on a metric space $(X, d)$ ,$\forall\, x, y \in X$ is a **Lipschitzian mapping** if there exists a real number $\alpha > 0$ such that $d(Tx, Ty) \le d(x, y)$.

**Contraction mapping** A mapping $f$ on a metric space $(X, d)$, $\forall\, x, y \in X$ is a **Contraction Mapping** if there exists a real number $\alpha$, $0 \le \alpha < 1$, such that $d(Tx, Ty) \le \alpha\, d(x, y)$.

**Non-expensive mapping** A mapping $f$ on a metric space $(X, d)$ , $\forall\, x, y \in X$ is a **Non-expensive mapping** if $d\,(Tx, Ty) \le d(x, y)$.

**Contractive Mapping:** A mapping $f$ on a metric space $(X, d)$ , $\forall\, x, y \in X$ is a contractive mapping if $d\,(Tx, Ty) < d(x, y)$.

It is important to note that

$contraction \Rightarrow non-expansive \Rightarrow Lipschitz \Rightarrow Contractive,$

While the opposite of what it implies is untrue.

**Example:**

- The identity mapping $I: X \to X$, is non-expansive but not contractive as $\forall\, x, y \in X, d(Ix, Iy) \le d(x, y)$.
- Mapping $f: X \to X$ defined by $f(x) = x + \frac{1}{x}, \forall\, x \in X$ is a contractive mapping while $f$ is not a contraction.
- Mapping $f: X \to X$ defined by $f(x) = 3x$, T is a Lipschitzian mapping for $M = 3$, while $f$ is not a contraction.

**Banach Contraction Principle**

**Theorem 14.1. Let $f$ be a contraction on a complete metric space $(X, d)$. Then $f$ has a unique fixed point $u \in X$.**

**Proof.** Notice first that if $u_1, u_2 \in X$ are fixed points of $f$,

then $d(u_1, u_2) = d(f(u_1), f(u_2)) \le \lambda d(u_1, u_2)$

hence $u_1 = u_2$.

We choose any $u_0 \in X$, and define the iterate sequence

$u_{n+1} = f(u_n)$.

By induction on $n$,

$d(u_{n+1}, u_n) \leq \lambda^n \ d(f(u_0), u_0)$ ……………………………..(1)

If $n \in \mathbb{N}$ and $m \geq 1$,

$d(u_{n+m}, u_n) \leq d(u_{n+m}, u_{n+m-1}) + \cdots + d(u_{n+1}, u_n)$

$\leq (\lambda^{n+m} + \cdots + \lambda^n) d(f(u_0), u_0)$…………(2)

$\leq \frac{\lambda^n}{1-\lambda} d(f(u_0), u_0)$.

Hence $\{u_n\}$ is a Cauchy sequence,

$X$ is complete $\Rightarrow \lim_{n\to\infty} u_n = u \ in \, X$.

$f$ is continuous then

$f(u) = \lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} u_{n+1} = u$.

Thus $u$ is a unique fixed point.

**Corollary** Let $X$ be a complete metric space and $Y$ be a topological space.

Let $f : X \times Y \to X$ be a continuous function. Consider that $f$ is a contraction on $X$ uniformly in , i.e.,

$d\big(f(u_1, v), f(u_2, v)\big) \leq \lambda d(u_1, u_2), \forall \, u_1, u_2 \in X, \forall \, v \in Y$

for some $\lambda < 1$. Then, for every fixed $y \in Y$, the mapping $x \to f(x, y)$ has a unique fixed point $\phi(y)$. Moreover, the function $y \to \phi(y)$ is continuous from $Y$ to $X$.

**Proof .** Using Banach contraction principle we only have to prove the continuity of $\phi$.

If $v, v_0 \in Y$ , we have

$d(\phi(v), \phi(v_0)) = d(f(\phi(v), v), f(\phi(v_0), v_0))$

$\leq d(f(\phi(v), v), f(\phi(v_0), v)) + d(f(\phi(v_0), v), f(\phi(v_0), v_0))$

$$\leq \lambda d(\phi(v), \phi(v_0)) + d(f(\phi(v_0), v), f(\phi(v_0), v_0))$$

$$\Rightarrow d(\phi(v), \phi(v_0)) \leq \frac{1}{1-\lambda} d(f(\phi(v_0), v), f(\phi(v_0), v_0)).$$

As the above right-hand tends to zero as $v \to v_0$, we have require condition of continuity.

**Example:**

Let $X = (0, 1]$ with the usual distance. Define $f : X \to X$ as $f(x) = \frac{x}{2}$.

**CHECK YOUR PROGRESS**

(CQ 1) State and Prove Banach Contraction theorem

ANSWER________________________

(CQ 2) What is fixed point?

ANSWER________________________

(CQ 3) Explain Lipshitz map?

ANSWER________________________

## *14.4 FURTHER EXTENSION OF CONTRACTION PRINCIPLE*

Numerous mathematicians, scientists, and researchers have used, generalized, and expanded the banach contraction Principle in numerous ways for single-valued and multi-valued mappings under various contractive conditions in diverse spaces. A significant advancement has been made in this area, which has a profound effect on all fields of mathematics.
Here, we highlight a few outcomes.

**Boyd-Wong's Fixed point theorem**

**Theorem 14.2.** **Let $X$ be a complete metric space, and $f : X \to X$ be a mapping. Let there exists a right-continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(r) < r$ if $r > 0$, and $d(f(u_1), f(u_2)) \leq \phi(d(u_1, u_2)), \forall\, u_1, u_2 \in X$.**
**Then $f$ has a unique fixed point $u \in X$.**
**Moreover, for any $u_0 \in X$ the sequence $f^n(u_0)$ converges to $u$.**
**Proof.** If $u_1, u_2 \in X$ are fixed points of $f$, then $d(u_1, u_2) = d(f(u_1), f(u_2)) \leq \phi(d(u_1, u_2)$
Hence $u_1 = u_2$
For the existence, we fix any $u_0 \in X$,
Now we define the iterate sequence $u_{n+1} = f(u_n)$.
Now we will try to prove that $u_n$ is a Cauchy sequence.
For $n \geq 1$, we define a positive sequence $\{x_n\}$ such that
$x_n = d(u_n, u_{n-1})$.
Clearly, $x_{n+1} \leq \phi(x_n) \leq x_n$;
Hence sequence $\{x_n\}$ an converges monotonically to some $x \geq 0$.
From the right-continuity of $\phi$, we conclude
$x \leq \phi(x)$, which imply $x = 0$.
If $\{x_n\}$ is not a Cauchy sequence, there exists $\varepsilon > 0$ and integers $m_i > n_i \geq i$ for every $i \geq 1$ such that
$d_i = d(u_{m_i}, u_{n_i}) \geq \varepsilon, \forall\, i \geq 1$.
Now we choose the smallest possible $m_i$, such that
$d(u_{m_{i-1}}, u_{n_i}) < \varepsilon$ for $large\ i$. (because $u_n \to 0$).
Therefore, for for $large\ i$,
$\varepsilon \leq d_k \leq d(u_{m_i}, u_{m_{i-1}}) + d(u_{m_{i-1}}, x_{n_i}) < x_{m_i} + \varepsilon$
$\Rightarrow d_i \to \varepsilon$ as $k \to \infty$.
Also, $d_i \leq d_{i+1} + d_{m_{i+1}} + x_{n_{i+1}} \leq \phi(d_i) + x_{m_{i+1}} + x_{n_{i+1}}$
Taking the limit as $i \to \infty$ we get
$\varepsilon \leq \phi(\varepsilon)$, which is a contradiction because $\varepsilon > 0$.
$\{x_n\}$ is a Cauchy sequence
$X$ be a complete metric space
$\lim_{n\to\infty} x_n = x \in X$.
$\Rightarrow f$ is continuous then

$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$

Thus $x$ is a unique fixed point.

**Caristi Fixed point theorem**

**Theorem 14.3. Let $X$ be a complete metric space, and let $f : X \to X$. Assume there exists a lower semicontinuous function $\psi : X \to [0,\infty)$ such that $d(x, f(x)) \leq \psi(x) - \psi(f(x)), \forall x \in X$. Then $f$ has (at least) a fixed point in $X$.**

**Proof.** We introduce a partial ordering on $X$, such that
$x \preccurlyeq y$ if and only if $d(x,y) \leq \psi(x) - \psi(y)$.
Let $\emptyset = Y \subset X$ be totally ordered, and consider a sequence $x_n \in Y$ such that $\psi(x_n)$ is decreasing to $\alpha := \inf\{\psi(x): x \in Y\}$.
If $n \in \mathbb{N}$and $m \geq 1$,
$d(u_{n+m}, u_n) \leq \sum_{i=0}^{m-1} d(x_{n+i+1}, x_{n+i})$
$\leq \sum_{i=0}^{m-1} \psi(u_{n+1}) - \psi(u_{n+i+1}) = \psi(u_{n+1}) - \psi(u_{n+m})$.
Hence $\{u_n\}$ is a Cauchy sequence, and having a limit $u \in X$, for $X$ is complete.
Since $\psi$ can only decreases (being lower semicontinuous), we also have $\psi(v) = \alpha$.
If $u \in Y$ and $d(u,v) > 0$, then it must be $u \preccurlyeq u_n$ for large n.
Also, $\lim_n \psi(u_n) = \psi(v) \leq \psi(u)$.
$\Rightarrow v$ is an upper bound for $Y$, and by the Zorn lemma
there exists a maximal element $\bar{u}$.
On the other hand, $u \preccurlyeq f(\bar{u})$
Hence due to the maximality of $\bar{u}$ we conclude that $\bar{u} = f(\bar{u})$.

We get to a **Ciric's fixed point theorem** by asserting the extension of Banach's contraction theorem.
Let $X$ be a complete metric space, and let $f : X \to X$ be such that $d\big(f(u_1), f(u_2)\big) \leq$
$\lambda \max\{ d(u_1,u_2), d(u_1, f(u_1)), d(u_2, f(u_2)), d(u_1, f(u_2)), d(u_1, f(u_1))\}$
for some $\lambda < 1$ and every $u_1, u_2 \in X$. Then $f$ has a unique fixed point $u \in X$.

We now focus on maps on metric spaces that are contraction-like without actually being contractions.

**Weak contractions :** Let $(X, d)$ be a metric space with a distance $d$. A mapping $f : X \to X$ is a weak contraction if
$d(f(u_1), f(u_2)) < d(u_1, u_2), \forall\, u_1 \neq u_2 \in X$.

The following straightforward example demonstrates that $f$ need not always be a weak contraction in order to have a fixed point.

**Example**
Assume the complete metric space $X = [1, +\infty)$, and consider $f : X \to X$ be defined as $f(x) = x + \frac{1}{x}$.
Easily we observe that $f$ is a weak contraction with no fixed points. However, when X is compact, the requirement proves to be sufficient.

**Theorem 14.4. Let $\boldsymbol{f}$ be a weak contraction on a compact metric space $\boldsymbol{X}$. Then $\boldsymbol{f}$ has a unique fixed point $\boldsymbol{u} \in \boldsymbol{X}$. Moreover, for any $\boldsymbol{u_0} \in \boldsymbol{X}$ the sequence $\boldsymbol{f^n}(\boldsymbol{u_0})$ converges to $\boldsymbol{u}$.**
**Proof.** $X$ is compact
$\Rightarrow$ the continuous function $x \to d(x, f(x))$ attains its minimum at some point $u \in X$.
If $u = f(u)$, we get
$d(u, f(u)) = \min_{x\in X} d(v, f(v)) \leq d(f(u), f(f(u))) < d(u, f(u))$
which is not possible.
Hence $u$ is the unique fixed point of $f$.
Now consider $v_0 = u$ be given, and we define
$d_n = d(f^n(v_0), u)$.
We see that
$d_{n+1} = d(f^{n+1}(v_0), f(u)) < d(f^n(v_0), u) = d_n$.
Thus $d_n$ is strictly decreasing, and have a limit $l \geq 0$.
Now assume $f^{n_k}(v_0)$ be a subsequence of $f^n(v_0)$ converging to some point $w \in X$.
Then

$l = d(w,u) = \lim_{k\to\infty} d_{n_k} = \lim_{k\to\infty} d_{n_{k+1}}$
$= \lim_{k\to\infty} d(f^{n_k}(v_0), u) = d(f(w), u),$
if $w \neq u$, then $d(f(w),u) = d(f(w), f(u)) < d(w,u)$.
Thus convergent subsequence of $f^n(v_0)$ has limit $v_0$,
Now $X$ is compact
$\Rightarrow f^n(v_0)$ converges to $v$.
The uniqueness is proved exactly as in the Banach contraction theorem.

## *14.5 CONVERSE TO THE BANACH CONTRACTION PRINCIPLE*

Assume we are given a mappiing $f : X \to X$ together with a set $X$. We are looking for a metric $d$ on $X$ such that $f$ is a contraction on $X$ and $(X,d)$ is a complete metric space. A required requirement is undoubtedly that each iterate $f_n$ has a distinct fixed point in view of Banach's Contraction principle. Surprisingly, the circumstance also proves to be adequate.

**Bessaga Fixed point theorem**

**Theorem 14.5. Let $X$ be an arbitrary set, and let $f : X \to X$ be a mapping such that $f^n$ has a unique fixed point $u \in X$ for every $n \geq 1$. Then for every $\varepsilon \in (0,1)$, there is a metric $d = d_\varepsilon$ on $X$ that makes $X$ a complete metric space, and $f$ is a contraction on $X$ with Lipschitz constant equal to $\varepsilon$.**

**Proof** First we choosing $\varepsilon \in (0,1)$.
Let $Y$ be the subset of $X$ consisting of all elements $v$ such that
$f^n(v) = u$ for some n ∈ N.
Now we define the corresponding equivalence relation on $X \setminus Y$
$u \sim w$ if and only if $f^n(u) = f^m(v)\ for\ some\ n,m \in \mathbb{N}$.
We observe that
if $f^n(u) = f^m(v)$ and $f^{n'}(u) = f^{m'}(v)$ then $f^{n+m'}(x) = f^{n'+m}(x)$
But since $x \in Y, \Rightarrow n + m' = n' + m$, i.e., $n - m = n' - m'$.
We now choose a member of each equivalence class.

We define the distance of $u$ from a $x \in X$ such that
$d(u,u) = 0, d(x,u) = \varepsilon^{-n}$ $if\ x \in Z$ with $x \neq u$where $n = \min\{m \in \mathbb{N} : f^m(x) = u\}$, and $d(x,u) = \varepsilon^{n-m}$ if $x \notin Z$, where $n, m \in \mathbb{N}$ such that $f^n(x^*) = f^m(x^*)$, where $x^*$ be the selected representative of the equivalence class $[x]$.
The discussion above has made the term clear.
Thus, for any $v, w \in X$, we have

$$d(x,y) = \begin{cases} d(v,u) + d(w,u) & if\ v \neq w, \\ 0 & if\ v = w, \end{cases}$$

It is simple to confirm that d is a metric. Observe that the only Cauchy sequences that do not eventually converge to $v$ are constants to demonstrate that $d$ is complete.
The only thing left to do is proof that $f$ is a contraction with a Lipschitz constant $\varepsilon$.
Let $v \in X,\ v \neq w$. If $v \in Y$ we have
$d(f(v), f(u)) = d(f(v), u) \leq \varepsilon^{-n} = \varepsilon\, \varepsilon^{-(n+1)} = \varepsilon d(v,u)$.
If $v \notin Y$ then
$d\big(f(v), f(u)\big) = d(f(v), u) = \varepsilon^{n-m} = \varepsilon\, \varepsilon^{n-(m+1)} = \varepsilon d(v,u)$
As $v \sim f(v)$.
Hence that $f$ is a contraction with a Lipschitz constant $\varepsilon$.

## *14.6 SEQUENCES OF MAPS AND FIXED POINTS*

Assume $(X, d)$ be a complete metric space. We assume problem of convergence of fixed points for a sequence of mappings $f_n : X \to X$.

**Theorem 14.5. Let each $\boldsymbol{f_n}$ has at least a fixed point $\boldsymbol{v_n = f_n(v_n)}$. Assume $\boldsymbol{f : X \to X}$ be a uniformly continuous mapping such that $\boldsymbol{f^k}$ is a contraction for some $\boldsymbol{k \geq 1}$. If $\boldsymbol{f_n}$ converges uniformly to $\boldsymbol{f}$, then $\boldsymbol{v_n}$ converges to $\boldsymbol{u = f(u)}$.**

**Proof** Let $f$ is a contraction.
Let $\lambda < 1$ be the Lipschitz constant of $f$.
For given $\varepsilon > 0$, we choose $n_0 = n_0(\varepsilon)$ such that

$d(f_n(v), f(v)) \leq \varepsilon(1-\lambda), \forall\, n \geq n_0, \forall\, v \in X$.
Then, for $n \geq n_0$,
$d(v_n, u) = d\big(f_n(v_n), f(u)\big) \leq d\big(f_n(v_n), f(v_n)\big) + d\big(f(v_n), f(u)\big)$
$\leq \varepsilon(1-\lambda) + \lambda d(xn, \bar{x})$.
Hence $d(v_n, u) \leq \varepsilon \Rightarrow$convergence.
We can observe that if
$d\big(f^k(v), f^k(w)\big) \leq \lambda^k d(v, w)$
for some $\lambda < 1$, we may define a new metric $d^*$ on $X$ equivalent to $d$ by taking
$d^*(v, w) = \sum_{i=0}^{k-1} \frac{1}{\lambda^i}\, d\left(f^i(v), f^i(w)\right)$
Also, as $f$ is uniformly continuous, $f_n$ converges uniformly to $f$ with respect to $d^*$.
Eventually, $f$ is a contraction with respect to $d^*$.
Infact, $d^*(v, w) = \sum_{i=0}^{k-1} \frac{1}{\lambda^i}\, d\left(f^{i+1}(v), f^{i+1}(w)\right)$
$= \lambda \sum_{i=0}^{k-1} \frac{1}{\lambda^{i+1}}\, d\left(f^i(v), f^k(w)\right) + \frac{1}{\lambda^{k+1}} d\big(f^k(v), f^k(w)\big)$
$\leq \lambda \sum_{i=0}^{k-1} \frac{1}{\lambda^i}\, d\left(f^i(v), f^i(w)\right) = \lambda d^*(v, w)$.
So the problem is reduced to case $k = 1$.

**Theorem 14.6. Let $X$ be locally compact. Consider that for each $n \in \mathbb{N}$ there is $k_n \geq 1$ such that $f_n^{k_n}$ is a contraction. Consider $f : X \to X$ be a mapping such that $f^k$ is a contraction for some $k \geq 1$. If $f_n$ converges pointwise to $f$, and $f_n$ is an equicontinuous family, then $v_n = f_n(v_n)$ converges to $u = f(u)$.**

**Proof** Let ε > 0 be sufficiently small such that
$M(u, \varepsilon) = \{x \in X : d(v, u) \leq \varepsilon \subset X$ is compact.
According to the Ascoli theorem,
$f_n$ converges to f uniformly on $M(u, \varepsilon)$.
As it is equicontinuous and pointwise convergent.
We choose $n_0 = n_0(\varepsilon)$ such that
$d\big(f_n^k(x), f^k(x)\big) \leq \varepsilon(1-\lambda), \ \forall\, n \geq n_0, \forall\, x \in M(u, \varepsilon)$

where λ < 1 is the Lipschitz constant of $f^k$.
Then, for $n \geq n_0$ and $v \in M(u, \varepsilon)$.
Now

$$d(f_n^k(v), u) = d\left(f_n^k(v), f^k(u)\right)$$
$$\leq d\left(f_n^k(v), f^k(v)\right) + d\left(f^k(v), f^k(u)\right)$$
$$\leq \varepsilon(1 - \lambda) + \lambda d(v, u) \leq \varepsilon.$$

Therefore $f_n^k(M(u, \varepsilon)) \subset M(u, \varepsilon)$ for all $n \geq n_0$.
As the mappings $f_n^{k_n}$ are contractions,
⇒for $n \geq n_0$, the fixed points $v_n$of $f_n \in M(u, \varepsilon)$,
i.e., $d(v_n, u) \leq \varepsilon$.

## *14.7 FIXED POINTS OF NON-EXPANSIVE MAPS*

First we understand some terms used in theorem which we discussed later.
A normed linear space, often known as a "normed space," is a real or complex vector space E where each vector x is connected to a real number $|x|$ known as its absolute value or norm, and holds following properties $|x| \geq 0$;

$(N1)|x| \geq 0$
$(N2)|x| = 0 \ iff \ x = 0$
$(N3) \ |kx| = |k||x|$;
$(N4)|x + y| \leq |x| + |y|$ (triangle inequality).

**Banach space:** A Banach space is a complete normed linear space.

Let $X$ be a Banach space, $C \subset X$ nonvoid, closed, bounded and convex, and let $f : C \rightarrow C$ be a non-expansive map. Whether $f$ admits a fixed point in $C$ is the problem. In general, the response is untrue.

**Browder-Kirk fixed point theorem**

**Theorem 14.7. Let $X$ be a uniformly convex Banach space and $C \subset X$ be nonvoid, closed, bounded and convex. If $f : C \to C$ is a non-expansive map, then f has a fixed point in $C$.**

**Proof.** Let $u^* \in C$ be fixed, and assume a sequence $x_n \in (0,1)$ converging to 1.

For each $n \in \mathbb{N}$, we define mapping $f_n : C \to C$ as

$f_n(u) = x_n f(u) + (1 - x_n)u^*$.

We observe that $f_n$ is a contractions on $C$,

therefore there is a unique $u_n \in C$ such that $f_n(u_n) = u_n$.

Since $C$ is weakly compact, $u_n$ has a subsequence weakly convergent to some $\bar{u} \in C$.

We shall prove $\bar{u}$ that is a fixed point of $f$.

Clearly $\lim_{n\to\infty}(||f(\bar{u}) - u_n||^2 - ||\bar{u} - u_n||^2) = ||f(\bar{u}) - \bar{u}||^2$.

As $f$ is non-expansive we have

$$||f(\bar{u}) - u_n|| \le ||f(\bar{u}) - f(u_n)|| + ||f(u_n) - u_n||$$
$$\le ||\bar{u} - u_n|| + ||f(u_n) - u_n||$$
$$= ||\bar{u} - u_n|| + (1 - x_n)||f(u_n) - u^*||.$$

As $x_n \to 1$ as $n \to \infty$ and $C$ is bounded,

Hence

$\lim_{n\to\infty} \sup(||f(\bar{u}) - u_n||^2 - ||\bar{u} - u_n||^2) \le 0$

$\Rightarrow f(\bar{u}) = \bar{u}$.

## *14.8 SUMMARY*

In this unit we discussed about some important theorems based on fixed point theory.

## *14.9 GLOSSARY*

1. Space- a set with some added structure.
2. metric- a notion of distance between its elements
3. Completeness- no "points missing" from it

4. Fixed point-point that does not change upon application of a map,

## *14.10 REFERENCES*

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## *14.11 SUGGESTED READINGS*

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## *14.12 TERMINAL QUESTION*

**Long Answer Questions**

(TQ 1) Define fixed point with example

(TQ 2) State and Prove Banach Contraction Principle

**Fill in the blanks**

(TQ 3) A mapping $f$ on a metric space $(X, d)$ ,$\forall\, x, y \in X$ is a ______________if there exists a real number $\alpha > 0$ such that $d(Tx, Ty) \leq d(x, y)$.

(TQ 4) Let $(X, d)$ be a metric space with a distance $d$. A mapping $f : X \rightarrow X$ is a weak contraction if_________________$\forall\, u_1 \neq u_2 \in X$.

## 14.13 ANSWERS

(TQ 3) Lipschitzian mapping (TQ 4) $d(f(u_1), f(u_2)) < d(u_1, u_2)$,


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