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Examination Session June-2022

(Fourth Semester)

MT-606

M.A./M.Sc. MATHEMATICS (MSCMT/MAMT)

[Analysis and Advanced Calculus - II]

Time : 2 Hours] **[Max. Marks : 40**

Note : This paper is of Forty (40) marks divided into two (02) Section A and B. Attempt the questions contained in these sections according to the detailed instructions given therein.

SECTION—A

(Long-Answer-Type Questions)

Note : Section 'A' contains five (05) long-answer-type questions of Ten (10) marks each. Learners are required to answer any two (02) questions only. 2×10=20

1. Prove that if T be an operator on a Hilbert space H, then a unique linear operator T^* on H s.t.,

   $(Tx, y) = (x, T^*y) \quad x, y \quad H$

   where T^* is the adjoint operator H.

2. Let X be a Banach space over the field K of scalars and let $f : [a, b] \to X$ and $g : [a, b] \to R$ be continuous and differentiable functions such that $\|Df(t)\| \quad Dg(t)$ at each point $t \quad (a, b)$. Then prove that :

   $\|f(b) \quad f(a)\| \quad g(b) \quad g(a)$

3. Prove that if T is an arbitrary operator on a finite dimensional Hilbert space H, then the eigenvalues of T constitute a non-empty finite subset of the complex plane. Furthermore, the number of points in this does not exceed the dimension n of the space H.

5. Let f be a continuous function on a compact interval $[a, b]$ of R into a Banach space X over K. Let F be the function $t \quad {}^{t}_{a}\, f$ on [a, b] into X. Let g be any differential function on $[a, b]$ into X such that $Dg = f$. Prove that F is differentiable, $DF = f$ and

   $${}^{b}_{a}\; f \quad F(b) \quad F(a) \quad g(b) \quad g(a)$$

6. Prove that an arbitrary operator T on a Hilbert space H can be uniquely expressed as $T = T_1 + iT_2$ and $T^* = T_1 - iT_2$, where T_1 and T_2 are self-adjoint operators.

7. Prove that if T_1 and T_2 are normal operators on H with the property that either commute with adjoint of the other, than $T_1 + T_2$ and T_1T_2 are normal.

8. Prove that a closed linear subspace M of a Hilbert space H is invariant under an operator $T \quad M$ is invariant under T^*.

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4. Prove that if f be a function defined on the interval $[a, b]$ of R into R such that f is m times differentiable in $[a, b]$ and (m + 1 times differentiable in interval (a, b). Then :

$$f(b) \quad f(a) \quad (b \quad a)Df(a) \quad \ldots \quad \frac{(b \quad a)^m}{m!} D_m f(a)$$

$$\frac{(b \quad a)^{m \quad 1}}{(m \quad 1)!} D^{m \quad 1} f(c)$$

where $c \quad (a,b)$.

5. If P is the projection on a closed linear subspace M of a Hilbert space H, then prove that :

(a) P is the projection on M of $H \Leftrightarrow I - P$ is the projection on $M^{\perp}$

(b) $x \quad M \qquad Px \quad x \qquad \|P_x\| \quad \|x\|$

SECTION—B

(Short-Answer-Type Questions)

4. Prove that if f be a function defined on the interval $[a, b]$ of R into R such that f is m times differentiable in $[a, b]$ and (m + 1 times differentiable in interval (a, b). Then :

$$f(b) \quad f(a) \quad (b \quad a)Df(a) \quad \ldots \quad \frac{(b \quad a)^m}{m!} D_m f(a)$$

$$\frac{(b \quad a)^{m \quad 1}}{(m \quad 1)!} D^{m \quad 1} f(c)$$

where $c \quad (a,b)$.

5. If P is the projection on a closed linear subspace M of a Hilbert space H, then prove that :

(a) P is the projection on M of $H \Leftrightarrow I - P$ is the projection on $M^{\perp}$

(b) $x \quad M \qquad Px \quad x \qquad \|P_x\| \quad \|x\|$

SECTION—B

(Short-Answer-Type Questions)

MT-606/5 (3) [P.T.O.]

Note : Section 'B' contains eight (08) short-answer-type questions of Five (05) marks each. Learners are required to answer any four (04) questions only. 4×5 = 20

1. Prove that if T is an operator on a Hilbert space H, then $(Tx, x) = 0 \quad x \quad H$ iff $T = 0$.

2. Prove that if T is an operator on a Hilbert space H, then T is normal iff its real and imaginary parts commute.

3. Prove that if T is normal operator on a Hilbert space H, then eigenspaces of T are pairwise orthogonal.

4. Prove that if P and Q are projections on closed linear subspaces M and N of a Hilbert space H, then

$$M \quad N \quad PQ \quad 0 \quad QP \quad 0$$

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$$M \quad N \quad PQ \quad 0 \quad QP \quad 0$$