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MT-506

Advanced Algebra-II

MA/M.Sc. Mathematics (MAMT/MSCMT-20)

2nd Year Examination, 2022 (June)

Time : 2 Hours] **Max. Marks : 40**

Note : This paper is of Forty (40) marks divided into two (02) Sections A and B. Attempt the questions contained in these sections according to the detailed instructions given therein.

SECTION–A

(Long Answer Type Questions)

Note : Section 'A' contains Five (05) long answer type questions of Ten (10) marks each. Learners are required to answer any Two (02) questions only.

(2×10=20)

1. Prove that if K be a Galois extension of a field F, Then there exists a one-to-one correspondence between the set of all subfields of K containing F and the set of all subgroups of G(K\F). Further, if E is any subfields of K which contains F, then

(a) [K : E] = o[G(K\E)] and [E : F] = index of G(K\E) in G(K\F).

(b) E is normal extension of F iff G(K\E) is normal subgroup of G(K\F).

(c) If E is a normal extension of F, then G(E\F) ≅ G(K\F)\G(K\E).

2. If A = $\{u_1, u_2, ..., u_n\}$ is any orthonormal set in any finite dimensional inner product space V, then prove that the following are equivalent

(a) Orthonormal set A is complete

(b) If $u \in$ V and $< u, u_i > = 0$ for $1 \leq l \leq n$, then $u = 0$.

(c) < A > = V, i.e. A generates V.

(d) If $u \in$ V then $u = \sum_{i=1}^{n} < u, u_i > u_i$.

(e) If $u, \in$ V then $<u, v> = \sum_{i=1}^{n} < u, u_i >< v, u_i >$.

(f) If $u \in$ V then $\| u \|^2 = \sum_{i=1}^{n} |< u, u_i >|^2$.

3. (a) Explain the Real inner product space with example.

(b) Prove that if V is an inner product space and $v \in$ V, then

(i) $\| v \| \geq 0$ and $\| v \| = 0$ if and only if $v = 0$

(ii) $\| \alpha v \| = | \alpha | \| v \|$.

4. (a) Define

(i) Rank of a Matrix.

(ii) Similar matrices.

(iii) Eigen values and eigen vectors of linear transformation.

(b) Let A be an $n \times n$ matrix over field F. Then prove that a non-zero vector $X \in F^n$ (or a column vector X) iff there exists a scalar $\lambda \in F$ such that $AX = X\lambda$.

5. (a) Prove that A linear transformation $t : V \to V$ is invertible iff matrix of t relative to some bases B of V invertible.

(b) Let V and V′ be finite dimensional vector spaces over a field F with the bases B and B′ respectively. If $t : V \to V'$ be a linear transformation, then prove that

$M_{B^*}^{B'^*}(t^*)\left[M_{B'}^{B}(t)\right]^T$, where t^* is the dual map of t and

B^* and B'^* are the bases dual to B and B′ respectively.

SECTION–B

(Short Answer Type Questions)

Note : Section 'B' contains Eight (08) short answer type questions of Five (05) marks each. Learners are required to answer any Four (04) questions only. (4×5=20)

1. (a) Prove that if F is a finite field of characteristic p, then $a \to a^p$ is an automorphism of F.

(b) Prove that Every field of Characteristic zero is perfect.

2. (a) Define

(i) Galois Extension.

(ii) Galois group.

(b) Let K be a Galois Extension of a field F and let characteristic of F be zero. Then prove that fixed field under the Galois group G(K\F) is F itself.

3. (a) Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $t(a, b) = (2a - 3b, a + b), \forall (a, b) \in \mathbb{R}^2$. Then find the matrix of t relative to the basis B = {(1,0), (0,1)}, B′ = {(2,3), (1,2)}.

(b) Let $V = \mathbb{R}^3$ and $t : V \to V$ be a linear transformation detlned by $t(x, y, z) = (x + z, -2x + y, -x + 2y + z)$, $\forall (a, b) \in \mathbb{R}^2$. What is the matrix t relative to the basis B = {(1,0,1), (–1,1,1), (0,1,1)}.

4. (a) Let V be a vector space over a field F and B its basis. Then prove that a linear Transformation. $T : V \to V$ is invertible iff matrix off relative to basis B is invertible.

(b) Let A be an $n \times n$ matrix over a field F, has n distinct eigenvalues λ_i, $i = 1,2,..., n$. then prove that there exists an invertible matrix P such that $P^{-1}AP = \text{diag}\ (\lambda_1, \lambda_2, ..., \lambda_n)$.

5. (a) Prove that a square matrix A of order n over a field F is invertible iff rank (A) = n.

(b) Find the characteristic roots and characteristic spaces

of the matrix $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

6. State and prove Cayley Hamilton theorem.

7. Let V be an inner product space. Prove that for any two vectors $u, v \in V$.

(a) $\|u + v\|^2 - \|u - v\|^2 = 4 < u, v >$.

(b) $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$.

8. Prove that If $T : V \to V$ is any map from an inner-product space V to itself such that

(a) $T(0) = 0$

(b) $\|T(u) - T(v)\| = \|u - v\|$.

Then T is an orthonormal linear transformation.
