MATHEMATICAL PHYSICS
UNIT – 3
HERMITE FUNCTION

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STRUCTURE OF UNIT

3.1 INTRODUCTION

Hermite polynomials were defined by Pierre-Simon Laplace in 1810, though in scarcely recognizable form, and studied in detail by Pafnuty Chebyshev in 1859.

They were consequently not new, although Hermite was the first to define the multidimensional polynomials in his later 1865 publications.

3.2 HERMITE'S EQUATION

The differential equation of the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \qquad \ldots(1)$$

Is called Hermite equation.

The solution of (1) is known as Hermite's polynomial.

3.3. SOLUTION OF HERMITE'S EQUATION

- Here, we have

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \qquad \ldots(1)$$

- Suppose its series solution is

$$y = a_0\, x^m + a_1\, x^{m+1} + a_2\, x^{m+2} + a_3\, x^{m+3} + \ldots\ldots + a_k\, x^{m+k}$$

or $$y = \sum_{k=0}^{\infty} a_k x^{m+k} \qquad \ldots(2)$$

$$\frac{dy}{dx} = \sum a_k \, (m+k)x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\Rightarrow \sum a_k (m+k)(m+k-1)x^{m+k-2} - 2x \sum a_k (m+k)x^{m+k-1} + 2n \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum a_k (m+k)(m+k-1)x^{m+k-2} - 2x \sum a_k (m+k)x^{m+k} + 2n \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum a_k (m+k)(m+k-1)x^{m+k-2} - 2x \sum a_k [(m+k)-n]x^{m+k} = 0 \qquad \ldots(3).$$

This equation holds good for k = 0 and all positive integer. By our assumption k cannot be negative.

To get the lowest degree term x^{m-2}, we put $k = 0$ in the first summation of (3) and we cannot have x^{m-2} from the second summation. Since $k \neq -2$.

The coefficient of x^{m-2} is

$$a_0\, m\,(m - 1) = 0 \Rightarrow m = 0,\ m = 1,\ \text{since } a_0 \neq 0 \qquad \ldots(4)$$

This is the indicial equation.

Now equating the coefficient of next lowest degree term x^{m-1}, zero in (3), we get (by putting $k = 1$ in the first summation and we cannot have x^{m-1} from the second summation since ($k \neq -1$).

$$a_1 m(m+1) = 0$$

$$\Rightarrow \left[\begin{array}{l} a_1 \; may \; or \; may \; not \; be \; zero \; when \; m = 0 \\ a_1 = 0, when \; m = 1 \end{array}\right. \qquad \left(\begin{array}{c} m + 1 \neq 0 \; as \; m \; is \\ already \; equal \; to \; zero \end{array}\right)$$

Again equating the coefficient of the general term x^{m-k} to zero, we get

$a_{k+2} (m + k + 2) (m + k + 1) – 2a_k (m + k – n) = 0$

$$a_{k+2} = \frac{2\,(m+k-n)}{(m+k+2)(m+k+1)} a_k \qquad \ldots(5)$$

If m = 0, then, $$a_{k+2} = \frac{2\,(k-n)}{(k+2)(k+1)} a_k \qquad \ldots(6)$$

If m = 1, then, $$a_{k+2} = \frac{2\,(k-1-n)}{(k+3)(k+2)} a_k \qquad \ldots(7)$$

Case I. When m = 0, $a_{k+2} = \frac{2\,(k-n)}{(k+2)(k+1)} a_k$

If k = 0, then, $a_2 = \frac{-2n}{2} a_0 = -n a_0$

If k = 1, then, $a_3 = \frac{2(1-n)}{6} a_1 = -2\,\frac{(n-1)}{3!}\, a_1$

If k = 2, then, $a_4 = \frac{2(2-n)}{12} a_2 = 2\,\frac{(2-n)}{12}\,(-n a_0) = (2)^2\,\frac{n(n-2)}{4!} a_0$

If k = 3, then, $a_5 = \frac{2(3-n)}{20} a_3 = 2\,\frac{2(3-n)}{20}\left(-\frac{2(n-1)}{3!} a_1\right) = (2)^2\,\frac{(n-1)(n-3)}{5!} a_1$

$$a_{2r} = \frac{(-2)^r n(n-2)(n-4)\ldots\ldots\ldots(n-2r+2)}{(2r)!} a_0$$

$$a_{2r+1} = \frac{(-2)^r (n-1)(n-3)\ldots\ldots\ldots(n-2r+1)}{(2r+1)!} a_1 = 0$$

When m = 0, then there are two possibilities

Possibility I. When, $a_1 = 0$, then $a_3 = a_5 = a_7 = a_{2r+1} = \ldots.. = 0$.

Possibility II. When $a_1 \neq 0$

$$y = \sum_{k=0}^{\infty} a_k x^k$$

i.e. $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots\ldots\ldots$

$= a_0 + a_2x^2 + a_4x^4 + \ldots\ldots\ldots\ldots\ldots\ldots + a_1x + a_3x^3 + a_5x^5.$ ...(8)

Putting the values of a_0, a_1, a_2, a_3, a_4 and a_5 in (8), we get

$$= a_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!}x^4 - \cdots + (-1)^r \frac{2}{(2r)!} n(n-2)\ldots(n-2r+2)x^{2r} + \cdots\right]$$

$$+a_1 x\left[1 - \frac{2(n-1)}{3!}x^2 + \frac{2^2(n-1)(n-3)}{5!} + (-1)^r \frac{2^r}{(2r+1)!}(n-1)(x-3)\ldots(n-2r+1)x^{2r} + \cdots\right] \quad \ldots.(9)$$

$$= a_0\left[1 + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{(2r)!} n(n-2)\ldots(n-2r+2)x^{2r}\right]$$

$$= a_0\left[x + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{(2r+1)}(n-1)(n-3)\ldots(n-2r+2)x^{2r+1}\right] \quad (\text{If } a_1 = a_0) \quad \ldots(10)$$

Case II. When m = 1, then $a_1 = 0$ and so by putting k = 0, 1, 2, 3, ………. In (7), we get

$$a_{k+2} = \frac{2(k+1-n)}{(k+3)(k+2)} a_k$$

$$a_2 = -\frac{2(n-1)}{3!} a_0$$

$$a_4 = \frac{2^2(n-1)(n-3)}{5!} a_0$$

……………………………………….

$$a_{2r}(-1)^r = \frac{2^r(n-1)(n-3)\dots(n-2r+1)}{(2r+1)!} a_0$$

Hence, the solution is

$$= a_0 x \left[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2(n-1)(n-3)}{5!} x^4 \dots + \frac{(-1)^r\, 2^r(n-1)(n-3)\dots(n-2r+1)}{(2r+1)!} x^{2r} + \cdots \right. \qquad \dots(11)$$

It is clear that the solution (11) is included in the second part of (9) except that a_0 is replaced by a_1 and hence in order that the Hermite equation may have two independent solutions, a_1 must be zero, even if m = 0 and then (9) reduce to

$$= a_0 x \left[\left[1 - \frac{2(n-1)}{3!}x^2 + \frac{2^2(n-1)(n-3)}{5!}x^4 \ldots + \frac{(-1)^r\, 2^r(n-1)(n-3)\ldots(n-2r+1)}{(2r+1)!}x^{2r} + \cdots\right]\right. \quad \ldots(12)$$

The complete integral of (1) is then given by

$$y = A\left[1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!}x^4 - \cdots\right] + B\left[1 - \frac{2(n-1)}{3!}x^2 + \frac{2^2(n-1)(n-3)}{5!}x^4 \ldots\right] \quad \ldots(13)$$

where A and B are arbitrary constants.

3.4. GENERATING FUNCTION OF HERMITE POLYNOMIALS (RODRIGUE FORMULA)

We know that

$$e^{x^2}\frac{\partial^n}{\partial^n}e^{\{-(1-x)^2\}} = H_n(x) + H_{n+1}(x)\, t + H_{n+2}(x).\, t^2 + \;\dots\dots\dots \qquad \dots(1)$$

Now differentiating $e^{\{-(1-x)^2\}}$ $w.r.t.,t$, we get

$$\frac{\partial}{\partial t}e^{\{-(1-x)^2\}} = -2(t-x)e^{\{-(t-x)^2\}}$$

Taking limit when $t \to 0$, we get

$$\lim_{t\to 0}\frac{\partial}{\partial t}e^{\left\{-(t-x)^2\right\}} = 2xe^{-x^2} \qquad \dots(2)$$

Again differentiating $e^{\{-(1-x)^2\}}$ w.r.t. 'x', we get

$$\frac{\partial}{\partial t} e^{\{-(1-x)^2\}} = (-1)^2 (t-x) e^{\{-(t-x)^2\}}$$

Taking limit when t → 0, we get

$$\lim_{t\to 0} \frac{\partial}{\partial x} e^{\left\{-(t-x)^2\right\}} = 2xe^{-x^2} \qquad \ldots(3)$$

From (2) and (3), we have

$$\lim_{t\to 0} \frac{\partial}{\partial t} e^{\left\{-(t-x)^2\right\}} = (-1)^1 \lim_{t\to 0} \frac{\partial}{\partial x} e^{\left\{-(t-x)^2\right\}}$$

Similarly,

$$\lim_{t\to 0}\frac{\partial^2}{\partial t^2}e^{\left\{-(t-x)^2\right\}} = (-1)^2\lim_{t\to 0}\frac{\partial^2}{\partial x^2}e^{\left\{-(t-x)^2\right\}}$$

....

....

$$\lim_{t\to 0}\frac{\partial^n}{\partial t^n}e^{\left\{-(t-x)^2\right\}} = (-1)^n\lim_{t\to 0}\frac{\partial^n}{\partial x^n}e^{\left\{-(t-x)^2\right\}} = (-1)^n\frac{d^n}{dx^n}e^{-x^2}$$

[differentiating *n* times] ...(4)

Putting t = 0 in (1), we get

$$\lim_{t\to 0}e^{x^2}\frac{\partial^n}{\partial t^n}e^{\left\{-(t-x)^2\right\}} = H_n(x) \qquad ...(5)$$

Putting the value of

$$\lim_{t \to 0} \frac{\partial^n}{\partial t^n} e^{\{-(t-x)^2\}}$$ from (4) in (5), we get

$$(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = H_n(x)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad \ldots(6)$$

n = 0

On putting n = 0 in (6), we get

$$H_0(x) = (-1)^0 e^{x^2} e^{-x^2} = 1$$

$$H_0(x) = 1$$

n = 1

On putting n = 1 in (6), we get

$$H_1(x) = (-1)^1 e^{x^2} \frac{d}{dx} e^{-x^2} = -e^{x^2}(-2x)e^{-x^2} = 2x$$

$$H_1(x) = 2x$$

n = 2

On putting n = 2 in (6), we get

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2} e^{-x^2} = e^{x^2} \frac{d}{dx}\left(-2xe^{-x^2}\right)$$

$$= e^{x^2}\left[-2e^{x^2} - 2x(-2x)e^{-x^2}\right.$$

$$= -2 + 4x^2$$

$$H_2(x) = 4x^2 - 2$$

n = 3

On putting n = 3 in (6), we get

$$H_3(x) = (-1)^3 e^{x^2} \frac{d^2}{dx^3}\left(e^{-x^2}\right) = -e^{x^2} \frac{d^2}{dx^3}\left(-2xe^{-x^2}\right)$$

$$= -e^{x^2} \frac{d}{dx}\left(-2xe^{-x^2} + (-2x)(-2x)e^{-x^2}\right)$$

$$= -e^{x^2} \frac{d}{dx}(-2 + 4x^2)e^{-x^2} = e^{x^2}\left[8xe^{-x^2} + (4x^2 - 2)(-2x)e^{-x^2}\right]$$

$$= -[8x + (4x^2 - 2)\ (-2x)] = -8x + 8x^3 - 4x = 8x^3 - 12x$$

$$H_3(x) = 8x^3 - 12x$$

Similarly $H_4(x) = 16x^4 - 48\, x^2 + 12$

$H_5(x) = 32x^5 - 160\, x^3 + 120x$

$H_6(x) = 64x^6 - 480\, x^4 + 720\, x^2 - 120$

$H_7(x) = 128x^7 - 1344\, x^5 + 3360\, x^3 - 1680x$

Example 1. Convert Hermite polynomial

$$2H_4(x) + 3\,H_3\,(x) - H_2\,(x) + 5\,H_1\,(x) + 6\,H_0$$

into ordinary polynomial.

Solution. Here, we have

$$2H_4(x) + 3\,H_3\,(x) - H_2\,(x) + 5\,H_1\,(x) + 6\,H_0$$

$$= 2[16x^4 - 48x^2 + 12] + 3\{8x^3 - 12x\} - (4x^2 - 2) + 5(2x) + 6(1)$$

$$= 32x^4 - 96x^2 + 24 + 24x^3 - 36x - 4x^2 + 2 + 10x + 6$$

$$= 32x^4 + 24x^3 - 100x^2 - 26x + 32$$

Example 2. Convert ordinary polynomial

$$64x^4 + 8x^3 - 32x^2 + 40x + 10$$

into Hermite polynomial.

Solution. Here, we have

Let $64x^4 + 8x^3 - 32x^2 + 40x + 10\ AH_4(x) + BH_3(x) + CH_2(x) + DH_1(x)\ EH_0(x)$

$$= A(16x^4 - 48x^2 + 12) + B(8x^3 - 12x) + C(4x^2 - 2) + D(2x) + E(1)$$

$$= 16Ax^4 + 8Bx^3\ (-48A + 4C)x^2 + (-12B + 2D)\ x + 12A - 2C + E$$

Equating the coefficients of like powers of x, we get

$$16A = 64 \Rightarrow A = 4$$

$$8B = 8 \Rightarrow B = 1$$

$$-48A + 4C = -32 \Rightarrow 4C = -32 + 192 \Rightarrow C = 40$$

$$-12B + 2D = 40 \Rightarrow -12 + 2D = 40 \Rightarrow 2D = 52 \Rightarrow D = 26$$

$$12A - 2C + E = 10 \Rightarrow 12 \times 4 - 2(40) + E = 10 \Rightarrow E = 42$$

The required Hermite polynomial is

$$4H_4(x) + H_3(x) + 40H_2(x) + 26H_1(x) + 42H_0(x)$$

3.5. ORTHOGONAL PROPERTY

The orthogonal property of Hermites polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0, & m \neq n \\ n^2 n! \sqrt{\pi}, & m = n \end{cases}$$

Solution. We know that

$$e^{\{x^2-(t_1-x)^2\}} = \sum \frac{H_n(x)}{n!} t_1^n \qquad \text{(generating function)} \qquad ...(1)$$

and

$$e^{\{x^2-(t_2-x)^2\}} = \sum \frac{H_m(x)}{m!} t_1^m \qquad ...(2)$$

Multiplying (1) and (2), we get

$$e^{\{x^2-(t_1-x)^2\}} e^{\{x^2-(t_2-x)^2\}} = \left[\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t_1^n\right]\left[\sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t_1^m\right]$$

$$= \sum_{m=0}^{\infty} \left[H_m(x)|H_m(x)|\right] \frac{t_1^n t_2^m}{n!m!}$$

Multiplying both the sides of this equation by e^{-x^2} and then integrating with the limits from $-\infty$ to ∞, we have

$$\sum_{nm}\left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx\right]\frac{t_1^n t_2^m}{n!m!} = e^{-x^2}\int_{-\infty}^{\infty} e^{\left\{x^2-(t_1-x)^2\right\}}.e^{\left\{x^2-(t_2-x)^2\right\}}dx$$

$$= e^{\left\{-\left(t_1^2+t_2^2\right)\right\}}\int_{-\infty}^{\infty} e^{\left\{-x^2+2x\left(t_1+t_2\right)\right\}}dx \qquad \text{...(3)}$$

We have already learnt that

$$\int_{-\infty}^{\infty} e^{\left\{-ax^2+2bx\right\}}dx = \sqrt{\frac{\pi}{2}}e^{\frac{b^2}{a}}$$ [standard formula] ... (4)

Replacing 2b by $(t_1 + t_2)$ and a by 1 in (4), we get

$$\int_{-\infty}^{\infty} e^{\left\{-x^2+2x\left(t_1+t_2\right)\right\}}dx = \sqrt{\pi}e^{(t_1+t_2)2} \qquad \text{...(5)}$$

Putting the value of

$$\int_{-\infty}^{\infty} e^{\left\{-x^2+2x(t_1+t_2)\right\}} dx$$

from (5) in R.H.S. of (3), we get

$$e^{\{-(t_1+t_2)^2\}}.\sqrt{\pi}e^{(t_1+t_2)^2} = \sqrt{\pi}e^{-t_1^2-t_2^2+t_1^2+t_2^2+2t_1+t_2}$$

$$= \sqrt{\pi}\left[1 + 2t_1 + t_2 + \frac{(2t_1t_2)^2}{2!} + \frac{(2t_1t_2)^3}{3!} + \cdots\right] = \sqrt{\pi}\sum\frac{(2t_1t_2)^n}{n!}$$

$$= \sqrt{\pi}\sum\frac{(2^n t_1^n t_2^n)}{n!} = \sqrt{\pi}\sum_{\substack{m=0\\ n=0}}^{\infty} 2^n t_1^n t_2^n \delta_{m.n} \;\; [t_2^n = t_2^m \delta_{m.n}]$$

From (3), we have

$$\sum_{nm}\left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx\right]\frac{t_1^n t_2^m}{n!m!} = \sqrt{\pi}\sum_{nm}\frac{2^n}{n!}t_1^n t_2^m \delta_{n.m}$$

On equating the coefficients of t_1^n, t_2^m on both sides, we get

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{H_n(x)H_m(x)}{n!m!} dx = \frac{\sqrt{\pi}\,2^n}{n!} \delta_{n.m}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \sqrt{\pi}\,2^n m!\delta_{n.m}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = \begin{cases} 0 & m \neq n \\ 2^n n!\sqrt{\pi}, & m = n \end{cases} \begin{bmatrix} \delta_{n,m} = 0, & if \;\; m \neq n \\ = 1, & if \;\; m = n \end{bmatrix} \sqrt{\pi}\,2^n m!\delta_{n.m}$$

Example 3. Find the value of

$$\int_{-\infty}^{\infty} e^{-x^2} H_2(x) H_3(x) dx.$$

Solution. We known that

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) = 0 \quad if \quad m \neq n$$

Here m = 2 and n = 3, $m \neq n$

Hence,

$$\int_{-\infty}^{\infty} e^{-x^2} H_2(x) H_3(x) = 0$$

Example 4. Find the value of

$$\int_{-\infty}^{\infty} e^{-x^2} \left[H_2(x)\right]^2 dx$$

Solution. We know that

$$\int_{-\infty}^{\infty} e^{-x^2} \left[H_n(x)\right]^2 dx = 2^n (n)! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \left[H_2(x)\right]^2 dx = 2^2 (2!) \sqrt{\pi} = 8\sqrt{\pi}$$

3.6. RECURRENCE FORMULAE FOR H_n (X) OF HERMITE EQUATION

Four recurrence Relations

- $2n\, H_{n-1}(x) = H'_n(x)$
- $2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x)$
- $H'_n(x) = 2x\, H_n(x) – H_{n+1}(x)$
- $H'_n(x) = x\, H'_n(x) + 2n\, H_n(x) = 0$

THANKS