MATHEMATICAL PHYSICS
UNIT - 8
Metric Tensor & Chriostofell Symbols

PRESENTED BY: DR. RAJESH MATHPAL
ACADEMIC CONSULTANT
SCHOOL OF SCIENCES
U.O.U. TEENPANI, HALDWANI
UTTRAKHAND
MOB:9758417736,7983713112

STRUCTURE OF UNIT

- 8.1. INTRODUCTION
- **8.2. RIEMANNIAN SPACE: METRIC TENSOR**
- **8.3. FUNDAMENTAL TENSORS g_{jk}, g^{jk} AND δ_k^j**
- **8.4. CHRISTOFELL'S 3-INDEX SYMBOLS**
- **8.5. GEODESICS**

8.1. INTRODUCTION

- In the mathematical field of differential geometry, one definition of a metric tensor is a type of function which takes as input a pair of tangent vectors v and w at a point of a surface (or higher dimensional differentiable manifold) and produces a real number scalar $g(v, w)$ in a way that generalizes many of the familiar properties of the dot product of vectors in Euclidean space. In the same way as a dot product, metric tensors are used to define the length of and angle between tangent vectors. Through integration, the metric tensor allows one to define and compute the length of curves on the manifold.

- A metric tensor is called *positive-definite* if it assigns a positive value $g(v, v) > 0$ to every nonzero vector v. A manifold equipped with a positive-definite metric tensor is known as a Riemannian manifold. On a Riemannian manifold, the curve connecting two points that (locally) has the smallest length is called a geodesic, and its length is the distance that a passenger in the manifold needs to traverse to go from one point to the other. Equipped with this notion of length, a Riemannian manifold is a metric space, meaning that it has a distance function $d(p, q)$ whose value at a pair of points p and q is the distance from p to q. Conversely, the metric tensor itself is the derivative of the distance function (taken in a suitable manner). Thus the metric tensor gives the *infinitesimal* distance on the manifold.
- While the notion of a metric tensor was known in some sense to mathematicians such as Carl Gauss from the early 19th century, it was not until the early 20th century that its properties as a tensor were understood by, in particular, Gregorio Ricci-Curbastro and Tullio Levi-Civita, who first codified the notion of a tensor. The metric tensor is an example of a tensor field.

- The components of a metric tensor in a coordinate basis take on the form of a symmetric matrix whose entries transform covariantly under changes to the coordinate system. Thus a metric tensor is a covariant symmetric tensor. From the coordinate-independent point of view, a metric tensor field is defined to be a nondegenerate symmetric bilinear form on each tangent space that varies smoothly from point to point.

8.2. RIEMANNIAN SPACE: METRIC TENSOR

An expression which express the distance between two adjacent point is called a *metric* or *line element*. In three dimensional space the line element, i.e., the distance between two adjacent points (x, y, z) and (x + dx, y + dy, z + dz) in Cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

In terms of general curvilinear coordinates, the line element becomes

$$ds^2 = \sum_{j=1}^{3}\sum_{k=1}^{3} g_{jk}\,du_j\,du_k = g_{jk}\,du_j\,du_k \qquad \text{(Using summation convention)}$$

This idea was generalised by Riemann to *n*-dimensional space.

The distance between two neighbouring points with coordinates x^j and $x^j + dx^j$ is given by

$$ds^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} g_{jk}\,dx^j dx^k = g_{jk}\,dx^j dx^k \qquad \ldots(8.1)$$

(Using summation convention)

where the coefficients g_{jk} are the functions of coordinates x^j, subject to the restriction g = determinant of g_{jk}, i.e, $|g_{jk}| \neq 0$.

The quadratic differential form $g_{jk}\, dx^j\, dx^k$ is independent of the coordinates system and is called the *Riemannian metric for n* dimensional space. The space which is characterised by Riemannian metric is called *Riemannian space*. Hence the quantities g_{jk} are the components of a covariant symmetric tensor of rank two, called the *metric tensor* or *fundamental tensor.*

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \ldots + (dx^n)^2 \text{ or } dx^j dx^k,$$

the space is called *n-dimensional Euclidean space*. It is now obvious that Euclidean spaces are the particular cases of Riemannian space.

In general theory of relativity (four dimensional space), the line element is given by

$$Ds^2 = g_{jk} dx^j dx^k \ (j, k = 1, 2, 3, 4).$$

In special theory of relativity, the line element is given by

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \ldots + (dx^n)^2 \text{ or } dx^j\, dx^k.$$

the space is called *n-dimensional Euclidean space*. It is now obvious that Euclidean spaces are the particular cases of Riemannian space.

In general theory of relativity (four dimensional space), the line element is given by

$$ds^2 = g_{jk} dx^i dx^j dx^k \ (j, k = 1, 2, 3, 4).$$

In special theory of relativity, the line element is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \qquad [\text{with } x^4 = ict,\ i = \sqrt{(-1)}]$$

$$= dx^j dx^j \ (j = 1, 2, 3, 4).$$

As $ds^2 = g_{jk}\, dx^j\, dx^k$ has been defined in general space (i.e., Riemannian space), it is independent of the coordinate system, i.e., $dx^2 = g_{jk}\, dx^j\, dx^k$ is an invariant.

8.3. FUNDAMENTAL TENSORS g_{jk}, g^{jk} AND δ^j_k

(i) Covariant fundamental tensor g_{jk}. The line element or interval *ds* in Riemannian space is given by

$$ds^2 = g_{jk}\, dx^j\, dx^k. \qquad ...(8.2)$$

As $dx^j\, dx^k$ are contravariant vectors and ds^2 is invariant for arbitrary choice of vectors dx^j and dx^k, it follows from quotient law that g_{jk} is a covariant tensor, we have

$$ds^2 = g_{jk}\, dx^j\, dx^k \quad \text{in system of variables } x^j$$

$$= \overline{g}_{\mu\nu}\, \overline{dx}^{\mu}\, \overline{dx}^{\nu} \quad \text{in system of variables } \overline{x}^{\mu}$$

i.e.,

$$= \overline{g}_{\mu\nu}\, \overline{dx}^{\mu}\, \overline{dx}^{\nu} = g_{jk}\, dx^j\, dx^k. \qquad ...(8.3)$$

Now applying *inverse transformation law to dx^j and dx^j, i.e.,*

$$dx^j = \frac{\partial x^j}{\partial \overline{x}^{\mu}}\, d\overline{x}^{\mu} \text{ etc.}$$

$$\overline{g}_{\mu\nu}\, \overline{dx}^{\mu}\, \overline{dx}^{\nu} = g_{jk} \frac{\partial x^j}{\partial \overline{x}^{\mu}} d\overline{x}^{\mu} \frac{\partial x^k}{\partial \overline{x}^{\nu}} d\overline{x}^{\nu}$$

$$= g_{jk} \frac{\partial x^j}{\partial \overline{x}^{\mu}} \frac{\partial x^k}{\partial \overline{x}^{\nu}} d\overline{x}^{\mu} d\overline{x}^{\nu}$$

i.e.,

$$\left\{\overline{g}_{\mu\nu} - g_{jk} \frac{\partial x^j}{\partial \overline{x}^{\mu}} \frac{\partial x^k}{\partial \overline{x}^{\nu}}\right\} d\overline{x}^{\mu} d\overline{x}^{\nu} = 0 \qquad ...(8.4)$$

As $d\overline{x}^{\mu}$ and $d\overline{x}^{v}$ are arbitrary contravarient vectors, we must have

$$\overline{g}_{\mu v} - c\frac{\partial x^j}{\partial \overline{x}^{\mu}}\frac{\partial x^k}{\partial \overline{x}^{v}} = 0$$

$$\overline{g}_{\mu v} = \frac{\partial x^j}{\partial \overline{x}^{\mu}}\frac{\partial x^k}{\partial \overline{x}^{v}} g_{jk}$$

Hence g_{jk} is a *covariant tensor of rank 2.*

g_{jk} may be expressed as

$$g_{jk} = \frac{1}{2}(g_{jk} + g_{kj}) + \frac{1}{2}(g_{jk} - g_{kj})$$

$$= A_{jk} + B_{jk}$$

...(8.5)

where $A_{jk} = \frac{1}{2}(g_{jk} + g_{kj})$ is symmetric tensor

and $B_{jk} = \frac{1}{2}(g_{jk} - g_{kj})$ is symmetric tensor ...(8.6)

∴ $ds^2 = g_{jk}\, dx^j\, dx^k = (A_{jk} + B_{jk})\, dx^j\, dx^k$. ...(8.7)

We have

$B_{jk}\, dx^j\, dx^k = B_{kj}\, dx^k\, dx^j$ (interchanging dummy indices *j* and *k*)

$= - B_{jk}\, dx^j\, dx^k$

(cince B_{jk} is antisymmetric i.e., $B_{jk} = - B_{kj}$)

i.e., $2B_{jk}\, dx^j\, dx^k = 0.$

As dx^j and dx^k are arbitrary vectors, we have

$$B_{jk} = 0$$

i.e., $\frac{1}{2}\left(g_{jk} + g_{kj}\right) = 0$

i.e., $g_{jk} + g_{kj}$

i.e., g_{jk} is symmetric.

So, we can write g_{jk} as

$$g_{\mu v}. = \frac{1}{2}\left(g_{\mu v} + g_{v\mu}\right)$$

Thus we have proved that the metric tensor g_{jk} is *covariant symmetric tensor of rank* 2. This is called *covariant fundamental tensor of rank* 2.

(ii) Contravariant fundamental tensor g^{jk}.

Let us define g^{jk} as

$$g^{jk}\frac{cofactor\ of\ g_{jk}\ in\ g}{g} \qquad \ldots(8.8)$$

where g is the determinant of g_{jk}, i.e.,

$$g = |g_{jk}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1n} \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} & g_{n2} & g_{n3} & \cdots & g_{nn} \end{vmatrix}$$

Since g_{jk} is symmetric, g is symmetric which implies cofactor of g_{jk} in g is symmetric and so g^{jk} is symmetric.

Let A^j be an arbitrary contravariant vector, then by quotient law,

$A_k = g_{jk} A^j$ …(8.9)

is an arbitrary covariant vector.

Now multiplying eqn. (8.9) by g^{kl}, we get

$$g^{kl} A_k = g_{jk} g^{kl} A^j. \qquad \ldots(8.10)$$

But $g_{jk} g^{kl} = g_{jk} \frac{cofactor\ of\ g_{kl}\ in\ g}{g}$

$= \delta_j^l$ (by theory of determinants).

Therefore equation (8.10) yields

$$g^{kl} A_k = \delta_j^l A^j = A^l \qquad \ldots(8.11)$$

i.e., the inner product of g^{kl} with an arbitrary covariant vector A_k yields a contravariant vector. Hence by quotient law g^{kl} is a *contravariant tensor* of rank 2.

Thus g^{jk} is symmetric contravariant tensor of rank two. This tensor is reciprocal of g_{jk} and is called *conjugate metric tensor* or *contravariant fundamental tensor* of rank 2.

(iii) Mixed fundamental tensor g_k^j or δ_j^l. we have

$$g_{jk}\, g^{kl} = \delta_j^l \qquad \text{...(8.12)}$$

As g_{jk} and g^{kl} are covariant and contravariant tensors of rank 2 recpectively, therefore, from quotient law δ_j^l is also a tensor of rank 2; it is a mixed tensor, contravariant in l and covarian in j and is known as *mixed fundamental tensor*. As important property of mixed fundamental tensor is that its components have the same value in all coordinate system, i.e., mixed fundamental tensor is invariant.

The three tensors g_{jk}, g^{jk} and δ_k^j are called the *fundamental tensors* and are of basic importance in general theory of relativity.

8.4. CHRISTOFELL'S 3-INDEX SYMBOLS

We now introduce two expressions (not tensors) formed of the fundamental tensors, known as *Christofell's symbols of first and second kind, namely :*

Christofell's symbol of first kind.

$$[jk, l] = \Gamma_{\mathrm{l,jk}} = \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \qquad \text{...(8.13)}$$

Christofell's symbol of second kind.

$$\begin{Bmatrix} l \\ jk \end{Bmatrix} = \Gamma^{\mathrm{l}}{}_{\mathrm{.jk}} = \frac{1}{2}\mathrm{g}^{\mathrm{lm}}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \qquad \text{...(8.14)}$$

From the symmetry property of g_{jk} it follows that

$$[jk, l] = [kj, l] \text{ or } \Gamma_{\mathrm{l,jk}} = \Gamma_{\mathrm{l.kj}} \qquad \text{...(8.15)}$$

and

$$\begin{Bmatrix} l \\ jk \end{Bmatrix} = \begin{Bmatrix} l \\ kj \end{Bmatrix} \text{ or } \Gamma^{\mathrm{l}}{}_{\mathrm{.jk}} = \Gamma^{\mathrm{l}}{}_{\mathrm{kj}} \qquad \text{...(8.16)}$$

thereby indicating that *Christofell's symbols* $\Gamma_{\mathrm{l.jk}}$ *and* $\Gamma^{\mathrm{l}}{}_{\mathrm{jk}}$ *are symmetrical with respect to indices j and k.*

Relations between Christofell's symbols of first and second kind.

(i) Replacing *l* by *m* in eqn. (8.13), we get

$$\Gamma_{\mathrm{m,jk}} = \frac{1}{2}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial x_{jk}}{\partial x^m}\right)$$

Multiplying both sides of above equation by g^{lm}, we get

$$g^{lm}\Gamma_{\mathrm{m,jk}} = \frac{1}{2} g^{lm}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \qquad \text{(since } g_{jm} = g_{mj}\text{)}$$

$$= \Gamma^{\mathrm{l}}_{\mathrm{jk}} \qquad \text{[Using (8.14)]}$$

i.e., $$\Gamma^{\mathrm{l}}_{\mathrm{jk}} = g^{lm}\Gamma_{\mathrm{m,jk}} \qquad \text{...(8.17)}$$

(ii) Interchanging *l* and *m* in eqn. (8.14), we get

$$\Gamma^{\mathrm{m}}_{\mathrm{jk}} = \frac{1}{2} g^{lm}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

Multiplying above equation by g_{lm}, we get

$$g_{lm}\Gamma^{\mathrm{m}}_{\mathrm{jk}} = \frac{1}{2} g_{lm}\, g^{lm}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \quad \text{(since } g_{lm}\, g^{ml} = \delta^l_l = 1\text{)}$$

$$= \Gamma_{\mathrm{l,jk}}$$

8.5. GEODESICS

In Euclidean three dimensional space the path of shortest distance between two fixed points is a straight line. Here we shall generalise this fundamental concept to Riemannian space.

The path of extremum **(maximum or minimum)** ***distance between any two points in Riemannian space is called the geodesic.*** Thus a geodesic is determined by the condition that the path between two fixed points A and B given by be extremum, *i.e.,*

$$\int_A^B ds$$

$\int_A^B ds$ extremum (or stationary), ...(8.18)

i.e.,

$$\delta \int_A^B ds = 0 \qquad \text{...(8.19)}$$

where δ represents the variation symbol.

In Riemannian space, we have

$$ds^2 = g_{jk}\, dx^j\, dx^k \qquad \text{...(8.20)}$$

Keeping the end points A and B fixed, let the path be deformed by giving every intermediate point an arbitrary infinitesimal displacement δx^m, so that expression (8.20) yields

$$2\, ds\, \delta\, (ds) = \delta\, (g_{jk})\, dx^j\, dx^k + g_{jk}\, \delta\, (dx^j)\, dx^k + g_{jk}\, dx^j\, \delta\, (dx^k)$$

$$= dx^j\, dx^k \frac{\partial g_{jk}}{\partial x^m} \delta x^m + g_{jk}\, dx^k\, \delta\, (dx^j) + g_{jk}\, dx^j\, \delta\, (dx^k).$$

Dividing both sides by 2 ds and using the relation

$$\delta \left(\frac{\partial x^j}{ds}\right) = \frac{d}{ds}(\delta x^j);$$

We get $(ds) = \frac{1}{2}\left\{\frac{dx^j}{ds} + \frac{dx^k}{ds} - \frac{\partial g_{jk}}{\partial x^m}\delta x^m + g_{jk}\frac{dx^k}{ds}\frac{d}{ds}(\delta x^j) + g_{jk}\frac{dx^j}{ds}\frac{d}{ds}(\delta x^k)\right\} ds.$

.....(8.21)

Substituting the value of $\delta(ds)$ from (8.21) in (8.19), we get

$$\frac{1}{2}\int_A^B \left\{\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m}\delta x^m + g_{jk}\frac{dx^j}{ds}\frac{d}{ds}\left(\delta x^k\right) + g_{jk}\frac{dx^k}{ds}\frac{d}{ds}\left(\delta x^j\right)\right\} ds = 0$$

On changing the dummy indices in the last two terms, we get

$$\frac{1}{2}\int_A^B \left\{ \frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m}\delta x^m + \left(g_{jm}\frac{dx^j}{ds} + g_{mk}\frac{dx^k}{ds} \right)\frac{d}{ds}.\left(\delta x^m\right) \right\} ds = 0.$$

Integrating the second term by parts and remembering that the variation δ is zero at the fixed end points A and B,

$$\frac{1}{2}\int_A^B \left\{ \frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m} - \frac{d}{ds}\left(g_{jm}\frac{dx^j}{ds} + g_{mk}\frac{dx^k}{ds} \right) \right\} \delta x^m ds = 0.$$

As the infinitesimal displacements δx^m are arbitrary, therefore for the integral to be stationary the coefficient of δx^m in the integrand must vanish at all points on the path, i.e.,

$$\frac{1}{2}\left[\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m} - \frac{d}{ds}\left(g_{jm}\frac{dx^j}{ds} + g_{mk}\frac{dx^k}{ds} \right) \right] = 0$$

i.e.,

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m} - \frac{1}{2}\frac{dg_{jm}}{ds}\frac{dx^j}{ds} - \frac{1}{2} g_{mk}\frac{d^2x^k}{ds^2} = 0.$$

$$-\frac{1}{2}\frac{dg_{mk}}{ds}\frac{dx^k}{ds} - \frac{1}{2} g_{mk}\frac{d^2x^k}{ds^2} = 0.$$

...(8.22)

But we have

$$\frac{dg_{jm}}{ds} = \frac{\partial g_{jm}}{\partial x^k}\frac{dx^k}{ds} \; and \; \frac{dg_{mk}}{ds} = \frac{\partial g_{mk}}{\partial x^j}\frac{dx^j}{ds}$$

With these substitutions, equation (8.21) becomes

i.e.,

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{mk}}{\partial x^j}\right) - \frac{1}{2}\left(g_{jm}\frac{d^2x^j}{ds^2} + g_{mk}\frac{d^2x^k}{ds^2}\right) = 0.$$

Replacing the dummy indices *j* and *k* and *l* in the second bracketed terms, we get

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{mk}}{\partial x^j}\right) - \frac{1}{2}\left(g_{jm}\frac{d^2x^l}{ds^2} + g_{mk}\frac{d^2x^l}{ds^2}\right) = 0.$$

Using symmetry property of g_{lm} (i.e., $g_{lm} = g_{ml}$) above equation mau be written as

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial g_{mj}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) + g_{lm}\frac{d^2x^l}{ds^2} = 0.$$

Now multiplying throughout by g^{mp}, we get

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}g^{mp}\left(\frac{\partial g_{mj}}{\partial x^k}+\frac{\partial g_{km}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^m}\right)+g_{lm}g^{mp}\frac{d^2x^l}{ds^2}=0$$

or

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}g^{mp}\Gamma_{m.jk}+\delta_l^p\frac{d^2x^l}{ds^2}=0$$

i.e.,

$$\frac{d^2x^p}{ds^2}+\frac{dx^j}{ds}\frac{dx^k}{ds}g^{mp}\Gamma_{m.jk}=0 \qquad \text{...(8.23)}$$

or

$$\frac{d^2x^p}{ds^2}+\frac{dx^j}{ds}\frac{dx^k}{ds}\Gamma_{jk}^p=0. \qquad \text{...(8.24)}$$

Equation (8.24) represents the required condition to be satisfied in order that the integral be stationary. Hence equation (8.24) represents the differential equation of a *geodesic*. For p = 1, 2, 3, 4 this equation gives four differential equations which determine a *geodesic*.

THANKS