MATHEMATICAL PHYSICS
UNIT – 6
LAPLACE TRANSFORM AND APPLICATION

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1. INTRODUCTION

The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782. Laplace transforms is an integral transform. It helps in solving the differential equations with boundary values without finding the general solution and values of the arbitrary constants. The method of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions.

2. OBJECTIVES

After studying this chapter we will learn about how Laplace transforms is useful for solving differential equations with boundary values without finding the general solution. With the use of different properties of Laplace transform and Inverse Laplace transform one can solve many important problem of physics with very simple way. Thus we will learn from this unit to use the Laplace transform for solving the differential equations.

3. LAPLACE TRANSFORM

Definition: The Laplace transform of a function *f(t)* is defined as follows

$$\mathrm{F}\,(\mathrm{s}) = \int_0^\infty \mathrm{e}^{-\mathrm{st}}\, f(t)dt$$

For all positive values of *t* and integral should exits. The Laplace transform is denoted by

$$\boldsymbol{L[f(t)] = F\,(s) = \int_0^\infty e^{-st} f(t)dt}$$

4. LINEARITY OF THE LAPLACE TRANSFORM

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\boldsymbol{L[af(t) + bg(t)] = aL[f(t)] + b[g(t)]}$$

Proof:

$$L[af(t) + bg(t)] = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$$

As we know that integration is a linear operation. So we can use the linearity property of integration in above equation

$$L[af(t) + bg(t)] = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt$$

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

5. CHANGE OF SCALE PROPERTY

If the Laplace transform *of* $f(t)$ is $F(s)$ then

$$\boldsymbol{L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}$$

Proof: From the definition of Laplace transform

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

$$put\ at = r \Longrightarrow dt = \frac{dr}{a}\ and\ also\ t = \frac{r}{a}$$

$$\Rightarrow L[f(at)] = \int_0^\infty e^{-\frac{sr}{a}} f(r)\frac{dr}{a} = \frac{1}{a}\int_0^\infty e^{-Sr} f(r)dr \quad \left[where\ S = \frac{s}{a}\right]$$

$$= \frac{1}{a}F(S) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

6. FIRST SHIFTING THEOREM

If $F(s)$ has the Laplace transform *of* $f(t)$ then

$$\boldsymbol{L[e^{at}f(t)] = F(s-a)}$$

Proof: Using the definition of Laplace transform

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-st+at} f(t)dt = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-st}\{e^{at} f(t)\}dt = L[e^{at} f(t)]$$

Alternative Method:

$$L[e^{at} f(t)] = \int_0^\infty e^{at} e^{-st} f(t)dt = \int_0^\infty e^{-st+at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt = \int_0^\infty e^{-ut} f(t)dt$$

Using $(s-a) = u$

$$= F(u) = F(s-a)$$

7. SECOND SHIFTING THEOREM (HEAVISIDE'S SHIFTING THEOREM)

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a), & for\ t > a \\ 0, & for\ 0 < t < a \end{cases}$

Then $L[g(t)] = e^{-as}F(s)$

Proof: As per the definition of Laplace transform

$$L[g(t)] = \int_0^\infty e^{-st}\, g(t)dt$$

$$L[g(t)] = \int_0^a e^{-st}\, g(t)dt + \int_a^\infty e^{-st} g(t)dt$$

Using the given condition $g(t) = 0\ for\ 0 < t < a\ and\ g(t) = f(t-a)\ for\ t > a$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)dt$$

$Now\ using\ (t-a) = r \Rightarrow dt = dr\ and\ t = (r+a)$ we get

$$L[g(t)] = \int_0^\infty e^{-s(r+a)} f(r)dr = e^{-sa}\int_0^\infty e^{-sr} f(r)dr = e^{-sa}F(s)$$

Hence $\Rightarrow$ $L[g(t)] = e^{-as}F(s)$

8. LAPLACE TRANSFORM OF THE DERIVATIVE OF $\boldsymbol{f(t)}$

If $L[f(t)] = F(s)$ and $f'(t)$ is the derivative of $f'(t)$ then

$$\boldsymbol{L[f'(t)] = sL[f(t)] - f(0)}$$

Proof: As we know

$L[f'(t)] = \int_a^\infty e^{-st} f'(t)dt$

Solving above equation using integration by parts we get

$$L[f'(t)] = [e^{-st} f(t)]_0^\infty - \int_0^\infty (-se^{-st}) f(t)dt$$

As we know that

$$e^{-\infty} = 0 \; and \; e^0 = 1 \Rightarrow e^{-st} f(t) = 0 \; when \; t = \infty \; and \; e^{-st} f(t) = f(0) when \; t = 0$$

$$\Rightarrow L[f'\ (t)] = -f(0) + s\int_0^\infty e^{-st} f(t)dt = -f(0) + sL[f(t)$$

$$\Rightarrow \boldsymbol{L[f'\ (t)] = sL[f(t)] - f(0)}$$

9. LAPLACE TRANSFORM OF THE DERIVATIVE OF ORDER *n*

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$$

Proof: As we know that the Laplace transform of derivative is given by

$$L[f'(t)] = sL[f(t)] - f(0) \ldots\ldots\ldots\ldots\ldots\ldots. \quad 1$$

Using this equation we can find the Laplace transform of $[f''(t)]$

$$L[f''(t)] = sL[f'(t)] - f'(0)$$

Using equation 1 we get

$$L[f''(t)] = s\{sL[f(t)] - f(0)\} - f'(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0) \ldots\ldots\ldots\ldots.2$$

Similarly we can find the value of $L[f'''(t)]$ by using equation 1 &2

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0) \ldots\ldots\ldots\ldots.3$$

Similarly, using above method we get

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$$

10. LAPLACE TRANSFORM OF THE INTEGRAL OF $f(t)$

If $L[f(t)] = F(s)$ and $f'(t)$ is the derivative of $f'(t)$ then

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

Proof: Let $g(t) = \int_0^t f(t)$ $\;and\; g(0) = 0$ then $g'(t) = f(t)$

As we know that $L[g'(t)] = sL[g(t)] - g(0)$

$$\Rightarrow L[g'(t)] = sL[g(t)] \quad as\; g(0) = 0$$

$$\Rightarrow \quad L[g(t)] = \frac{1}{s}L[g'(t)]$$

Using the value of $g(t) = \int_0^t f(t)$ $\;and\; g'(t) = f(t)$ we will get

$$\Rightarrow L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

$$\Rightarrow L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

11. LAPLACE TRANSFORM OF SOME IMPORTANT FUNCTIONS

1. $\boldsymbol{L(1) = \frac{1}{s}}$

Proof: From the definition of Laplace transform, the Laplace transform of *L(1)* can be written as

$$L(1) = \int_0^\infty 1.e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$$

$$[As\ e^{-\infty} = 0\ and\ e^0 = 1]$$

2. $\boldsymbol{L(e^{at}) = \frac{1}{s-a}}$ $where\ s > a$

Proof: As per the definition of Laplace transform

$$L(e^{at}) = \int_0^\infty e^{at}.e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a} \qquad [As\ e^{-\infty} = 0\ and\ e^0 = 1]$$

3. $L(\sin at) = \frac{a}{s^2+a^2}$

Proof: $L(\sin at) = L\left(\frac{e^{iat} - e^{-iat}}{2i}\right) = \frac{1}{2i}\left[L\left(e^{iat}\right) - L\left(e^{-iat}\right)\right]$

$$\text{Since } \sin\theta = \frac{1}{2i}\left[e^{i\theta} - e^{-i\theta}\right]$$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\sin at) = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{(s+ia)-(s-ia)}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2}$$

4. $L(\cos at) = \frac{s}{s^2+a^2}$

Proof: $L(\cos at) = L\left(\frac{e^{iat} + e^{-iat}}{2}\right) = \frac{1}{2}\left[L\left(e^{iat}\right) + L\left(e^{-iat}\right)\right]$

$$\text{Since } \cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\cos at) = \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2}\left[\frac{(s+ia)+(s-ia)}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2+a^2}\right] = \frac{s}{s^2+a^2} \qquad [\text{as we know } i^2 = -1]$$

5. $\boldsymbol{L(\sinh at) = \frac{a}{s^2 - a^2}}$

Proof: $L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2i}\right) = \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$

Since $\sinh\theta = \frac{1}{2}\left[e^{\theta} - e^{-\theta}\right]$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\sin at) = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right] = \frac{a}{s^2 - a^2}$$

6. $\boldsymbol{L(\cosh at) = \frac{s}{s^2 - a^2}}$

Proof: $L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$

Since $\cos\theta = \frac{1}{2}\left[e^{\theta} + e^{-\theta}\right]$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\cosh at) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right] = \frac{s}{s^2 - a^2}$$

7. $\boldsymbol{L(t^n) = \frac{n!}{s^{n+1}}}$ ***where n and s are positive***

Proof: $\boldsymbol{L(t^n)} = \int_0^\infty t^n . e^{-st} dt$

$$now\ using \quad st = u \Rightarrow t = \frac{u}{s} \quad \Rightarrow dt = \frac{du}{s}$$

We will get

$$L(t^n) = \int_0^\infty \frac{u^n}{s^n} . e^{-u} \frac{du}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du$$

$$we\ know\ that \int_0^\infty e^{-u} u^n \, du = \Gamma(n+1) = n!$$

Hence we have $L(t^n) = \frac{n!}{s^{n+1}}$

Example 1: Find the Laplace transform of $sin^3 2t$

Solution: we have given $f(t) = sin^3 2t$

And we also know that $\sin 3\theta = 3\sin\theta - 4sin^3\theta$

From above equation $sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 6t]$

Hence $L[sin^3 2t] = \frac{1}{4}[3\,\mathrm{L}(\sin 2t) - \mathrm{L}(\sin 6t)]$

$$= \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right]$$

$$As\ we\ know\ that\quad L(\sin at) = \frac{a}{s^2+a^2}$$

$$= \frac{6}{4}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

Example 2: Find the Laplace transform of $sin\ 2t\ sin3t$

Solution: we have given $f(t) = sin\ 2t\ sin3t$

$$Using\ relation\ \rightarrow sin\ A\ sinB = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\gg sin\ 2t\ sin3t = \frac{1}{2}[\cos t - \cos 5t]$$

So $L(sin\ 2t\ sin3t) = \frac{1}{2}[L(\cos t) - \mathrm{L}(\cos 5t)]$

Now using relation $L(\cos at) = \frac{s}{s^2+a^2}$

We have $L(sin\ 2t\ sin3t) = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right] = \frac{12s}{(s^2+1)(s^2+25)}$

Example 3: Show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$ *Given that* $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$

Solution: Suppose $F(t) = \left(2\sqrt{\frac{t}{\pi}}\right)$ *then*

$$F'(t) = \frac{1}{\sqrt{\pi t}} \text{ and also we can see that } F(0) = 0$$

Now we know that $L[F'(t)] = sL[F(t)] - F(0)$

Hence $L\left(\frac{1}{\sqrt{\pi t}}\right) = sL\left(2\sqrt{\frac{t}{\pi}}\right) - 0 = s.\frac{1}{s^{3/2}} - 0$

$$\Rightarrow L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$$

Example 4: Find the Laplace transform of $t + t^2 + t^3$

Solution: we have given $f(t) = t + t^2 + t^3$

Now using the relation $L(t^n) = \frac{n!}{s^{n+1}}$

We have$L[f(t)] = L(t) + L(t^2) + L(t^3) = \frac{1}{s^2} + \frac{2}{s^3} + \frac{6}{s^4}$

Example 5: Find the Laplace transform of $t\ cosh\ at$

Solution: we have given $f(t) = t\ cosh\ at$

We know that $L(cosh\ at) = \frac{s}{s^2-a^2}$

Now using the relation $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$

We will get $L(t\ cosh\ at) = -\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right) = -\frac{(s^2-a^2).1-s.2s}{(s^2-a^2)^2}$

$$= -\frac{(s^2-a^2-2s^2}{(s^2-a^2)^2}$$

$$= \frac{(s^2+a^2)}{(s^2-a^2)^2}$$

12. LAPLACE TRANSFORM OF $\frac{1}{t}f(t)$

If $L[f(t)] = F(s)$ **then If** $L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} f(s)\,ds$

Proof: As per the Laplace transform

$$L[f(t)] = F\,(s) = \int_0^{\infty} e^{-st} f(t)dt$$

Integrating with respect to s, we get

$$\int_s^{\infty} F(s)ds = \int_s^{\infty}\left[\int_0^{\infty} e^{-st} f(t)dt\right]ds$$

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}ds\right]dt = \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= -\int_0^{\infty}\frac{f(t)}{t}[e^{-st}]_0^{\infty}dt = -\int_0^{\infty}\frac{f(t)}{t}[e^{-\infty} - e^{-st}]dt$$

$$= -\int_0^{\infty}\frac{f(t)}{t}[0 - e^{-st}]dt = \int_0^{\infty} e^{-st}\left[\frac{1}{t}f(t)\right]dt = L\left[\frac{1}{t}f(t)\right]$$

$$\Rightarrow L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} F(s)ds \qquad \textbf{Proved}$$

Similarly: **LAPLACE TRANSFORM OF** $t.f(t)$

$$L[t^n.f(t)] = (-1)^n\frac{d^n}{ds^n}F(s)$$

13. LAPLACE TRANSFORM OF $\boldsymbol{t.f(t)}$

$$L[t^n.f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example 6: Find the Laplace transform of $\frac{\sin 2t}{t}$

Solution: $L(\sin 2t) = \frac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4} ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty = \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$=\cot^{-1}\frac{s}{2}$$

Example 7: Find the Laplace transform of the function

$$f(t) = te^{-t}\sin 2t$$

Solution: $L[\sin 2t] = \frac{2}{s^2+4}$

$$L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+4} = F(s) \qquad say$$

$$L[e^{-t}\sin 2t] = -F'(s) = \frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right] = \frac{2.2(s+1)}{[(s+1)^2+4]^2} = \frac{4(s+1)}{[(s+1)^2+4]^2} \qquad \text{Ans.}$$

14. UNIT STEP FUNCTION

The unit step function is defined as follows:

$$\boldsymbol{u(t-a)} = \begin{cases} \mathbf{0}, & \boldsymbol{when\ t} < a \\ \mathbf{1}, & \boldsymbol{when\ t \geq a} \end{cases} \qquad \boldsymbol{where\ a \geq 0}$$

15. LAPLACE TRANSFORM OF UNIT STEP FUNCTION

$$\boldsymbol{L[u(t-a)] = \frac{e^{-as}}{s}}$$

Proof: Using the definition of Laplace transform, we have

$$\boldsymbol{L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)dt}$$

Now using the condition of unit step function

$$L[u(t-a)] = \int_0^a e^{-st} u(t-a)dt + \int_a^\infty e^{-st} u(t-a)dt$$

$$= \int_0^a e^{-st} 0.dt + \int_a^\infty e^{-st} 1.dt = 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

Example 8: Convert the following function in terms of unit step function and then find the Laplace Transform

$$f(t) = \begin{cases} 6, & when\ t < 2 \\ 4, & when\ t \geq 2 \end{cases}$$

Solution: Given that

$$f(t) = \begin{cases} 6, & when\ t < 2 \\ 4, & when\ t \geq 2 \end{cases}$$

This further can be written as

$$f(t) = \begin{cases} 6+0, & when\ t < 2 \\ 6-2, & when\ t \geq 2 \end{cases} = 6 + \begin{cases} 0, & when\ t < 2 \\ -2, & when\ t \geq 2 \end{cases}$$

$$= 6 + (-2)\begin{cases} 0, & when\ t < 2 \\ 1, & when\ t \geq 2 \end{cases} = 6 - 2u(t-2)$$

[Using the condition of unit step function]

$$L[f(t)] = 6L(1) - 2L[u(t-2)] = \frac{6}{s} - 2\frac{e^{-2s}}{s}$$

16. PERIODIC FUNCTIONS

If $f(t)$ be a periodic function with period $T, \Rightarrow f(t+T) = f(t)$ then

$$\boldsymbol{L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}}$$

Proof: As we know

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

This can be written as in the following manner

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \cdots$$

Now substituting $t = u + T,\ t = u + 2T,\ \ldots and\ dt = du$ in second integral, third integral, and so on respectively, we will get

$$L[f(t)] = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \cdots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^T e^{-su} f(u+T)du + e^{-2sT} \int_0^T e^{-su} f(u+2T)du + \cdots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^T e^{-su} f(u)du + e^{-2sT} \int_0^T e^{-su} f(u)du + \cdots$$

As $f(u)$ be a periodic function with period $T, \Rightarrow f(u+T) = f(u+2T) = \cdots = f(u)$

Now we can write

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^T e^{-st} f(t)du + e^{-2sT} \int_0^T e^{-st} f(u)dt + \cdots$$

$$= \int_0^T e^{-st} f(t)dt[1 + e^{-sT} + e^{-2sT} + \cdots]$$

Now using the condition $[1 + x + x^2 + x^3 + \ \ldots = \frac{1}{1-x}$ we have

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

Example 8: Find the Laplace transform of the waveform

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3.$$

Solution: $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)dt$

$$L\left[\frac{2t}{3}\right] = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st} \left(\frac{2}{3}t\right) dt = \frac{1}{1-e^{-3s}} \frac{2}{3} \left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3} \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] = \frac{2}{3} \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s(1-e^{-3s})} + \frac{2}{3s^2}$$

17. SOME IMPORTANT FORMULAE OF LAPLACE TRANSFORM

S.No.	$f(t)$	$F(s)$
1	e^{at}	$\frac{1}{s-a}$
2	$\sin at$	$\frac{a}{s^2+a^2}$
3	$\cos at$	$\frac{s}{s^2+a^2}$
4	$\sinh at$	$\frac{a}{s^2-a^2}$
5	$\cosh at$	$\frac{s}{s^2-a^2}$
6	t^n	$\frac{n!}{s^{n+1}}$
7	$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$
8	$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2+a^2}$
9	$\frac{t}{2a}\sin at$	$\frac{s}{(s^2+a^2)^2}$
10	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

18. INVERSE LAPLACE TRANSFORM

If $F(s)$ is the Laplace Transform of a function $f(t)$, then $f(t)$ is known as Inverse Laplace Transform.

$$f(t) = L^{-1}[F(s)]$$

The Inverse Laplace Transform is very useful to solving the differential equations without finding the general solution and arbitrary constants.

19.SOME IMPORTANT FORMULAE OF INVERSE LAPLACE TRANSFORM

S.No.	$F(s)$	$f(t) = L^{-1}[F(s)]$
1	$\frac{1}{s-a}$	e^{at}
2	$\frac{a}{s^2+a^2}$	$\sin at$
3	$\frac{s}{s^2+a^2}$	$\cos at$
4	$\frac{a}{s^2-a^2}$	$\sinh at$
5	$\frac{s}{s^2-a^2}$	$\cosh at$
6	$\frac{n!}{s^{n+1}}$	t^n
7	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\sin at$
8	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
9	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$
10	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
11	$\frac{1}{s}$	1

Example 9: Prove that $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$

Solution: we know that $L^{-1}\left[\frac{1}{s^n}\right] = \left[\frac{t^{n-1}}{(n-1)!}\right] = \left[\frac{t^{n-1}}{\Gamma n}\right]$

Using above relation we can write $L^{-1}\left[\frac{1}{s^{1/2}}\right] = \left[\frac{t^{\frac{1}{2}-1}}{\Gamma\frac{1}{2}}\right]$

$= \left[\frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}\right]$ $\quad Since\ \Gamma\frac{1}{2} = \sqrt{\pi}$

$$\Rightarrow L^{-1}\left[\frac{1}{s^{1/2}}\right] = \left[\frac{1}{\sqrt{\pi t}}\right] \quad \Rightarrow \left[\frac{1}{s^{1/2}}\right] = L\left[\frac{1}{\sqrt{\pi t}}\right]$$

Example 10: Find the inverse Laplace Transform of the following:

(i) $\frac{1}{s-3}$ (ii) $\frac{1}{s^2-25}$ (iii) $\frac{s}{s^2+16}$

(iv) $\frac{1}{s^2+9}$ (v) $\frac{1}{(s-2)^2+1}$ (vi) $\frac{s-1}{(s-1)^2+4}$

Solution.

(i) $L^{-1}\frac{1}{s-3} = e^{3t}$ $\left[since\ L^{-1}\frac{1}{s-a} = e^{at}\right]$

(ii) $L^{-1}\frac{1}{s^2-25} = L^{-1}\frac{1}{5}\left\{\frac{5}{s^2-(5)^2}\right\} = \frac{1}{5} sinh 5t$ $\left[since\quad L^{-1}\frac{a}{s^2-a^2} = sinh\ at\right]$

(iii) $L^{-1}\frac{s}{s^2+16} = L^{-1}\frac{s}{s^2+(4)^2} = \cos 4t$ $\left[since\ since\quad L^{-1}\frac{s}{s^2+a^2} = cos\ at\ \right]$

(iv) $L^{-1}\frac{1}{s^2+9} = L^{-1}\frac{1}{s^2+(3)^2} = \frac{1}{3}\sin 3t$ $\left[since\ since\quad L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a} sin\ at\ \right]$

(v) $L^{-1}\frac{1}{(s-2)^2+1} = e^{2t}\sin t$ $\left[since\ since\quad L^{-1}\frac{1}{(s-b)^2+a^2} = e^{bt} sin\ at\right]$

(vi) $L^{-1}\frac{s-1}{(s-1)^2+4} = L^{-1}\frac{s-1}{(s-1)^2+(2)^2} = e^{t}\cos 2t$ $\left[since\ since\quad L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at} cos\ bt\right]$

Example 11: Find $L^{-1}\frac{s^2+3s+8}{s^3}$

Solution: Here, we have

$$L^{-1}\frac{s^2+3s+8}{s^3} = L^{-1}\left[\frac{1}{s}+\frac{3}{s^2}+\frac{8}{s^3}\right]$$

$$= 1+\frac{3t}{1!}+\frac{8}{2!}t^2 \qquad \left[\, since \;\; L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}\right]$$

$$= 1+3t+4\,t^2$$

20. MULTIPLICATION BY S

$$L^{-1}[sF(s) \quad] = \frac{d}{dt} f(t) + f(0)\delta(t)$$

Example 12: Find the Inverse Laplace Transform of $\frac{s}{s^2+4}$

Solution: we know that $L^{-1}\frac{1}{s^2+a^2} = \sin at$

Hence $\quad L^{-1}\frac{1}{s^2+4} = \sin 2t$

And $\quad L^{-1}\frac{s}{s^2+4} = \frac{d}{dt}(sin\ 2t) + \sin(0)$

$$\delta(t) = 2\cos 2t \qquad Ans.$$

21. DIVISION BY s (MULTIPLICATION BY $\frac{1}{s}$)

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t [\ L^{-1}\ |F(s)|]\,dt = \int_0^t f(t)dt$$

Example 13: Find the Inverse Laplace Transform of

(i) $\frac{1}{s(s+a)}$ (ii) $\frac{1}{s(s^2+1)}$

Solution:

(i) $since\ \ L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt$$

$$= \int_0^t e^{-at}\,dt = \left[\frac{e^{-at}}{-a}\right]_0^t = \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1}{a}\left[1 - e^{-at}\right]$$

(ii) *we know that* $L^{-1}\frac{1}{(s^2+1)} = \sin t$

$$L^{-1}\frac{1}{s}\left(\frac{1}{s^2+1)}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+1}\right) = \int_0^t \sin t\, dt =$$

$$= [-\cos t]_0^t = [-\cos t + 1] = [1 - \cos t]$$

22. FIRST SHIFTING PROPERTY

If the inverse Laplace transform of $F(s)$ is $f(t)$ such that

$$L^{-1}[F(s)] = f(t)$$

Then $$L^{-1}F(s+a) = e^{-at}\, L^{-1}[F(s)]$$

Example 14: Find the Inverse Laplace Transform of

(i) $\frac{1}{(s+4)^4}$ (ii) $\frac{s}{s^2+4s+13}$ (iii) $\frac{1}{9s^2+6s+1}$

Solution:

(i) $we\ know\ that$ $L^{-1}\left[\frac{1}{s^4}\right] = \frac{t^3}{3!}$

then $L^{-1}\left[\frac{1}{(s+4)^4}\right]$

$= e^{-4t}L^{-1}\left[\frac{1}{s^4}\right]$ $using\ first\ shifiting\ property$

$= e^{-4t}\frac{t^3}{3!} = \frac{1}{6}e^{-4t}t^3$

Solution (ii) $L^{-1}\left(\frac{s}{s^2+4s+13}\right) = L^{-1}\frac{s+2-2}{(s+2)^2+(3)^2}$

$$= L^{-1}\frac{s+2}{(s+2)^2+(3)^2} - L^{-1}\frac{2}{(s+2)^2+(3)^2}$$

Using First shifting property $\Rightarrow L^{-1}F(s+a) = e^{-at}\,L^{-1}[F(s)]$

$$= e^{-2t}L^{-1}\frac{s}{(s)^2+(3)^2} - e^{-2t}L^{-1}\frac{2}{3}\left(\frac{3}{s^2+3^2}\right)$$

$$= e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t$$ *Ans.*

Solution (iii) $L^{-1}\frac{1}{9s^2+6s+1} = L^{-1}\frac{1}{(3s+1)^2}$

$$= \frac{1}{9}L^{-1}\frac{1}{(s+\frac{1}{3})^2}$$

$$= \frac{1}{9}e^{-t/3}L^{-1}\frac{1}{s^2}$$ [*Using First shifting property*]

$$= \frac{1}{9}e^{-t/3}t = \frac{te^{-t/3}}{9}$$

23. SECOND SHIFTING PROPERTY

SECOND SHIFTING PROPERTY

$$L^{-1}[\, e^{-as}\ F\,(s)\,] = f(t-a)u(t-a)$$

Example 15: Obtain Inverse Laplace Transform of

(i) $\frac{e^{-\pi s}}{(s+3\,)}$ $\quad(ii)$ $\frac{e^{-s}}{(\,s+1\,)^3}$

Solution:

(i) As we know that

$$L^{-1}\ \frac{1}{s+3} = e^{-3t}$$

Now using second shifting theorem we can find the inverse Laplace transform of

$$L^{-1}\,\frac{e^{-\pi s}}{(s+3)} = e^{-3(\,t-\pi)}u(t-\pi) \qquad since\ [L^{-1}[\, e^{-as}\ F\,(s)] = f(t-a)u(t-a)]$$

(ii) *As we know that* $L^{-1}\frac{1}{s^3} = \frac{t^2}{2!}$

Then

$L^{-1}\frac{1}{(s+1)^3} e^{-t}\frac{t^2}{2!}$ $[using\ first\ shifiting\ property]$

Hence $L^{-1}\frac{e^{-s}}{(s+1)^3} =$

$e^{-(t-1)}\frac{(t-1)^2}{2!}\ u(t-1)$ $[using\ second\ shifiting\ property]$

24. INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

$$L^{-1}\left[\frac{d}{ds}\ F\ (s)\right] = -tL^{-1}[F(s)] = -tf(t)$$

$$\Rightarrow L^{-1}\ [F(s)] = -\ \frac{1}{t}L^{-1}\left[\frac{d}{ds}\ F\ (s)\right]$$

Example 16: Find $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$.

Solution: $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(\frac{s+1}{s-1})\right]$

using inverse laplace transform of derivatives

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+1) - \frac{d}{ds}\ \log(s-1)\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{t}[e^{-t} - e^{t}] = \frac{1}{t}[e^{t} - e^{-t}] \qquad Ans.$$

25. INVERSE LAPLACE TRANSFORM OF INTEGRALS

$$L^{-1}\left[\int_s^\infty F(s)ds\right] = \frac{f(t)}{t} = \frac{1}{t}L^{-1}[F(s)]$$

$$\Rightarrow L^{-1}[F(s)] = L^{-1}\left[\int_s^\infty F(s)ds\right]$$

Example17: Find the Inverse Laplace Transform of $\frac{2s}{(s^2+1)^2}$

Solution: we have to find $L^{-1}\left(\frac{2s}{(s^2+1)^2}\right)$

We will solve this using inverse Laplace transform of integrals

$$L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = tL^{-1}\int_s^\infty \frac{2sds}{(s^2+1)^2}$$

$$tL^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty = tL^{-1}\left[-0+\frac{1}{s^2+1}\right] = tL^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= t \sin t$$

26. INVERSE LAPLACE TRANSFORM BY PARTIAL FRACTION METHOD

Example 18: Find the Inverse Laplace Transform of $\frac{1}{s^2-5s+6}$.

Solution: Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}$$

Example 19: Find the Inverse Laplace Transform of $\frac{s+1}{s^2-6s+25}$.

Solution:

$$L^{-1}\left[\frac{1}{s^2-6s+25}\right] = L^{-1}\left[\frac{1}{(s-3)^2+(4)^2}\right] = L^{-1}\left[\frac{s-3+4}{(s-3)^2+(4)^2}\right]$$

$$= L^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right] + L^{-1}\left[\frac{4}{(s-3)^2+(4)^2}\right]$$

$$= e^{3t}\cos 4t + e^{3t}\sin 4t \qquad [Using\ first\ shifiting\ property]$$

27. SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants. The method wil be clear from the following examples:

Let us now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t), \qquad y(0) = K_0 ,$$
$$and\ y'(0) = K_1$$

where *a* and *b* are constant. Here is the given **input** (*driving force*) applied to the mechanical or electrical system and is the **output** (*response to the input*) to be obtained.

In Laplace's method we do three steps:

Step 1. Setting up the subsidiary equation.

This is an algebraic equation for the transform $Y = L(y)$ obtained by transforming the given differential equation using the Laplace transform of derivatives,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

Where $R(s) = L(r)$

Now collecting the *Y*-terms, we have the subsidiary equation as follows

$$(s^2 + as + b)Y = (s + 1)y(0) + y'(0) + R(s)$$

Step 2. Solution of the subsidiary equation by algebra.

We divide by and use the so-called **transfer function**

$$Q(s) = \frac{1}{(s^2 + as + b)}$$

This gives the solution

$$Y = (s + 1)y(0)Q(s) + y'(0)Q(s) + R(s)Q(s)$$

Note that ***Q depends neither on r*(t) *nor on the initial conditions*** (but only on *a* and *b*).

Step 3. Inversion of Y to obtain $\boldsymbol{y = L^{-1}Y}$

Now take the inverse Laplace transform to get the solution of differential equations.

Example 20: Solve the following equation by Laplace transform

$$y'' - y = t\,; \qquad y(0) = 1 \; and \; y'(0) = 1$$

Solution: The given equation is

$$y'' - y = t\,; \qquad y(0) = 1 \; and \; y'(0) = 1$$

Step 1: Taking the Laplace transform of the given equation, we get the subsidiary equation

$$L(y'') - L(y) = L(t)$$

Now using the condition of Laplace transform of derivatives with $L(y) = Y$

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$$

We get

$$s^2 Y - sy(0) - y'(0) - Y = \frac{1}{s^2}$$

$$(s^2 - 1)Y = sy(0) + y'(0) + \frac{1}{s^2} = s + 1 + \frac{1}{s^2}$$

$$As\ given y(0) = 1 \; and \; y'(0) = 1$$

Step 2: Transfer function is given by

$$Q(s) = \frac{1}{(s^2-1)} \qquad \text{and hence}$$

$$Y = (s+1)Q(s) + \frac{Q(s)}{s^2} = \frac{(s+1)}{(s^2-1)} + \frac{1}{s^2(s^2-1)}$$

On simplifying

$$Y = \frac{1}{(s-1)} + \left\{\frac{1}{(s^2-1)} - \frac{1}{s^2}\right\}$$

Step 3: Now taking the inverse Laplace transform to get the solution of differential equation

$$y(t) = L^{-1}Y = L^{-1}\left\{\frac{1}{(s-1)}\right\} + L^{-1}\left\{\frac{1}{(s^2-1)} - \frac{1}{s^2}\right\}$$

$$y(t) = L^{-1}\left\{\frac{1}{(s-1)}\right\} + L^{-1}\left\{\frac{1}{(s^2-1)}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$y(t) = e^t + \sinh t - t \qquad Ans.$$

28. SELF ASSESSMENT QUESTION

- **Self Assessment Question (SAQ) 1:** Find the Laplace transform of $2sin\,2t\cos 4\,t$
- **Self Assessment Question (SAQ) 2:** Find the Laplace transform of $t^{\frac{1}{2}}$
- **Self Assessment Question (SAQ) 3:** Find the Laplace transform of $F(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & 1 \leq t < 2 \\ t^2, & 2 \leq t < \infty \end{cases}$
- **Self Assessment Question (SAQ) 4:** Find the Laplace transform of $1 + sin\,2t$
- **Self Assessment Question (SAQ) 5:** Find the Laplace transform of $sinh^3\,t$

- **Self Assessment Question (SAQ) 6:** Find the Laplace transform of $t\, cost$
- **Self Assessment Question (SAQ) 7:** Find the Laplace transform of $\frac{1}{t} sin^2 t$
- **Self Assessment Question (SAQ) 8:** Find the Laplace transform of

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 0 & elsewehre \end{cases}$$

- **Self Assessment Question (SAQ) 9:** Find the Laplace transform of the periodic function

$$f(t) = e^t \quad for\ 0 < t < 2\pi$$

- **Self Assessment Question (SAQ) 10:** Find the Inverse Laplace transform of $\frac{1}{s-5}$
- **Self Assessment Question (SAQ) 11:** Find the Inverse Laplace transform of $\frac{2s-5}{9s^2-25}$
- **Self Assessment Question (SAQ) 12:** Find the Inverse Laplace transform of $\frac{s^2}{s^2+a^2}$
- **Self Assessment Question (SAQ) 13:** Find the Inverse Laplace transform of $\frac{1}{s(s^2+a^2)}$

- **Self Assessment Question (SAQ) 14:** Find the Inverse Laplace transform of $\frac{s}{(s+7)^4}$
- **Self Assessment Question (SAQ) 15:** Find the Inverse Laplace transform of $\frac{e^{-s}}{(s+2)^3}$
- **Self Assessment Question (SAQ) 16:** Find the Inverse Laplace transform by partial fraction method of $\frac{1}{s^2-7s+12}$
- **Self Assessment Question (SAQ) 17:** Solve the differential equation using Laplace transform method

$$\frac{d^2y}{dx^2} + y = 0, \qquad where\ y = 1\ and\ \frac{dy}{dx} = -1\ at\ x = 0$$

- **Self Assessment Question (SAQ) 18:** Solve the differential equation using Laplace transform method
- $y'' + 4y' + 4y = 6e^{-t}, \qquad where\ y(0) = -2\ and\ y'(0) = 8$

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THANKS