MATHEMATICAL PHYSICS
UNIT – 2
BESSEL'S EQUATION

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2.1 INTRODUCTION

We find the Bessel's equation while solving Laplace equation in polar coordinates by the needed of separation of variables. This equation has a number of applications in engineering.

Bessel's function are involved in

- The Oscillatory motion of a hanging chain
- Euler's theory of a circular membrane
- The studies of planetary motion
- The propagation of waves
- The Elasticity
- The fluid motion
- The potential theory
- Cylindrical and spherical waves
- Theory of plane waves
- Bessel's function are also known as cylindrical and spherical function.

2.2 BESSEL'S EQUATION

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - x^n)y = 0$$

is called the Bessel's differential equation, and particular solutions of this equation are called Bessel's fraction of order *n*.

2.3 SOLUTION OF BESSEL'S EQUATION

$$x^2 \frac{d^2y\prime}{dx^2} + x\frac{dy}{dx} + \ (x^2 - x^n)y = 0. \qquad \ldots(1)$$

Let $\sum_{r=0}^{\infty} a_r x^{m+r}$ $or\ y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots$ …(2)

So that $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(m+r)\, x^{m+r-1}$

and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2}$

Substituting these values in (1), we get

$$x^2 \sum_{r=0}^{\infty} a_r(m\underline{+}r)(m\underline{+}r-1)x^{m+r-2} + x\sum_{r=0}^{\infty} a_r(m+\underline{r})x^{m+r-1} + (x^2 - n^2)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1)x^{m+r} + \sum_{r=0}^{\infty} a_r (m+r)x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r [(m+r)(m+r-1) + (m+r) - n^2] x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r [(m+r)^2 - n^2] x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0.$$

Equating the coefficient of lowest degree term of x^m in the identity (3) to zero, by putting $r = 0$ in the first summation we get the indicial equation.

$$a_0[m+0)^2 - n^2] = 0. \qquad (r = 0)$$

$$\Rightarrow \qquad m^2 = n^2 \text{ i.e. } m = n, m = -n \qquad a_0 \neq 0$$

Equating the coefficient of the next lowest degree term x^{m+1} in the identity (3), we put r = 1 in the first summation

$$a_1\,[m+1)^2 - n^2] = 0 \text{ i.e. } a_1 = 0, \text{ since } m+1)^2 - n^2 \neq 0$$

Equating the coefficient of x^{m+r+2} in (3) to zero, to find relation in successive coefficients, we get

$$a_{r+2}[(m+r+2)^2 - n^2] + a_r = 0$$

$$\Rightarrow \quad a_{r+2} = -\frac{1}{(m+r+2)^2 - n^2} \cdot a_r$$

Therefore, $a_3 = a_5 = a_1 = \ldots. = 0$, since $a_1 = 0$

If r = 0, $a_2 = -\frac{1}{(m+2)^2 - n^2} \cdot a_0$

If r = 2, $a_4 = -\frac{1}{(m+4)^2 - n^2}\, a_2 = \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2}\, a_0$ and so on.

On substituting the values of the coefficients a_1, a_2, a_3, a_4 …….. in (2), we have

$$y = a_0 x^m = -\frac{a_0}{(m+2)^2 - n^2} x^{m+2} + \frac{a_0}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^{m+4} + \cdots$$

$$y = a_0 x^m = \left[1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^4 - \cdots\right]$$

For $m = n$

$$y = a_0 x^n \left[1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2.2!(n+1)(n+2)} x^4 - \cdots\right]$$

where a_0 is an arbitrary constant.

For $m = -n$

$$y = a_0 x^{-n} \left[1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2.2!(-n+1)(-n+2)} x^4 - \cdots\right]$$

2.4 BESSEL'S FUNCTIONS, J_n (x)

The Bessel's equation is $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \ (x^2 - x^n)y = 0.$ ...(1)

Solution of (1) is

$$y = a_0x^{-n}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots + (-1)^r \frac{x^{2r}}{(2^r r!).2^r(n+1)(n+2)\ldots(n+r)} + \cdots\right]$$

$$= a_0 x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r}.r!(n+1)(n+2)\ldots(n+r)}$$

where a_0 is an arbitrary constant.

If $a_0 = \frac{1}{2^n \sqrt{(n+1)}}$

The above solution is called Bessel's function denoted by J_n (x).

Thus $J_n(x) = \frac{1}{2^n\sqrt{(n+1)}}\sum(-1)^r \frac{x^{n+2r}}{2^{2r}.r!(n+1)(n+2)...(n+r)}$ $\quad (\sqrt{n+1} = n!)$

$$\Rightarrow \quad J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{\sqrt{(n+1)}} - \frac{1}{1!\sqrt{(n+2)}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\sqrt{(n+3)}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\sqrt{(n+4)}}\left(\frac{x}{2}\right)^6 + \cdots\right\}$$

$$\Rightarrow \quad J_n(x) = \frac{x^n}{2^n\sqrt{n+1}}\left[1 - \frac{x^2}{2.(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} + \cdots\right] \quad ...(2)$$

$$\Rightarrow \quad J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\sqrt{(n+r+1)}}\left(\frac{x}{2}\right)^{n+2r} \Rightarrow J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\sqrt{(n+r)!}}\left(\frac{x}{2}\right)^{n+2r}$$

If n = 0, $J_0(x) = \sum\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r} \Rightarrow J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots$

If n = 1, J_1 (x) $= \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \cdots$

We draw the length of these two functions. Both the functions are oscillatory with a varying period and a decreasing amplitude.

Replacing *n* by – *n* in (2), we get J_{-n} (x) $= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{(-n+r+1)}} \left(\frac{x}{2}\right)^{-n+2r}$

Case I. If *n* is not integer or zero, then complete solution of (1) is

Case II. If n = 0, then $y_1 = y_2$ and complete solution of (1) is the Bessel's function of order zero.

Case III. If *n* is positive integer, then y_2 is not solution of (1). And y_1 fails to give a solution for negative values of n. Let us find out the general solution when *n* is an integer.

2.5 Bessel's function of the second kind of order *n*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad \ldots(1)$$

Let $y = u(x)\, J_n(x)$ be the second of the Bessel's equation when *n* integer.

$$\frac{dy}{dx} = u' J_n + u J'_n$$

$$\frac{d^2y}{dx^2} = u'' J_n + 2u' J'_n + u J''_n$$

Substituting these values of y, y', y^n in (1), we get

$$x^2 (u'' J_n + 2u' J'_n + u J''_n) + x(u' J_n + u.J'_n) + (x^2 - n^2)\, u J_n = 0$$

$$\Rightarrow \quad u\,[x^2 J^n{}_n + x J'_n + (x^2 - n^2) J_n] + x^2 u'' J_n + 2x^2 u' J_n + x u' J_n = 0 \qquad \ldots(2)$$

$$\Rightarrow \quad x^2 J''_n + x J'_n + (x^2 - n^2) J_n = 0 \qquad [\text{Since } J_n \text{ is a solution of (1)}]$$

(2) becomes $x^2 u'' J_n + 2x^2 u' J'_n + xu' J_n = 0 \qquad \ldots(3)$

Dividing (3) by x^2 u' J_n, we have

$$\frac{u^n}{u'} + 2\frac{J_n'}{J_n} + \frac{1}{x} = 0$$

(4) Can also be written as …(4)

$$\frac{d}{dx}[\log u' + 2\frac{d}{dx}[\log J_n] + \frac{d}{dx}(\log x) = 0$$

$$\Rightarrow \quad \frac{d}{dx}[\log u' + 2\log J_n + \log x] = 0$$

$$\Rightarrow \quad \frac{d}{dx}[\log(u'.J_n^2 x) = 0 \quad \text{…(5)}$$

Integrating (5), we get

$$\log u'.J_n^2.x = \log C_1$$

$$\Rightarrow \quad u'.J_n^2.x = C_1 \quad \Rightarrow \quad u' = \frac{C_1}{J_n^2.x} \quad \text{…(6)}$$

On integrating (6), we obtain

$$u = \int \frac{C_1}{J_n^2.x} dx + C_2$$

Putting the value of u in the assumed solution y = u (x). $J_n^2(x)$, we get

2.6 RECURRENCE FORMULAE

These formulae are very useful in solving the questions. So, they are to be committed to memory.

1.	$x J_n' = nJ_n - x J_{n+1}$
2.	$x J_n' = -nJ_n + x J_{n-1}$
3.	$2 J_n' = J_{n-1} - J_{n+1}$
4.	$2n J_n = x(J_{n-1} + J_{n+1})$
5.	$\frac{d}{dx}(x^{-n}J_n) = -x^{-n} J_{n+1}$
6.	$\frac{d}{dx}(x^{-n}J_n) = x^n J_{n-1}$

Formula I. x $J_n' = nJ_n - xJ_{n+1}$

Proof. We know that

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating with respect to x, we get

$$J_n' = \sum \frac{(-1)^r(n+2r)}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}$$

$$\Rightarrow \quad xJ_n' = n\sum \frac{(-1)^r}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r} + x\sum \frac{(-1)^r.2r}{2.r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= xJ_n + x\sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\sqrt{(n+s+2)}} \left(\frac{x}{2}\right)^{n+2s-1} \qquad \text{[Putting r – 1 = s]}$$

$$= nJ_n - x\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\sqrt{[(n+1)+s+1]}} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$xJ_n' = nJ_n - xJ_{n+1}$$

Proved.

Formula II. $xJ_n' = -nJ_n + xJ_{n-1}$

Proof. We know that $J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{(n+r+2)}} \left(\frac{x}{2}\right)^{n+2r}$

Differentiating w.r.t. 'x', we get $J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}$

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r [(2n+2r)-n]}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r} - n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2}{r!\sqrt{(n+r)}} \left(\frac{x}{2}\right)^{n+2r} - nJ_n$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{[(n-1)+r+1]}} \left(\frac{x}{2}\right)^{(n-1+2r)} - nJ_n$$

$\Rightarrow$ $\boldsymbol{xJ_n' = xJ_{n-1} - nJ_n}$

Formula III. $2J'_n = J_{n-1} - J_{n+1}$

Proof.

We know that

$$xJ'_n = J_n - xJ_{n+1} \quad \ldots(1) \quad \text{(Recurrence formula I)}$$

$$xJ'_n = -nJ_n + xJ_{n-1} \quad \ldots(2) \quad \text{(Recurrence formula II)}$$

Adding (1) and (2), we get

$$2xJ'_n = -xJ_{n+1} + xJ_{n-1} \;\Rightarrow\; 2J'_n = J_{n-1} - J_{n+1}$$

Formula IV. $2nJ_n = x\,(J_{n-1} + J_{n+1})$

Proof.

We know that

$$xJ_n' = nJ_n - xJ_{n+1} \quad \ldots(1) \quad \text{(Recurrence formula I)}$$

$$xJ_n' = -nJ_n + xJ_{n-1} \quad \ldots(2) \quad \text{(Recurrence formula II)}$$

subtracting (2) from (1), we get

$$0 = 2\,n\,J_n - xJ_{n+1} - xJ_{n-1}$$

$$\Rightarrow \quad 2\,n\,J_n = x\,(J_{n-1} + J_{n+1}) \quad \ldots(3)$$

Formula V. $\frac{d}{dx}(x^{-n}. J_n) = -x^{-n} J_{n+1}$

Proof. We know that $xJ'_n = nJ_n - xJ_{n+1}$

(Recurrence formula I)

Multiplying by x^{-n-1}, we obtain $x^{-n} J'_n = nx^{-n-1} J_n - x^{-n} J_{n+1}$

i.e., $x^{-n} J'_n = nx^{-n-1} J_n =- x^{-n} J_{n+1}$

$\Rightarrow \qquad \frac{d}{dx}(x^{-n} J_n) = - x^{-n} J_{n+1}$

Formula VI. $\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$

Proof.

We know that $x^{-n} J_n' = -nJ_n + x J_{n-1}$ (Recurrence formula II)

Multiplying by x^{n+1}, we have

$$x^n J_n' = -nx^{n-1} J_n + x^n J_{n-1} \quad \text{i.e., } x^n J_n' + nx^{n-1} J_n = x^n J_{n-1}$$

$$\Rightarrow \quad \frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

2.7 ORTHOGONALITY OF BESSEL FUNCTION

Proof. We know that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\alpha^2x^2-n^2)y=0 \qquad \ldots(1)$$

$$\Rightarrow \quad x^2\frac{d^2z}{dx^2}+x\frac{dz}{ddx}+(\beta^2x^2-n^2)z=0 \qquad \ldots(2)$$

Solution of (1) and (2) are y = $J_n(\alpha x)$, z = $J_n(\beta x)$ respectively.

Multiplying (1) by $\frac{z}{x}$ and (2) by $-\frac{y}{x}$ and adding, we get

$$x\left(z\frac{d^2y}{dx^2}-y\frac{d^2z}{dx^2}\right)+\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)+(\alpha^2-\beta^2)xyz=0.$$

$$\Rightarrow \frac{d}{dx}\left[x\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right]+(\alpha^2-\beta^2)xyz=0 \qquad \ldots(3)$$

Integrating (3) w.r.t. 'x' between the limits 0 and 1, we get

$$\frac{d}{dx}\left[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_0^1 + (\alpha^2 - \beta^2)\int_0^1 x\, y\, z\, dx = 0$$

$$\Rightarrow \quad (\beta^2 - \alpha^2)\int_0^1 x\, y\, z\, dx = \left[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_0^1 = \left[z\frac{dy}{dx} - y\frac{dz}{dx}\right]_{x=1} \qquad \ldots(4)$$

Putting the values of $y = J_n(\alpha x), \frac{dy}{dx} = \alpha J_n'(\alpha x), z = J_n(\beta x), \frac{dz}{dx} = \beta, J_n'(\beta x)$ in (4), we get

$$(\beta^2 - \alpha^2)\int_0^1 xJ_n(\alpha x).J_n(\beta x)dx = [\alpha J_n'(\alpha x)J_n(\beta x) = \beta J_n'(\beta x)J_n(\alpha x)]_{x=1}$$

$$= \alpha J_n'(\alpha)J_n(\beta) - \beta J_n'(\beta)J_n(\alpha) \qquad \ldots(5)$$

Since α, β are the roots of $J_n(x) = 0$, so $J_n(\alpha) = J_n(\beta) = 0$

Putting the values of $J_n(\alpha) = J_n(\beta) = 0$ in (5), we get

$$(\alpha^2 - \beta^2) \int_0^1 x\, J_n(\alpha x).\, J_n(\beta x)\, dx = 0$$

$\Rightarrow$ $\int_0^1 x\, J_n(\alpha x).\, J_n(\beta x)\, dx = 0$ **Proved.**

We also know that $J_n(\alpha) = 0$. Let β be a neighboring value of α, which tends to α.

Then

$$\lim_{\beta\to\alpha} \int_0^1 xJ_n(\alpha x).J_n(\beta x)dx = \lim_{\beta\to\alpha} \frac{0+\alpha J_n^{'}(\alpha).J_n(\beta)}{\beta^2-\alpha^2}$$

As the limit is of the form $\frac{0}{0}$, we apply L' Hopital's rule

$$\int_0^1 xJ_n^2(\alpha x)dx = \lim_{\beta\to\alpha} \frac{0+\alpha J_n^{'}(\alpha).J_n^{'}(\beta)}{2\beta} = \frac{1}{2}\left[J_n^{'}(\alpha)\right]^2 \left[\because \alpha = \beta\right]$$

Proved.

2.8 A GENERATING FUNCTION FOR $J_n(x)$

Prove that $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$

Proof. We know that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

$$e^{\frac{xz}{2}} = 1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots \qquad \ldots(1)$$

$$e^{\frac{x}{2z}} = 1 - \left(\frac{x}{2z}\right) + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots \qquad \ldots(2)$$

On multiplying (1) and (2), we get

$$e^{\frac{x}{2}\left(z\frac{1}{z}\right)} = \left[1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \frac{1}{3!}\left(\frac{xz}{2}\right)^3 + \cdots\right] \times \left[1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots\right] \ldots(3)$$

The coefficient of z^n in the product of (3), we get

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots = J_n(x)$$

Similarly, coefficient of z^{-n} in the product of (3) = $J_{-n}(x)$

$$\therefore \quad e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + z J_1 + z^2 J_2 + z^3 J_3 + \ldots + z^{-1} J_{-1} + z^{-2} J_{-2} + z^{-3} J_{-3} + \ldots$$

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

For this reason $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ is known as the generating function of Bessel's functions.

Proved.

2.9 SOME EXAMPLES

Example 1. *Show that Bessel's Function $J_n(x)$ is an even function when n is even and is odd function when n is odd.*

Solution. We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{x}{2}\right)^{n+2r} \qquad \ldots(1)$$

Replacing x by – x in (1), we get

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{-x}{2}\right)^{n+2r} \qquad \ldots(2)$$

Case I. If *n* is even, then n + 2r is even $\Rightarrow \left(\frac{-x}{2}\right)^{n+2r} = \left(\frac{x}{2}\right)^{n+2r}$

Thus (2), becomes

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{x}{2}\right)^{n+2r}$$

$= J_n(x)$ $\quad \begin{bmatrix} For\ even\ function \\ f(-x) = f(x) \end{bmatrix}$

Hence, $J_n(x)$ is even function

Case II. If *n* is odd, then $n + 2r$ is odd $\Rightarrow \left(\frac{-x}{2}\right)^{n+2r} = -\left(\frac{x}{2}\right)^{n+2r}$

Thus (2). Becomes

$$J_n(-x) = -\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{x}{2}\right)^{n+2r}$$

$= -J_n(x)$ $\quad \begin{bmatrix} For\ odd\ function \\ f(-x) = -f(x) \end{bmatrix}$

Proved.

Hence, $J_n(x)$ os odd function.

Example 2. *Prove that:*

$$\lim_{x\to 0}\frac{J_n(x)}{x^n}=\frac{1}{2^n\sqrt{n+1}};(n>-1).$$

Solution. From the equation (2) of Article 29.3 on page 798, we know that

$$J_n(x)=\frac{x^n}{2^n\sqrt{n+1}}\left[1-\frac{x^2}{2.(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}-\cdots\right]$$

On taking limit on both sides when x → 0, we get

$$\lim_{x\to 0}\frac{J_n(x)}{x^n}=\lim_{x\to 0}\frac{1}{2^n\sqrt{n+1}}\left[1-\frac{x^2}{2.(2n+2)}+\frac{x^4}{2.4.(2n+2)(2n+4)}-\ldots\right]$$

$$=\frac{1}{2^n\sqrt{n+1}}$$

Example 3. *Find the value of* $J_{-1}(x) + J_1(x)$.

Solution. By using Recurrence relation IV for $J_n(x)$ is

$$2n J_n = x (J_{n-1} + J_{n+1})$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

Put $n = 0$

$$J_{-1}(X) + J_1(x) = 0$$

Example 4. Prove that

Formula V. $\frac{d}{dx}(x^{-n}. J_n) = -x^{-n} J_{n+1}$

Proof. We know that $xJ_n' = nJ_n - xJ_{n+1}$ (Recurrence formula I)

Multiplying by x^{-n-1}, we obtain $x^{-n} J_n' = nx^{-n-1} J_n - x^{-n} J_{n+1}$

i.e., $x^{-n} J_n' = nx^{-n-1} J_n =- x^{-n} J_{n+1}$

$\Rightarrow \quad \frac{d}{dx}(x^{-n} J_n) = - x^{-n} J_{n+1}$

THANKS