CLASSICAL MECHANICS

BLOCK – 4 : MECHANICS OF SYSTEM OF PARTICLES

PRESENTED BY: DR. RAJESH MATHPAL

ACADEMIC CONSULTANT

SCHOOL OF SCIENCES

U.O.U. TEENPANI, HALDWANI

UTTRAKHAND

MOB:9758417736,7983713112

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4.1 INTRODUCTION

- We have study that the applications of Newton's laws of motion require the specification of all the forces acting on the body at all instants of time.
- But in practical situations, when the constraint forces are present, the applications of the Newtonian approach may be a difficult task.
- You will see that the greatest drawback with the Newtonian procedure is that the mechanical problems are always tried to resolve geometrically rather than analytically.
- When there is constrained motion, the determination of all the insignificant reaction forces is a great bother in the Newtonian approach.

- In order to resolve these problems, methods have been developed by D'Alembert, Lagrange and others.
- The techniques of Lagrange use the generalized coordinates which will be discussed and used in this unit.
- You will see that in the Lagrangian formulation, the generalized coordinates used are position and velocity, resulting in the second order linear differential equations

4.2 OBJECTIVES

After studying this unit, you should be able to-

- understand degrees of freedom, constraints and generalized coordinates
- understand and use D'Alembert principle
- understand and use Lagrange's equation of motion
- solve the problems based on D'Alembert principle and Lagrange's equation of motion

4.3 SYSTEM OF PARTICLES

- We know that a system of particles means a group of particles inter-related. The equations for a system of particles can be readily used to develop those for a rigid body. An object of ordinary size known as a "macroscopic" system — contains a huge number of atoms or molecules. A very important concept introduced with a system of particles is the center of mass.
- Let us consider a system consisting of N point particles, each labeled by a value of the index i which runs from 1 to N. Each particle has its own mass m_i and (at a particular time) is located at its particular place r_i. The center of mass (CM) of the system is defined by the following position vector-

Center of mass $\overrightarrow{r_{CM}} = (1/M)\sum_i m_i \vec{r}_i$ .....(1)

where M is the total mass of all the particles.

- This vector locates a point in space which may or may not be the position of any of the particles. It is the mass-weighted average position of the particles, being nearer to the more massive particles. Let us consider a simple example to understand this. Let us consider a system of only two particles, of masses m and 2m, separated by a distance l. Let us choose the coordinate system so that the less massive particle is at the origin and the other is at x= l, as shown in Figure (1). Then we have $m_1 = m$, $m_2 = 2m$, $x_1 = 0$, $x_2 = l$. (The y and z coordinates are zero of course.) By the definition $x_{CM} = (1/3m)(m \times 0 + 2m \times l) = (2/3)l$

- With time, the position vectors of the particles $\vec{r_i}$ generally change, therefore, in general the center of mass (CM) moves.
- The velocity of its motion is the time derivative of its position-

$$\overrightarrow{v_{CM}} = (1/M)\sum_{i=1}^{N} m_i \vec{v}_i \quad .....(2)$$

- But the sum on the right side is just the total linear momentum of all the particles. Solving for this, we find an important result-

Total linear momentum of a system $\overrightarrow{p_{tot}} = M\overrightarrow{v_{CM}}$ .....(3)

- This is just like the formula for the linear momentum of a single particle. Therefore, we see that the total linear momentum of a system is the same as if all the particles were located together at the CM and moving with its velocity.

- The total force on the ith particle consists of the (net) external force on it, plus the net force due to the interactions with other particles in the system, which we call the internal forces. We can write this out as follows-

$$\vec{F_i} = \overrightarrow{F_i^{ext}} + \sum_j \vec{F}_{ij} \qquad .....(4)$$

where $\vec{F}_{ij}$ denotes the force exerted on the ith particle by the jth particle.

- To get the total force on the whole system, we simply add up all these forces-

$$\overrightarrow{F_{tot}} = \sum_i \vec{F}_i^{ext} + \sum_i \sum_{j\neq 1} \vec{F}_{ij} \qquad .....(5)$$

- In the double sum on the right the terms cancel in pairs by Newton's 3rd law

(for example, $\overrightarrow{F_{12}} + \overrightarrow{F_{21}}$ =0). The double sum thus gives zero, therefore-

$$\overrightarrow{F_{tot}} = \sum_i \vec{F}_i^{ext} \qquad .....(6)$$

- The total force on the system is the sum of only the external forces. Since there are very many internal forces, the fact that they give no net contribution to the total force is why it is possible for many applications to treat an object of macroscopic size as a single particle. The internal forces do play important roles in determining some aspects of the system, such as its energy.
- For each particle individually, we have Newton's 2nd law-

$$\overrightarrow{F_i} = \frac{\overrightarrow{dp_i}}{dt} \qquad .....(7)$$

- Adding these for all the particles, and using the above result for $\overrightarrow{p_{tot}}$, we find two forms of the 2nd law as it applies to systems of particles-

Newton's 2nd law for systems-

$$\overrightarrow{F_{tot}^{ext}} = \frac{\overrightarrow{dp_{tot}}}{dt} = M\,\overrightarrow{a_{CM}} \qquad .....(8)$$

Here $\overrightarrow{a_{CM}} = \frac{\overrightarrow{dv_{CM}}}{dt}$ is the acceleration of the CM. We see that the total external force produces an acceleration of the center of mass, as though all the particles were located there.

- This is not the only possible effect of the external forces. They can also cause rotational motion about the CM. But the internal forces do not change the motion of the CM.
- From the first formula above we find one of the most important laws of mechanics i.e. law of conservation of momentum. If the total external force on a system is zero, the total momentum of the system is conserved. In physics, the term “is conserved” means “remains constant in time

- The total kinetic energy of all the particles is

$$\frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i(\vec{v}_i + \vec{v}_{CM})^2 = \frac{1}{2}\sum_i m_i\left[v_i^2 + v_{CM}^2 + 2\vec{v}_i.\vec{v}_{CM}\right]$$

$$= \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\sum_i m_i v_i^2 + \vec{v}_{CM}.\sum_i m_i\vec{v}_i$$

- But the value of last term is zero, therefore we have-

$$\frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\sum_i m_i v_i^2$$

- We can interpret the terms on the right as-
    - The first term is what the kinetic energy would be if all the particles really were at the CM and moving with its speed. We often call this the kinetic energy of the CM motion.
    - The second term is the total kinetic energy as it would be measured by an observer in the CM reference frame. We call this the kinetic energy relative to the CM, or sometimes the internal kinetic energy.
- The total kinetic energy is the sum of these two terms-

  Kinetic energy of a system, K = $\frac{1}{2}Mv_{CM}^2 + K(rel. to\ CM)$
- This breakup of the kinetic energy into that of the CM plus that relative to the CM is an example of the general property. We will see that this property also holds for angular momentum. It holds for linear momentum too, but the second part, the total linear momentum relative to the CM, is always zero.

- Newton’s second law of motion says that the total (external) force is equal to the rate of change of total (linear) momentum. It follows that

$$d\vec{p}_{tot} = \vec{F}_{tot}dt$$

- Integrating both sides, we find for the net change in momentum-

$$\Delta\vec{p}_{tot} = \int_{t_0}^{t_1} \vec{F}_{tot}dt$$

This integral (over time) of the force is called the impulse. We have shown a theorem:

- We know that according to Impulse-momentum theorem, “The impulse of the total force is equal to the change of the total linear momentum.”

- One use of this relation is to define the average force that acts during a specified time interval. Let the force act for time Δt, producing a net change $\Delta\vec{p}$ in the total momentum. The average force is given by-

  Average force $\vec{F}_{av} = \frac{\Delta\vec{p}_{tot}}{\Delta t}$

- This is useful in cases where the force is an unknown function of time and we would like to describe its average effect over some specific time interval without having to investigate the detailed behaviour.

4.4 DEGREE OF FREEDOM

- "The minimum number of independent variables or coordinates required to specify (or define) the position of a dynamical system, consisting of one or more particles, is called the number of degrees of freedom of the system."
- Let us consider the example of the motion of a particle, moving freely in space. This motion can be described by a set of three coordinates (x, y, z) and hence the number of degrees of freedom possessed by the particle is three.
- Similarly, a system of two particles moving freely in space needs two sets of three coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) i.e. six coordinates to specify its position. Thus, the system has six degrees of freedom. If a system consists of N particles, moving freely in space, we require 3N independent coordinates to describe its position. Hence, the number of degrees of freedom of the system is 3N.

- The configuration of the system of N particles, moving freely in space, may be represented by the position of a single in 3N dimensional space which is known as configuration space of the system. The configuration space for a system of one freely moving particle is 3-dimensional and for a system of two freely moving particles, it is six dimensional.
- The Number of coordinates required to define (or specify) a dynamical system, becomes smaller, when the constraints are present in the system. Therefore, the degrees of freedom of a dynamical system is defined as the minimum number of independent coordinates or variables required to specify the system well-matched with the constraints.
- If there are n independent variables, say q_1, q_2 ,............ q_n and n constants C_1, C_2, ..., C_n such that $\sum_{i=1}^{n} C_i dq_i = 0$ at any position of the system, then we must have-

  $C_1 = C_2 = ...C_n = 0$

4.5 CONSTRAINTS

- Generally, the motion of a particle or system of particles is restricted by one or more conditions. The restrictions on the motion of a system are called constraints and the motion is said to be constrained motion.
- A constrained motion cannot proceed arbitrarily in any manner. For example, a particle motion is restricted to occur only along some specified path, or on a surface (plane or curved) arbitrarily oriented in space. The motion along a specified path is the simplest example of a constrained motion.

4.5.1 Holonomic Constraints and Nonholonomic Constraints

- The constraints that can be expressed in the form $f(x_1, y_1, z_1: x_2, y_2, z_2; x_n, y_n, z_n; t) = 0$, where time t may occur in case of constraints which may vary with time, are called holonomic and the constraints not expressible in this way are termed as non-holonomic.
- The motion of the particle placed on the surface of sphere under the action of gravitational force is bound by non-holonomic constraint $(r^2 - a^2) \geq 0$

4.5.2 Scleronomic Constraint and Rheonomic Constraints

- If the constraints are independent of time, they are called as scleronomic but if they contain time explicity, they are termed as rheonomic. A bead sliding on a moving wire is an example of rheonomic constraint.

4.6 FORCES OF CONSTRAINT

- You should know that the constraints are always related to forces which restrict the motion of the system. These forces are known as "forces of constraint". For example, the reaction force on a sliding particle on the surface of a sphere is the force of constraint.
- If we consider the case of a rigid body, the inertial forces of action and reaction between any two particles are the forces of constraint. In a simple pendulum, the force of constraint is the tension in the string. Similarly, in the case of a bead sliding on the wire is the reaction by the wire exerted on the bead at each point.
- These forces of constraint are elastic in nature and generally appear at the surface of contact because the motion due to external applied forces is slowed down by the contact. Newton has not given any direction to calculate these forces of constraint. Generally, the forces of constraint act in a direction perpendicular to the surface of constraints while the motion of the object is parallel to the surface. In such cases, the work done by the forces of constraint is zero. These constraints are known as workless and may be called as ideal constraints.

4.7 GENERALIZED COORDINATES

- The smallest possible number of variables to describe the configuration of a system are called "Generalized coordinates". Thus the name generalized coordinates is given to a set of independent coordinates sufficient in number to describe completely the state of configuration of a dynamical system.
- The coordinates are represented as q_1, q_2, q_3,............q_k, ...q_n; where n is the total number of generalized coordinates. Thus, we should know that these are the minimum number of coordinates required to describe the motion of the system.

- Let us consider some examples of generalized coordinates. For a particle constrained to move on the circumference of a circle, only one generalized coordinate $q_1 = \theta$ is enough and two generalized coordinates $q_1 = \theta$ and $q_2 = \varphi$ for a particle moving on the surface of a sphere.
- The number of generalized coordinates for a system of N particles, constrained by m equations, are $n = 3N - m$. It is not necessary that these coordinates should be rectangular, spherical or cylindrical.

Let us see the generalized notations for displacement, velocity, acceleration, momentum, force, potential energy in terms of generalized co-ordinates as follows-

- **Generalized Displacement:** δq_j are called generalized displacement or arbitrary displacements. If q_j is an angle coordinate, δq_j is an angular displacement.
- **Generalized Velocity:** Generalized velocity may be described in terms of time derivative $\dot{q}_j$ of the generalized coordinate q_j.
- **Generalized Acceleration:** The double derivative of q_j i.e. $\ddot{q}_j$ is known as generalized acceleration.
- **Generalized Momentum:** The momentum associated with generalized coordinate q_k is called the generalized momentum p_k associated with a coordinate q_k.
- Generalized momentum $p_k = \frac{\partial T}{\partial \dot{q}_k}$, where T is the kinetic energy of a system
- **Generalized Force:** $Q_j = \sum_{i=1}^{i=N} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_j}$, here Q_j is called the generalized force associated with a coordinate q_j and N , the number of free particles in the system.

4.8 PRINCIPLE OF VIRTUAL WORK

- Let us consider a system of N particles. We know that an infinitesimal virtual displacement of ith particle of the system is represented by $\delta\vec{r}_i$. This is the displacement of position coordinates only and does not involve variation of time i.e.

$$\delta\vec{r}_i = \delta\vec{r}_i\ (q_1, q_2, q_3, \ldots\ldots\ldots q_n) \qquad \ldots\ldots(9)$$

- Let us assume that the system is in equilibrium, then we know that the total force on any particle is zero i.e.

$$\vec{F}_i = 0, \qquad i = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots N$$

- The virtual work of the force $\vec{F}_i$ in the virtual displacement $\delta\vec{r}_i$ will also be zero i.e.

$$\delta W_i = \vec{F}_i . \delta\vec{r}_i = 0$$

- Similarly, the sum of virtual work done for all the particles should be zero i.e.

$$\delta W = \sum_{i=1}^{i=N} \vec{F}_i . \delta\vec{r}_i = 0 \qquad \ldots\ldots(10)$$

- The equation (10) represents the "Principle of Virtual Work". It states that the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium.
- The total force $\vec{F}_i$ on the ith particle can be expressed in the following way-

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i \qquad .....(11)$$

- Here, $\vec{F}_i^a$ is the applied force and $\vec{f}_i$ is the force of constraint. Therefore, equation (10) takes the form as-

$$\delta W = \sum_{i=1}^{i=N} (\vec{F}_i^a + \vec{f}_i) . \delta\vec{r}_i = 0$$

or

$$\sum_{i=1}^{i=N} \vec{F}_i^a . \delta\vec{r}_i + \sum_{i=1}^{i=N} \vec{f}_i . \delta\vec{r}_i = 0 \qquad .....(12)$$

- Now we limit ourselves to the systems where the virtual work of the forces of constraints is zero (i.e. in the case of rigid body), then from equation (12), we have-

$$\sum_{i=1}^{i=N} \vec{F}_i^a . \delta\vec{r}_i = 0 \qquad .....(13)$$

i.e. for the equilibrium of a system, the virtual work of applied forces is zero. It is obvious that the principle of virtual work deals with the statics of a system of particles.

4.9 D'ALEMBERT'S PRINCIPLE

- **D'Alembert's principle**, alternative form of Newton's second law of motion, stated by the 18th-century French polymath Jean le Rond d'Alembert. In effect, the principle reduces a problem in dynamics to a problem in statics.
- The Newton's second law states that the force *F* acting on a body is equal to the product of the mass *m* and acceleration *a* of the body, or *F* = *ma;* in D'Alembert's form, the force *F* plus the negative of the mass *m* times acceleration *a* of the body is equal to zero: *F* - *ma* = 0.
- In other words, the body is in equilibrium under the action of the real force *F* and the fictitious force -*ma*. The fictitious force is also called an inertial force and a reversed effective force.
- This method is based on the principle of virtual work. We know that according to Newton's second law of motion, the force acting on the ith particle is given by-

$$\vec{F}_i = \frac{\overrightarrow{dp_i}}{dt} = \dot{\vec{p}}_i$$

Or $$\vec{F}_i - \dot{\vec{p}}_i = 0$$ where i = 1, 2, 3.......N

- It is clear from the above equation that any particle in the system is in equilibrium under a force which is equal to the actual force $\vec{F}_i$ plus a reversed effective force $\dot{\vec{p}}_i$. Therefore, for virtual displacement $\delta\vec{r}_i$, we can write- $$\sum_{i=1}^{i=N}\left(\vec{F}_i - \dot{\vec{p}}_i\right).\delta\vec{r}_i = 0 \quad .....(14)$$
- But $\vec{F}_i = \vec{F}_i^a + \vec{f}_i$, therefore from equation (14), we have-

$$\sum_{i=1}^{i=N}\left(F_i^a - \dot{\vec{p}}_i\right).\delta\vec{r}_i + \sum_{i=1}^{i=N}\vec{f}_i.\delta\vec{r}_i = 0 \quad .....(15$$

- Again, the system is restricted for which the virtual work of the constraints is zero, i.e. $\sum_{i=1}^{i=N} \vec{f}_i . \delta\vec{r}_i = 0$. Therefore, above equation (15) reduces as-

$$\sum_{i=1}^{i=N} \left(F_i^a - \dot{\vec{p}}_i\right) . \delta\vec{r}_i = 0 \qquad .....(16)$$

This is known as D'Alembert's principle.

- In general, we can write-

$$\sum_{i=1}^{i=N} \left(\vec{F}_i - \dot{\vec{p}}_i\right) . \delta\vec{r}_i = 0 \qquad .....(17)$$

4.10 LAGRANGE'S EQUATION OF MOTION

- Let us consider a system of N particles. The coordinate transformation equations are-

$\vec{r}_i = \vec{r}_i\ (q_1, q_2, q_3, \ldots\ldots, q_k \ldots\ldots\ldots, q_n, t)$
.....(18)

where 't' is the time and q_k (k = 1, 2, 3, ...n) are the generalized coordinates.

- Differentiating equation (18) with respect to t, we get-

$$\frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots \ldots \ldots + \frac{\partial \vec{r}_i}{\partial q_k}\frac{dq_k}{dt} + \cdots \ldots \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

Or $$\vec{v}_i = \dot{\vec{r}}_i = \sum_{k=1}^{k=n} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \quad .....(19)$$

where $\dot{q}_k$ are the generalized velocities.

- The virtual displacement is given by-

$$\delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial\vec{r}_i}{\partial q_2}\delta q_2 + \ldots\ldots\ldots\ldots + \frac{\partial\vec{r}_i}{\partial q_k}\delta q_k + \ldots\ldots. + \frac{\partial\vec{r}_i}{\partial q_n}\delta q_n$$

Or

$$\delta\vec{r}_i = \sum_{k=1}^{k=n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k \qquad \text{.....(20)}$$

(since by definition, the virtual displacements do not depend on time)

- According to D'Alembert's principle-

$$\sum_{i=1}^{i=N}(\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \qquad \text{.....(21)}$$

- Here $\sum_{i=1}^{i=N}(\vec{F}_i).\delta\vec{r}_i = \sum_{i=1}^{i=N}(\vec{F}_i).\sum_{k=1}^{k=n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k$

$$= \sum_{k=1}^{k=n}\sum_{i=1}^{i=N}\left[F_i.\frac{\partial\vec{r}_i}{\partial q_k}\right]\delta q_k = \sum_{k=1}^{k=n}G_k\,\delta q_k \qquad \text{.....(22)}$$

- where $G_k = \sum_{i=1}^{i=N} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_k} = \sum_{i=1}^{i=N} \left\{ F_{xi} \frac{\partial x_i}{\partial q_k} + F_{yi} \frac{\partial y_i}{\partial q_k} + F_{zi} \frac{\partial z_i}{\partial q_k} \right\}$ .....(23)

are known as the components of generalized force associated with the generalized coordinates q_k.

- Now, $\sum_{i=1}^{i=N} (\dot{\vec{p}}_i). \delta \vec{r}_i = \sum_{i=1}^{i=N} m_i \ddot{\vec{r}}_i . \sum_{k=1}^{k=n} \frac{\partial r_i}{\partial q_k} \delta q_k$

$$= \sum_{k=1}^{k=n} \left\{ \sum_{i=1}^{i=N} m_i \ddot{\vec{r}}_i . \frac{\partial r_i}{\partial q_k} \right\} \delta q_k \qquad .....(24)$$

and $\sum_{i=1}^{i=N} m_i \ddot{\vec{r}}_i . \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{i=N} \left\{ \frac{d}{dt} (m_i \dot{\vec{r}}_i . \frac{\partial \vec{r}_i}{\partial q_k} \right\} - m_i \dot{\vec{r}}_i . \frac{d}{dt} (\frac{\partial \vec{r}_i}{\partial q_k})$ .....(25)

- We can write the above equation (25) as-

$\sum_{i=1}^{i=N} m_i \ddot{\vec{r}}_i . \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{i=N} \left\{ \frac{d}{dt} (m_i \vec{v}_i . \frac{\partial \vec{v}_i}{\partial q_k} \right\} - m_i \vec{v}_i . (\frac{\partial \vec{v}_i}{\partial q_k})$ .....(26)

- Putting equating (26) in equation (24), we get-

$$\sum_{i=1}^{i=N}(\dot{\vec{p}}_i).\delta\vec{r}_i = \sum_{k=1}^{k=n}\left\{\sum_{i=1}^{i=N}\left\{\frac{d}{dt}(m_i\vec{v}_i.\frac{\partial\vec{v}_i}{\partial q_k}\right\} - m_i\vec{v}_i.(\frac{\partial\vec{v}_i}{\partial q_k})\right\}\delta q_k$$

$$= \sum_{k=1}^{k=n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{q}_k}\left(\sum_{i=1}^{i=N}\frac{1}{2}m_i(v_i.v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{i=N}\frac{1}{2}m_i(v_i.v_i)\right\}\right]\delta q_k$$

$$= \sum_{k=1}^{k=n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k \qquad .....(27)$$

- Using equations (22) and (27) in equation (21), we get-

$$\sum_{k=1}^{k=n}G_k\,\delta q_k - \sum_{k=1}^{k=n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k = 0$$

Or
$$\sum_{k=1}^{k=n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k - \sum_{k=1}^{k=n}G_k\,\delta q_k = 0$$

Or
$$\sum_{k=1}^{k=n}\left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right\} - G_k\right]\delta q_k = 0$$

- The constraints are holonic, i.e. any virtual displacement δq_k is independent of δq_j. Therefore, $\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right\} - G_k = 0$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad .....(28)$$

The above equation represents the general form of Lagrange's equations.

- For a conservative system-

$$\vec{F}_i = \nabla_i V = -\hat{\imath}\,\frac{\partial V}{\partial x_i} - \hat{\jmath}\,\frac{\partial V}{\partial y_i} - \hat{k}\,\frac{\partial V}{\partial z_i} \qquad .....(29)$$

- Comparing equations (23) and (29), we get-

$$G_k = -\frac{\partial V}{\partial q_k} \qquad .....(30)$$

- From equation (28)-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = - \frac{\partial V}{\partial q_k}$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad .....(31)$$

- Or we can write –

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_k} - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad .....(32)$$

(since the scalar potential V is the function of generalized coordinates q_k only not depending on generalized velocities.)

- We define a new function as-

$$L = T - V \qquad .....(33)$$

which is called the Lagrangian of the system i.e. the difference of kinetic energy and potential energy is the Lagrangian.

- Now we can write the above equation as-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad .....(34)$$

- These equations are known as Lagrange's Equations for conservative system

4.11 CYCLIC COORDINATES

- Cyclic Coordinate is the coordinate , on which the physical parameter (like momentum) doesn't depend or moreover , one can conclude that this physical parameter will remain conserve when the motion is being taken in that coordinate . These coordinates are also known as ignorable coordinates.
- This is quite useful to determine the conservative nature of motion . Let us take ,

$$L = m/2 \, (x'^2 + y'^2) + a(2x)$$

So , here the motion is in xy plain but our potential term (2ax) is y coordinate independent , hence by the use of cyclic coordinate , one can conclude that the momentum in y direction will be conserve , however this may not be the only conserve quantity.

- **Example 1:** Consider the motion of a particle of mass m. Using Cartesian coordinates as generalized coordinates, deduce Newton's equation of motion from Lagrange's equations.
- Solution: We know the general form of the Lagrange's equations-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k$$

Here $q_1 = x$, $q_2 = y$, $q_3 = z$ and generalized force components $G_1 = F_x$, $G_2 = F_y$, $G_3 = F_z$

The kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

- For x-coordinate, we can write the Lagrange's equation as-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x$$

- But $\frac{\partial T}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial T}{\partial x} = 0$

Therefore, $\frac{d}{dt}(m\dot{x}) - 0 = F_x$ or $F_x = \frac{d}{dt}(m\dot{x}) = \frac{dp_x}{dt}$

Where p_x is the x-component of the momentum.

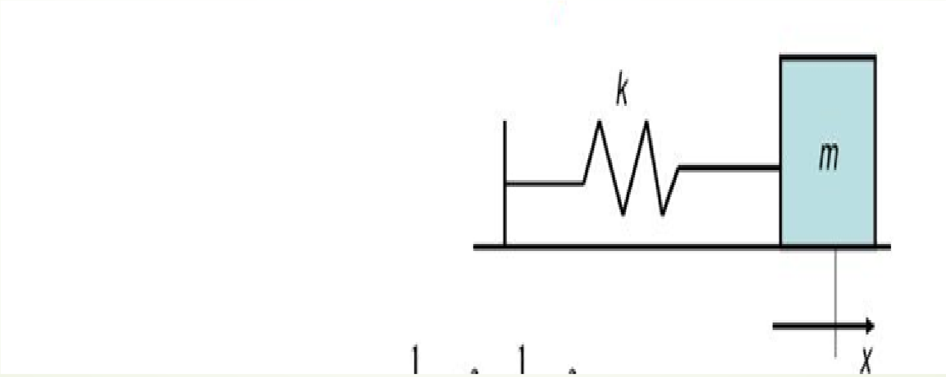
- Similarly for y and z- components-

$$F_y = \frac{dy}{dt}, \qquad F_z = \frac{dp_z}{dt}$$

- Thus, $\vec{F} = \frac{d\vec{p}}{dt}$

This is Newton's equation of motion.

- **Example 2:** A mass M is attached a spring of force constant k. The other end of the spring is fixed in a wall. The entire system is placed on a frictional less surface. If the mass is slightly displaced by linear displacement x, deduce the equation of motion and hence find out the expression for time period of the system.
- Solution:



- We know the Lagrangian's equation of motion-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

.....(1)

- In this case, only one generalized coordinate i.e. x is required, thus $q_k = x$.
- The above equation can be written as-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

.....(2)

- The kinetic energy of the system, T = (1/2) M $\dot{x}^2$
- The potential energy of the system, V= (1/2) k x^2
- Thus, we can write Lagrangian as-

$$L = T - V$$

$$= (1/2)\, M\, \dot{x}^2 - (1/2)\, k\, x^2 \qquad .....(3)$$

$\frac{\partial L}{\partial \dot{x}} = M\, \dot{x}$ and $\frac{\partial L}{\partial x} = -kx$

- From equation (2), we have-

$$\frac{d}{dt}(M\, \dot{x}) - (-kx) = 0$$

Or

$$M\, \ddot{x} + kx = 0 \qquad .....(4)$$

This is required equation.

- From above equation, $\ddot{x}$ = - (k/M) x

Standard equation of Simple Harmonic Motion, $\alpha = -\omega^2 x$

- Comparing the above equation with standard equation of motion of linear mass spring system (Simple Harmonic Motion), we get-

$$\omega^2 = (k/M) \quad \text{or } \omega = \sqrt{\frac{k}{M}}$$

Or

$$\frac{2\pi}{T} = \sqrt{\frac{k}{M}}$$

Or

$$T = 2\pi\sqrt{\frac{M}{k}}$$

- This is the expression for time period of linear mass spring system.

4.12 REFERENCES

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THANKS