LILAVATI (BHASKARA II BOOK)

Dr. Shivangi Upadhyay

Uttrakhand Open University, Haldwani (Uttarakhand)

supadhyay@uou.ac.in

Abstract

- Lilavati (*Līlāvatī*) was composed by renowned astronomer and mathematician Bhaskaracarya (Bhaskara II) (b. 1114 AD) in 1150.
- This is the first volume of his main work Siddhanta Shiromani (”Crown of treatises”) alongside Bijaganita, Grahaganita and Goladhyaya.
- Lilavati is a comprehensive exposition of arithmetic, algebra, geometry, mensuration, number theory and related topics.

- Lilavati is the most celebrated work in the tradition of Mathematics in India.
- This highly important textbook has been used for almost 85 years and is still used as a textbook in many Sanskrit institutions in India.
- This articulate, scholarly and legendary presentation has been translated into several languages of the world.
- This textbook deals mainly with 'Arithmetic' and consists of 279 verses written in Sanskrit in poetic form (terse verses) with a prose commentary.

- This book contains a number of interesting, poetic problems, which give a flavour of ancient Indian school problems.
- The concepts and methods developed in Lilavati are significant even today and are beneficial to school-going children, sophomores, teachers, scholars, historians and those working for the cause of mathematics.
- This book is also useful to students appearing in various competitive examinations to degree and professional courses in India, where calculator is not allowed.
- The aim of this talk is to present some highlights and discuss certain significant aspects Lilavati in 21st Century.

Introduction

- In ancient India mathematics was family-based. Mathematical education was largely restricted within the family and there was not much scope of innovation. Father passed on the commentaries to his son, who pursued it.
- The role of women in mathematics during those times was highly restricted or even probably non-existent. Religion also played a key role.
- Mathematical beliefs were tantamount to religious beliefs and changing religious beliefs was not acceptable. The role of commentaries was important because mathematicians wrote commentaries on their own work and the work of their predecessors.
- These commentaries got transferred in disclipinic succession.

- Bhaskara II (1114 - 1185 A.D.) also known as Bhaskharacharya was born in Maharashtra.
- He is considered as an outstanding poet-mathematician of his times.
- He was the head of the astronomical observatory at Ujjain, where other famous Indian mathematicians including Brahmagupta had studied and worked previously. He worked on algebra, number systems, and astronomy.
- He wrote beautiful texts illustrated with mathematical problems and he provided the best summary of the mathematics and astronomy of the classical period.

- He made fundamental contributions to the development of number theory, the theory of indeterminates infinite series expressions for sine, cosine and tangent, computational mathematics, etc. 200 years after Bhaskara did any significant work happened in Indian Mathematics.
- Bhaskara was a great poet and had mastered eight volumes on Grammar, six on medicine, six on logic, five on mathematics, four vedas, a triad of three ratnas and two Mimamsas.

- Bhaskara produced six works during his lifetime: Lilavati, Bijaganita, Siddhantasiromani, Vasanabhasya of Mitaksara, Brahmatulya, and Vivarana.
- These were all books on math or astronomy, with some of them being commentaries on his own works or that of others.
- Bhaskara was excellent at arithmetic, including a good understanding of negative and zero numbers. He was also good at solving equations and had an understanding of mathematical systems, years ahead of his European peers.

- Bhaskaracarya himself never gave any derivations of his formulae.
- Emperor Akbar commissioned a Persian translation of the Lilavati in 1587 and Faizi (brother of Akbars vizier, Abul Fazl) executed it.
- According to Faizi, Lilavati was Bhaskaracharya's daughter.

- Lilavati is the first part of Bhaskaracharyas work Siddhantashiromani that he wrote at the age of 36. Siddhantashiromani consists of four parts namely
- 1. Lilavati
- 2. Algebra
- 3. Planetary motions and
- 4.Astronomy.
- Lilavati has an interesting story associated with how it got its name.

- Bhaskaracharya created a horoscope for his daughter Lilavati, and predicted that she would remain both childless and unmarried.
- To avoid this fate, he ascertained an auspicious moment for his daughter's wedding and to alert his daughter at the correct time, he placed a cup with a small hole at the bottom of a vessel filled with water, arranged so that the time at which the cup sank was the optimum time Lilavati was to get married (water-clock).
- He put the device in a room with a warning to Lilavati not go near it.
- In her curiosity though, she went to look at the device and a pearl from her bridal dress accidentally dropped into it, thus upsetting it and blocking the hole and the lucky hour passed without the cup sinking.

- The auspicious moment for the wedding thus passed unnoticed leaving a devastated Bhaskaracharya. Lilavati was then doomed never to wed.
- The wedding went on as planned but her husband died soon after the ceremony.
- To console his daughter, who remained a widow the rest of her life, Bhaskara promised to name a book after her and named it Lilavati (The Beautiful), one that would remain till the end of time as a good name is akin to a second life.
- Even now, it is used in some Sanskrit schools. It should be remembered that Lilavati served as a textbook for nearly 800 years in different parts of India until the British system of education was introduced.
- This is something worth emulating in order to make education of mathematics interesting, exciting and also fun.

- Many of the problems are addressed as questions to his beloved daughter, Lilavati herself who must have been a very bright young woman.
- For Example ”Oh Lilavati, intelligent girl, if you understand addition and subtraction, tell me the sum of the amounts 2, 5, 32, 193, 18, 10, and 100, as well as [the remainder of] those when subtracted from 10000.” and ”Fawn-eyed child Lilavati, tell me, how much is the number [resulting from] 135 multiplied by 12, if you understand multiplication by separate parts and by separate digits. And tell [me], beautiful one, how much is that product divided by the same multiplier”

- Leelavati was written in 1150, before the days of printing, when the material and equipment required for making permanent written records were not abundant.
- Therefore, like almost all of the scientific and philosophical works written in Sanskrit, Leelavati is also composed in verse form so that pupils could memorise the rules without the need to refer to written texts.

Contents of Lilavati.

- There are certain verses which deal with Mensuration (measurement of various geometrical objects), Volume of pyramid, cylinders, heaps of grains etc., wood cutting, shadows, trigonometric relations and also on certain elements of Algebra such as finding an unknown quantity subject to certain constraints using the method of supposition.
- The book contains thirteen chapters and covers many branches of mathematics, arithmetic, algebra, geometry, and a little trigonometry and mensuration.

- More specifically the contents include: definitions, properties of zero (including division, and rules of operations with zero), extensive numerical work, including use of negative numbers and surds, estimation of π,arithmetical terms, methods of multiplication, squaring, inverse rule of three, and rules of 3, 5, 7, 9, and 11, interest computation, arithmetical and geometrical progressions, plane geometry, solid geometry, the shadow of the gnomon, the kuttaka - a method to solve indeterminate equations, integer solutions (first and second order) and combinations.
- His contributions to this topic are particularly important, since the rules the given are (in effect) the same as those given by the renaissance European mathematicians of the 17th century, yet his work was of the 12th century.

- Bhaskara's method of solving was an improvement of the methods found in the work of Aryabhata and subsequent mathematicians. Bhaskara II gives the value of π as $\frac{22}{7}$ in the book but suggest a more accurate ratio of $\frac{3927}{1250}$ for use in astronomical calculations.
- Also according to the book, the largest number is the parardha equal to one hundred thousand billion.
- Lilavati includes a number of methods of computing numbers such as multiplications, squares and progressions with examples using kings and elephants, objects that a common man could understand.

- Excerpt from Lilavati (Appears as an additional problem attached to stanza 54, Chapter 3. Translated by T N Colebrook)
  Whilst making love a necklace broke.
  A row of pearls mislaid.
  One sixth fell to the floor.
  One fifth upon the bed.
  The young woman saved one third of them.
  One tenth were caught by her lover.
  If six pearls remained upon the string
  How many pearls were there altogether?

- 1971: Professor Phadke translated Lilavati of Bhaskaracharya (A Treatise of Mathematics of Vedic Tradition) in Marathi titled Lilavati Punardarshana - A New Light in 1971.
- He took great pains in writing the excellent Marathi translation with comments and explanation of the rationale of Lilavati in terms of modern mathematics.
- Krishnaji Shankara Patwardhan, Somashekhara Amrita Naimpally and Shyam Lal Singh [1] translated their Marathi work in to English.
- 1816: John Taylor, Lilawati: or A Treatise on Arithmetic or Geometry by Bhascara Acharya
- 1817: Henry Thomas Colebrooke, Algebra, with Arithmetic and mensuration, from the Sanscrit of Brahmegupta and Bhscara, Page 24, chap 2/3.

- 1971: Professor Phadke , ”Lilavati of Bhaskaracharya (A Treatise of Mathematics of Vedic Tradition)” in Marathi titled Lilavati Punardarshana - A New Light in 1971.
- He took great pains in writing the excellent Marathi translation with comments and explanation of the rationale of Lilavati in terms of modern mathematics.
- 1975: K. V. Sarma, Lilavati of Bhaskaracarya with Kriya-kramakari, Hoshiarpur: VVBIS & IS, Panjab University.
- 2001: K. S. Patwardhan, S. A. Naimpally and S. L. Singh. Lilavati of Bhaskaracarya: a treatise of mathematics of Vedic tradition: with rationale in terms of modern mathematics largely based on N.H. Phadke’s Marathi translation of Lilavati.

- He was the first to recognize that certain types of quadratic equations could have two solutions.
- His Chakrawaat method of solving indeterminate solutions preceded European solutions by several centuries.
- Bhaskaracharya wrote this work by selecting good parts from Sridharacharya's Trishatika, Mahaviracharya's Ganitasarasamgraha, some parts of Brahmagupta, & Padmanabha and adding material of his own.
- However his work is outstanding for its systemisation, improved methods and the new topics that he has introduced. Furthermore the Lilavati contained excellent recreative problems and it is thought that Bhaskara's intention may have been that a student of 'Lilavati' should concern himself with the mechanical application of the method.

- Lilavati became quite popular in India during the time it was first composed. Handwritten copies of Lilavati replaced most other prevalent texts of those times and eventually reached other countries.
- Lilavati does not contain much of algebra, at least not explicitly. Also, the verses, which deal with Permutations, are separate from that of Combinations. The former appears before the verses on Geometry and the later appears after.
- In the Lilavati the eight mathematical operations (parikarmastaka), addition, subtraction, multiplication, division, squaring, cubing, extraction of square and cube-roots are dealt with first. The operations with zero (sunyaparikarma) follow.

- Then come vyastavidhi (method of inversion), istakarma (unitary method), sankramana (finding a & b when a+b and a-b are known i.e., the method of elimination), vargasankramana (finding a & b from $a - b$ and $a^2 - b^2$, vargakarma (finding a & b so that $a^2 + b^2 - 1$ and $a^2 - b^2 - 1$ may be perfect squares), mulagunaka (problems involving square roots i.e. those which lead to quadratic equations); [trairasika (rule of three); bhandapratibhandaka (barter), misravyavalara (mixtures), srenivyavahara (series); ankapasa (permutations and combinations) and kuttaka (indeterminate analysis).
- In fact, some of these topics like series, permutations and combinations and indeterminate analysis more properly come under algebra.
- The section on geometry (ksetraganita) opens with the enunciation of the theorem of the square of the hypotenuse.

- But the enunciation is algebraic rather than geometrical and leads on to the solution of rational right triangles and height and distance problems.
- The condition for given lengths of sides forming the sides of a closed rectilinear figure is then given. The rules for calculating the attitude, area, etc., of triangles and different types of quadrilaterals come next.
- After criticising Brahmagupta's rule for finding the diagonals of quadrilaterals, he gives his method of getting a rational quadrilateral by the juxtaposition of rational right triangles and shows how the diagonals are then easily found.
- Circles are dealt with next, a very satisfactory approximate formula for calculating the arc in terms of the chord and vice versa were given, so also are given the correct expressions for the volume and surface of a sphere.

- Though Sridhara before Bhaskara gave the correct expression for the volume, Bhaskara's is in more general terms. The sides of rectilinear figures with 3,4,.,9 sides are calculated next.
- Khatavyavahara (section on excavations) and krakacavyavahara (shadow problems) cover some interesting problems.
- Except for the section on permutations and combinations, the scheme is the same as that in the mathematical chapters of the Brahmasphuta siddhanta.
- But Bhaskara's treatment is always richer and more comprehensive.

Arithmetic

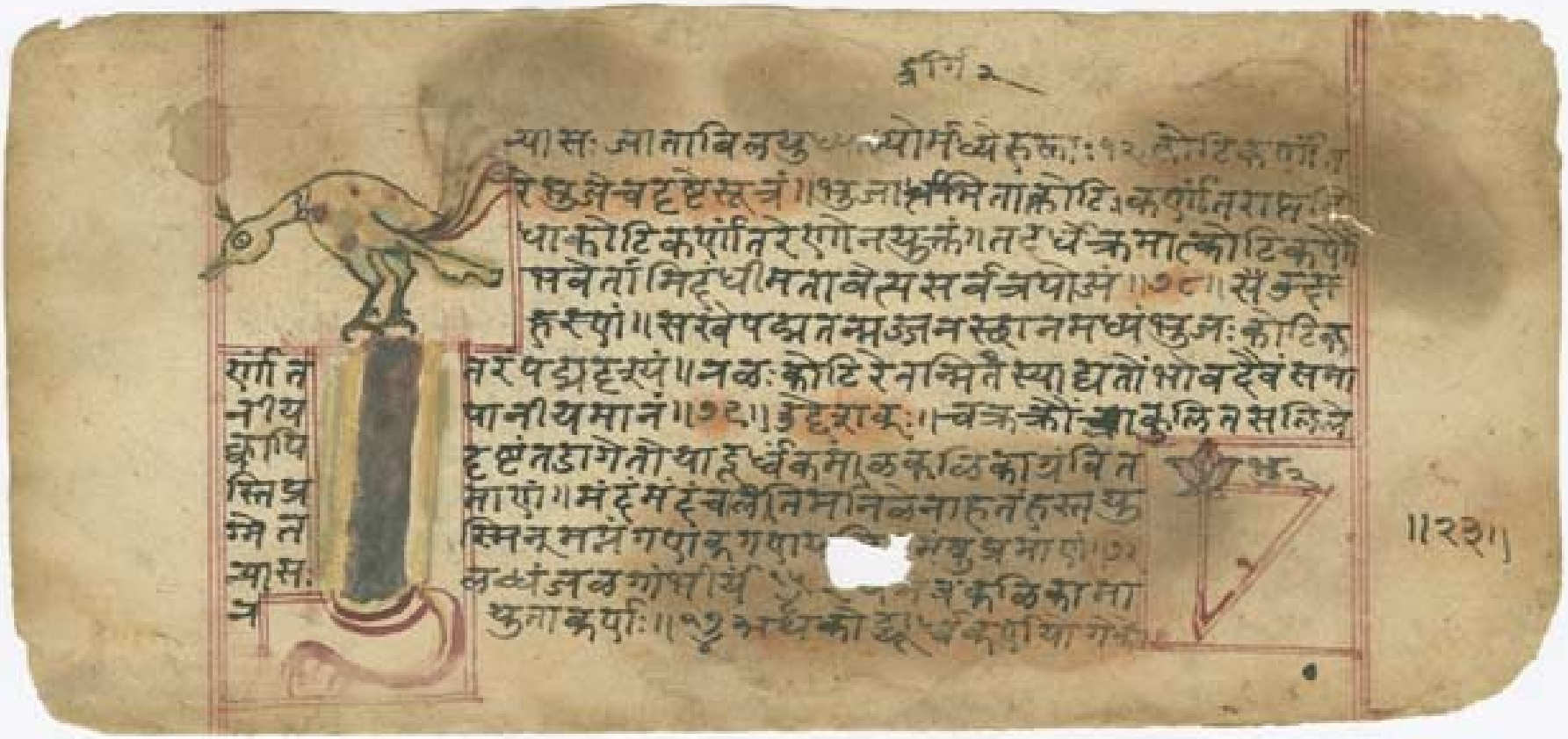
- The first verse of Lilavati is an invocatory verse on Lord Ganesha, the god of wisdom, as it was customary in those days before the beginning of any auspicious event. Lilavati begins:
- ”Salutation to the elephant-headed Being who infuses joy into the minds of his worshippers, who delivers from every difficulty those who call upon him and hose feet are reverenced by the gods.”
- Here the word Pati is used for Arithmetic.
- It literally means slate mathematics.

- The verse also claims that discriminating people because of its clarity, brevity as well as literary flavour love this work. This shows the literary significance of the work.
- Following the invocation, the next 9 verses deal with definitions of measurement units for various things.
- Firstly, he defines the various units of money, which were in vogue during those days.
- This is followed by measures of gold, units of length, measures of grain in volume and lastly the measure of time. This indicates that the text is quite formal in treatment.
- Also, the fact that Arithmetic is directly related to commerce is evident.
- It sets the tone of the work not as an abstract piece but rather one of practical significance in day-to-day applications.

- This contrasts with other works of the time where mathematical texts were not necessarily justifying their use in everyday life.
- There is a reference to the measure of grains as being Turkish in verse 9.
- It appears that his verse must be an inserted verse since there was no influence of Muslims either in the north or in the south during Bhaskaracharyas times and that it was unlikely that they were in common use.
- The next two verses define the all-important positional notation of digits and their values.

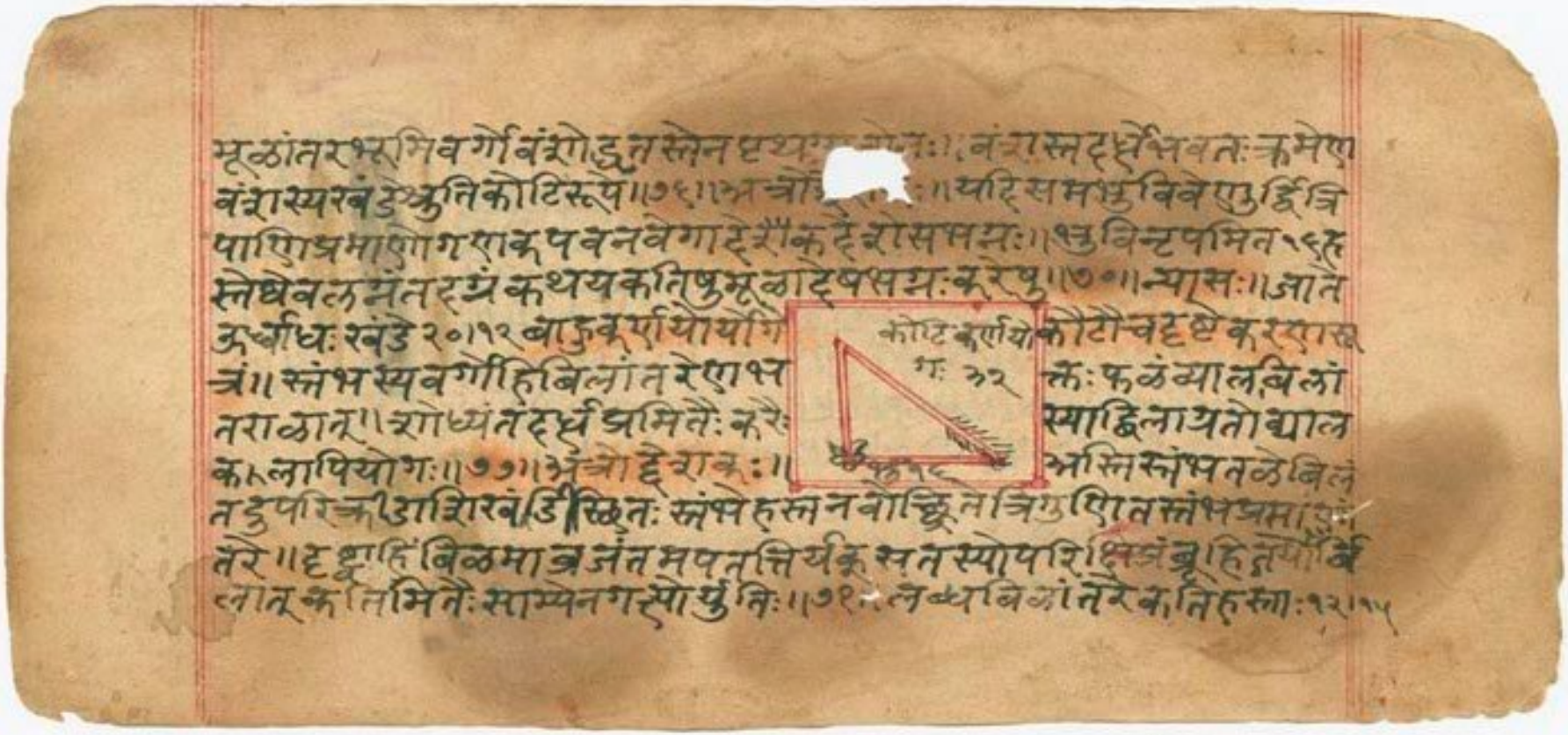
- It clearly states that the value of digits increase by a factor ten from right to left.
- The highest value defined is parardha = 10^{17}. The fact that Bhaskaracharya was an astronomer probably justifies the need for such high magnitudes, which may be required for describing large celestial distances.
- The author claims that the highest value defined in Sanskrit (not in this work) is 10^{140}.
- It is not clear as to what would be the use of such a high number because the number of atoms in the universe is only about 10^{80}.
- It is also interesting to note that absolutely no reference to numerals of any kind is explicitly mentioned, though it is known that Brahmi numerals that were invented in the first century A.D. were used.

- This is a page from a manuscript of the Lilavati of Bhaskara II (1114-1185). This manuscript dates from 1650. The rule for the problem illustrated here is in verse 151, while the problem itself is in verse 152:



- 151: The square of the pillar is divided by the distance between the snake and its hole; the result is subtracted from the distance between the snake and its hole. The place of meeting of the snake and the peacock is separated from the hole by a number of hastas equal to half that difference. 152: There is a hole at the foot of a pillar nine hastas high, and a pet peacock standing on top of it. Seeing a snake returning to the hole at a distance from the pillar equal to three times its height, the peacock descends upon it slantwise. Say quickly, at how many hastas from the hole does the meeting of their two paths occur? (It is assumed here that the speed of the peacock and the snake are equal.)

- These verses and much else from the Lilavati may be found in Kim Plofker, ”Mathematics in India”, in Victor Katz, ed., The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook (Princeton: Princeton University Press, 2007), pp. 385-514.


कोटिकर्णयो
गः ३२

- This page from the Lilavati gives another illustration of the Pythagorean Theorem. Frank J. Swetz and Victor J. Katz, ”Mathematical Treasures - Lilavati of Bhaskara,” Loci (January 2011)

Addition and Subtraction

- The next few verses describe the method (or technique or algorithm) to add two numbers in the positional system.
- This is the characteristic of the entire work - the verses describe the method in an algorithmic fashion. The verse itself is terse and there are no elaborate explanations of any kind.
- This is very contrasting to the scientific and mathematical works of the Greeks that are usually verbose.
- An important thing to observe is that the verse claims that both addition and subtraction could be performed place-wise either from right to left or vice-versa.
- There also seems to be no mention of a carry-over whenever the sum of digits exceeded 9 (it was a base-10 system).

- However, it is possible that these were left as an exercise to the student to workout the details. It is also possible that there was an intermediate step instead of a carry-over. An example is demonstrated here:
- tens units

  2 6

  + 8 7

  =10, 13
- As it can be seen, in the above addition (either from left-to-right or right-to-left), carry- over is not employed.
- The sum for both the units and tens positions have exceed 9, but are retained as it is.

- There is a comma placed in order to identify the positional notation.
- This makes intuitive sense because 10 in the tens place is not allowed, but all it means is $10 \times 10 = 100$.
- Similarly, 13 in the units place is not allowed, but it means just 13.
- Hence, we perform the addition of these two as the next step:
-

hundreds	tens	units
1	0	0
+ 0	1	3
= 1	1	3

- This procedure is repeated until the final result has each digit < 10 in the positional notation.
- In the above example, we stop after the second step.
- Although this seems cumbersome and requires more steps, there are two advantages:
- 1. This procedure could be performed either from left-to-right or from right-to-left and would ultimately yield the same result.
- This is not true with the modern carry-over method. In fact, in this method, addition can be arbitrarily performed starting from any positional value and in any order. The result would be the same.

- The carry-over method requires the operation to be performed strictly from right to left.
- 2. For subtraction, a slight modification is necessary to incorporate this method.
- Assume that we wish to subtract 36 from 10000. We will have to split 10000 as 9999 + 1.
- We could perform 9999 - 36 in the same way as above to yield 9963. We then perform 9963+1 = 9964 to arrive at the final answer.
- It is quite possible that Bhaskaracharya performed such splitting because elsewhere in giving methods for multiplication, he performs such splitting of numbers to simplify the procedure.
- This is purely a speculation part. The illustrious Bhaskaracharya and his students may have been aware of the carryover method.

Multiplication and Division

- Bhaskaracharya gives 5 different methods for multiplication.
- They involve various tricks like splitting the multiplier into two convenient parts and multiplying the multiplicand by each of the two parts and adding the results; factoring the multiplier, multiplying by each factor and then summing up the results etc.
- Another interesting characteristic of each of these methods is that Bhaskaracharya challenges the student following the introduction of each method by giving a problem/s to test the understanding of the student.
- Solutions are not provided but these examples are straight forward.

- He typically expects all the methods to be used to ensure that the solution is consistent. The verses are also quite poetic and beautiful.
- The use of poetic language typically involves the use of such adjectives and similies as - 'O! Auspicious girl with lovable eyes of a young dear', 'Oh Friend!', 'My beloved', 'Deer-eyed', 'Fickle-eyed', 'Oh! you intelligent girl Lilavati' etc.
- This clever use of language is partly teasing but also engaging and challenging the student intellectually.
- This is quite contrasting to modern mathematical textbooks, which are always in prose form and quite dry.
- Elsewhere, Bhaskaracharya employs humour quite effectively which is also lacking in present day textbooks.
- Bhaskaracharya gave two methods of multiplication in his Lilavati.

- 1.To multiply 325 by 243 Bhaskaracharya writes the numbers thus:
  243 243 243
  3 2 5
  ——————-
- Now working with the rightmost of the three sums he computed 5 times 3 then 5 times 2 missing out the 5 times 4 which he did last and wrote beneath the others one place to the left.
- Note that this avoids making the ”carry” in ones head.
  243 243 243
  3 2 5
  ——————-
  1015
  20
  ——————-

- Now add the 1015 and 20 so positioned and write the answer under the second line below the sum next to the left.

243 243 243
3 2 5

1015
20

1215

- Work out the middle sum as the right-hand one, again avoiding the ”carry”, and add them writing the answer below the 1215 but displaced one place to the left.

243 243 243
3 2 5
———————-
4 6 1015
8 20
———————-
1215
486

- Finally work out the left most sum in the same way and again place the resulting addition one place to the left under the 486.

243 243 243
3 2 5
———————-
6 9 4 6 1015
12 8 20
———————-
1215
486
729
———————-

- Finally add the three numbers below the second line to obtain the answer 78975.

243 243 243
3 2 5
———————-
6 9 4 6 1015
12 8 20
———————-
1215
486
729
———————-
78975

- Despite avoiding the ”carry” in the first stages, of course one is still faced with the ”carry” in this final addition.
- 2.The second of Bhaskaracharya’s methods proceeds as follows:
  325
  243
  ———
- Multiply the bottom number by the top number starting with the left-most digit and proceeding towards the right.
- Displace each row one place to start one place further right than the previous line. First step
  325
  243
  ———
  729

- Second step
  325
  243
  ———
  729
  486
- Third step, then add
  325
  243
  ———
  729
  486
  1215
  ———
  78975

- Bhaskaracharya gives only one method for division, which is the same as the modern method of finding the product of the divisor with the largest integer such that it can be subtracted from the extreme left hand digits of the dividend.
- This integer is the first digit of the quotient.
- He also talks about removing any common factors of the dividend and divisor.
- This shows that he probably knew the Fundamental Theorem of Arithmetic that every integer can be factored uniquely into the product of prime factors.
- He does not however mention primes explicitly, but he does say that if it is possible to factorize the number and so on indicating that he probably had some idea about primes.

- Moreover, Euclids work on primes including the famous proof of infinitude of primes was well documented in his magnum opus Elements, which was written in 300 B.C.
- In dealing with numbers Bhaskaracharya, handled efficiently arithmetic involving negative numbers.
- He is sound in addition, subtraction and multiplication involving zero. the zero used by Bhaskaracharya in his rule (a.0)/0 = a, given in Lilavati, is equivalent to the modern concept of a non-zero ”infinitesimal”.
- Although this claim is not without foundation, perhaps it is seeing ideas beyond what Bhaskaracharya intended.

Square root and n^{th} root

- The algorithm for square root is nearly the same, which is taught, in modern mathematics.
- The only difference is that Bhaskaracharya groups the digits only one at a time instead of doing two at a time as modern mathematics does.
- The advantage of Bhaskaracharyas method is that the same procedure is extendible to n^{th} root.
- This shows that the methods which Bhaskarcharya and his predecessors like Brahmagupta and Aryabhata developed were not just aimed at solving the specific instance but they were also interested in methods which could be generalized.
- Sometimes speediness was traded for generality as will be evident in later examples also.

- These methods are rarely used these days, thanks to the invention of Logarithms that makes these operations far easier. Logarithms were invented only later.
- Bhaskaracharya, like many of the Indian mathematicians, considered squaring of numbers as special cases of multiplication, which deserved special methods. He gave four such methods of squaring in Lilavati.
- Here is an example of explanation of inverse proportion taken from Chapter 3 of the Lilavati. Bhaskaracharya writes:-
- In the inverse method, the operation is reversed.

- That is the fruit to be multiplied by the augment and divided by the demand.
- When fruit increases or decreases, as the demand is augmented or diminished, the direct rule is used.
- Else the inverse. Rule of three inverse: If the fruit diminish as the requisition increases, or augment as that decreases, they, who are skilled in accounts, consider the rule of three to be inverted.
- When there is a diminution of fruit, if there be increase of requisition, and increase of fruit if there be diminution of requisition, then the inverse rule of three is employed.

- As well as the rule of three, Bhaskaracharya discusses examples to illustrate rules of compound proportions, such as the rule of five (Pancarasika), the rule of seven (Saptarasika), the rule of nine (Navarasika), etc.
- Bhaskaracharya's examples of using these rules are discussed in [15].
- An example from Chapter 5 on arithmetical and geometrical progressions is the following:-
- Example: On an expedition to seize his enemy's elephants, a king marched two yojanas the first day.
- Say, intelligent calculator, with what increasing rate of daily march did he proceed, since he reached his foe's city, a distance of eighty yojanas, in a week?

- Bhaskaracharya shows that each day he must travel 22/7 yojanas further than the previous day to reach his foe's city in 7 days.
- An example from Chapter 12 on the kuttaka method of solving indeterminate equations is the following:-
- Example: Say quickly, mathematician, what is that multiplier, by which two hundred and twenty-one being multiplied, and sixty-five added to the product, the sum divided by a hundred and ninety-five becomes exhausted.
- Bhaskaracharya is finding integer solution to $195x = 221y + 65$. He obtains the solutions $(x, y) = (6, 5)$ or $(23, 20)$ or $(40, 35)$ and so on.

- In the final chapter on combinations Bhaskaracharya considers the following problem. Let an n-digit number be represented in the usual decimal form as
- $d_1 d_2 ... d_n$ $(\star)$ where each digit satisfies $1 \leq d_j \leq 9$, $j = 1, 2, ..., n$.
- Then Bhaskaracharya's problem is to find the total number of numbers of the form $(\star)$ that satisfy $d_1 + d_2 + ... + d_n = S$.

Fractions

- It seems that Bhaskaracharya did not know about decimals.
- However, he knew about fractions and made an extensive study of them. He gives eight operations on fractions.
- He also knew about Least Common Multiple (LCM) and Highest Common Factor (HCF), which are needed for various operations on fractions.

- He has a humourous example of a miser giving a beggar, $(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5} \times \frac{1}{16} \times \frac{1}{4})^{th}$ part of a dramma (16 drammas make one niska which is one silver coin) which amounts to $\frac{6}{7680} = \frac{1}{1280}$ which is equivalent to one Kavadi or Cowrie, the lowest denomination possible thus justifying the misers virtue (or vice versa).
- Bhaskara gives methods for multiplication of fractions, addition of fractions, divisions, square root, cube root, squares, cubes etc. These methods are more or less extensions of the earlier ones on integers, only suitably modified.

On Infinity and Eight rules of Zero

- In his book Lilavati, he reasons: ”In this quantity also which has zero as its divisor there is no change even when many [quantities] have entered into it or come out [of it], just as at the time of destruction and creation when throngs of creatures enter into and come out of [him, there is no change in] the infinite and unchanging [Vishnu]”.[22]
- Bhaskaracharya seemed to have known the importance of zero, not just in positional notation, but also as a number.
- He has special verses describing the peculiar properties of zero. He lists eight rules such as $a + 0 = 0$, $0^2 = 0$, $\sqrt{0} = 0$, $a \times 0 = 0$ etc.
- The interesting aspect of this verse is the definition of infinity or Khahara as a fraction whose denominator is zero.
- In other words, $\frac{a}{0} = \infty$.

- He knew that whenever a number was divided by zero, it lead to a problem.
- He gives a verse on the nature of this infinity as follows (copied from translation):
- There is no change in infinite (khahara) figure if some thing is added to or subtracted from the same.
- It is like there is no change in infinite Visnu (Almighty) due to dissolution or creation of abounding living beings.
- This definition of infinity is definitely wrong but the properties are right (as against what modern mathematics defines).

- During his time, there were other astronomers and mathematicians who did not believe in the immutability of infinity.
- For example, Jnanaraja (1503 A.D.) says that infinity does not remain immutable when something is added to it or subtracted from it.
- However they used Khahara for infinity. It seems that they did not have any other symbol or notation for infinity.
- They used the $\frac{a}{0}$ as a notation for infinity quite safely for further calculations or reversing the process of mathematical operations as and when needed.

Reverse process and the concept of limits

- Bhaskaracharya elaborates inverse operations for back-calculations for obtaining an unknown quantity from known ones in a mathematical expression.
- As an example, in order to find a certain number which when added by 10, 4 subtracted from $\frac{2}{5}^{th}$ of the sum so obtained to yield a result of 12, the inverse process is employed.
- 4 is first added with 12 to yield 16 and then 16 is divided by $\frac{2}{5}$ to yield 40.
- Finally, 10 is added to 40, to yield the original number 30 as the correct answer.

- This reverse method can be applied to all mathematical operations.
- Bhaskaracharya specifically mentions that if a certain number is multiplied by zero and also divided by zero, it should not be treated as either zero or infinity.
- It should be retained as is for further calculations. He demonstrated this with an example:
- A certain number is multiplied by 0 and added to half of the result. If the sum so obtained is first multiplied by 3 and then divided by 0, the result is 63. Find the original number. It is obtained by reverse (Viloma) process.

- For the above example, the expected solution is as follows: $\frac{((x\times 0+\frac{1}{2}\times 0)\times 3)}{0} = 63$ implies $\frac{9x}{2} \times \frac{0}{0} = 63$ which implies $x = 14$.
- The author claims that this closely borders the definition of a limit, because strictly speaking the last step of the above process is similar to saying $\lim_{h\to 0} \frac{((x\times 0+\frac{1}{2}\times 0)\times 3)}{0} = 63$ which implies $x = 14$.
- This is the concept of the limit in traditional language and this was definitely a step forward in the right direction.
- However, the concept of limits and the theory of Calculus needed much more insight and machinery

Ratio and Proportions

- Bhaskara extensively treats direct and inverse proportions.
- He cites several problems where the rule of three is applicable.
- This states that if $a : b :: c : d$ where a and c are of the same kind and b and d are of the same kind.
- d, the unknown can be found by the formula $d = \frac{(b \times c)}{a}$.
- This holds only for direct proportion and Bhaskara explicitly mentions this and the inverse proportion relation also. He extends this rule for the rule of five, seven, nine etc.

Simple Interest, Progressions

- Bhaskara deals with simple interest with apparent ease.
- He does not say anything about compound interest.
- He also gives a thorough treatment of arithmetic progressions (A.P) and geometric progression but not harmonic progression.
- He gives the direct and inverse formulae for finding the sum of series, the last term, the constant difference for an A.P.

Algebra

- Bhaskaracharya does not explicitly use any language for mathematics.
- As a result of this, algebra suffered greatly.
- The sheer manipulation of symbols can produce new results as we witness in modern mathematics and algebra.
- This was solely lacking in the Indian system because of the lack of a mathematical language.
- It is quite commendable that Bhaskaracharya and other Indian mathematicians developed all of their mathematics by intuition and empirical understanding rather than by random manipulations of symbols.

- Bhaskara and others of the classical Indian tradition never provided proofs.
- However, it is quite clear that they did have a certain rationale in their minds when they developed these results.
- They probably did not consider it important to show the proof because the method was quite obvious and self-evident.
- This is similar to the way mathematics was pursued by Gauss and his contemporaries.
- Gauss believed that intuitive ideas, which helped a proof, were like scaffolding of a building under construction and these necessarily had to be removed after the building was completed.

Squares and Cubes

- Bhaskaracharya knew the elementary algebraic identities such as $(a+b)^2 = a^2 + b^2 + 2ab$, $a^2 - b^2 = (a+b)(a-b)$, formula for a cube, and many other complicated relationships.
- Pascals triangle was also known. It was called as Khandameru.

Trigonometry and Geometry

- Bhaskaracharya deals with the quadratic equation $ax^2 + bx + c = 0$.
- Specifically he chose $a = 1$ and $c < 0$.
- This is because the discriminant $b^2 - 4ac$ turns out to be positive and the roots of the equation are real.
- Bhaskaracharya was not familiar with imaginary numbers (first known as impossible numbers).
- However, for the positive discriminant case, Bhaskaracharya gives the correct expression for the roots.
- In this section, Bhaskaracharya gives a very interesting puzzle from the epic Mahabharata where Arjuna uses a certain number of arrows (say x) to destroy the horses of Karna, a certain number to destroy his chariot, flag, and bow and to cut off his head.

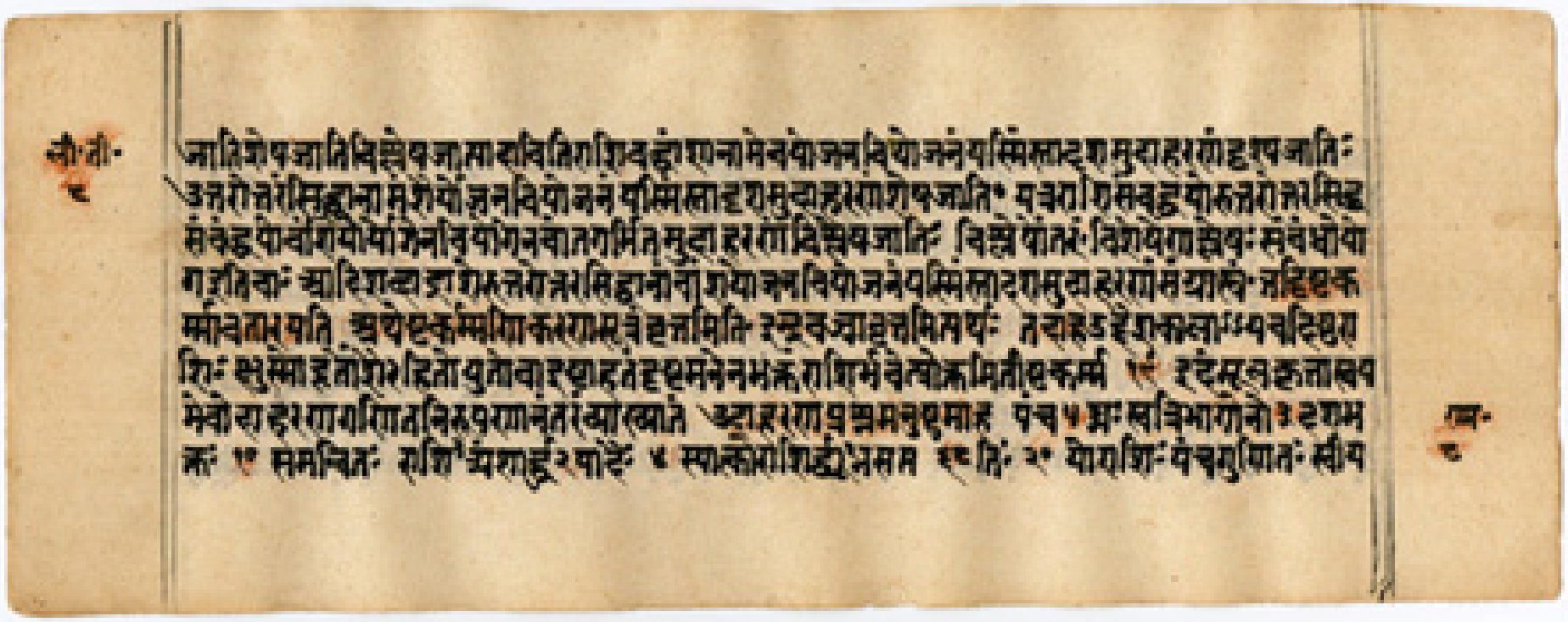
- The solution of the puzzle is the root of a quadratic equation.
- It is interesting to note that, Bhaskaracharya always solves for x^2 first and not x directly.
- This is because there was no symbol for surds (square root) and so he takes the square root of x^2 to obtain the value for x.
- He always takes the positive value of square root.
- It is not clear whether he knew that a square root had negative values.
- There are no other methods on algebra though there is a use of some algebra in some of the methods he describes on trigonometry and geometry.

- Bhaskara starts with the definition of the sides of a right-angled triangle. He then states the Pythagoras theorem (without proof).
- It was known in India since the time of Sulvasutrakaras (3000-800 B.C.) whereas Pythagoras published it in 560 B.C. Bhaskaracharya gives a method for finding an approximate square root of a number that is not a perfect square.
- To compute (approximately) $\sqrt{\frac{a}{b}}$, choose a large square number x.
- Then compute approximately $\sqrt{abx}$ and divide by $b\sqrt{x}$.

- Bhaskara then deals with various problems related to different situations of right-angled triangles such as finding the two sides when the hypotenuse is given.
- Bhaskara states that it is impossible for one side of a triangle to be greater than sum of the other two sides.
- This Triangle Inequality is valid only in Euclidean geometry.

Mensuration of Polygons and other objects

- Bhaskara gives formula for finding the area of a quadrilateral (both cyclic and acyclic).
- He also gives the formula for finding the diagonal of a quadrilateral.
- He also deals with trapezium, disk, sphere etc. He deals with chords of a circle extensively.



Volume

- Bhaskara gives methods for determining volume of a pyramid and its frustrum and the volume of a prism.
- He also gives practical applications of finding the cost for cutting wood in a particular shape (frustrum of a cone) and its area calculation.
- He also gives methods to calculate volume of a heap of grain.
- An important observation to make is that there are no diagrams or figures for illustrating any of these geometrical methods.
- This is most peculiar because geometry is treated in the same way as arithmetic and algebra. One can infer that the lack of diagrams was another handicap for the development of Indian mathematics, especially geometry. He also deals with lengths of shadows.

Discrete Mathematics

- Bhaskara gives a fair amount of treatment of what we call today as Discrete Mathematics.
- He correctly gives the formulae for n!, ${}^nP_r$, ${}^nC_r$ etc.
- He specifically mentions that the study of combinatorial analysis is useful in prosody to discover all possible meters (he also gives a puzzle on similar lines), in architecture, medical sciences, Khandameru (Pascals triangle), chemical composition etc.
- He further claims that he is omitting these applications for the sake of brevity.
- This shows that Bhaskara was very much interested in applications of Mathematics and was not a pure mathematician.

- In those days, it was probably uncommon to pursue mathematics for its own sake.
- This also shows that Bhaskara knew the importance of mathematics in a wide variety of applications.
- Whether he applied it to other disciplines (other than astronomy) is not known.
- Bhaskara illustrates the principle of finding the number of permutations by an interesting puzzle:
  Lord Shiva holds ten different weapons, namely a trap, a goad, a snake, a drum, a potsherd, a club, a spear, a missile, an arrow and a bow in his hands.

- Find the number of different Shiva idols. Similarly, solve the problem for Vishnu idols; Vishu has four objects: a mace, a disc, a lotus and a conch.
- For the first example, there are 10! = 3628800 possible Shiva idols and in the case of Lord Vishu, there are 4! = 24 different idols.
- The author makes an interesting observation that in the daily ritual (sandhyavandanam), there are 24 different names of Lord Vishnu.
- Bhaskara also deals with repeated digits and their combinations.
- He also gives an elementary problem in Partitions. This shows the diversity of topics covered by Bhaskara.

Kuttaka or Indeterminate analysis

- Indeterminate analysis is the problem of finding integer solutions to x and y in the equation $ax + by = c$ where a, b and c are all integers.
- The method to do this was called as Kuttaka which means ”to beat the problem into powder”.
- In other words, the solution, which was developed by Brahmagupta and later by Bhaskaracharya, involved successively simplifying the problem in an iterative process and then solving it.
- This reminds of powdering a larger object into smaller pieces first before powdering these pieces into finer pieces and so on.

- These are also known as Diphontine Equations.
- Bhaskaracharya provides a method for finding the solution, which makes use of the Euclidean Algorithm for finding the Greatest Common Divisor (G.C.D).
- The Kuttaka method is said to be an important contribution of Bhaskaracharya.
- Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave.
- The one remaining monkey was climbing up a tree. What is the total number of monkeys?
- We have $(\frac{x}{5} - 3)^2 + 1 = x$, whose two roots are 50 and 5.
- The answer 5 is inadmissible; so there are 50 monkeys.

- Out of the swans in a certain lake, ten times the square root of their number went away to Manasa Sarovara when rains started, and one eighth the number went away to the forest Sthala Padmini. Three pairs of swans remained in the tank, engaged in water sports. What is the total number of swans?

- On a pillar 9 cubits high is perched a peacock. From a distance of 27 cubits, a snake is coming to its hole at the bottom of the pillar. Seeing the snake, the peacock pounces upon it. If their speeds are equal, tell me quickly at what distance from the hole is the snake caught?
  $9^2 + x^2 = (27 - x)^2$. So $x = 12$.
- The questions in Leelavati are known for their variety, story-like problems and the challenge they offer to students.

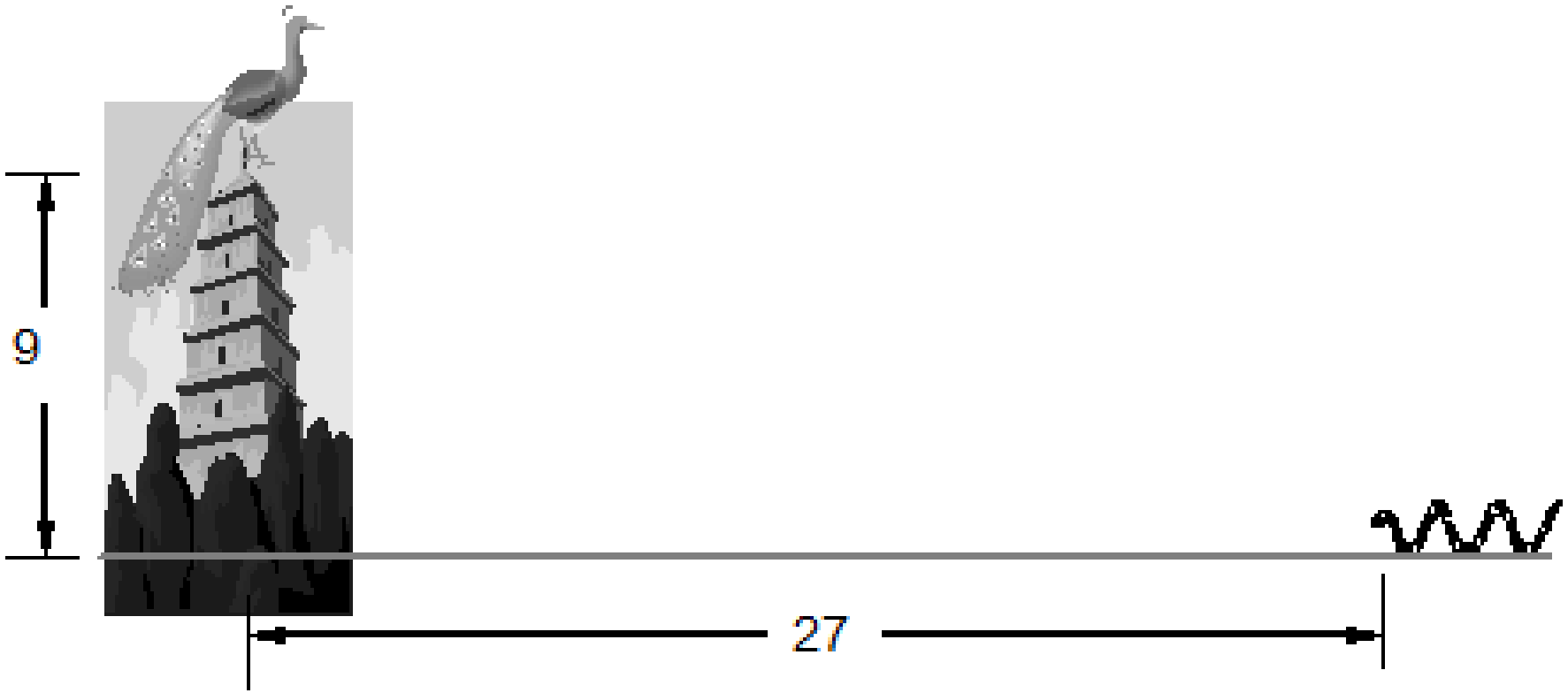
- Some examples from Leelavati are listed below:
- 1. Of a group of elephants, half and one third of the half went into a cave, One sixth and one seventh of one sixth was drinking water from a river. One eight and one ninth of one eighth were sporting in a pond full of lotuses The lover king of the elephants was leading three female elephants; how many elephants were there in the flock?
- 2. Oh! You auspicious girl with enchanting eyes of a fawn, Lilavati, If you have well understood the above methods of multiplication What is the product of 135 and 12. Also, tell me what number will you obtain when the product is divided by 12.

- Example 1. A traveller, engaged in a pilgrimage, gave half ($\frac{1}{2}$) his money at Prayaaga; two-ninths ($\frac{2}{9}$) of the remainder at Kaashi (Benares); a quarter ($\frac{1}{4}$) of the residue in payment of taxes on the road; six-tenths ($\frac{6}{10}$) of what was left at Gaya; there remained sixty-three (63) Nishkas (gold coins) with which he returned home. Tell me the amount of his original stock of money, if you have learned the method of reduction of fractions of residues.

- Here the rule is: Divide the product of the denominators minus the numerators by the product of the denominators; and by the quotient obtained, divide the product of the known quantity multiplied by the assumed.
- Thus: Let 1 be the number assumed.
- Subtract the numerators from the denominators; thus: 2-1 =1; 9-2 = 7; 4-1 =3; 10-6 =4
- Product of denominators minus the numerators = $1 \times 7 \times 3 \times 4 = 84$, and
- Product of the original denominators = $2 \times 9 \times 4 \times 10 = 720$.

- Dividing the first product by the second, we get $\frac{84}{720} = \frac{7}{60}$.
- Product of the known quantity by the assumed = $63 \times 1 = 63$
- Therefore, his original stock of money = $\frac{63}{\frac{7}{60}} = 540$.
- The interested reader may verify the result by working out the problem from first principles.

- Example 2
- A snakes hole is at the foot of a pillar of nine cubits height, and a peacock is perching on its summit.
- Seeing a snake, at a distance of thrice the pillar, gliding towards his hole, he pounces obliquely upon him.
- Say quickly at how many cubits from the snake's hole do they meet, both proceeding an equal distance?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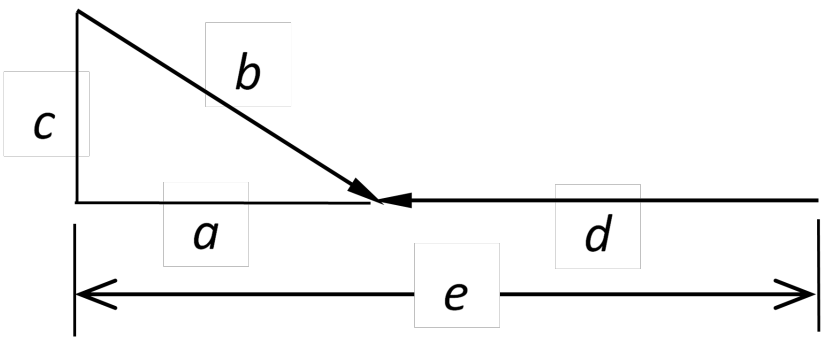
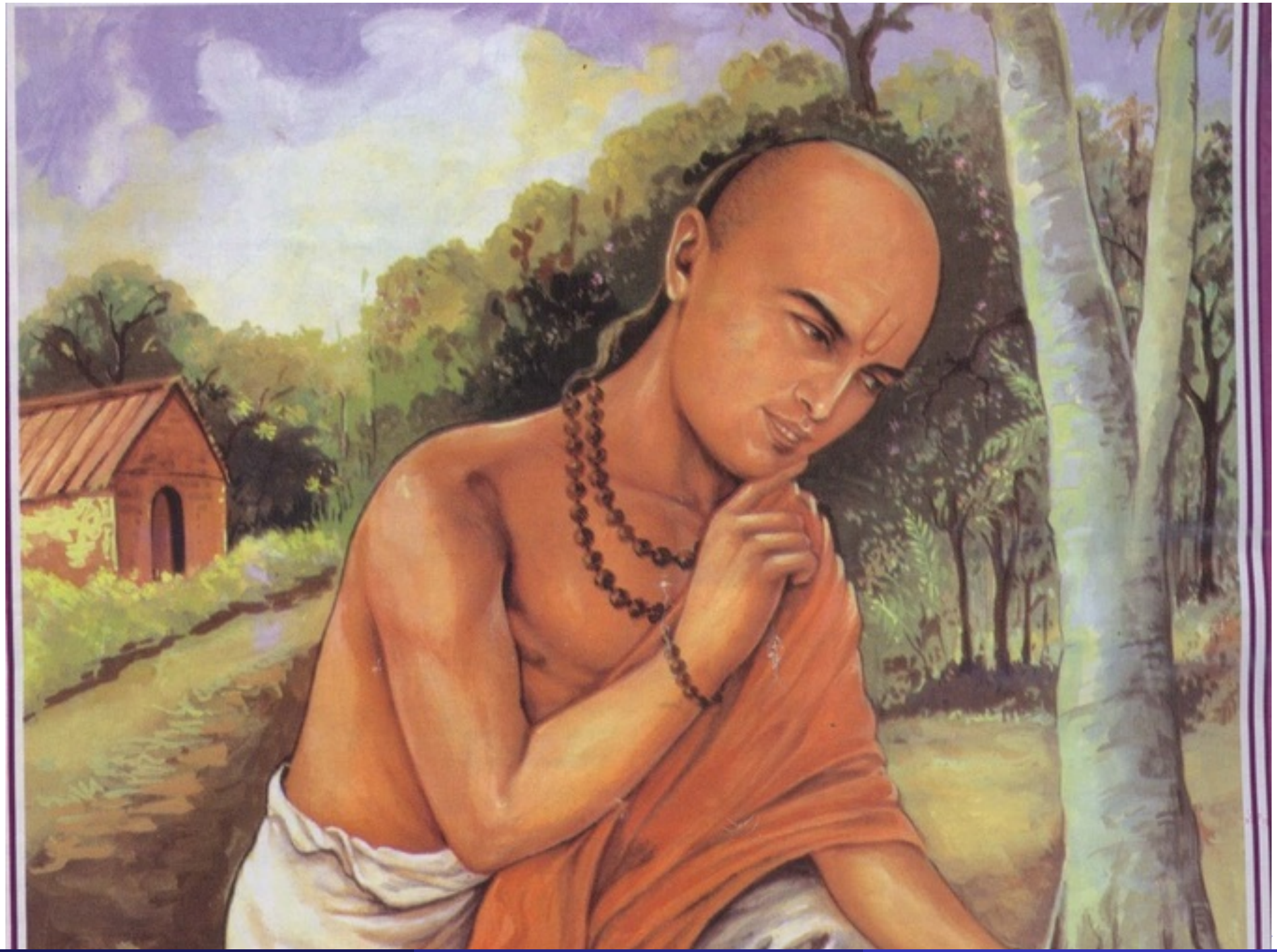
27

- Rule: The square [of the height] of the pillar is divided by the distance of the snake from his hole;
- the quotient is to be subtracted from that distance.
- The meeting of the snake and the peacock is, from the snakes hole, half the remainder, in cubits.
- Referring to the figure, $a = \frac{[e(c\frac{2}{e})]}{2} = \frac{[279\frac{2}{27}]}{2} = 12$



c = 9; a+d = e =27; b = d
[Answer: a = 12]



Discussion and concluding remarks

- Lilavati, Bhaskaracharya's monumental work is not only a literary treatise but also occupies a distinguished and honored place in the history of Mathematics.
- It is a testimony to the extra-ordinary mathematical acumen of Bhaskaracharya, who is regarded as one of the most innovative mathematician of India of his era.
- He was also an excellent teacher as indicated by the teasing and pleasing verses through which he tests his disciple's abilities to solve mathematical problems.
- Bhaskaracharya may not know anything about what a Proof is, which is the most important part of any modern mathematical work.

- However, his terse verses which contain the algorithmic rules and the generalizations which he recommends indicate that there may have been a certain rationale or reasoning in the minds of the mathematicians of those times.
- Moreover, his methods are not vague by any means and are quite precise.
- They probably relied more on intuition and did not feel the necessity of providing the rationale they had, if any.
- This is best exemplified by the genius Srinivasa Ramanujan of our times whos intuitive leaps of imagination were closer to Bhaskaracharya in spirit.

- Ramanujans idea of a proof was very sketchy and when Hardy and Littlewood asked him why a particular result of his was true, Ramanujan would reply that it was true because he knew it (or rather it occurred to him in a flash).
- This was by no means a proof, but it shows the power of intuition, the ability to find new patterns without the shackles of formalism.
- However, this was also the limitation of Indian Mathematics. It did not flourish to its full potential.

- There are several reasons for this. There were no symbols or notation invented to handle mathematical objects and this was a huge handicap. This is a very important process, which leads to the unreasonable effectiveness of mathematics.
- The language in which it was written - Sanskrit - was difficult and only the very well learned scholars cold decipher the poetic verses.
- The other reason for its loss of popularity was the lack of a proof or elaboration of the rationale behind the methods.
- Western mathematics and science very much demanded not just the result but why the result was true, if it was true at all.

- This may seem too much to ask for, but it was useful, because it guaranteed the determinism of Mathematics.
- All mathematical knowledge became sacrosanct, thanks to the rigorous demand of proof.
- It became sacred and deterministic to the extent that many people would take to it because they could not face the realism of the world and other sciences.
- Paul Erdos, the Hungarian mathematical genius of our times was an example of this kind.

- He took to mathematics because it was the only thing in this world that was guaranteed to be true, whatever truth meant to him.
- Nothing else appealed to him because of their innate uncertainty barring mathematics.
- Another huge handicap of Indian mathematics was the lack of illustrations or diagrams.
- As a result of this, Indian treatment of geometry was only very elementary.
- It is possible that they used some diagrams while solving problems, but this was never communicated in their works.

- What these ancient Indian mathematicians seem to excel was in their intuitive way of thinking and the algorithmic approach to mathematics.
- This is quite useful, especially in the computer age where such algorithmic approach would have received a huge support and would have been an advantage.
- Another advantage of Indian mathematics was the conciseness of the documentation.
- This is a very good example of compression of mathematical ideas.
- This was partly the reason that it was easily copied and transferred either by hand or orally.
- Remember, there was no printing available during those times.

- The last verse of Lilavathi demonstrates the poetic brilliance of Bhaskara:
- ” (Lass) Lilavati is born in a respectable family, stands out in any group of enlightened persons, has mastered idioms and proverbs. Whomsoever she embraces will be happy and prosperous.
- However, the same verse could also be interpreted as:
- ”This Lilavati clearly explains fractions, simple fractions, multiplication etc. It also beautifully describes problems in day-to-day transactions.Rules are transparent and examples are beautifully worded. Those who master this Lilavati will be happy and prosperous.”
- Bhaskaracharya was given the title of ’Ganakacakrachudamani’ which means ’A Crest Jewel among Mathematicians’. Now we know why.

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Thank you for your kind attention!