Dr. Kamlesh Bisht (Mathematics)

BOOK: Differential Equation, Calculus of Variations and Special Functions
UNIT – III : Partial Differential Equation Of Second Order, Monge's Method

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Objectives

After studying this unit, you should be able to-

- Discuss partial differential equations of order two with variable coefficients.
- Learn how a large class of second order partial differential equation may be solved by using the methods applicable for solving ordinary differential equation.
- Study Monge's method for solution of some special type of second order partial differential equation.

Introduction

A partial differential equation (P.D.E) is said to be of order two, if it involves at least one of the differential coefficients *r*, *s*, *t* and none of order higher than two. The general form of second order partial differential equation in two independent variable *x*, *y* is given as:

$$F(x, y, z, p, q, r, s, t) = 0 ;$$

where $$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y},\ t = \frac{\partial^2 z}{\partial y^2}$$

The most of the general linear partial differential equation of second order in two independent variable x and y with variable coefficient is given as:

$$Rr + Ss + Tt + Pp + Qq + Zz = F$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

Solution of P.D.E of second order by inspection

Before taking up the genetral equation of second degree P.D.E., we discus the solution of simple problems which can be integrated merely by inspection. On two successive integral of given P.D.E., we get the general solution which is relation in x, y, z. To understand this we discuss the following problems.

Question: Solve t+ s+ q = 0

Solution: We can write the given problem as

$$\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial y}=0$$

Integrating with respect to y, treating x as constant, we get

$$\frac{\partial z}{\partial y}+\frac{\partial z}{\partial x}+z=f(x) \qquad \text{or} \qquad p+q=f(x)-z$$

which is the form of standard Lagrange's linear equation $Pp + Qq = R$, so the auxiliary equation will be

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}$$

from first two terms, we obtain

$$x - y = c_1 \text{ (constant)} \qquad .....(1)$$

and from first and last terms, we have

$$\frac{dz}{dx} + z = f(x) \qquad .....(2)$$

which is linear differential equation of first order having integrative factor e^x.

Hence the solution of (2) will be

$$z \cdot e^x = \int f(x)\, e^x dx + c_2 \text{ (constant)}$$

Therefore the required solution of given equation will be (by using (1)]

$$ze^x - \phi(x) = \psi(x - y)$$

where c_2 is a function of c_1 or of $(x - y)$.

Question: Show that a surface passing through the circle $z=0$, $x^2+y^2=1$ and satisfying the differential equation $s=8xy$ is $z=(x^2+y^2)^2-1$

Solution: We can write the given differential equation as

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 8xy$$

Integrating with respect to x, we get

$$\frac{\partial z}{\partial y} = 4x^2 y + f(y)$$

Again integrating with respect to y, we obtain

$$z = 2x^2 y^2 + \int f(y)\,dy + \phi_1(x)$$

or $$z = 2x^2 y^2 + \phi_2(y) + \phi_1(x) \qquad \text{..........} (5)$$

where $$\phi_2(y) = \int f(y)\,dy$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

Now given circle is

$$x^2 + y^2 = 1, \quad z = 0$$

Putting $z = 0$ in (5), we get

$$2x^2y^2 + \phi_2(y) + \phi_1(x) = 0 \qquad \text{..... (6)}$$

Now, $x^2 + y^2 = 1 \Rightarrow (x^2 + y^2)^2 = 1^2$

or $$2x^2y^2 + x^4 + y^4 = 1 \qquad \text{......(7)}$$

On comparing (6) with (7), we get

$$\phi_2(y) + \phi_1(x) = x^4 + y^4 - 1$$

Substituting this in (5), we obtain

$$z = 2x^2y^2 + x^4 + y^4 - 1$$

or $$z = \left(x^2 + y^2\right)^2 - 1$$

Hence the result.

Monge's Method For Solving Equation Of The Type Rr+Ss+Tt = V

Monge's gives a method for solving p.d.e. of second order of the type

$$Rr + Ss + Tt = V \qquad \text{.....(1)}$$

where R, S, T and V are, in general, functions of x, y, z, p and q. Indeed this a equation of first degree in r, s and t. To solve such type of equations, first we determine the intermediate integrals. For this we have

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

or $$dp = rdx + sdy \qquad \text{....(2)}$$

hence $$r = \frac{dp - sdy}{dx} \qquad \text{....(3)}$$

Similarly $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$$

or $$dq = sdx + tdy \qquad \text{....(4)}$$

hence $$t = \frac{dq - sdx}{dy} \qquad \text{....(5)}$$

Now, r and t are eliminated from equation (1) with the help of (3) and (5). Thus we get an equation in s as

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V$$

or $$\left(Rdpdy + Tdqdx - Vdydx\right) - s\left(Rdy^2 - Sdydx + Tdx^2\right) = 0 \qquad \text{....(6)}$$

Equation (6) will be identically satisfied if we take

$$Rdpdy + Tdqdx - Vdydx = 0 \qquad \text{....(7)}$$

and $$Rdy^2 - Sdydx + Tdx^2 = 0 \qquad \text{....(8)}$$

which are called **Monge's subsidiary equations** and will provide us the intermediate integrals. Here we note that the equation (8) is quadratic for the ratio $dy : dx$ and therefore can be decomposed into two linear equations in dx and dy of the form

$$dy - m_1 dx = 0 \quad \text{and} \quad dy - m_2 dx = 0$$

Now combining equations $dy - m_1 dx = 0$ and (7) with $dz = pdx + qdy$, two integrals $u_1 = u_1(x, y, z, p, q)$ and $v_1 = v_1(x, y, z, p, q)$ can be obtained. Then we get $u_1 = f_1(v_1)$ as the first intermediate integral. Similarly on combining equations $dy - m_2 dx = 0$ and (7) with $dz = pdx + qdy$, and following the above procedure, the second intermediate integral $u_2 = f_2(v_2)$ can be obtained.

From these two intermediate integrals, the values of p and q may be obtained in terms of x and y and then substituting them in $dz = pdx + qdy$ and integrating it, the complete integral of (1) is obtained.

Question: Solve $r = a^2 t$ by Monge's method

Answer: Comparing the given equation with Rr+Ss+Tt=V, we get R=1, S=0, T=-a2, V=0. The Monge's subsidiary equation are given by

$$Rdpdy + Tdqdx - Vdydx = 0$$

and $$Rdy^2 + Sdydx + Tdx^2 = 0$$

Substituting the values of R, S, T and V, the subsidiary equations will be

$$dpdy - a^2 dqdx = 0 \qquad \text{....(9)}$$

$$dy^2 - a^2 dx^2 = 0 \qquad \text{....(10)}$$

Equation (10) may be factorised as

$$(dy - adx) = 0 \qquad \text{....(11)}$$

and $$(dy + adx) = 0 \qquad \text{....(12)}$$

Combining equation (11) with subsidiary equation (9), we get

$$dp(adx) - a^2 dqdx = 0$$

or $$dp - adq = 0 \qquad (\because dx = 0, \text{ gives trivial solution}) \qquad \text{....(13)}$$

Now from (11) and (13) we obtain

$$y - ax = c_1, \; p - aq = c_2$$

therefore the first intermediate integral is

$$(p - aq) = f_1(y - ax) \qquad \text{....(14)}$$

Similarly combining $(dy + adx) = 0$ with subsidiary equation (9), we get the second intermediate integral as

$$(p + aq) = f_2(y + ax) \qquad \text{....(15)}$$

Now from above two intermediate integrals (14) and (15) we deduce the value of p and q as.

$$p = \frac{1}{2}\left[f_1(y - ax) + f_2(y + ax)\right]$$

$$q = \frac{1}{2a}\left[f_2(y + ax) - f_1(y - ax)\right]$$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \left(\frac{dy + adx}{2a}\right) f_2(y + ax) - \left(\frac{dy - adx}{2a}\right) f_1(y - ax)$$

On integration, we have

$$z = \frac{1}{2a}\phi_2(y + ax) - \frac{1}{2a}\phi_1(y - ax)$$

Hence the required solution is

$$z = F_1(y + ax) + F_2(y - ax)$$

Monges' Method For Solving Equation Of The Type *Rr+Ss+Tt+U(rt-s²)=V*

Prof G. Monge gave a method for solving equation

$$Rr + Ss + Tt + U(rt - s^2) = V \qquad \text{.....(1)}$$

where R, S, T, U and V are, in general, functions of x, y, z, p and q.

We know that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

or $$dp = r dx + s dy$$

or $$r = \frac{dp - s dy}{dx} \qquad \text{.....(2)}$$

Similarly

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$$

therefore $$t = \frac{dq - S dx}{dy} \qquad \text{......(3)}$$

Putting the values of r and t from (2) and (3) in (1), we get

$$R\left(\frac{dp-sdy}{dx}\right)+Ss+T\left(\frac{dq-sdx}{dy}\right)+U\left\{\frac{dp-sdy}{dx}\cdot\frac{dq-sdx}{dy}-s^2\right\}=V$$

or $(Rdpdy + Tdqdx + Udpdq - Vdxdy) - s\,(Rdy^2 - Sdxdy + Tdx^2$

$$+ Udpdx + Udqdy) = 0 \qquad .....(4)$$

Equation (4) will be identically satisfied if we take

$$Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \qquad .....(5)$$

and

$$Rdy^2 - S\,dxdy + Tdx^2 + Udpdx + Udqdy = 0 \qquad .....(6)$$

These simultaneous equations (5) and (6) are known as Monge's subsidiary equations.

Here the equation (6) can not be factorized. So we will try to factorize

$$\left(Rdy^2 - S\,dxdy + Tdx^2 + Udpdx + Udqdy\right) + \lambda\left(Rdpdy + Tdq\,dx + Udpdq - Vdxdy\right) = 0 \qquad .....(7)$$

where λ is some multiple and is determined later.

Let us suppose that the factors of (7) are

$$\left(Rdy+m_1Tdx+m_2Udp\right)\left(dy+\frac{1}{m_1}dx+\frac{\lambda}{m_2}dq\right)=0 \qquad .....(8)$$

On comparing (7) with (8), we obtain

$$\frac{R}{m_1}+m_1T=-\left(S+\lambda V\right), \quad m_2=m_1, \quad \frac{R\lambda}{m_2}=U \qquad .....(9)$$

The last two relations gives $m_1=\frac{R\lambda}{U}$. Putting this in the first relation of (9), we obtain

$$\lambda^2\left(UV+RT\right)+\lambda SU+U^2=0 \qquad .....(10)$$

This equation is called **λ–equation**, where λ, in general, is a function of x, y, z, p and q.

Now since equation (10) is quadratic in λ so suppose that it is satisfied by two values of λ say λ_1 and λ_2 then the factors corresponding to these values will be

$$\left(Rdy+\frac{R\lambda_1}{U}Tdx+R\lambda_1dp\right)\left(dy+\frac{U}{R\lambda_1}dx+\frac{U}{R}dq\right)=0$$

as $$m_1 = m_2 = \frac{R\lambda_1}{U}$$

or $$\left(Udy + \lambda_1 Tdx + \lambda_1 Udp\right)\left(Udx + \lambda_1 Rdy + \lambda_1 Udq\right) = 0 \qquad .....(11)$$

Similarly corresponding to λ_2, we can obtain

$$\left(Udy + \lambda_2 Tdx + \lambda_2 Udp\right)\left(Udx + \lambda_2 Rdy + \lambda_2 Udq\right) = 0 \qquad .....(12)$$

Now one factor from (11) and one from (12) will be combined in pairs to get intermediate integrals in the form $u = f(v)$. We can combine factors as

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$$

$$Udx + \lambda_2 Rdy + \lambda_2 Udp = 0$$

and $$Udx + \lambda_1 Rdy + \lambda_1 Udp = 0$$

$$Udy + \lambda_2 Tdx + \lambda_2 Udp = 0$$

These two pairs will give intermediate integrals provided these total differential equations are integrable, from which the values of p and q can be determined. Substituting these values of p and q in $dz = pdx + qdy$, we get the general solution on integration.

Question: Solve 3r+4s+t+(rt-s²)=1

Solution: Comparing the given equation with Rr+Ss+Tt+U(rt-s²)=V, we have R=3, S=4, T=1, U=1, V=1. Then λ- quadratic equation

$$\lambda^2(UV+RT)+\lambda SU+U^2=0$$

becomes $$4\lambda^2+4\lambda+1=0$$

or $$(2\lambda+1)^2=0 \Rightarrow \lambda_1=\lambda_2=-\frac{1}{2}$$

Hence there is only one intermediate integral given by the equations

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0$$

and $$Udx+\lambda_2 Rdy+\lambda_2 Udq=0$$

On putting above values, we get

$$dy + \left(-\frac{1}{2}\right)dx + \left(-\frac{1}{2}\right)dp = 0$$

and $$dx + \left(-\frac{1}{2}\right)3dy + \left(-\frac{1}{2}\right)dq = 0$$

or $$-2dy + dx + dp = 0$$

and $$3dy - 2dx + dq = 0$$

On integration, we obtain

$$-2y + x + p = c_1 \quad \text{.....(13)}$$

and $$3y - 2x + q = c_2 \quad \text{.....(14)}$$

Hence the only intermediate integral is

$$-2y + x + p = f(3y - 2x + q)$$

where f is any arbitrary function

Now solving (13) and (14) for p and q, we get

$$p = 2y - x + c_1$$

$$q = -3y + 2x + c_2$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2y - x + c_1)dx + (-3y + 2x + c_2)dy$$

or $$dz = 2(ydx + xdy) - xdx - 3ydy + c_1dx + c_2dy$$

On integrating, we obtain the general solution as

$$z = 2xy - \frac{1}{2}x^2 - \frac{3}{2}y^2 + c_1x + c_2y + c_3$$

where c_1, c_2, c_3 are arbitrary constants.

References:

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