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BOOK: Differential Equation, Calculus of Variations and Special Functions

UNIT – IV : Classification of Linear PDE of Second Order, Cauchy Problem and Method of Separation of Variables

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Objectives

After studying this unit, you should be able to-

- Identify and classify partial differential equations (PDE).
- have an idea of Cauchy problem.
- solve the partial differential equations by method of separation of variables.

INTRODUCTION

The importance of partial differential equations among the topics of applied mathematics has been recognized for many years. However, the increasing complexity of today's technology is demanding of the mathematician, the engineer and the scientists, an understanding of the subject previously attained only by specialists. This unit of partial differential equations (PDE) comprises identification and classification of PDE. It also presents the principal technique method of separation of variables for constructing solution to partial differential equation problems. The solved and supplementary problems have the vital role of apply-ing reinforcing and sometimes expanding the theoretical concepts.

Classification of PDE of Second Order

Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{....(1)}$$

where R, S and T are continuous functions of x and y only possessing continuous partial derivatives. The PDE can be classified into three categories depending on nature of values of the discriminant $S^2 - 4RT$. Thus (1) is known as

Hyperbolic if $S^2 - 4RT > 0$

Parabolic if $S^2 - 4RT = 0$

Elliptic if $S^2 - 4RT < 0$

***Ex.* 1 :** Consider the one dimensional Laplace's equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ *i.e.* $r + t = 0$. Comparing it with equation (1), we have $R = 1$, $S = T = 0$. Hence $S^2 - 4RT = 0$ and so given equation is parabolic.

***Ex.* 2 :** Consider the one dimensional diffusion equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y}$ *i.e.* $r - q = 0$. Comparing it with equation (1), we have $R = 1$, $S = 0$ and $T = -1$. Hence $S^2 - 4RT = 4 > 0$ and so given equation is hyperbolic.

Classification of a Second Order PDE in More Than Two Independent Variables

A linear second order partial differential equation having more than two independent variables can suitably be reduced, in general, to a canonical form only when the coefficients are constants. Let x, x_2,..., x_n be n independent variables and u be the dependent variable, then such a second order PDE may be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad \text{....(1)}$$

where a_{ij}, b_i and c are constants and $a_{ij} = a_{ji}$. Now we consider a one-one transformation

$$\xi_i = \xi_i (x_1, x_2, \ldots.., x_n),\ i = 1, 2, \ldots.., n \qquad \text{....(2)}$$

Then the equation (1) transforms to

$$\sum_{k=1}^{n} \sum_{l=1}^{n} A_{kl}\, u_{\xi_k \xi_l} + F\left(\xi_1, \xi_2, \ldots, \xi_n; u, u_{\xi_1}, u_{\xi_2}, \ldots, u_{\xi_n}\right) = 0 \qquad \text{.....(3)}$$

where
$$A_{kl} = a_{ij} (\xi_k)_{x_i} (\xi_i)_{x_j} \qquad \text{...(4)}$$

The characteristic quadratic $Q(\alpha)$ associated with equation (1) in this case is

$$Q(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \alpha_i \alpha_j \qquad \text{...(5)}$$

The associated real symmetric matrix in this case will be

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix} \qquad \text{....(6)}$$

and the characteristic roots "eigenvalues" will be given by

$$|M - \alpha I| = 0 \qquad \text{...(7)}$$

and their nature and signs will determine the type of the given PDE.

Case I : Elliptic PDE : If all the eigenvalues are nonzero and of the same sign, the given PDE is of elliptic type.

Case II : Hyperbolic PDE : If all the eigenvalues are nonzero and have the same sign except precisely one of them, the given PDE is of hyperbolic type.

Case III : Ultra Hyperbolic PDE ($n \geq 4$) : If all the eigenvalues are nonzero and atleast two of them are positive and two negative then the given PDE is of ultra hyperbolic type.

Case IV : Parabolic PDE : If any of the eigenvalues is zero, the given PDE is of parabolic type.

Ex. 1. Determine the nature of following PDE $\dfrac{\partial^2 z}{\partial x^2} = x^2 \dfrac{\partial^2 z}{\partial y^2}$

Sol. $$\frac{\partial^2 z}{\partial x^2} - x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Comparing with standard second order PDE, we h $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$

$$R = 1,\ S = 0,\ T = -x^2$$

$$S^2 - 4RT = 0 - 4(-x^2) = 4x^2$$

Since $x^2 > 0$, therefore given PDE is hyperbolic.

Ex. 2. Classify the following PDE as hyperbolic, parabolic or elliptic :

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial x^2} = 0$$

Sol. On comparing it with equation (1), we have

$$R=1, S=2, T=1$$

Hence the value of discriminant

$$S^2 - 4RT = 0$$

Therefore given PDE is parabolic in nature.

Ex. 3. Show that the equation $\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial x^2} = 0$ ***is elliptic for values of x and in the region $x^2 + y^2 < 1$, parabolic on the boundary and hyperbolic outside this region.***

Sol. Given equation is

$$\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x \partial y} + \left(1-y^2\right)\frac{\partial^2 z}{\partial y^2} = 0$$

Obviously $\quad R = 1,\ S = 2x,\ T = 1 - y^2$

Now discriminant is

$$S^2 - 4RT = 4x^2 - 4(1 - y^2) = 4(x^2 + y^2 - 1)$$

Given equation is elliptic in nature if

$$S^2 - 4RT < 0$$

or $\quad 4(x^2 + y^2 - 1) < 0 \Rightarrow x^2 + y^2 < 1 \quad$ (inside boundary)

Given equation is parabolic in nature if

$$S^2 - 4RT = 0$$

or $\quad 4(x^2 + y^2 - 1) = 0 \Rightarrow x^2 + y^2 = 1 \quad$ (on boundary)

Given equation is hyperbolic in nature if

$$S^2 - 4RT > 0$$

or $\quad 4(x^2 + y^2 - 1) > 0 \Rightarrow x^2 + y^2 > 1 \quad$ (outside the boundary)

Cauchy Problem

The Cauchy problem is a boundary value problem dealing with the unique solution of a second order quasi–linear PDE when its initial value and slope are specified.

Statement : Determine the solution $z = z(x, y)$ of the PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{...(1)}$$

where R, S and T are in general functions of x, y, z, p and q such that the solution takes on a given space curve C, having the parametric equation

$$x = x(t), y = y(t), z = z(t) \qquad \text{...(2)}$$

prescribed value of z and $\frac{\partial z}{\partial n}$ where n is the distance measured along the normal to the curve.

The latter set of boundary conditions is equivalent to assuming that the values of x, y, z, p, q are determined on the curve, but it be noted that the values of p and q can not be assigned arbitrarily along the curve.

Method of solution : The solution of eq. (1) will be some surface, called **integral surface**, passing through C. Hence at each point of *C*, by relations (2) we have

$$\dot{z}_0 = p\dot{x} + q\dot{y}_0 \qquad \text{.....(3)}$$

which shows that p_0 and q_0 are not independent.

Thus, the Cauchy problem finds the solution of (1) passing through the integral strip of the first order formed by the planar elements $(x_0, y_0, z_0, p_0, q_0)$ of the curve *C*. At every point of the integral strip $p_0 = p_0(t)$, $q_0 = q_0(t)$, so that of we differentiate these equation w.r.t. 't' we find that

$$\dot{p}_0 = r\dot{x}_0 + s\dot{y}_0, \ \dot{q}_0 = s\dot{x}_0 + t\dot{y}_0 \qquad \text{....(4)}$$

Knowing $R, S, T, f, \ \dot{x}_0, \dot{y}_0, p_0, q_0, \dot{p}_0, \dot{q}_0$ at each point of *C*, we may regard equations (1) and (4) as linear simultaneous equations for the unknowns *r*, *s*, *t* at each point of *C*. Solving by Cramer's's rule, we get

$$\frac{r}{\Delta_1} = \frac{-s}{\Delta_2} = \frac{t}{\Delta_3} = -\frac{1}{\Delta} \qquad \text{.....(5)}$$

where

$$\Delta_1 = \begin{vmatrix} S & T & f \\ \dot{y}_0 & 0 & -\dot{p}_0 \\ \dot{x}_0 & \dot{y}_0 & -\dot{q}_0 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} R & T & f \\ \dot{x}_0 & 0 & -\dot{p}_0 \\ 0 & \dot{y}_0 & -\dot{q}_0 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} R & S & f \\ \dot{x}_0 & \dot{y}_0 & -\dot{p}_0 \\ 0 & \dot{x}_0 & -\dot{q}_0 \end{vmatrix} \qquad .....(6)$$

$$\Delta = \begin{vmatrix} R & S & T \\ \dot{x}_0 & \dot{y}_0 & 0 \\ 0 & \dot{x}_0 & \dot{y}_0 \end{vmatrix} \qquad .....(7)$$

If $\Delta \neq 0$, we can easily calculate the expressions for second order derivatives r_0, s_0 and t_0 along C.

The third order partial differential coefficient of z can similarly be calculated at every point of C by differentiating (1) w.r.t. x and y respectively, making use of

$$\dot{r}_0 = z_{xxx}\dot{x}_0 + z_{xxy}\dot{y}_0 \qquad .....(8)$$

etc. and solving as above.

Proceeding in this manner, we can calculate partial derivatives of every order of the points of C. The values of the function z at neighbouring points, can be obtained by using Taylor's Theorem for functions of two independent variables. Thus the Cauchy problem possesses a solution as long as $\Delta \neq 0$. In the elliptic case $4RT - S^2 > 0$, so that $\Delta \neq 0$ always holds and the derivatives, of all orders, of z are uniquely determined.

If $\Delta = 0$, then the Cauchy's method of solution breaks down. This critical case leads to the condition

$$R\dot{y}^2 - S\dot{x}\dot{y} + T\dot{x}^2 = 0$$

or $$Rdy^2 - Sdydx + Tdx^2 = 0 \qquad \text{....(9)}$$

At each point $(x, y, 0)$ of Γ (which is orthogonal projection of the curve C on the plane $z = 0$) the eq. (9) would give a pair of directions along which $\Delta = 0$. These directions are called as **characteristics.**

Thus curves known as **characteristic base curves,** may be drawn through every point $(x, y, 0)$ of the base curve Γ simply by approximating them by straight line segments whose directions are taken to coincide with those of the tangents given by the roots of (9), viz.

$$\frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - 4RT}}{2R} \qquad \text{.....(10)}$$

Thus a curve Γ in the xy plane satisfying (10) is called a characteristic base curve of the PDE (1), and the curve C of which it is the projection is called a **characteristics curve** of the same equation.

Note that **characteristics** are :

Real and distinct if $S^2 - 4RT > 0$

Coincident if $S^2 - 4RT = 0$ and

Imaginary if $S^2 - 4RT < 0$

Hence these are two families of characteristics if the given PDE is hyperbolic, one family if it is parabolic and none if it is elliptic. Thus the Cauchy problem will fail to have unique solution if an arc element of the base curve coincides with the characteristics.

Characteristic equations :

Corresponding to (1), consider λ–quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{....(11)}$$

when $S^2 - 4RT \geq 0$, eq. (11) has real roots. Then, the ordinary differential equation

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

are called the characteristic equations.

Again the solution of (11) will be characteristic curves or simply the characteristic of the second order PDE (1).

Method of Separation of Variables

For given linear second order partial differential equation

$$Rr + Ss + Tt + Pp + Qq + Zz = F(x,y) \qquad \text{....(1)}$$

where R, S, T, P, Q, Z and F are functions of independent variables x and y only. Let $Z(x,y)$ be solution of (1).

The method of separation of variables for this problem is a powerful tool and begins with assumption that $Z(x,y)$ is of the form $X(x) \,.\, Y(y)$ i.e.

$$Z(x,y) = X(x) \,.\, Y(y) \qquad \text{....(2)}$$

where X is function of independent variables x only and Y is function of independent variables y only.

On substituting (2) in (1) we have

$$\frac{1}{X} f(D) X = \frac{1}{Y} g(D') Y \qquad \text{....(3)}$$

where f(D) and g(D′) are quadratic functions of $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$ respectively. This has the effect of replacing the single PDE with two second order linear ordinary differential equations since LHS of (3) is function of x alone and the RHS is function of y alone. Since x and y are independent variables, the two sides of (3) will be equal only if each side is a constant (say λ) be

$$\frac{1}{X} f(D) X = \frac{1}{Y} g(D') Y = \lambda$$

or $$f(D) X = \lambda X \quad \text{and} \quad g(D') Y = \lambda Y \qquad .....(4)$$

which can be solved by the methods of ordinary differential equation.

The theory of eigenfunction expansions enters into the treatment of any in homogenous aspect of the problem. The general solution of equation (4) will depend on the choice of λ positive or negative or zero. In practical problems, the nature of the boundary conditions determine the nature of λ and it becomes an eigenvalue problem.

The method of separation of variables can be employed in a similar manner for more than two independent variables also.

In the application of ordinary linear differential equation, we first find the general solution and then determine the arbitrary constant from the initial values, But the same method is not applicable to problem involving PDE In method of separation of variables right from the beginning we try to find the particular solution of PDE which satisfy all or some of the boundary conditions and then the remaining conditions are also satisfied. The combination of these particular solutions gives the solution of the problem.

Ex. 1. *Find the characteristics of* $y^2r - x^2t = 0$.

Sol. Given $$y^2r - x^2t = 0 \qquad \text{....(5)}$$
Comparing (5) with
$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have} \qquad \text{.....(6)}$$

$$R = y^2,\ S = 0 \text{ and } T = -x^2$$

Hence $$S^2 - 4RT = 0 - 4y^2(-x^2) = 4x^2y^2 > 0$$

and thus (1) is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$. The λ quadratic is
$$R\lambda^2 + S\lambda + T = 0 \text{ or } y^2\lambda^2 - x^2 = 0 \qquad \text{....(7)}$$

Solving (7), we get $\lambda = \dfrac{x}{y}, -\dfrac{x}{y}$ (two district real roots)

Corresponding characteristic equations are

$$\frac{dy}{dx} + \frac{x}{y} = 0 \text{ and } \frac{dy}{dx} - \frac{x}{y} = 0$$

or $$xdx + ydy = 0 \text{ and } xdx - ydy = 0$$

Integrating, we get
$$x^2 + y^2 = C_1 \quad \text{and} \quad x^2 - y^2 = C_1$$
which are required families of characteristics.

Here these are families of circles and hyperbolas respectively.

Ex. 2. Find the characteristics of $x^2r + 2xys + y^2t = 0$ .....(8)

Sol. Comparing (8) with (6) we have

$$R = x^2,\ S = 2xy \text{ and } T = y^2$$

Hence

$$S^2 - 4RT = 0$$

and hence (3) is parabolic everywhere. The quadratic is

$$\lambda^2 x^2 + 2xy + y^2 = 0$$

Solving it we get

$$(\lambda x + y)^2 = 0 \quad \text{or } \lambda = -\frac{y}{x}, -\frac{y}{x} \text{ (two equal roots)}$$

The characteristic equations is

$$\frac{dy}{dx} - \frac{x}{y} = 0 \quad \text{or} \quad \frac{1}{y}dy - \frac{1}{x}dx = 0$$

Integrating, we get

$$\frac{y}{x} = c_1 \quad \text{and} \quad y = c_1 x \qquad ....(9)$$

which is the required family of characteristics. (9) represents a family of straight lines passing through the origin.

Ex. 3. Use the method of separation of variables to solve the equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \text{ given that } u(x,0) = 6e^{-3x}$$

Sol. Let $u(x, t) = X(x)T(t)$ be solution of given PDE where X is a function of x only and T is a function of t only.

Now $$\frac{\partial u}{\partial x} = T\frac{dX}{dx} \; and \; \frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

On substituting these values in given PDE, we get

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

Dividing by XT, we have

$$\frac{X'}{X} = \frac{2T'}{T} + 1 = -n^2 \text{ (say)}$$

Now we have two ordinary differential equations.

$$\frac{X'}{X} = -n^2 \quad \text{and} \quad \frac{2T'}{T} + 1 = -n^2$$

or $$\frac{dX}{dx} + n^2 X = 0, \; and \; \frac{T'}{T} = -\left(\frac{n^2+1}{2}\right)$$

Sloving these equations, we find that

$$X = c_1 e^{-n^2 x} \quad \text{and} \quad T = c_2 e^{-\left(\frac{n^2+1}{2}\right)t}$$

Hence $$u(x,t) = X(x)T(t) = c_1 c_2 e^{-n^2 x - \left(\frac{n^2+1}{2}\right)t}$$

Under given condition we get $6e^{-3x} = c_1 c_2 e^{-n^2 x}$

$\Rightarrow$ $$c_1 c_2 = 6 \text{ and } n^2 = 3$$

Thus the required solution of the problem is $u(x,\ t) = 6\ e^{-3x-2t}$

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