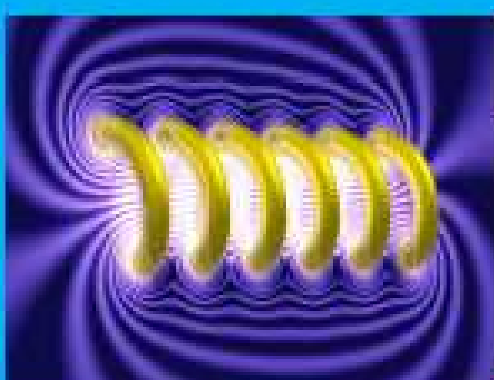
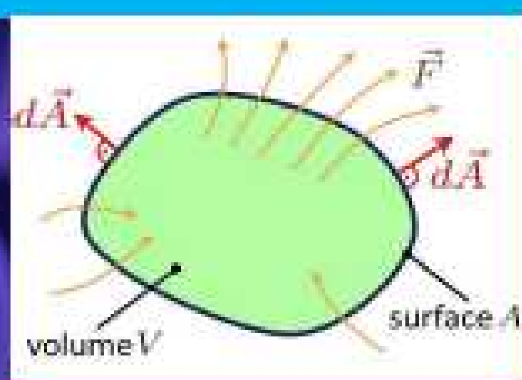
MSCPH508

M. Sc. IInd Semester

ELECTRODYNAMICS





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Electrodynamics

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Course: Electrodynamics **Course code: MSCPH508**

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UNIT 1: ELECTROSTATICS

Structure

1.1 INTRODUCTION

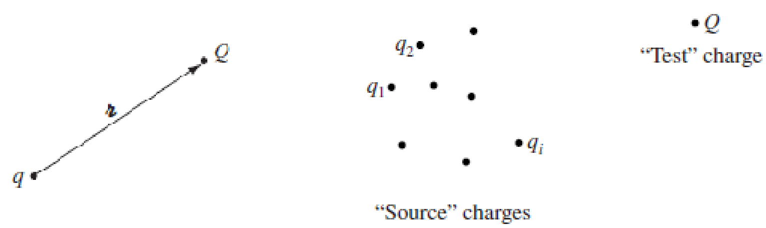
The electrostatics laterally contains two terms electro and statics thereby meaning charge in stationary state. Therefore, electrostatics deals with the attraction or repulsion between the source charges, which creates electric field but stationary and test charges, which may be moving. The unit starts with the revision of basic laws and definitions and then finally discusses the Gauss law and its application, Laplace and Poisson's equation and boundary value problems.

1.2 OBJECTIVES

- define Coulomb's law
- define electric field, potential and dipoles
- application of gauss's law
- how to solve the problems using gauss's law
- Laplace's equation in one and two dimensions

1.3 COULOMB'S LAW

If an isolated source charge (q) is placed in the space, then it induces an electric field around it. If a test charge (Q) is placed in this induced field then it experiences a force, as shown in figure 1(a). According to Coulomb's law that the force exerted by source charge (q) on any other charge (say test charge Q, except itself) placed in it given by;



$$\vec{F}=\frac{1}{4\pi\varepsilon_0}\frac{q.Q}{r^2}\hat{r}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

The constant ε_0 is called the permittivity of free space**.** In SI units, where force is in Newton (N), distance in meters (m), and charge in coulombs (C),

$$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/N.m^2$$

In words, the force is proportional to the product of the charges and inversely proportional to the square of the separation distance. As always, $r$is the separation vector fromthe location of q ($\acute{r}$) to the location of Q (R);

$$r = R - \acute{r}$$

r is its magnitude, and $\hat{r}$is its direction. The force points along the line from q to Q; it is repulsive if q and Q have the same sign, and attractive if their signs are opposite. Coulomb's law and the principle of superposition constitute the physical input for electrostatic. If many source chargesq_1, q_2, q_3 … … are present in the space then the total force exerted on the test charge (Q) will be the vector sum of the forces exerted by all the charges present in that space, as shown in Figure 1(b).

1.4 ELECTRIC FIELD

If we have several point charges q_1,q_2, q_3 .... at distances r_1,r_2,r_3............... from Q then total force on Q can be given by;

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{\boldsymbol{r}}_1 + \frac{q_2 Q}{r_2^2}\hat{\boldsymbol{r}}_2 + \ldots\right)$$
$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{\boldsymbol{r}}_1 + \frac{q_2}{r_2^2}\hat{\boldsymbol{r}}_2 + \frac{q_3}{r_3^2}\hat{\boldsymbol{r}}_3 + \ldots\right), \quad \text{------------------ (1.2)}$$

Or it can be written as;

$$\mathbf{F} = Q\mathbf{E}, \quad \text{-------------------- (1.3)}$$

Where E replaces the following expression,

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\boldsymbol{r}}_i. \quad \text{---------------------- (1.4)}$$

E is called electric field of source charges and physically defined as the force exerted on the unit test charge. Electric field is the function of r and how the source charges are configured.

In explaining the electric field, we have assumed that the charges are discrete. If the charge is distributed continuously over some region, then the electric field is given by,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\,dq$$ ---------------- (1.5)

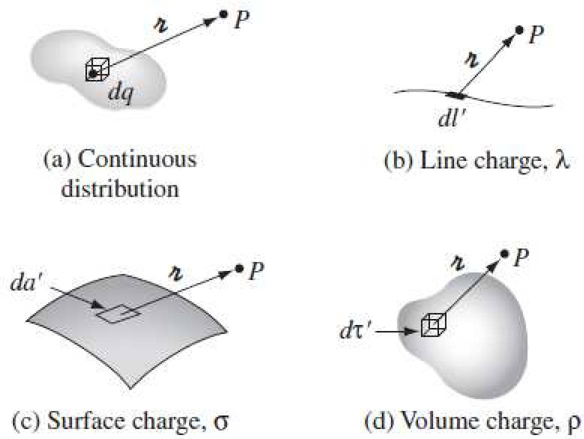


Figure 1.2 pictorial representations of charge densities

(i) If the charge is spread out along a *line* (Fig. 5b), with charge-per-unit-length ρ_l,

$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)}$$ ------------(1.6)

where *dl* is an element of length along the line. Also the total charge over the line is given by;

$$Q = \int_l \rho_\ell \, dl \quad \text{(C)}$$ ------------- (1.7)

(ii) If the charge is smeared out over a *surface* (Fig. 5c), with charge-per-unit-area ρ_s,

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad \text{(C/m}^2\text{)}$$ --------- (1.8)

where *ds*is an element of area on the surface. Then total charge can be written as;

$$Q = \int_S \rho_s \, ds \quad \text{(C)}$$ -------------(1.9)

(iii) If the charge fills a *volume* (Fig. 5d), with charge-per-unit-volume ρ_v defined as;

$$\rho_v = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \quad \text{(C/m}^3\text{)}$$ -------------------(1.10)

where *dv*is an element of volume, and total charge can be expressed as;

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \quad \text{(C)}$$ ------------- (1.11)

Consider again, a single point charge q, situated at the origin then the electric field,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \quad \text{------------------ (1.12)}$$

which falls of as $1/r^2$, radially from the origin, can be represented by the electric lines, as shown by figure 12. The direction of electric lines towards the origin if placed charge is positive and away from the centre if negative. If two charged are placed at some distance, then depending upon the nature of the charges the shape of the electric lines are shown in figure below.

The magnitude of the field is indicated by the density of the filed lines. It is strong near the centre and weaker as we move radially outwards. Field lines begin with the positive charge and end on negative charge. Moreover, the field lines never cross each other.

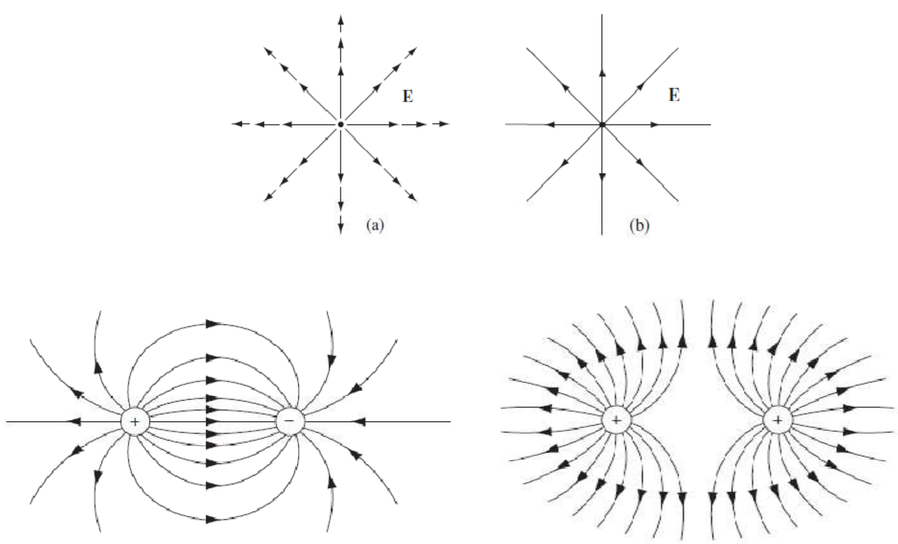


Figure 1.3 electric field lines for different charges

1.5 ELECTRIC POTENTIAL

Suppose a point charge q is placed at the origin, as shown in figure 1.4 below, then induced electric field at a distance r is given by equation 1.12. The line integral of this field from point a to point b, which are at distances r_a and r_b from the origin, can be written as;

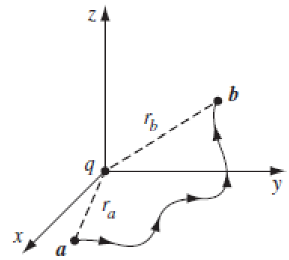


Figure 1.4

$$\int_a^b \mathbf{E} \cdot d\mathbf{l}.$$

dl in spherical coordinates, $, d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}},$

then,

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr. \qquad \text{-------------------- (1.13)}$$

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

The integral around a closed path is evidently zero when $r_a = r_b$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0, \qquad \text{-------------------- (1.14)}$$

Now, according to Stake's theorem,

$$\nabla \times \mathbf{E} = 0. \qquad \text{------------------ (1.15)}$$

Moreover, the origin is just a reference point, the equation of still valid for any position of charge. If many charges are present then according to superposition theorem, the total electric field is the vector sum of their individual fields;

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots, \qquad \text{----------------- (1.16)}$$

Therefore,

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = \mathbf{0}.$$ ------------- (1.17)

These equations hold for any static charge. Now, any vector whose curl is zero is equal to a gradient of some scalar, therefore,

$$\mathbf{E} = -\nabla V$$ ------------ (1.18)

V is defined as an electric potential,

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'.$$ ---------------- (1.19)

In general, the electric potential V of a point charge q at a distance r is given as,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}},$$ --------------- (1.20)

Similarly, the electric potential due to collection of charges, by applying superposition principle,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$ ---------------- (1.21)

And for continuous charge distribution,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}} dq.$$ -------------------- (1.22)

1.6 ELECTRIC DIPOLE, ELECTRIC QUADRUPOLE AND MULTIPOLES

If you are very far away from a localized charge distribution, it "looks" like a point charge, and the potential is to good approximation is given by equation 1.20, where *Q* is the total charge. We have often used this as a check on formulas for *V*. But what You might reply that the potential is then approximately zero, and of course, you're right, in a sense (indeed, the potential at large *r* is *pretty small* even if *Q* is *not* zero). But we're looking for something a bit more informative than that.

A (physical) electric dipole consists of two equal and opposite charges (±*q*) separated by a distance *d*. Then electric potential can be written as, Figure 1.5 shows the schematic of this.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$ --------------- (1.23)

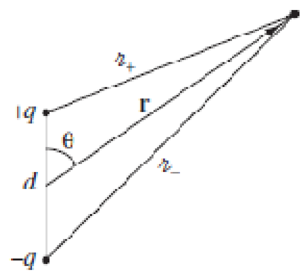


Figure 1.5

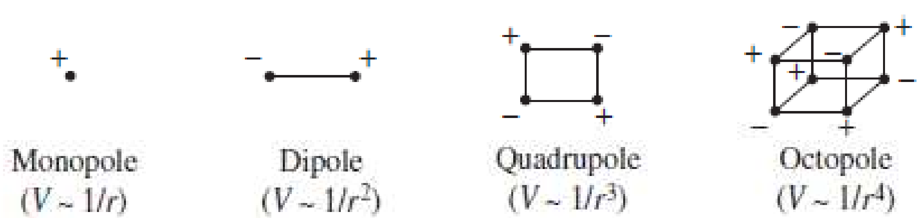
Then we can write the following expressions;

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$ ------------------- (1.24)



The multipole expansion of electric potential (V) in the power of 1/r

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right].$$ --------- (1.25)

1.7 GAUSS'S LAW

1.7.1 Flux of an Electric Filed through a Surface

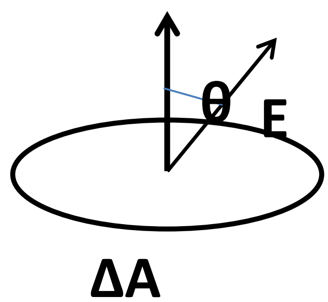


Figure 1.6

Consider a hypothetical surface of area ΔA and suppose a uniform electric field $\vec{E}$exists in the space. A line perpendicular to surface as shown in figure 1.6, upward is considered as a positive normal. Suppose, the electric field $\vec{E}$ makes an angle θ with the positive normal to the surface. Then the quantity, $\Delta\emptyset = E.\Delta A.cos\theta$ or $\Delta\emptyset = \vec{E}.\overrightarrow{A}$, is called the flux of the electric field through the chosen surface.

If surface is made up of many small surfaces, then the total flux can be written as;

$\emptyset = \int \vec{E}.\ \vec{A}$ ------------------- (1.26)

The surface under consideration may be closed one, enclosing a volume, such as a spherical surface. When the flux through a closed surface is required then flux is written as;

$\emptyset = \oint \vec{E}.\ \vec{A}$-------------- (1.27)

This suggests that the flux through any close surface is a measure of total charge inside. For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside (Figure 1.3). On the other hand, a charge *outside* the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other (Figure). This is the *essence* of **Gauss's law**. The statement of the Gauss's law, "The flux of the net electric field through a closed surface equals to the net charge enclosed by the surface (q_{in}) divided by ε_0". The expression may be written as;

$$\oint \vec{E}.\ \vec{A} = \frac{q_{enc}}{\varepsilon_0} \text{-------------- (1.28)}$$

In the case of a point charge *q* at the origin, the flux of **E** through a spherical surface of radius *r* is,

$$\oint \vec{E}.\ \vec{A} = \oint \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.r^2 sin\theta d\theta d\emptyset dr\hat{r} = \frac{q}{\varepsilon_0} \text{------------ (1.29)}$$

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\oint \vec{E}.\ \vec{A} = \sum_{i=1}^{n} \frac{q_i}{\varepsilon_0} \text{------------------- (1.30)}$$

As it stands, Gauss's law is an *integral* equation, but we can easily turn it into a *differential* one, by applying the divergence theorem:

$$\oint \vec{E}.\ \vec{A} = \int (\nabla\ .E)\ dv \text{------------------- (1.31)}$$

Rewriting Q_{enc} in terms of the charge density ρ, we have,

$$q_{enc} = \int \rho\ dv \text{---------------------------- (1.32)}$$

Therefore, Gauss's law becomes,

$$\int (\nabla\ .E)\ dv = \int \frac{\rho}{\varepsilon_0}\ dv \text{ ---------------------- (1.33)}$$

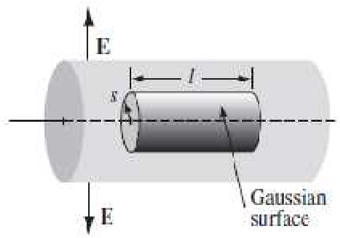
And since this holds for *any* volume, the integrands must be equal,

$$\nabla. E = \frac{\rho}{\varepsilon_0} \text{----------------------------------- (1.34)}$$

1.7.2 Applications of Gauss's Law

Example 1 A long cylinder (Figure below) carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

Solution Draw a Gaussian cylinder of length l and radius s. For this surface, Gauss's law states:



$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{enc}$$

The enclosed charge is,

$$Q_{enc} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \frac{2}{3}\pi k l s^3.$$

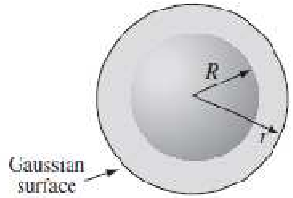
$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi s l$$

$$|\mathbf{E}| \, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3$$

$$\mathbf{E} - \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$$

Example 2 Find the field outside a uniformly charged solid sphere of radius Rand total charge q.

Solution Imagine a spherical surface at radius $r > R$, as shown below; this is called a Gaussian surface in the trade. Gauss's law says that



$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{enc},$$

and in this case Qenc = q. At first glance this doesn't seem to get us very far, because the quantity we want (**E**) is buried inside the surface integral. Luckily, symmetry allows us to extract **E** from under the integral sign: **E** certainly points radially outward,5 as does $d\mathbf{a}$, so we can drop the dot product,

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \int_S |\mathbf{E}|\, da,$$

and the magnitude of **E** is constant over the Gaussian surface, so it comes outside the integral:

$$\int_S |\mathbf{E}|\, da = |\mathbf{E}| \int_S da = |\mathbf{E}|\, 4\pi r^2.$$

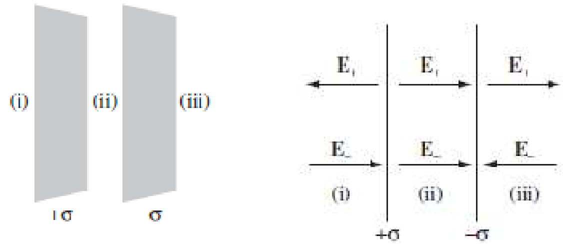
$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

The field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the centre.

Example 3 Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

Solution The left plate produces a field $(1/2\varepsilon_0)\ \sigma$, which points away from it (as shown in figure below) to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\varepsilon_0)\ \sigma$, which points toward it to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). Conclusion: The field between the plates is σ/ε_0, and points to the right; elsewhere it is zero.



1.8 LAPLACE AND POISSON EQUATIONS

Electrostatics primarily deals with the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, in the form of the following equation;

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'. \qquad \text{-------------------- (1.35)}$$

Integrals of this type can be difficult to calculate for any but the simplest charge configurations. Occasionally we can get around this by exploiting symmetry and using Gauss's law. However, best strategy is first to calculate the potential, V, by using the following equation:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'. \qquad \text{----------------------- (1.36)}$$

Still, it is often too tough to handle analytically. Moreover, if charge is free to move inside the conductors then ρ itself may not be known in advance. In such cases, it is useful to recast the problem in differential form, using Poisson's equation,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \qquad \text{-------------------------------------- (1.37)}$$

which, together with appropriate boundary conditions. Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. (If $\rho = 0$ everywhere,of course, then $V = 0$, but still there may be plenty of charge elsewhere,but we're considering places where there is no charge.) In this case, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0, \quad \text{------------------------------------- (1.38)}$$

or, written out in Cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad \text{-------------------- (1.39)}$$

This formula is so fundamental to the subject that one might almost say electrostatics *is* the study of Laplace's equation. At the same time, it, appears in such diverse branches of physics as gravitation and magnetism, the theory of heat, and the study of soap bubbles. In mathematics, it plays a major role in analytic function theory. The general solutions of Laplace's equations are called harmonic functions. The one- and two dimensional solutions will be described later. Before that the boundary value problems will be discussed

1.9 BOUNDARY VALUE PROBLEMS

The solutions of Laplace's equations do not depend on time, initial conditions are irreverent and only boundary conditions are specified. There are three basic types of boundary conditions that are usually associated Laplace's equation. They are as follows;

(a) If the solution $u\ (x, y)$ to Laplace equation in a domain Ω is specified on the boundary ∂Ω i.e., u (x, y) = f (x, y) on ∂Ω, then it is known as Dirichlet boundary conditions. Then the Dirichlet boundary value problems for Laplace equation is of the form;

$$\nabla^2 u\ (x, y) = 0 \text{ in } \Omega;$$

$$\text{and } u\ (x, y) = f\ (x, y) \text{ on } \partial\Omega.$$

(b) If the directional derivative along the outward normal to the boundary is specified on ∂Ω that is

$$\frac{\partial u}{\partial n}(x, y) = g(x, y) \quad \text{for } (x, y) \in \partial\Omega.$$

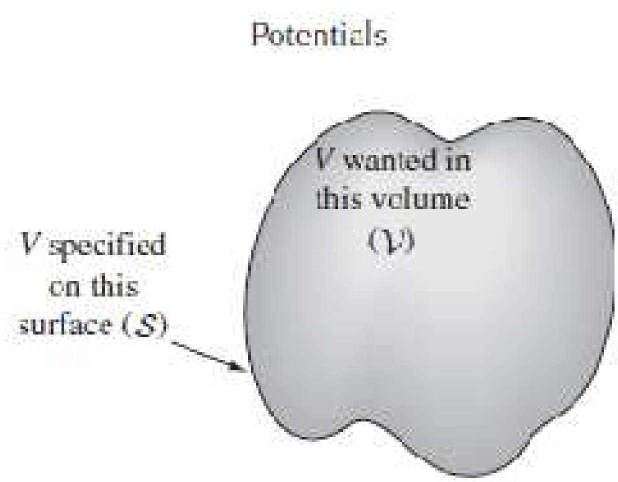
then it is called as Neumann boundary value problem. In physical terms, the normal component of the solution gradient is known on the boundary. Laplace's equations together with Neumann boundary conditions are known as Neumann boundary value problem and defined as below.

$$\nabla^2 u = 0 \text{ in } \Omega; \quad \frac{\partial u}{\partial n}(x,y) = g(x,y) \quad \text{for } (x,y) \in \partial\Omega$$

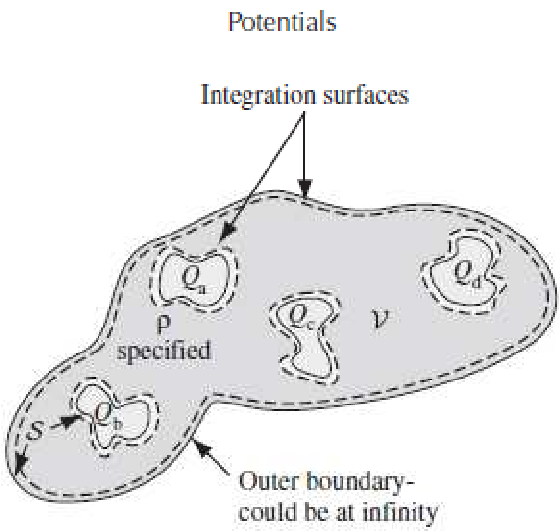
(c) If Dirichlet BC are specified on part of the boundary $\partial\Omega$ and Neumann type BC are specified on the remaining part of the boundary $\partial\Omega$, then it is defined as Robin's or mixed type boundary conditions.

Now, regarding the solution of the Laplace's equations we have two uniqueness theorem, which are stated as follows;

First Uniqueness theorem: The solutions to the Laplace's equation in some volume V (as shown in figure below) is uniquely determined if V is specified on the boundary surface S.



Second Uniqueness theorem: In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given,



$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho.$$

1.10 LAPLACE'S EQUATION in One DIMENSION

Suppose V depends on only one variable, x. Then Laplace's equation becomes;

$$\frac{d^2V}{dx^2} = 0.$$

The general solution is the equation for a straight line.

$$V(x) = mx + b,$$

It contains two undetermined constants (m and b), as is appropriate for a second-order (ordinary) differential equation. They are fixed, in any particular case, by the boundary conditions of that problem. For instance, it might be specified that V = 4 at x = 1, and V = 0 at x = 5. In that case, m = −1 and b = 5, so V = −x + 5 (figure 1.7 below).

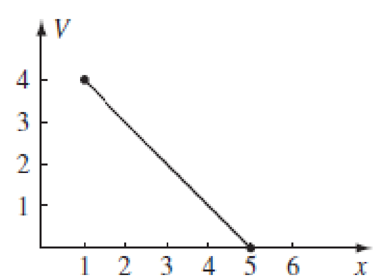


Figure 1.7

1. *V(x)* is the *average* of *V (x + a)* and *V (x − a)*, for any *a*:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)].$$

Laplace's equation is a kind of averaging instruction; it tells you to assign to the point *x* the average of the values to the left and to the right of *x*. Solutions to Laplace's equation are, yet fit the end points properly.

2. Laplace's equation tolerates no local maxima or minima; extreme values of V must occur at the end points. Since Laplace's equation requires, on the contrary, that the second derivative is zero, it seems reasonable that solutions should exhibit no extrema.

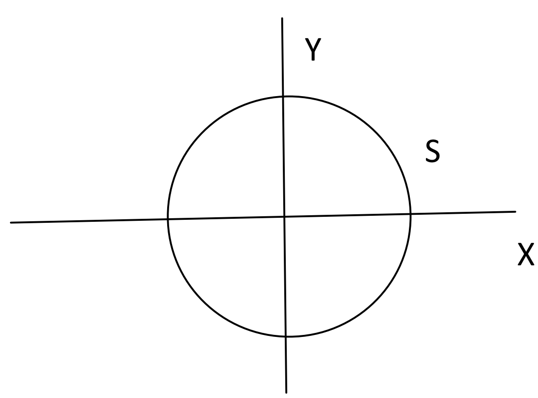
1.8 Laplace's Equation in Two Dimensions

If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

This is no longer an ordinary differential equation (that is, one involving ordinary derivatives only); it is a partial differential equation.

Now consider the simple surface, a circle in x-y plane and we know the value at the boundary $V= V_0$ then what is about the inside the surface. Now the imagine all possible solutions, which this equation can satisfied as mentioned below,



$V = 1$ or x or y or xy or x^2-y^2 or x^3-3xy^2

If you put any of these values of combinations, then it will satisfy the Laplace's equation. The we can imagine that a simple complex function $(x+iy)^n$ can be a solution

$V(x,y)$ = Real/Imaginary $(x+iy)^n$ is the general solution inside the surface.

If you put n=1 then solution will be either $V(x,y) = x$ or $V(x,y) = y$

If you put n=2 then solution will be either $V(x,y) = x^2-y^2$ or $V(x,y) = 2xy$

And so on.......

Now in terms of polar coordinates we can write,

$x = r\cos\theta$ and $y = r\sin\theta$

then $V(x,y) \rightarrow V(r,\theta)$ = Real/Imaginary $[r^n(\cos\theta + i\sin\theta)^n]$

= Real/Imaginary $r^n e^{in\theta}$

So the general solution of Laplace' s equation in 2-dimensional can be written as;

$$V(r,\theta) = \sum_{n=0}^{\infty} a_n\, r^n cos\theta + \sum_{n=0}^{\infty} b_n\, r^n\, sin\theta$$

Where a_n and b_n are coefficients depending upon boundary conditions.

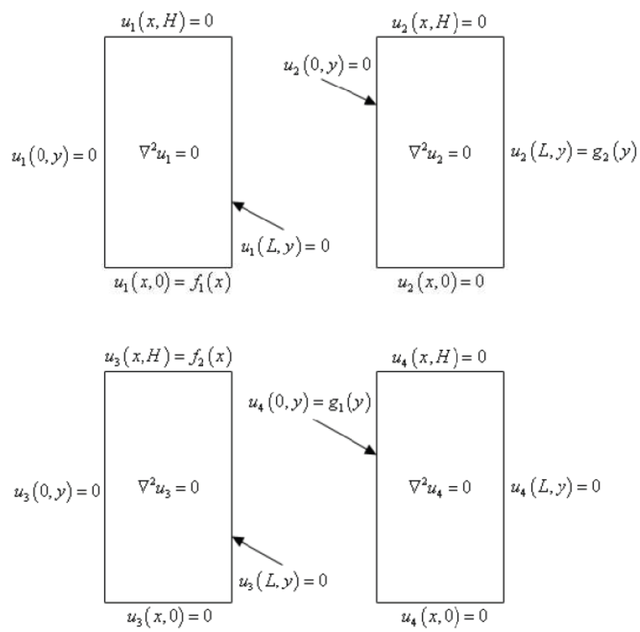
Now consider another example, how we solve Laplace's equation will depend upon the geometry of the 2-D object Let's start out by solving it on the rectangle given by $0 \leq X \geq L$, and $0 \leq Y \geq H$. For this geometry Laplace's equation along with the four boundary conditions will be,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0,y) = g_1(y) \qquad u(L,y) = g_2(y)$$

$$u(x,0) = f_1(x) \qquad u(x,H) = f_2(x)$$

. Both variables are spatial variables and each variable occurs in a 2nd order derivative and so we'll need two boundary conditions for each variable. Next, let's notice that while the partial differential equation is both linear and homogeneous the boundary conditions are only linear and are not homogeneous. This creates a problem because separation of variables requires homogeneous boundary conditions. To completely solve Laplace's equation, we're in fact going to have to solve it four times. Each time we solve it only one of the four boundary conditions can be nonhomogeneous while the remaining three will be homogeneous. The four problems are probably best shown with a quick sketch so let's consider the following sketch.



Because we know that Laplace's equation is linear and homogeneous and each of the pieces is a solution to Laplace's equation then the sum will also be a solution.

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

Also, this will satisfy each of the four original boundary conditions.

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0) = f_1(x) + 0 + 0 + 0 = f_1(x)$$

In each of these cases the lone nonhomogeneous boundary condition will take the place of the initial condition in the heat equation problems that we solved a couple of sections ago. We will apply separation of variables to each problem and find a product solution that will satisfy the differential equation and the three homogeneous boundary conditions. Using the Principle of Superposition, we'll find a solution to the problem and then apply the final boundary condition to determine the value of the constant(s) that are left in the problem. The process is nearly identical in many ways to what we did when we were solving the heat equation.

Example: Find a solution to the following partial differential equation.

$$\nabla^2 u_4 = \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$$

$$u_4(0,y) = g_1(y) \qquad u_4(L,y) = 0$$

$$u_4(x,0) = 0 \qquad u_4(x,H) = 0$$

Solution: We'll start by assuming that our solution will be in the form,

$$u_4(x,y) = h(x)\varphi(y)$$

and then recall that we performed separation of variables. So from that problem we know that separation of variables yields the following two ordinary differential equations that we'll need to solve.

$$\frac{d^2h}{dx^2} - \lambda h = 0 \qquad \frac{d^2\varphi}{dy^2} + \lambda\varphi = 0$$

$$h(L) = 0 \qquad \varphi(0) = 0 \qquad \varphi(H) = 0$$

Note that in this case, unlike the heat equation we must solve the boundary value problem first. Without knowing what is there is no way that we can solve the first differential equation here with only one boundary condition since the sign of will affect the solution. Let's also notice that we solved the boundary value problem in Example 1 of Solving the Heat Equation and so there is no reason to resolve it here. Taking a change of letters into account the eigenvalues and Eigen functions for the boundary value problem here are,

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad \varphi_n(y) = \sin\left(\frac{n\pi y}{H}\right) \qquad n = 1,2,3,\ldots$$

Now that we know what the eigenvalues are let's write down the first differential equation with λ plugged in.

$$\frac{d^2h}{dx^2} - \left(\frac{n\pi}{H}\right)^2 h = 0$$
$$h(L) = 0$$

Because the coefficient of h(x) in the differential equation above is positive we know that a solution to this is,

$$h(x) = c_1 \cosh\left(\frac{n\pi x}{H}\right) + c_2 \sinh\left(\frac{n\pi x}{H}\right)$$

However, this is not really suited for dealing with the boundary condition. So, let's also notice that the following is also a solution.

$$h(x) = c_1 \cosh\left(\frac{n\pi(x-L)}{H}\right) + c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

You should verify this by plugging this into the differential equation and checking that it is in fact a solution. Applying the lone boundary condition to this "shifted" solution gives,

$$0 = h(L) = c_1$$

The solution to the first differential equation is now,

$$h(x) = c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

and this is all the farther we can go with this because we only had a single boundary condition. That is not really a problem however because we now have enough information to form the product solution for this partial differential equation. A product solution for this partial differential equation is,

$$u_n(x,y) = B_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right) \qquad n = 1, 2, 3, \ldots$$

The Principle of Superposition then tells us that a solution to the partial differential equation is,

$$u_4(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

and this solution will satisfy the three homogeneous boundary conditions. To determine the constants all we need to do is apply the final boundary condition.

$$u_4(0,y) = g_1(y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

Now, in the previous problems we've done this has clearly been a Fourier series of some kind and in fact it still is. The difference here is that the coefficients of the Fourier sine series are now,

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right)$$

instead of just. We might be a little more tempted to use the orthogonality of the sines to derive formulas for the, however we can still reuse the work that we've done previously to get formulas for the coefficients here.

Remember that a Fourier sine series is just a series of coefficients (depending on *n*) times a sine. We still have that here, except the "coefficients" are a little messier this time that what we saw when we first dealt with Fourier series. So, the coefficients can be found using exactly the same formula from the Fourier sine series section of a function on we just need to be careful with the coefficients.

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right) = \frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{2}{H\sinh\left(\frac{n\pi(-L)}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy \qquad n = 1,2,3,\ldots$$

The formulas are a little disordered this time in comparison to the previous.

1.12 SUMMARY

1. Coulomb's Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q.Q}{r^2}\hat{r}$$

2. Electric field

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\imath_i^2}\hat{\boldsymbol{\imath}}_i.$$

3. Electric potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

4. The multipole expansion of electric potential (V) in the power of 1/r

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right].$$

For monopole V (r) ~ 1/r

For dipole V (r) ~ $1/r^2$

For Quadrupole V (r) ~ $1/r^3$

For octopole V (r) ~ $1/r^4$

5. Gauss's Law

$$\oint \vec{E}\,.\,\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

Or $\nabla. E = \frac{\rho}{\varepsilon_0}$

6. Poisson equation

$$\nabla^2 V = 0,$$

7. Laplace equation

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho,$$

1.13 PROBLEMS

1. A uniform electric field of magnitude E= 100Newton/Coulomb exists in the space in x-direction. Calculate the flux of this field through a plane square area of edge 10 cm placed in the y-z plane. Take the normal along the positive x-axis to be positive.

2. The electric field in a region is radially outwards with magnitude E=Ar. Find the charge contained in a sphere of radius a centred at the origin. Take A= 100 V/m^3 and a= 20.0 cm.

3. Find a solution to the following partial differential equation.

$$\nabla^2 u_3 = \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0$$

$$u_3(0,y) = 0 \qquad u_3(L,y) = 0$$

$$u_3(x,0) = 0 \qquad u_3(x,H) = f_2(x)$$

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UNIT 2: ELECTROSTATICS FIELD IN MATTERS

Structure

2.1 INTRODUCTION

In this unit, we shall study electric fields in matter, with reference to two large classes as conductors and dielectrics. Conductors are substances that contain an "unlimited" supply of charges that are free to move about through the material. On the other hand, in dielectrics, all charges are attached to specific atoms or molecules—they are on a tight bound state, and all they can move a bit within the atom or molecule. Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behaviour of dielectric materials. There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: stretching and rotating.

Below, few simple points regarding the conductors and insulators are mentioned for revision;

1. A perfect dielectric to have σ = 0 (for good insulators $10^{-17} \leq \sigma \geq 10^{-10}$ S/m)
2. A perfect conductor to have σ= ∞ (for good conductors $10^{6} \leq \sigma \geq 10^{7}$ S/m)

Perfect dielectric: $\mathbf{J} = 0$ since $\mathbf{J} = \sigma\mathbf{E}$ and $\sigma = 0$

Perfect conductor: $\mathbf{E} = 0$ since $\mathbf{E} = \mathbf{J}/\sigma$ and $\sigma = \infty$

3. When an electric field is applied to a conductor, conduction current flow in the same direction as the electric field.
4. When a dielectric material is subject to an electric field, the atoms or molecules of the material become polarized. When a dielectric material is subject to an electric field, the atoms or molecules of the material become polarized.

When no field is present, the electron cloud is symmetrical about the nucleus (a) in Figure 2.1. In a dielectric, when the field is applied, as shown in Figure 2.1(b), a shift occurs and E is said to polarize the atoms and create a dipole. The dipole creates its own electric field known as the polarization field, P. Molecules such as water have a permanent dipole moment, but the dipoles are randomly aligned until an applied field is applied.

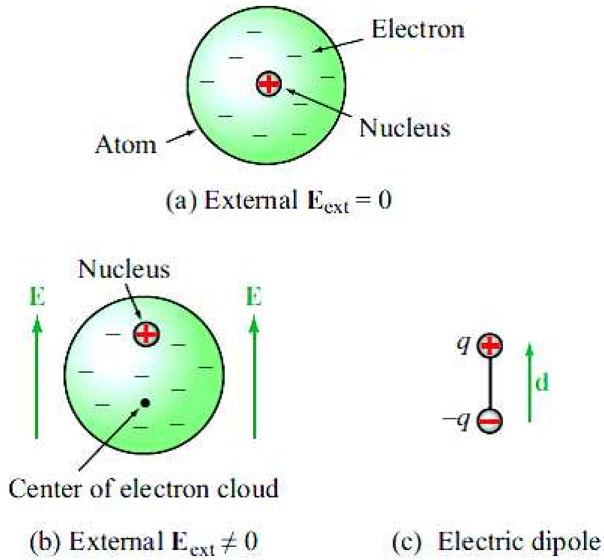


Figure 2.1. Effect on nucleus when (a) no electric field and (b) external field is applied

2.2 OBJECTIVES

After studying this unit, you should be able to-

- explain Polarization
- define dielectrics constant, permittivity
- how to solve the dielectric problems
- apply Gauss's Law in the Presence of Dielectrics

2.3 INDUCED DIPOLES

The atom as a whole is electrically neutral; there is a positively charged core (the nucleus) and a negatively charged electron cloud surrounding it. Now, when a neutral atom it is placed in an electric field E, the two regions of charge (nucleus and orbiting electrons) within the atom are influenced by the field: the nucleus is pushed in the direction of the field, and the electrons the opposite way. In principle, if the field is large enough, it can pull the atom apart completely causing ionization of it and the substance then becomes a conductor. With less strong fields, however, equilibrium is soon established, for if the center of the electron cloud does not coincide with the nucleus, these positive and negative charges attract one another, and that holds the atom together. The two opposing forcespulling the electrons and nucleus apart, their mutual attraction drawing them back together reach a balance, leaving the atom polarized, with plus charge shifted slightly one way, and minus the other. The atom now has a tiny dipole moment p, which points in the same direction as E.

Typically, this induced dipole moment is approximately proportional to the field if the field is not too strong:

$$\mathbf{p} = \alpha \mathbf{E}. \quad \text{---------------------------- (2.1)}$$

The constant of proportionality α is called atomic polarizability.Its value depends on the detailed structure of the atom of consideration and direction as well.

For molecules, the situation is not quite different and complex, because frequently molecules can polarize more readily in some directions than in others. When the field is applied at some angle to the axis, then it must be resolved into parallel and perpendicular components, and then each component will be multiply by the related polarizability. So the total dipole moment can be written as;

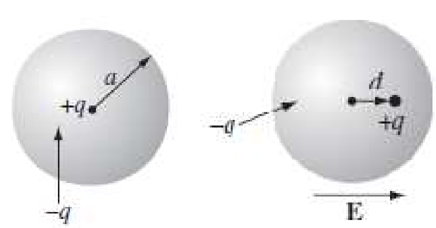
$$\mathbf{p} = \alpha_{\perp}\mathbf{E}_{\perp} + \alpha_{\parallel}\mathbf{E}_{\parallel}. \quad \text{---------------- (2.2)}$$

In this case, the induced dipole moment may not even be in the same direction as E. However, the most general linear relation between E and p is given by equation 2.3 along x-, y- and z-directions.

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \quad \text{---------------- (2.3)}$$

The set of nine constants α_{ij} (i= x,y,z and j= x,y,z) constitute the polarizability tensor for the molecule. Their values depend on the orientation of the axes we use, though it is always possible to choose "principal" axes such that all the off-diagonal terms (α_{xy}, α_{zx}, etc.) vanish, leaving just three non-zero polarizabilities: α_{xx}, α_{yy}, and α_{zz}.

Now consider a primitive model for an atom consists of a point nucleus (+*q*) surrounded by a uniformly charged spherical cloud (−*q*) of radius *a* (Figure 2.2). Then the atomic polarizability of such an atom can be calculated as follows,



in the presence of an external field E, the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Figure 2.2. Because the actual displacements involved are extremely small then it is reasonable to assume that the electron cloud retains its spherical shape. Equilibrium occurs when the nucleus is displaced a distance *d* from the center of the sphere. At that point, the external field pushing the nucleus to the right, exactly balances the internal field pulling it to the left: $E = E_e$, where $E_e$is the field produced by the electron cloud. Now the field at a distance *d* from the center of a uniformly charged sphere is,

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$ -------------------- (2.4)

At equilibrium, then,

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}, \quad \text{or } p = qd = (4\pi\epsilon_0 a^3)E.$$ ------------------------- (2.5)

The atomic polarizability is therefore

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v,$$ -------------------------------- (2.6)

where *v* ($4\pi a^3/3$)is the volume of the atom. Although this atomic model is extremely crude, but the result (Equation 2.6) is quite reasonable, it is accurate to within a factor of four or so for many simple atoms.

2.4 ALIGNMENT OF POLAR MOLECULES

The neutral atom discussed in figure 2.2 had no dipole moment to start with p. Only applied electric field has induceddipole moment. Some molecules have built-in, permanent dipole moments. In the water molecule, for example, the electrons tend to cluster around the oxygen atom (Figure 2.3), and since the molecule is bent at 105°, this leaves a negative charge at the vertex and a net positive charge on the opposite side. The dipole moment of water is unusually large and approximately equals to 6.1 $\times 10^{-30}$ C · m. Due to this dipole moment it is consider as an effective solvent. When such molecules, called polar molecules, are placed in a uniform electric field, then the force on the positive end, $F_+ = qE$, exactly cancels the force on the negative end, $F_- = -qE$(Figure 2.3). However, there will be a torque which tries to rotate the molecule along the direction of applied field. The magnitude of the torque (N) can be given as;

$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) = [(\mathbf{d}/2) \times (q\mathbf{E})] + [(-\mathbf{d}/2) \times (-q\mathbf{E})] = q\mathbf{d} \times \mathbf{E}.$$ ------ (2.7)

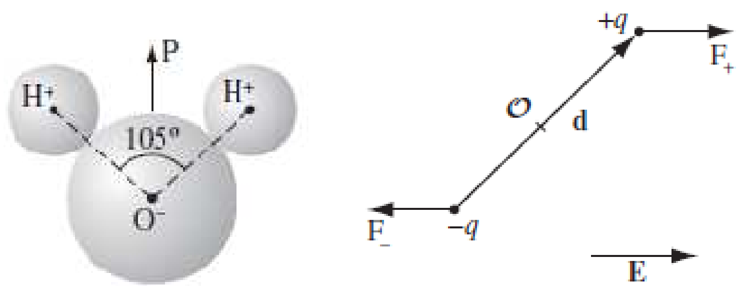


Figure 2.3. Configuration of water molecule and formation of dipole, when it is placed in an external electric field

Thus a dipole p = $q$din a uniform field **E** experiences a torque;

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}. \qquad \text{---------------------- (2.8)}$$

Notice that, **N** is in such a direction as to line **p** up parallel to **E**; a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

If the field is non-uniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$, there will be a net force on the dipole, in addition to the torque. Of course, Emust change rather abruptly for there to be significant variation in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics. Nevertheless, the formula for the force on a dipole in a non-uniform field is of some interest:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E}), \qquad \text{------------------ (2.9)}$$

where ΔErepresents the difference between the field at the plus end and the field at the minus end. Assuming the dipole is very short, we can approximate the small change in E_x:

$$\wedge F_x \equiv (\nabla F_x) \cdot \mathbf{d}, \qquad \text{----------------------- (2.10)}$$

with corresponding formulas for $E_y$and E_z. More compactly,

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E},$$

------------------- (2.11)

Therefore,

$$F = (p \cdot \nabla)E.$$ -- (2.11)

For a dipole of infinitesimal length, called as a perfect dipole, equation 2.8 gives the torque about the center of the dipole even in a non-uniform field. At any other point torque is given by N = (p × E) + (r × F).

2.5 POLARIZATION

When a piece of dielectric material is placed in an electric field and if the substance consists of neutral atoms (or non-polar molecules), the field will induce in each a tiny dipole moment, pointing in the same direction as the field. If the material is made up of polar molecules, in which each possess a permanent dipole, it will experience a torque, tending to rotate along the field direction. Although random thermal motions compete with this process of alignment of molecule, so the alignment is never complete, especially at higher temperatures, and disappears as soon as field is removed. These two mechanisms produce the same basic result which infers the creation of a lot of little dipoles pointing along the direction of the field anf then polarization of the material.

A convenient measure of this effect is P ≡ dipole moment per unit volume, which is called the polarization. Even in polar molecules there will be some polarization by displacement although, generally it is easier to rotate a molecule than to stretch it, so the second mechanism, which tries to rotate the polar molecule dominates. However, even in some materials it is possible to “freeze in” polarization. “Freeze in” indicates that the polarization persists even after the field is removed.

2.6 THE FIELD OF A POLARIZED OBJECT

2.6.1 Bound Charges

Suppose, we have a piece of polarized material—that is, an object containing a lot of microscopic dipoles lined up. The dipole moment per unit volume Pis given. Then to calculate the field produced by polarized object (not the field that may have caused the polarization), we can imagine that the material is made up of infinitesimal dipoles. We know that, the field produced by an individual dipole, and then we can integrate it to get the total. However, it is easier to work with the potential. For a single dipole p, the electric potential at some distance r can be written as;

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2},$$ ------------------- (2.12)

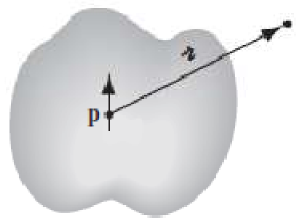


Figure 2.4 Field produced by an individual dipole at a distance r

where ris the vector from the dipole to the point at which we are evaluating the potential, as shown in Figure 2.4. In the present context, we have a dipole moment p = P dτ′ in each volume element dτ′, so the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'.$$ ----------------------------- (2.13)

Now, we can write,

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2},$$ ------------------- (2.14)

where the differentiation is with respect to the source coordinates (**r**′), we have

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'.$$ ------------------------------- (2.15)

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}}\nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)d\tau' - \int_{\mathcal{V}}\frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'\right]$$ ------------------------- (2.16)

------------------ (2.17) $$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}}\frac{1}{\mathscr{r}}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'.$$

The first term in the equation (2.17) resembles with the potential of a surface charge as;

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$$ ----------------------------------- (2.18)

where $\hat{n}$is the normal unit vector, while the second term looks like the potential of a volume charge as;

$$\rho_b = \nabla \cdot \mathbf{P}.$$
--------------------------- (2.19)

Therefore, we can write electric potential at a distance r as,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\, d\tau'$$ -------------------- (2.20)

It indicates that the potential and hence also the field of a polarized object is the same as that produced by a volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$plus a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{n}$. Instead of integrating the contributions of all the infinitesimal dipoles, we could first find those bound charges, and then calculate the fields they produce, in the same way we calculate the field of any other volume and surface charges (for example, using Gauss's law). It is found that the field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges," σ_b and ρ_b. Bound charges are in some sense "fictitious" and only accounting to facilitate the calculation of fields.

Now, suppose we have a long string of dipoles, as shown in Figure 2.5 below. Along the line, the head of one effectively cancels the tail of its neighbour, but at the ends there are two charges left over plus at the right end and minus at the left. It is like a movement of an electron from one end to the other end, though in fact single electron has not made the complete movement. In fact, a lot of tiny displacements add up to one large one. So we call the net charge at the ends a bound charge to remind ourselves that it cannot be removed.

Figure 2.5 Simplification of bound charge as a string

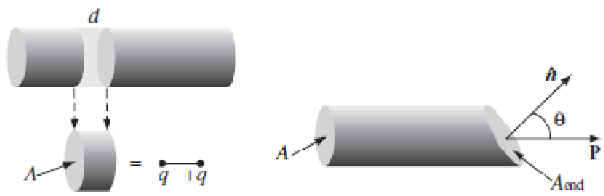


Figure 2.6 bound charge across a cross sectional area (a) perpendicular and (b) inclined with some angle to the direction of polarization.

in a dielectric every electron is attached to a specific atom or molecule. But apart from that, bound charge is no different from any other kind. To calculate the actual amount of bound charge resulting from a given polarization, examine a "tube" of dielectric parallel to P. The dipole moment of the tiny chunk shown in figure 2.6(a) is P(Ad), where Ais the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge (q) at the end, this same dipole moment can be written qd. The bound charge that piles up at the right end of the tube is therefore, if the ends have been sliced off perpendicularly, the surface charge density is

$$q = PA. \quad \text{------(2.21)}$$

$$\sigma_b = \frac{q}{A} = P \quad \text{------(2.22)}$$

For an oblique cut (as shown in figure 2.6(b)), the charge is still the same, but $A = A_{end} \cos\theta$, so

$$\sigma_b = \frac{q}{A_{end}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}. \quad \text{------(2.23)}$$

The effect of the polarization, then, is to distribute a bound charge $\sigma_b = \mathbf{P} \cdot \hat{n}$ over the surface of the material. If the polarization is non-uniform, we get accumulations of bound charge within the material, as well as on the surface. Figure 2.7 indicates that a diverging Presults in a pileup of negative charge at the centre. Indeed, the net bound charge $\rho_b\, d\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface. The latter is $\mathbf{P} \cdot \hat{n}$per unit area, so

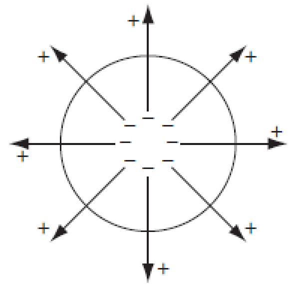


Figure 2.7 Polarization direction is diverged from the centre

$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla \cdot \mathbf{P}) \, d\tau. \quad \text{-------------------------- (2.24)}$$

------------------------------ (2.25) $$\rho_b = -\nabla \cdot \mathbf{P},$$

There is another way of analyzing the uniformly polarized sphere, which nicely illustrates the idea of a bound charge. What we have, really, is two spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly upward (the z direction), and all the minus charges move slightly downward (Figure 2.8). The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is precisely the bound surface charge σ_b.

Figure 2.8

The field in the region of overlap between two uniformly charged spheres is calculated as,

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}}{R^3},$$

---------------------------------- (2.26)

where q is the total charge of the positive sphere, d is the vector from the negative center to the positive center, and R is the radius of the sphere. We can express this in terms of the polarization of the sphere, p = qd = $(4/3)\,\pi\, R^3$P, as

$$\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}. \qquad \text{-------------------------------------- (2.27)}$$

Meanwhile, for points outside,it is as though all the charge on each sphere were concentrated at the respective center. We have, then, a dipole, with potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}. \qquad \text{--------------- (2.28)}$$

2.7 THE FIELD INSIDE A DIELECTRIC

Suppose I want to calculate the macroscopic field at some point **r** within a dielectric (Figure 16). I know I must average the true (microscopic) field over an appropriate volume, so let me draw a small sphere about r, of radius R, say, a thousand times the size of a molecule. The macroscopic field at r, then, consists of two parts: the average field over the sphere due to all charges *outside*, plus the average due to all charges *inside:*

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}} \qquad \text{--------------------------- (2.29)}$$

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\int_{\text{outside}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'. \qquad \text{--------------- (2.30)}$$

$$\mathbf{E}_{\text{in}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \qquad \text{----------------------------- (2.31)}$$

$$\mathbf{E}_{\text{in}} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

------------------------------ (2.32)

Figure 2.9 Microscopic field inside the dielectric at point r by drawing a sphere of radius R centred about r and then taking contributions of dipoles lying inside and outside the spheres.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau',$$ ---------------------- (2.33)

The average field (over a sphere), produced by charges outside, is equal to the field they produce at the center, so E_{out} is the field at r due to the dipoles exterior to the sphere. The dipoles inside the sphere are too close to treat in this fashion. But fortunately all we need is their average field, and that is regardless of the details of the charge distribution within the sphere. The only relevant quantity is the total dipole moment, p= $(4/3)\,\pi\, R^3$P

Now, by assumption, the sphere is small enough that Pdoes not vary significantly over its volume, so the term left out of the integral in equation 2.30 corresponds to the field at the center of a uniformly polarized sphere, to wit: $-(1/3\varepsilon_0)$P. But this is precisely what E_{in} in equation 2.32. The macroscopic field, then, is given by the potential where the integral runs over the entire volume of the dielectric. Notice that the average field over *any* sphere (due to the charge inside) is the same as the field at the center of a uniformly polarizedsphere with the same total dipole moment. This means that no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field. Incidentally, while the argument ostensibly relies on the spherical shape I chose to average over, the macroscopic field is certainly independent of the geometry of the averaging region, and this is reflected in the final answer, equation 2.33. Presumably one could reproduce the same argument for a cube or an ellipsoid or whatever—the calculation might be more difficult, but the conclusion would be the same.

2.8 THE ELECTRIC DISPLACEMENT

2.8.1 Gauss's Law in the Presence of Dielectrics

Previously we found that the effect of polarization is to produce accumulations of bound charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \hat{n}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything else which is called as a free charge density, ρ_f. The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; precisely any charge that is nota result of polarization is included in free charge density. Within the dielectric, the total charge density can be written:

-- $\rho = \rho_b + \rho_f,$ ---------------------------------- (2.34)

and then Gauss's law takes the form as below,

$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f,$ ------------------- (2.35)

$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$ --- (2.36)

where E is now the total field including to cause polarization. It is convenient to combine the two divergence terms:

$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},$ ------------------------------ (2.37)

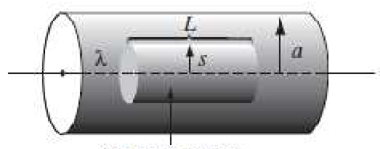
The expression in parentheses, designated by the letter D, is known as the electric displacement. In terms of D, Gauss's law can be written as;

$\nabla \cdot \mathbf{D} = \rho_f,$ ----------------------------- (2.38)

or, in integral form,

$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}},$ ---------------------------- (2.39) where Q_{fenc} denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because it makes reference only to free charges, and free charge is the things we can control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues and this polarization produces the bound charge. In any typical problem, we know ρ_f, but we do not initially know ρ_b. Therefore, we can calculate D using equation 2.39. If the symmetry of the system is known then, we can immediately calculate D by the standard Gauss's law methods.

Now, consider a long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius *a,* as shown in figure below. Find the electric displacement.



On applying equation 2.39, we find

$$D(2\pi s L) = \lambda L.$$

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}.$$

Notice that this formula holds both within the insulation and outside it. In the latter region, **P** = 0, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know **P**.

We cannot apply Gauss's law precisely at the surface of a dielectric, for here $\rho_b$blows up, taking the divergence of Ewith it. But everywhere elsethe logic is sound, and in fact if we picture the edge of the dielectric as having some finite thickness, within which the polarization tapers off to zero (probably a more realistic model than an abrupt cut-off anyway), then there *is* no surface bound charge; $\rho_b$varies rapidly but smoothly within this "skin," and Gauss's law can be safely applied everywhere. At any rate, the integral form (Equation. 2.39) is free from this "defect.

2.9 LINEAR DIELECTRICS

2.9.1 Susceptibility, Permittivity, Dielectric Constant

We know that the polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles. For many substances, in fact, the polarization is proportional to the field, provided E is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \qquad \text{----------------------- (2.40)}$$

The constant of proportionality, χ_e, is called the electric susceptibility of the medium (a factor of ε_0 has been extracted to make $\chi_e$dimensionless). The value of $\chi_e$depends on the microscopic structure of the substance in question and also on external conditions such as temperature). Therefore, the materials that obey equation 2.40 are called as linear dielectrics.The constant $\varepsilon_0$is called the permittivity of the material.

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \qquad \text{---------------------------------- (2.41)}$$

In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity is ε_0. That's why ε_0 is called the permittivity of free space. In another words, we can say that the vacuum is just a special kind of linear dielectric, in which the permittivity happens to have the value 8.85×10^{-12} $C^2/N \cdot m^2$.) If you remove a factor of ε_0, the remaining dimensionless quantity is called the relative permittivity ε_r, or dielectric constant, of the material.

Note that Ein equation 2.40 is the totalfield; it may be due in part to free charges and in part to the polarization itself. If, for instance, we put a piece of dielectric into an external field E_0, we cannot compute Pdirectly from equation 2.40; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization. The simplest approach is to begin with the displacement,at least in those cases where Dcan be deduced directly from the free charge distribution.

In linear media we have,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}, \qquad \text{----------------- (2.42)}$$

so D is also proportional to E:

$$\mathbf{D} = \epsilon \mathbf{E},$$

-- (2.43)

where

$$\epsilon \equiv \epsilon_0(1 + \chi_e).$$ -- (2.44)

Since P and D are now proportional to E, but still their curls, like E's, will not vanish. The line integral of Paround a closed path that spans the boundary between one type of material and another need not be zero, even though the integral of Earound the same loop must be. The reason is that the proportionality factor $\varepsilon_0\chi_e$is different on the two sides. For instance, at the interface between a polarized dielectric and the vacuum, as shown in figure 2.10), Pis zero on one side but not on the other. Around this loop

$$\oint \mathbf{P} \cdot d\mathbf{l} \neq 0,$$ ------------------- (2.45)

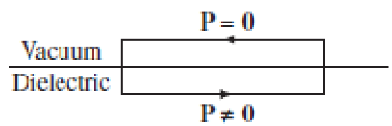


Figure 2.10

Loop of polarization in dielectric and vacuum media

and hence, by Stokes' theorem, the curl of **P** cannot vanish everywhere within the loop, in fact, it is infinite at the boundary. Of course, if the space is entirelyfilled with a homogeneous10 linear dielectric, then this objection is void; in this rather special circumstance.

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = \mathbf{0},$$ ------------------ (2.46)

so **D** can be found from the free charge just as though the dielectric were not there

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}},$$ ------------------------------------ (2.47)

where **E**vac is the field the same free charge distribution would produce in the absence of any dielectric. Therefore,

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}.$$ -------------------------------- (2.48)

When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant. Actually, it is not necessary for the dielectric to fill all space: in regions where the field is zero anyway, it can hardly matter whether the dielectric is present or not, since there's no polarization in any event.

This unit described polar and non-polar molecules and then conductors and dielectrics in terms of dipoles. Many terms are defined like, induced dipoles, polarization, bound and free charges etc. The expression of electric field inside the dielectric materials is also included. In the presence of free charges and bound charges, Gauss' law modifies. In the linear medium, it gives expression between the displacement vector, dielectric constant and applied electric field. Important expressions are mentioned below to summarize the unit.

2.10 SUMMARY

Dipole moment

$$\mathbf{p} = \bar{\alpha}\mathbf{E}.$$

Atomic polarizability

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v,$$

P ≡ dipole moment per unit volume

Field inside the dielectric

$$\mathbf{E}_{in} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

Surface charge density

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$$

Volume charge density

$$\rho_b \equiv -\nabla\cdot\mathbf{P}.$$

Displacement vector

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P},$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1+\chi_e)\mathbf{E},$$

Gauss's law in terms of displacement vector

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}},$$

2.11 PROBLEMS

1. A parallel-plate capacitor is filled with insulating material of dielectric constant ε_r. What effect does this have on its capacitance?

2. A metal sphere of radius *a* carries a charge *Q*. It is surrounded, out to radius *b*, by linear dielectric material of permittivity ε. Find the potential at the center (relative to infinity).

Hint: To compute V, we need to know E; to find E, we might first try to locate the bound charge; we could get the bound charge from P, but we can't calculate P unless we already know E (Eq. 30). We seem to be in a bind. What we do know is the free charge Q, and fortunately the arrangement is spherically symmetric, so begin by calculating D

3. A sphere of radius R carries a polarization **P**(**r**) = k**r**, where k is a constant and **r** is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

4. Find the electric field produced by a uniformly polarized sphere of radius *R.*

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UNIT 3 MAGNETOSTATICS

3.1 INTRODUCTION

Magneto statics is the branch of electromagnetic studies involving magnetic fields produced by steady non-time varying currents. Evidently, currents are produced by moving charges undergoing translational motion. An effective current (called magnetization current) is also produced if magnetic dipoles are non-uniformly distributed. The magnetic effects can be produced by a magnet or by a current carrying conductor. In the Magneto statics, we mainly study the magnetic fields in systems where the currents are steady. It is the magnetic analogue of electrostatics, where the charges are stationary. The magnetization need not be static; the equations of magneto statics can be used to predict fast magnetic switching events that occur on time scales of nanoseconds or less. Magneto statics is even a good approximation when the currents are not static — as long as the currents do not alternate rapidly.

In the present unit, you will study the force on a moving charge in simultaneous electric and magnetic fields, Biot-Savart law, magnetic force between current elements, Ampere's circuital law and its applications. Along this you will also study the different types of magnetic substances, Hysteresis curve and its importance.

3.2 OBJECTIVES

After studying this unit, you should be able to-

- Understand magnetic induction
- Understand Lorentz force
- Apply Biot-Savart law
- Apply Ampere's circuital law
- Solve problems using Biot-Savart law and Ampere's circuital law
- Understand curl, divergence of B, understand vector potential and magnetic flux
- Understand Intensity of magnetization
- Calculate the magnetic intensity
- Understand magnetic susceptibility
- Understand Curie's law
- Understand the magnetic substances
- Understand hysteresis and calculate the energy loss due to hysteresis

3.3 MAGNETIC INDUCTION

The region around a magnet or current carrying conductor, where a magnetic needle experiences a deflection in a definite direction, is known 'magnetic field'. A magnetic field is said to existed when a point charge moving through a deflecting force. This field is represented by a vector quantity $\vec{B}$, called magnetic field or magnetic induction. The magnetic induction can be defined in terms of lines of induction as the number of lines of induction passing through a unit area placed normal to the lines measures the magnitude of magnetic induction or magnetic flux density $\vec{B}$. The unit of magnetic induction is the tesla (T). Obviously, in a region smaller is the relative spacing of the lines of induction, the greater is the magnetic induction. The tangent to the line of induction at any point gives the direction of magnetic induction $\bar{B}$ at that point.

Just as the electric field vector E is the basic quantity in electrostatics, the magnetic-flux density or magnetic induction B plays a fundamental role in magneto statics. Another fundamental quantity is the magnetic dipole μ, which plays a role not unlike the electric charge in electrostatics. It is, however important to realize that, as far as we know, there does not exist any magnetic monopoles (or charges). The two quantities, B and μ, were early on linked through simple relations. For example, the torque N exerted by a magnetic-flux density on a test dipole (i.e., small enough not to alter the magnetic-flux) is given by

$$N = \mu \times B$$

It was also established that there is a connection between electrical currents and magnetic fields. As will soon be shown, a current density J (or simply a current I) is a source of magnetic-flux density. Since the current density is defined as the amount of charge that flows through a cross-section per unit of time (units of Coulombs per square meter second), the conservation of charge requires that the so-called continuity equation be satisfied

$$\frac{\partial \rho}{\partial t} + \nabla . J = 0$$

The above equation implies that any decrease (increase) in charge density within a small volume must be accompanied by a corresponding flow of charges out of (in) the surface delimiting the volume. Because magneto statics is concerned with steady-state currents, we will limit ourselves (at least in this chapter) to the following equation

$$\nabla . J = 0$$

3.4 LORENTZ FORCE

Letusconsiderachargedparticleofchargeqwhichismovingwithvelocity $\bar{v}$ inamagnetic field $\bar{B}$, then the magnetic force acting on that charged particle is given by-

$\bar{F} = q(\bar{v} \times \bar{B})$ (3.1)

The direction of $\bar{F}$ will be perpendicular to both the direction of velocity $\bar{v}$ and the direction of magnetic field $\bar{B}$. Its exact direction is given by the law of vector product of two vectors.

The magnitude of magnetic force is given as-

$F = qvB\sin\theta$ (3.2)

Where θ is the angle between velocity $\bar{v}$ and magnetic field $\bar{B}$.

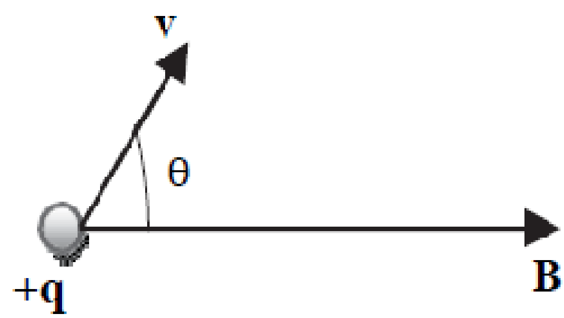

Figure 3.1

Case 1: If velocity v and magnetic field B are at right angle. The angle between velocity $\bar{v}$ and magnetic field $\bar{B}$ is 90^0

$F_{max} = qvB \sin 90^0 = qvB$

The magnetic force acting on the charged particle is maximum that is equal to qvB.

Case 2: If the charged particle is moving parallel to the magnetic field. The angle between velocity $\bar{v}$ and magnetic field $\bar{B}$ is 0^0 or 180^0

$F = qvB \sin 0^0 = 0$

The magnetic force acting on the charged particle is maximum that is equal to 0.

If v = 0, then F = 0. This means that if the charged particle is at rest in the magnetic field, then it does not experience any force.

If a charged particle is moving in space where both an electric field $\bar{E}$ and a magnetic field $\bar{B}$ are present, then the total force acting on the charged particle is called the Lorentz force.

The electric force acting on charged particle, $Fe = qE$ (3.3)

The magnetic force acting on the charged particle,

$$F_m = q(v. \times B) \quad (3.4)$$

Total force acting on the charged particle,

$F = Fe + Fm$

So by using equation 3.3 and 3.4 the total force acting on the charge particle can be given as

$$F = q\bar{E} + q(\bar{v} \times \bar{B})$$

$$\vec{F}=q[\vec{E}+(\vec{v}\times\vec{B})] \quad (3.5)$$

The force given by equation (3.5) is called the Lorentz force and the equation is known as Lorentz force equation.

If a charged particle enters perpendicular to both the electric and magnetic fields, then it may cancel each other and therefore, the charged particle will pass undeflected. In this situation,

$$q[\vec{E}+(\vec{v}\times\vec{B})]=0$$

$$\bar{E}=-(\bar{v}\times\bar{B})$$

In magnitude, $\bar{E}=(\bar{v}\times\bar{B})$

$$v=\bar{E}/\bar{B} \quad (3.6)$$

Thus a charged particle entering in simultaneous electric and magnetic field may pass undeflected. Such an arrangement of simultaneous electric and magnetic fields is called velocity- selector. Because the charged particle of only specified velocity given by v = E/B can pass undeflected. The particle of velocity v < E/B will be deflected towards electric force and those with velocity v > E/B will be deflected towards magnetic force.

3.5 BIOT-SAVARTLAW

Biot-Savart law is used to calculate the magnetic field produced by a current carrying conductor.

According to Biot-Savart law, the magnetic field dB produced due to this current element at point Pat a distance r from the element is-

(i) directly proportional to the current flowing in the element i.e. $dB \propto i$

(ii) directly proportional to the length of element i.e. $dB \propto dl$

(iii) directly proportional to sin of angle between current element and the line joining current element to point P i.e. $dB \propto \sin\theta$

(iv) inversely proportional to the square of the distance of the element from point P i.e. $dB \propto \frac{1}{r^2}$

Combining these, we get-

$dB \propto \frac{idl\sin\theta}{r^2}$

or $dB = \frac{\mu}{4\pi}\frac{idl\sin\theta}{r^2}$ (3.7)

where, $\frac{\mu}{4\pi}$ is a dimensional constant of proportionality whose value depends upon the units used for the various quantities. It depends on the medium between the current element and point of observation (P). Here, μ is called the permeability of medium. Equation (3.7) is called Biot-Savart law. The product of current i and the length of element dl i.e. idl is called the current element. Current element is a vector quantity; its direction is along the direction of current.

If you place the conductor in vacuum or air, then μ is replaced by $\mu_0$and thus Biot-Savart law can be written as-

$dB = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2}$ (3.8)

μ_0 is called the permeability of free space or air. Its value in the SI system is assigned as-

$\mu_0 = 4\pi \times 10^{-7}$ weber/ampere-meter ($WbA^{-1}m^{-1}$)

Thus, $\frac{\mu_0}{4\pi} = 10^{-7}\ WbA^{-1}m^{-1}$

μ_0 or $\frac{\mu_0}{4\pi}$ may also be expressed in Newton/Ampere^2 (N/A^2).

The direction of magnetic field is perpendicular to the plane containing current element and the line joining point of observation to current element. Therefore, in vector form, Biot-

Savart law can be expressed as- $d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{l}\times\vec{r}}{r^3}$ (3.9)

The resultant magnetic field at P due to the whole conductor can be found by integrating equation (3.9) over the entire length of the conductor. Thus

$\vec{B} = \int d\vec{B}$

Direction of magnetic field dB: The direction of magnetic field $d\vec{B}$ is perpendicular to both the current element $id\vec{l}$ and the position vector $\vec{r}$ of point P relative to current element and may be found by the law of vector cross product or by Maxwell's right hand screw rule. Thus in figure 2 the direction of magnetic field at point P is shown by × (cross) i.e. vertically inward (downward perpendicular to the plane of the paper) and at point P', the direction of magnetic field is shown by •(dot) i.e. vertically outward (upward perpendicular to the plane of the paper).

3.5.1 Definition of Ampere:

The force of attraction or repulsion between two long, parallel and straight conductors in vacuum has been used to define ampere.

$F/l = \frac{\mu_0}{4\pi}\frac{2i_1 i_2}{r}$ (3.10)

Let $i_1 = i_2 = 1$ Amp. and r = 1 meter, then

$F/l = \frac{\mu_0}{4\pi}\frac{2i^2}{r} = 1 \times 10^{-7} \times \frac{2\times(1)^2}{1}$ (3.11)

$= 2 \times 10^{-7}$ N/meter

Thus, 1 ampere is the current which when flowing in each of two infinitely long parallel conductors 1 meter apart in vacuum produces between them a force of exactly 2×10^{-7} N/meter of length.

3.6 AMPERE'S CIRCUITAL LAW

According to Ampere's circuital law, "The line integral of magnetic induction around a closed path is equal to μ_0 times the net current enclosed by the path" i.e. $\oint \vec{B}.\vec{dl} = \mu_0 i$ (3.12)

Where i is the current enclosed by the path.

Let us suppose that the magnetic field induction B arises due to a long wire carrying a current of i ampere. Now let us consider a circular path of radius r centered on this current carrying wire.

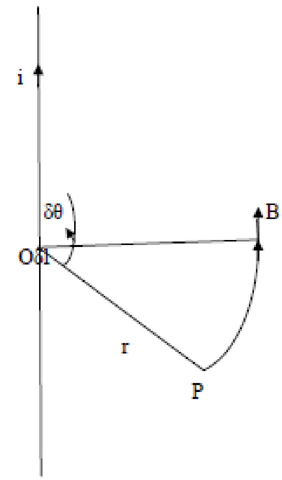

Figure 3.2

The magnitude of magnetic induction at any point P on the circular path is given by-

$$B = \frac{\mu_0}{4\pi}\frac{2i}{r} \tag{3.13}$$

For all points on the circular path, the magnetic induction $\vec{B}$ has the same magnitude given by equation (3.13) and it is parallel to the tangent to the circular path. Therefore, the line integral of the magnetic induction B around the circular path centered on the current carrying wire is given by-

$$\oint \vec{B}dl = \oint \vec{B}dl = \oint \frac{\mu_0}{4\pi}\frac{2i}{r} rd\theta$$

$$= \frac{\mu_0}{4\pi} 2i \oint \delta\theta$$

$$= \frac{\mu_0}{4\pi} 2i\,(2\pi) = \mu_0 i$$

Thus we have- $\oint \vec{B}.\vec{dl} = \mu_0 i$

The sign of integral depends upon the direction in which the current is enriched. The sign is positive if the path followed for line integral is parallel to B and negative if the path followed is anti-parallel.

If the path enclosing the current is not circular but is irregular of any shape, then we divide the path into large number of small elements. Ampere's law holds for closed path of any shape.

3.6.1 Differential form of Ampere's Law

Ampere's circuital law can be expressed in terms of magnetic field intensity ($\vec{H}$). We know that-

$\vec{B} = \mu_0\vec{H}$

Therefore, from equation (3.12) we have-

$\oint \mu_0\vec{H}.\vec{dl} = \mu_0 i$

Or $\oint \vec{H}.\vec{dl} = i$

(3.14)

But current $i = \iint \vec{J}.\vec{dS}$

(3.15)

Where $\vec{J}$ is the current density and $\vec{dS}$ is small element of area at the point of current density $\vec{J}$ inside the closed path.

Therefore, equation takes the form as-$\oint \vec{H}.\vec{dl} = \iint \vec{J}.\vec{dS}$

(3.16)

Using Stoke's theorem, we have-

$\oint \vec{H}.\vec{dl} = \iint \text{curl } \vec{H}.\vec{dS}$

Therefore, equation (3.16) becomes-

$\iint \text{curl } \vec{H}.\vec{dS} = \iint \vec{J}.\vec{dS}$

i.e. $\iint (\text{curl}\vec{H} - \vec{J}).\vec{dS} = 0$ (3.17)

As the surface is arbitrary, therefore integrand must vanish i.e.

$\text{curl}\vec{H} - \vec{J} = 0$

or $\text{curl } \vec{H} = \vec{J}$ (3.18)

Multiplying both sides by μ_0 in equation (9.18), we get-

$\mu_0 \text{curl } \vec{H} = \mu_0\vec{J}$

or $\text{curl } \mu_0\vec{H} = \mu_0\vec{J}$

or $\text{curl } \vec{B} = \mu_0\vec{J}$ (3.19)

Equation (3.18) or (3.19) is the differential form of Ampere's circuital law. The above relation (3.19) indicates that the magnetic induction at a point is derived from the given value of $\vec{J}$ at that point by integration. However this equation is not enough to derive $\vec{B}$ at a point because for the same value of $\vec{J}$ at the point another term may be added to $\vec{B}$. We, therefore, need another condition.

3.7 CURL OF $\vec{B}$

The curl of a vector field at any point is defined as a vector quantity whose magnitude is equal to the maximum line integral per unit area along the boundary of an infinitesimal test area at that point and whose direction is perpendicular to the plane of the test area. The curl of vector field is sometimes called circulation or rotation.

According to Ampere's circuital law, "The line integral of magnetic induction around a closed path is equal to μ_0 times the net current enclosed by the path" i.e.

$$\oint \vec{B}.\vec{dl} = \mu_0 i \qquad (3.20)$$

Where i is the current enclosed by the path.

Let us consider a region in which there is a steady flow of charge. The current density in this region remains constant i.e. it does not change with time however its value may vary from place to place. Now let us consider a closed path. The total current enclosed by this path is the flux of current density through the surface bounded by closed path i.e. the total current enclosed by the path given as-

$$i = \iint \vec{J}.\vec{dS} \qquad (3.21)$$

where $\vec{J}$ is the current density and $\vec{dS}$ is small element of area at the point of current density $\vec{J}$ inside the closed path.

Putting the value of i from equation (3.21) in equation (3.20), you get-

$$\oint \vec{B}.\vec{dl} = \mu_0 \left[\iint \vec{J}.\vec{dS}\right]$$

Using Stoke's theorem, you can convert line integral into surface integral as-

$$\iint \text{curl } \vec{B}.\vec{dS} = \mu_0 \left[\iint \vec{J}.\vec{dS}\right]$$

$$\iint [\text{curl } \vec{B} - \mu_0\vec{J}].\ \vec{dS} = 0$$

As the surface is arbitrary, therefore you have-

$$\text{curl}\vec{B} - \mu_0\vec{J} = 0$$

$$\text{curl}\vec{B} = \mu_0\vec{J} \qquad (3.22)$$

Thus the curl of $\vec{B}$ is equal to μ_0 times current density. The above equation (3.22) is the differential form of Ampere's circuital law. The above relation indicates that the magnetic induction at a point is derived from the given value of $\vec{J}$ at that point by integration. However this equation is not enough to derive $\vec{B}$ at a point because for the same value of $\vec{J}$ at the point another term may be added to $\vec{B}$. We, therefore, need another condition.

3.8 DIVERGENCE OF $\vec{B}$

The divergence of a vector function at certain point is defined as the outward flux of the vector field per unit volume enclosed through an infinitesimal closed surface surrounding the point. The divergence of a vector function is scalar quantity. It should be noted that the divergence itself is simply an operator and has no physical meaning in itself. After operating on suitable physical vector functions, it represents various significant physical scalar quantities. If the divergence of any vector function in a region is zero, it means that the flux of the vector function entering any element of this region is equal to that leaving it.

According to Biot-Savart law the magnetic field at a point due to a current element $i\overrightarrow{dl}$ at a point having position vector $\vec{r}$ relative to current element is given by-

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{i\,\overrightarrow{dl}\times\vec{r}}{r^3} \qquad (3.23)$$

The magnetic field due to complete circuit current is given as-

$$\vec{B} = \frac{\mu_0}{4\pi}\oint\frac{i\,\overrightarrow{dl}\times\vec{r}}{r^3} \qquad (3.24)$$

Taking divergence on both sides, you get-

$$\text{div}\vec{B} = \nabla.\vec{B} = \nabla.\left\{\frac{\mu_0}{4\pi}\oint\frac{i\,\overrightarrow{dl}\times\vec{r}}{r^3}\right\} \qquad (3.25)$$

$$\text{or div}\,\vec{B} = \frac{\mu_0}{4\pi}\oint\nabla.\left\{\frac{i\,\overrightarrow{dl}\times\vec{r}}{r^3}\right\}$$

$$\text{But}\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

Hence the above relation can be written as-

$$\text{div}\vec{B} = -\frac{\mu_0}{4\pi}\oint\nabla.\left\{i\overrightarrow{dl}\times\nabla\left(\frac{1}{r}\right)\right\}$$

Using vector identity $\nabla.\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\nabla\times\vec{A}\right)-\vec{A}.\left(\nabla\times\vec{B}\right)$, the above expression becomes-

$$\text{div}\vec{B} = -\frac{\mu_0}{4\pi}\oint\nabla\left(\frac{1}{r}\right).\left(\nabla\times i\overrightarrow{dl}\right)-\left(i\overrightarrow{dl}\right).\left\{\nabla\times\nabla\left(\frac{1}{r}\right)\right\} \qquad (3.26)$$

Now let us interpret the result. You that the magnetic field is specified at field point and the current element $i\overrightarrow{dl}$ is due to source point. The field point depends on variables (x,y,z) but on the other hand the field source $i\overrightarrow{dl}$ does not depend on variables (x,y,z), therefore it is obvious that

$\nabla \times (i\vec{dl}) = 0$ (3.27)

Also you know that the curl of gradient of a scalar function is always zero i.e.

curl grad $\left(\frac{1}{r}\right) = 0$ or $\nabla \times \nabla \left(\frac{1}{r}\right) = 0$ (3.28)

Now using relation (3.27) and (3.28) in equation (3.26), you get-

$$\text{div}\vec{B} = -\frac{\mu_0}{4\pi}\oint \nabla\left(\frac{1}{r}\right).0 - (i\vec{dl}).\{0\} = 0$$

i.e. $\text{div}\vec{B} = 0$ (3.29)

The above condition holds for all superposition of such fields or for the field of any distribution of currents. The equation (3.29) implies that the magnetic field is solenoidal.

3.9 VECTOR POTENTIAL

The vector identity div curl $\vec{A} \equiv 0$ shows that the solution of the equation div $\vec{B} = 0$ can be represented in the form as-

$\vec{B} = \text{curl}\,\vec{A}$ (3.30)

The vector field $\vec{A}$, the curl of which is equal to the magnetic field $\vec{B}$ is known as vector potential of a magnetic field $\vec{B}$.

$\vec{A}$will be specified uniquely only if its divergence as well as its curl is given. We choose

$\text{div}\vec{A} = 0$ (3.31)

This choice is called Lorentz gauge- the gauging condition for the potential. The arbitrariness in the choice of the vector potential indicates that the vector potential plays only an auxiliary role and cannot be measured experimentally.

Let us derive equation for vector potential. We know that $\nabla \times \vec{B} = \mu_0\vec{J}$

Putting the value of $\vec{B}$ from equation (3.30), the above equation becomes-

$\nabla \times (\nabla \times \vec{A}) = \mu_0\vec{J}$

Using vector identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla.\vec{A}) - (\nabla.\nabla)\vec{A}$, the above equation becomes-

$\nabla(\nabla.\vec{A}) - (\nabla.\nabla)\vec{A} = \mu_0\vec{J}$

Or grad div $\vec{A} - \nabla^2\vec{A} = \mu_0\vec{J}$

Using Lorentz gauge given by equation (3.31), the above relation becomes-

$0 - \nabla^2\vec{A} = \mu_0\vec{J}$

Or$\nabla^2\vec{A} = -\mu_0\vec{J}$ (3.32)

In terms of Cartesian components of $\vec{A}$, we can write-

$$\left.\begin{aligned}\nabla^2 A_x &= -\mu_0 J_x\\ \nabla^2 A_y &= -\mu_0 J_y \qquad (3.33)\\ \nabla^2 A_z &= -\mu_0 J_z\end{aligned}\right\}$$

Each component of the vector potential thus satisfies Poisson's equation ($\nabla^2\varphi = -\frac{\rho}{\varepsilon_0}$) which has the solution as-

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(r')}{|\vec{r}-\vec{r'}|}dV' \qquad (3.34)$$

If all currents are concentrated in a finite region of space, then by analogy with equation (3.34), the solution of equations (3.33) can be written as-

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{J_i(r')}{|\vec{r}-\vec{r'}|}dV' \qquad (3.35)$$

Where i stands for x, y and z. In vector form, we have-

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}(r')}{|\vec{r}-\vec{r'}|}dV' \qquad (3.36)$$

In a case of a filamentary current i through a differential length dl' along the wire, we have-

$$J\,dV' = (i/S)(Sdl') = i\,dl'$$

Now the above equation becomes-

$$d\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{i(\vec{r'})dl'}{|\vec{r}-\vec{r'}|} \qquad (3.37)$$

Summing up overall volume elements of the filament, we get-

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{i(r')}{|\vec{r}-\vec{r'}|}dV' \qquad (3.38)$$

The components of $\vec{A}$ vary as 1/r, like electric potential, which does not diverge with in a charge distribution. As divergence of a curl of a vector is always zero and div $\vec{B} = 0$ can be written as a curl of a vector and thus $\vec{A}$ is a vector. Due to these reasons $\vec{A}$ is called by the name of vector potential.

Gauge invariance:

We can write electric field in terms of a scalar potential as given below:

$E = -\nabla\varphi$ Provided that $\nabla \times E = 0$

However, we have just found that in the presence of a changing magnetic field the curl of the electric field is non-zero. In other words, E is not, in general, a conservative field. Does this mean that we have to abandon the concept of electric scalar potential? Fortunately, no. It is still possible to define a scalar potential which is physically meaningful.

As we know that: $\nabla.\mathrm{B}=0$, which is valid for both time-varying and non-time-varying magnetic fields. Since the magnetic field is solenoidal, we can write it as the curl of a vector potential:

$$\mathrm{B}=\nabla\times\mathrm{A}$$

By substituting this equation in the above equation, we will obtain:

$$\nabla\times E=-\frac{\partial\nabla\times A}{\partial t},$$

By arranging the terms, we can get:

$$\nabla\times(E+\frac{\partial A}{\partial t})=0$$

We know that a curl-free vector field can always be expressed as the gradient of a scalar potential, so we can write it as:

$$E+\frac{\partial A}{\partial t}=-\nabla\varphi$$

Or

$$E=-\nabla\varphi-\frac{\partial A}{\partial t}$$

This equation tells us that scalar potential φ only describes the conservative electric field generated by electric charges. The electric field induced by time-varying magnetic fields is non-conservative, and is described by the magnetic vector potentialA.

The electric and magnetic fields are obtained from the vector and scalar potentials are important, because they determine the electromagnetic forces exerted on charged particles. Note that the above prescription does not uniquely determine the two potentials. It is possible to make the following transformation, known as a gauge transformation, which leaves the fields unaltered.

$$\varphi\rightarrow\varphi+\frac{\partial\psi}{\partial t}$$

$$A\rightarrow A-\nabla\psi$$

Where $\psi\ (r,t)$ is the general scalar field. It is necessary to adopt some form of convention, generally known as a gauge condition, to fully specify the two potentials. This is the Lorentz gauge condition.

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t}+\nabla.\mathrm{A}=0$$

We can also write this equation in the Lorentz invariant form as:

$$\partial_\mu \varphi^\mu = 0$$

This implies that if the Lorentz gauge holds in one particular inertial frame then it automatically holds in all other inertial frames. A general gauge transformation can be written

$$\varphi^\mu \to \varphi^\mu + c\partial^\mu \psi$$

Note that even after the Lorentz gauge has been adopted, the potentials are undetermined to a gauge transformation using a scalar field, ψ, which satisfies the source less wave equation.

3.10 MAGNETIC FLUX

Let us consider a plane placed in a magnetic field. The magnetic flux linked with that plane is defined as the dot(scalar) product of magnetic field ($\vec{B}$) and the area of the plane ($\vec{A}$) i.e.

$$\text{The magnetic flux } \varphi = \vec{B}.\vec{A} \qquad (3.39)$$

If the perpendicular to the plane makes an angle θ with the direction of magnetic field, then-

$$\text{The magnetic flux } \varphi = BA\cos\theta \qquad (3.40)$$

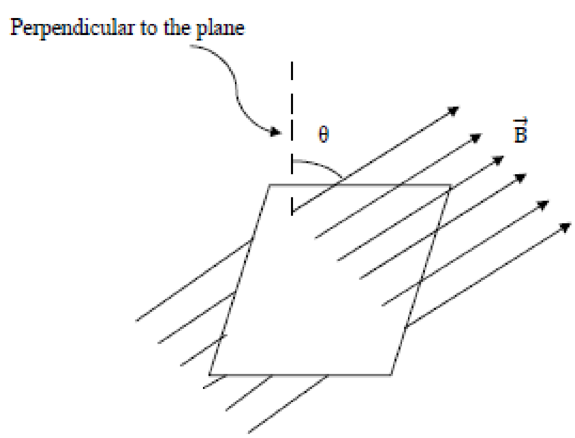

Figure 3.3

From equation (3.40), you can write-

$$\varphi = (B\cos\theta)\, A \qquad (3.41)$$

= component of the magnetic field perpendicular to the plane× area of the plane

Thus, you can define the magnetic flux as the product of the component of the magnetic field perpendicular to the plane and the area of the plane.

If you consider the plane perpendicular to the uniform magnetic field, then the product of the magnitude of the field and the area of the plane is called the magnetic flux φ linked with the plane i.e.

$\Phi = BA$ (since $\theta = 0^0$) (3.42)

If infinitesimal small surface area ($\overrightarrow{dS}$) is considered, then magnetic flux linked with that surface area is given as-

$d\varphi = \vec{B}.\overrightarrow{dS}$(3.43)

The total magnetic flux linked with the entire surface-

$$\varphi = \iint \vec{B}.\overrightarrow{dS}$$

(3.44)

φ is positive if the outward normal to the plane is in the same direction as $\vec{B}$ and is negative if the outward normal is opposite to $\vec{B}$.

The SI unit of the magnetic flux φ is weber (Wb).

Since from equation (3.42), you have-

$$B = \varphi/A$$

Thus the unit of magnetic flux is also expressed in weber/meter^2 (Wb/meter^2). That is why the magnetic field induction B is also called the magnetic flux density.

The CGS unit of magnetic flux is Maxwell.

$$1 \text{ weber} = 10^8 \text{ maxwell}$$

The magnetic flux is a scalar quantity while magnetic flux density is a vector quantity.

You may also express the magnetic flux in terms of the magnetic lines of force. We can represent a magnetic field by magnetic lines of force. If you draw limited lines of force so that in a magnetic field B = 1 Wb/meter^2 only one line of force passes per meter^2 through an area perpendicular to $\vec{B}$in a magnetic field B = 2 Wb/meter^2 only two lines of force pass per meter^2 perpendicular to B, and so on, then these lines are called the lines of flux. In a magnetic field the number of lines of flux passing per meter^{2}through an area perpendicular to the magnetic field is equal to the magnetic flux linked with that plane.

If $\theta = 90^0$ i.e. the plane is parallel to the magnetic field, then no flux-line will pass through it and the magnetic flux linked with that plane will be zero.

3.11 MAGNETIC SUSCEPTIBILITY

Magnetic susceptibility is a measure of how easily a substance is magnetized in a magnetizing field. Normally, for magnetic substances, the intensity of magnetization $(\overrightarrow{M})$ is directly proportional to the magnetic intensity $(\overrightarrow{H})$ of the magnetizing field i.e.

$\vec{M} \alpha \vec{H}$

Or $$\vec{M} = \chi_m \vec{H}$$

Where χ_m is a constant known as magnetic susceptibility of the substance. It is defined as the ratio of the intensity of magnetization to the magnetic intensity of the magnetizing field.

$\chi_m = \frac{M}{H}$

χ_m is unit less quantity. The value of χ_m is zero in vacuum, because there can be no magnetization in vacuum. Normally the substances can be classified in terms of χ_m as follows-

χ_m = + ve, substance is paramagnetic

χ_m = -ve, substance is diamagnetic

χ_m = +ve and very large, substance is ferromagnetic

However, for them, the magnetization $\vec{M}$ is not accurately proportional to $\vec{H}$ and therefore, χ_m is not strictly constant.

3.12 RELATION BETWEEN RELATIVE PERMEABILITY AND MAGNETIC SUSCEPTIBILITY

We have already read that when a substance is kept in a magnetizing field, it becomes magnetized. The total magnetic flux density B within the substance is the flux density that would have been produced by the magnetizing field in vacuum plus the flux density due to the magnetization of the substance. If M be the intensity of magnetization of the substance, then, we know the relation which make the relation between B, H, M is given as:

$$B = \mu_0 (H + M)$$

where H is the magnetic intensity, M is the intensity of magnetization and B is total magnetic flux density. From the above equation:

$\chi_m = \frac{M}{H}$

or $$M = \chi_m H$$

Putting the value of M in the above expression, we get:

$$B = \mu_0 (H + \chi_m H)$$

Or $$B = \mu_0 H (1+\chi_m)$$

The relation between B and H is given as: $B = \mu H$

By substituting the value of B in above equation, we get-

$\mu H = \mu_0 H (1+\chi_m)$

or $$\mu = \mu_0 (1+\chi_m)$$

or $\frac{\mu}{\mu_0} = 1 + \chi_m$

Since $\frac{\mu}{\mu_0} = \mu_r$, is known as the relative permeability, therefore-

$\mu_r = 1 + \chi_m$

This is the relation between relative permeability and magnetic susceptibility.

3.13 MAGNETIC SUBSTANCES

Normally the substances show magnetic properties are known as magnetic substances. Magnetic substances may be solids, liquids and gases. We can classify these substances on the basis of their magnetic behaviour.

- Diamagnetic Substances
- Paramagnetic Substances
- Ferromagnetic Substances

3.13.1 Diamagnetic Substances

The substances, when placed in an external magnetic field, are softly magnetized opposite to the direction of the magnetizing field are called diamagnetic substances and their magnetism is called the diamagnetism. Some examples of diamagnetic substances are Bismuth, zinc, copper, lead, gold, silver, water, hydrogen, sodium chloride, nitrogen, mercury etc.

Diamagnetic substances have the following properties

1. These substances have negative magnetic susceptibility and independent of temperature.
2. The flux density in a diamagnetic substance placed in a magnetizing field is slightly less than that in the free space.
3. The relative permeability of these substances is less than 1 i.e. $\mu_r < 1$ for diamagnetic substances.

4. A diamagnetic gas, when allowed to ascend in between the poles of a magnet, spreads across the magnetic field.

5. In a non-uniform magnetic field, a diamagnetic substance tends to move from the stronger to the weaker part of the magnetic field. If we take a diamagnetic liquid in a watch glass placed on two magnetic poles very near to each other, then the liquid is depressed in the middle, where the magnetic field is strongest. Now, if the distance between the poles is increased, the liquid rises in the middle, because now the magnetic field is strongest near the poles (Figure 3.4).

Figure 3.4

3.13.2 Paramagnetic Substances

The substances, when placed in a magnetic field, are softly magnetized in the direction of the magnetizing field are called paramagnetic substances and their magnetism is called paramagnetism. These substances, when brought close to a pole of a powerful magnet, are attracted towards the magnet. The examples of paramagnetic substances are aluminum, antimony, copper chloride, sodium, platinum, manganese, liquid oxygen, solutions of salts of iron and nickel etc.

Paramagnetic substances have the following properties:

1. These substances have positive magnetic susceptibility and susceptibility of paramagnetic substances varies inversely as the kelvin temperature of the substance i.e.

$$\chi_m \alpha \frac{1}{T}$$

This is known as Curie's law.

2. The flux density in a paramagnetic substance placed in a magnetizing field is slightly greater than that in the free space.

3. The relative permeability of these substances is greater than 1 i.e. $\mu_r > 1$ for paramagnetic substances.

4. A paramagnetic gas, when allowed to ascend in between the poles of a magnet, spreads along the magnetic field.

5. In a non-uniform magnetic field, a paramagnetic substance tends to move from the weaker to the stronger part of the magnetic field. If we take a paramagnetic liquid in a watch glass placed on two magnetic poles very near to each other, then the liquid rises in the middle, where the magnetic field is strongest. Now, if the distance between the poles is increased, the liquid depresses in the middle and rises near the edges, because now the magnetic field is strongest near the poles (Figure 3.5).

Figure 3.5

8. When a rod of paramagnetic material is suspended freely between two magnetic poles, then its axis becomes parallel to the magnetic field. The poles produced at the ends of the rod are opposite to the nearer magnetic poles (Figure 3.6).

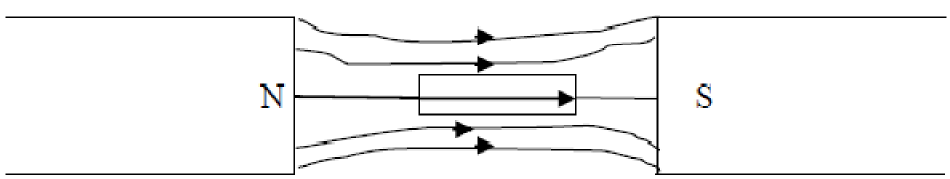

Figure 3.6

3.13.3 Ferromagnetic Substances

Some substances, when placed in a magnetic field, are strongly magnetized in the direction of the magnetizing field. These materials are attracted fast towards a magnet when brought close to either of the poles of the magnet. These are called ferromagnetic substances and their magnetism is called ferromagnetism. Iron, cobalt, nickel, magnetite etc. are some ferromagnetic substances

Ferromagnetic substances have the following properties:

1. These substances have positive and very large magnetic susceptibility.
2. The relative permeability of for ferromagnetic substances is very-very greater than 1 i.e. $\mu_r >> 1$.
3. These substances show all the properties of paramagnetic substances to a much high degree.
4. Ferromagnetism decreases with increase in temperature. If you heat a ferromagnetic substance, then at a definite temperature the ferromagnetic property of the substance suddenly disappears and the substance becomes paramagnetic. The temperature above which a ferromagnetic substance becomes paramagnetic is known as Curie temperature of the substance. The Curie temperature of iron is 770^0 C and that of nickel is 358^0 C.

Note: It is to be noted that, every substance is diamagnetic in nature. In paramagnetic or ferromagnetic substances, the diamagnetic property is masked by the stronger paramagnetic or ferromagnetic properties

3.14 CURIE'S LAW

In 1895, Curie discovered experimentally that the magnetization or intensity of magnetization of a paramagnetic substance is directly proportional to the magnetic intensity H of the magnetizing field and inversely proportional to the Kelvin temperature T i.e.

$$M \, \alpha \, \frac{H}{T}$$

Or $$M = C\frac{H}{T}$$

where C is a constant. This equation is known as Curie's law and the constant C is called the Curie constant. The law, however, holds so long the ratio $\frac{H}{T}$ does not become too large.

M cannot increase without limit. It approaches a maximum value corresponding to the complete alignment of all the atomic magnets contained in the substance.

You can express Curie's law in an alternative form. You know that the magnetic susceptibility χ_m is defined as-

$$\chi_m = \frac{M}{H}$$

Putting the value of M from above equation in the above equation, we get-

$$\chi_m = \frac{C\frac{H}{T}}{H} = \frac{C}{T}$$

or $$\chi_m \, \alpha \, \frac{1}{T}$$

i.e. the magnetic susceptibility is inversely proportional to Kelvin temperature. This is known as the Curie's law.

3.15 HYSTERESIS

As we have already read that for a ferromagnetic substance the magnetic flux density (B) is not a linear function of magnetic intensity H. For this case, the relative magnetic permeability is not constant however, it is a function of H. Additionally, we can say that there is no unique value of relative magnetic permeability for a particular ferromagnetic substance. The relationship between magnetic flux density B and corresponding magnetic intensity H for such a material initially unmagnetised is represented by a typical curve as shown in figure (3.7), known as the magnetization curve or B-H curve.

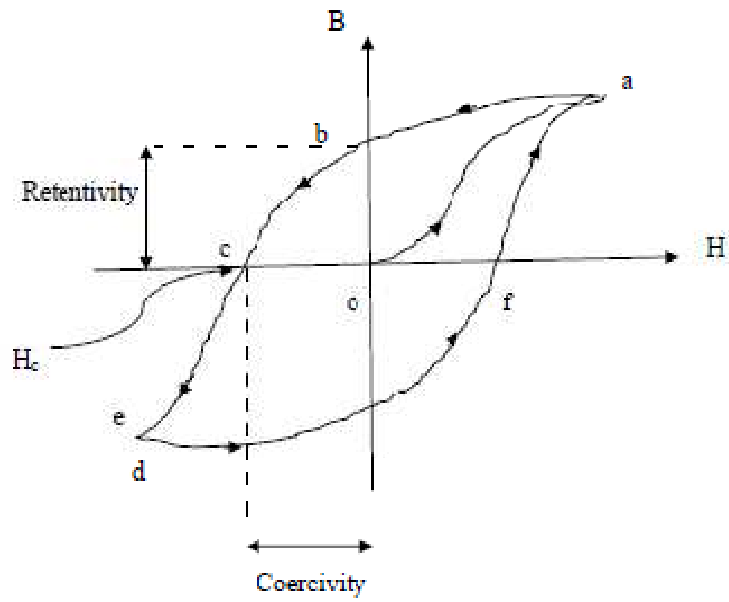

Figure 3.7

The variation of B with variation in H is represented in Figure 3.7. It is also known as hysteresis curve. The point O represents in the initial unmagnetised state of the substance (B =0) and a zero magnetic intensity (H =0) (Figure 3.7). As H is increased, B increases non-uniformly along curved path oa. At a, the substance acquires a state of magnetic saturation. Any further increase in H does not produce any increase in B. Now the value of B becomes practically constant.

If now the magnetizing magnetic field H is decreased, the magnetic flux density B of the substance also decreases following a new path ab, not the original path ao. Thus B lags behind H. When H becomes zero, B still has a value equal to ob. The magnetic flux density in the substance is seen to depend upon not on the magnetic intensity alone but on the magnetic history of the substance as well. At point b, the specimen has become a permanent magnet since magnetization is still present even though the magnetizing field H has been cut off. The magnetization remaining in the substance when the magnetizing field is reduced to zero is called the 'residual magnetism'. The power of retaining this magnetization is called the 'retentivity' or the 'remanence' of the substance. In this way, the retentivity of a substance is a measure of the magnetization remaining in the substance when the magnetizing field is removed. In the above figure, ob represents the retentivity of the substance.

If now the magnetizing field H is increased in the reverse direction, the magnetic flux density B decreases along path bc, still lagging behind H, until it becomes zero at point c where H equals oc. This value of His denoted by H_c. This value oc of the magnetizing field is called the 'coercive force' or 'coercivity' of the substance. Thus, the coercivity of a substance is a measure of the reverse magnetizing field required to destroy the residual magnetism of the substance. When we increase H beyond oc, the substance is increasingly magnetized in the opposite direction along cd and a reverse induction is set up in the substance which quickly attains the saturation value. At point d, the substance is again magnetically saturated.

By taking H back from its maximum negative value, through zero, to its original maximum positive value, a symmetrical curve defa is obtained. At point e where the substance is magnetized in the absence of any external magnetizing field, it is said to be a permanent magnet.

In this way, we found that the magnetization and also the magnetic flux density B always lags behind the magnetizing field H. The lagging of B behind H is called 'Hysteresis'. The closed curve or loop, abcdefa which represents a cycle of magnetization of the substance is known as the 'hysteresis curve or loop' of the substance. On repeating the process, the same closed curve is traced again but the portion oa is never obtained.

3.15.1 Importance of Hysteresis curve

By using the hysteresis curve of various ferromagnetic materials, we can select the material which gives minimum hysteresis curve when placed to the cycle of magnetization. From hysteresis curve, an idea of the magnetic properties like susceptibility, permeability, retentivity, coercivity of a ferromagnetic material can be made.The choices of a magnetic material for the construction of a permanent magnets, electromagnets, cores of transformer and magnetic shielding can be decided from the hysteresis curve of the sample.

3.15.2 Energy Loss due to Hysteresis

According to molecular theory of magnetization, the molecules of magnetized or unmagnetised magnetic substance are themselves complete magnets. When we apply a magnetized field, the molecular magnets align themselves in the direction of the field. During this process, work is done by the magnetizing field in turning the molecular magnets against the mutual attractive forces. This energy required to magnetize a substance is not completely recovered when the magnetizing field is turned off, since the magnetization does not become zero. The specimen retains some magnetization because some of the molecular magnets remain aligned in the new formation due to the group forces. To destroy them out completely, a coercive force in the reverse direction has to be applied. In this way, there is a loss of energy in taking a magnet through a cycle of magnetization. This loss of energy or heat is called 'hysteresis loop'. Now let us calculate this loss of energy.

Let us consider a magnetic material having n molecular magnets per unit volume. Let m be the magnetic moment of each elementary magnet and θ the angle which its axis makes with the direction of magnetizing field H, then magnetic moment per unit volume parallel to the magnetic field is-

$$\mu_M = \Sigma\, m \cos\theta \qquad (3.45)$$

The magnetic moment per unit volume perpendicular to the magnetizing field is $\Sigma\, m \sin\theta$ and this is equal to zero since there can be no magnetization perpendicular to H.

Now, the torque due to the magnetizing field acting on the dipole of moment m when it is inclined at an angle θ to the field is-

$\tau = \mu_0\, m\, H \sin\theta$(3.46)

and the work done when it moves through a small angle from θ to $\theta + d\theta = -\,\tau\, d\theta$

$= -\,\mu_0\, m\, H \sin\theta d\theta$

Here minus sign comes in because the work has to be done against the magnetic field in increasing θ by dθ.

Hence, the work done per unit volume of the material

$$dW = -\,\mu_0\, m\, H \sin\theta d\theta \qquad (3.47)$$

As θ increases by dθ, the intensity of magnetization M also increases by dM obtained from equation (3.45) as-

$dM = d\,(\Sigma\, m \cos\theta)$

$$= -\,\Sigma\, m \sin\theta d\theta \qquad (3.48)$$

From equations (3.47) and (3.48), we get-

$$dW = \mu_0 H\, dM \qquad (3.49)$$

Thus, the work done by the magnetizing field per unit volume of the material for completing a cycle is-

$$W = \oint dW$$

$= \oint \mu_0 H\, dM = \mu_0 \oint H\, dM$

$= \mu_0 \times$ Area of M-H loop

(3.50)

Since we know that-

$B = \mu_0 (H + M)$

Or $dB = \mu_0 (dH + dM)$

Or $dM = \frac{dB}{\mu_0} - dH$

(3.51)

Putting for dM in equation (3.50), we get-

$$W = \mu_0 \oint H\left(\frac{dB}{\mu_0} - dH\right)$$

$$= \mu_0 \oint H \frac{dB}{\mu_0} - \mu_0 \oint H\, dH$$

$$= \oint H\, dB - \mu_0 \oint H\, dH \quad (3.52)$$

But $\oint H\, dH = 0$, because the plot of H against H is a straight line and the area enclosed by it is zero. Thus equation (3.52) gives-

$$W = \oint H\, dB$$

$$= \text{Area of B-H loop} \quad (3.53)$$

Thus, the work done per unit volume of the material per cycle is equal to the area of μ_0 times the area of M-H loop or the area of B-H loop. The unit of this work is Joule/meter^3 per cycle and is dissipated in the form of heat.

3.16 SUMMARY

In the present unit, you have studied about Lorentz force and Biot-Savart law. Along this you have also studied that a current carrying conductor produces magnetic field around it. You have seen that the conductors attract each other if currents in them are in the same direction and repel each other if currents are in opposite directions. You have analyzed that according to Ampere's circuital law, the line integral of magnetic induction around a closed path is equal to μ_0 times the net current enclosed by the path. You have also seen that Ampere's law holds for closed path of any shape. You have known about divergence and curl of magnetic field, scalar and vector potential, magnetic susceptibility and hysteresis. To present the clear understanding and to make the concepts of the unit clear, many solved examples are given in the unit. To check your progress, self-assessment questions (SAQs) are given place to place.

Self-Assessment Question (SAQ) 1: A current of 2 Amp is passed through a winding of 20 turns per cm. If the magnetic induction is 1.2 Wb/meter^2, calculate the intensity of magnetic field and the intensity of magnetization.

Self-Assessment Question (SAQ) 2: Choose the correct option- The unit of intensity of magnetization is-

(a) Amp/meter^2 (b) weber/meter^2 (c) Amp/meter (d) Amp× meter

Self-Assessment Question (SAQ) 3: Choose the correct option- The magnetic permeability of a substance is a measure of –

(a) its conduction of magnetic lines of force through it

(b) its conduction of electric lines of force through it

(c) its conduction of electricity through it

(d) none of these

Self-Assessment Question (SAQ) 4: Choose the correct option- The magnetic permeability of vacuum, in SI units, is-

(a) 1 (b) infinite (c) zero (d) $4n\times10^{-7}$

Self-Assessment Question (SAQ) 5: Choose the correct option- Magnetism in substances is caused by-

(a) orbital motion of electrons only (b) spin motion of electrons only

(c) due to spin and orbital motion of electrons both (d) none of these

Example 1: A substance has magnetic susceptibility equal to 2. Calculate the relative permeability.

Solution: Given, $z_m = 2$ we know-

$\mu_r = 1 + z_m = 1 + 2 = 3$

Example 2: The relative permeability for a material is 3. What will be its magnetic susceptibility?

Solution: Given μ_r= 3 we know that-$\mu_r = 1 + z_m$

Or$z_m = \mu_r – 1$ = 3-1 =2

Self-Assessment Question (SAQ) 1: Choose the correct option- Diamagnetic substance when placed in a magnetic field is-

(a)Weakly attracted (b)strongly attracted (c)repelled (d) none of these

Self-Assessment Question (SAQ) 2: Choose the correct option- The magnetic susceptibility of a diamagnetic material is-

(a) Large and positive (b) large and negative (c) small and positive (d) small and negative

3.17 GLOSSARY

Magnetic field - The region surrounding a magnetic

Magnetic induction -Vector that specifies magnitude and direction of magnetic field

Simultaneous - Coincident

Electric force - Force experienced by a charge placed at a point in an electric field Magnetic force - Force experienced by a charge in a magnetic field

Infinitesimal - Tiny

Vacuum - Empty

Characteristics - Features

Steady -Stable

Flow- Current

Divergence- Deviation

Magnetic flux – Surface integral of the magnetic field over that surface

Magnetic flux density – Vector that specifies the magnitude and direction of magnetic field

Aligned- Bring into line

Magnetic field- The region surrounding a magnet in which the force of the magnet can be detected

Induction- Stimulation

Magnetization- The magnetic state of any substance is described by a quantity

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3.20 TERMINAL QUESTIONS

1. Explain the magnitude and direction of the force acting on a charge moving in a magnetic field. When is the force maximum and when minimum?
2. Explain Biot Savart law.
3. Establish the expression for magnetic force acting between two long, parallel and straight current carrying conductors.
4. Both the electric and magnetic field can deflect an electron. What is the difference between these deflections?
5. Explain Ampere’s circuital law. Give its significance. Derive its differential form.
6. Explain Maxwell’s correction in Ampere’s circuital law.
7. Explain the concept of Maxwell’s displacement current and show how it led to the modification of the Ampere’s law.

8. Obtain the generalized form of Ampere's circuital law. Comment on the concept of the displacement.
9. Throw the light on characteristics of displacement current.
10. Using Ampere's circuital law, establish the expression of magnetic field due to a long current carrying wire.
11. Give a comparison between Coulomb's law and Biot-Savart law.
12. Prove that the curl of $\bar{B}$ is equal to μ_0 times current density.
13. What is the difference between the torque acting on a magnetic dipole and an electric dipole?

UNIT 4: ELECTRODYNAMICS

STRUCTURE

4.1 INTRODUCTION

In this chapter, we will discuss about Maxwell's Equations. Maxwell's equations are a set of four differential equations that form the theoretical basis for describing classical electromagnetism: Gauss's law: Electric charges produce an electric field. The electric flux across a closed surface is proportional to the charge enclosed.

4.2OBJECTIVES

- To learn about Maxwell's Equations
- To learn about equation of continuity
- To learn about displacement current
- To discuss Maxwell's equation in free space
- To discuss Maxwell equation for harmonically varying field

4.3 EQUATION OF CONTINUITY

A continuity equation in physics is an equation that describes the transport of some quantity. It is particularly simple and powerful when applied to a conserved quantity, in electromagnetic theory, the continuity equation is an empirical law expressing (local) charge conservation. Mathematically it is an automatic consequence of Maxwell equation, although charge conservation is more fundamental than Maxwell's equations. It states that the divergence of the current density J (in amperes per square metre) is equal to the negative rate of change of the charge density ρ (in coulombs per cubic metre),

$$\mathrm{div}\vec{J} = -\frac{d\rho}{dt}$$

Let S be the surface enclosing a volume V and let ds be a small element of this surface. The direction of ds is taken to be taken to be that of outward normal.

Current is the movement of charge. The continuity equation says that if charge is moving out of a differential volume (i.e. divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge.

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must charge with time in a manner that the time rate of decrease of charge within the volume equals the net rate of flow of charge out of the volume.

Volume charge density

$$\rho = \frac{dq}{dv}$$

$$dq = \rho dv$$

Integrating –

$$\int dq = \int \rho dv$$

$$q = \int_v \rho dv$$

By definition of current,

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_v \rho dv \qquad \text{- - - - - ①}$$

As charge is decreasing –

$$I = \frac{-dq}{dt} = \frac{-d}{dt}\int_v \rho dv \qquad \text{- - - - - ②}$$

If J is current density

$$J = \frac{dI}{dS}$$

$$dI = JdS \qquad \vec{J} \cdot \overrightarrow{ds} = Jds \ (o)$$

$$dI = \vec{J} \cdot \overrightarrow{ds} \qquad \vec{J} \,||\, \overrightarrow{ds} \theta = 0$$

Integrating

$$\oint dI = \oint_s \vec{J} \cdot \overrightarrow{ds}$$

$$I = \oint_s \vec{J} \cdot \overrightarrow{ds} \qquad \text{- - - - - ③}$$

Now equating equation ②&③

$$\overrightarrow{\oint J} \cdot \overrightarrow{ds} = -\frac{d}{dt}\int_v \rho dv$$

Using Gauss Divergence theorem

$$\oint \vec{J} \cdot \overrightarrow{ds} = \int \text{div } J \text{ div}$$

$$\int \text{div } J \, dv = -\frac{d}{dt}\int \rho dv$$

or

$$\int \text{div } J \, dv = -\int \frac{d}{dt} \rho dv$$

$$\int \left(\text{div } J + \frac{d\rho}{dt}\right) dv = 0$$

dv is arbitrary taken dv $\neq 0$

$$\boxed{\mathbf{div\vec{J} + \frac{d\rho}{dt} = 0}}$$

J = Current density

ρ = Charge density

This is the continuity equation. H is the mathematical expression for the conservation of change. It state that the total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that the change cannot be created or destroyed, i.e. no source and sinks are present in that volume.

In case of stationary current charges density at any point within the region remains constantρ = constant

$$\frac{d\rho}{dt} = 0$$

So that div J = 0

or

$$\vec{\nabla} . J = 0$$

This expresses the fact that there is no net outward flux of current density.

4.4 DISPLACEMENT CURRENT

In electromagnetism, displacement current density is the quantity $\partial \boldsymbol{D}/\partial t$ appearing in Maxwell equations that is defined in terms of the rate of change of $\boldsymbol{D}$, the electric displacement field. Displacement current density has the same units as electric current density, and it is a source of the magnetic field just as actual current is. However, it is not an electric current of moving charges, but a time-varying electric field. In physical materials (as opposed to vacuum), there is also a contribution from the slight motion of charges bound in atoms, called dielectric polarization.

Ampere's Circuital Law states the relationship between the current and the magnetic field created by it. This law states that the integral of magnetic field density (B) along an imaginary closed path is equal to the product of current enclosed by the path and permeability of the medium.

$$\int \vec{B} \cdot dl = \mu_0 I$$

$$I = \int \vec{J} \cdot \overrightarrow{ds}$$

Now,

$$\int \vec{B} \cdot dl = \mu_0 \int \vec{J} \cdot \overrightarrow{ds}$$

$$\therefore \vec{B} = \mu_0 \vec{H}$$

$$\int \mu_0 \vec{H} \cdot \overrightarrow{dl} = \mu_0 \int \vec{J} \cdot \overrightarrow{ds}$$

$\int \vec{H} \cdot \vec{dl} = \int \vec{J} \cdot \vec{ds}$

Using Stokes theorem to convert line integral into surface integral

$\int \vec{H} \cdot \vec{dl} = \int \text{curl}\,\vec{H} \cdot \vec{ds}$

Now,

$\int \text{curl}\,\vec{H} \cdot \vec{ds} = \int \vec{J} \cdot \vec{ds}$

$$\int (\text{curl}\,\vec{H} - \vec{J})\vec{ds} = 0$$

$\vec{ds} \neq 0$ {ds is taken as arbitrary const}

So,

Curl $\vec{H} - \vec{J} = 0$

Curl H = J - - - - - ①

Now from equation of continuity

$\text{div}J + \frac{d\rho}{dt} = 0$

Substituting value from equation ①

$\text{div}\,(\text{curl}\vec{H}) + \frac{d\rho}{dt} = 0$

Since, $\text{div}\,(\text{curl}\,\vec{H}) = 0$

$\frac{d\rho}{dt} = 0$

Equation ① holds for steady condition only in which charge density is not changing, i.e. not a function of time. Therefore, for time dependent electric fields (changing) equation ① should be modified. Maxwell added an extra term J'.

$\text{curl}\vec{H} = \vec{J} + \vec{J}'$

Taking divergence both side –

$\text{div}\,(\text{curl}\vec{H}) = \text{div}\vec{J} + \text{div}\vec{J}'$

$0 = \text{div}\vec{J} + \text{div}\vec{J}'$

$\text{div}\,J' = -\,\text{div}\,J$

$= -\left(-\frac{d\rho}{dt}\right)$

$\mathbf{div J' = \frac{d\rho}{dt}}$

$$\left\{\begin{array}{l} \text{div}\vec{J}' + \\ -\frac{d\rho}{dt} = 0 \\ \text{div}\,J = \frac{d\rho}{dt} \end{array}\right\}$$

From Maxwell first equation, we have –

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{E_0}$

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$$

$$\epsilon_0 \vec{E} = D$$

$$\vec{\nabla} \cdot \vec{D} = \rho$$

Now,

$$\mathrm{div}\vec{J}' = \frac{d\rho}{dt}$$

$$= \frac{d}{dt}\left(\vec{\nabla} \cdot \vec{D}\right)$$

$$= \vec{\nabla} \cdot \frac{d\vec{D}}{dt}$$

$$\mathrm{div}\vec{J}' = \mathrm{div}\,\frac{d\vec{D}}{dt}$$

$$\vec{J}' = \frac{d\vec{D}}{dt}$$

Now,

$$\mathrm{curl}\vec{H} = \vec{J} + \vec{J}'$$

$$\mathrm{curl}\ H = \vec{J} + \frac{d\vec{D}}{dt}$$

This is modified form of Ampere Circuital Law.

J' arises due to variation of electric displacement with time. H's termed as displacement current density.

4.5 MAXWELL EQUATIONS

Maxwell's Equations, formulated around 1861 by James Clerk Maxwell, describe the interrelation between electric and magnetic fields. They were a synthesis of what was known at the time about electricity and magnetism, particularly building on the work of Michael Faraday, Charles-Augustin Coulomb, Andre-Marie Ampere, and others. These equations predicted the existence of electromagnetic waves, giving them properties that were recognized to be properties of light, leading to the (correct) realization that light is an electromagnetic wave. Other forms of electromagnetic waves, such as radio waves, were not known at the time, but were subsequently demonstrated by Heinrich Hertz in 1888. These equations are considered to be among the most elegant edifices of mathematical physics.

They are usually formulated as four equations (but later we will see some particularly elegant versions with only two), and the equations are usually expressed in differential form, that is, as Partial differential equation involving the divergence and curl operators. They can also be expressed with integrals. They are often expressed in terms of four vector fields: E, B, D, and H, though the simpler forms use only E and B.

4.5.1 MAXWELL FIRST EQUATION

The electric flux through any closed surface is equal to the electric charge enclosed by the surface. Gauss's law describes the relation between an electric charge and the electric field it produces. This is often pictured in terms of electric field lines originating from positive charges and terminating on negative charges, and indicating the direction of the electric field at each point in space.

Maxwell first equation is the Gauss Law in electrostatic A/c to Gauss law electric flux passing through any close surface is $\frac{1}{\epsilon_0}$ times the charge enclosed within it.

$$\phi = \frac{q}{\epsilon_0}$$

$$\phi = \int \vec{E} \cdot \overrightarrow{ds}$$

$$\int \vec{E} \cdot \overrightarrow{ds} = \frac{q}{\epsilon_0}$$

∴ Volume charge density

$$\rho = \frac{dq}{dv}$$

$$dq = \rho\ dv$$

Now integrating

$$\int dq = \int \rho\ dv$$

$$q = \int \rho\ dv$$

$$\boxed{\textbf{div E} = \frac{\rho}{\epsilon_0}}$$ Gauss law of differential form

$$\int \vec{E} \cdot \overrightarrow{ds} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho dv$$

$$\int \vec{E} \cdot \overrightarrow{ds} = \frac{\rho}{E_0}\ dv$$

Gauss divergence theorem –

$$\int \vec{E} \cdot \overrightarrow{ds} = \int \text{divE}\ dv$$

$$\int \text{div Edv} = \int \frac{\rho}{\epsilon_0}\ dv$$

$$\int \left(\text{divE} - \frac{\rho}{\epsilon_0}\right) dv = 0$$

dv is an arbitrary taken dv ≠ 0

$$\text{div E} - \frac{\rho}{\epsilon_0} = 0$$

$$\text{div E} = \frac{\rho}{\epsilon_0}$$

This is Maxwell first equation.

or

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$$

or

$$\boxed{\vec{\nabla} \cdot \vec{\mathbf{D}} = \boldsymbol{\rho}}$$

This Gauss law in Magneto statics.

Self-Assessment Question (SAQ) 1: Which theorem converts line integral into surface integral?

4.5.2 MAXWELL SECOND EQUATION

The second of the equations is just like the first, but for the magnetic field. The divergence of B must be the spatial density of magnetic monopoles. Since they have never been observed (though various Grand Unified Theories might allow for them), the value is zero.

This wasn't formulated initially in terms of monopoles, but was actually a statement that magnetic "lines of force" (the lines that intuitively describe the field) never end. They just circulate around various conductors carrying electric current. In contrast to this, lines of the electric field can be thought to "begin" and "end" on charged particles.

The magnetic field flux through any closed surface is zero. This is equivalent to the statement that magnetic field lines are continuous, having no beginning or end. Any magnetic field line entering the region enclosed by the surface must also leave it. No magnetic monopoles, where magnetic field lines would terminate, are known to exist.

$$\phi_B = 0$$

$$\int \vec{B} \cdot \overrightarrow{ds} = 0$$

$$\int \text{div } B \, dv = 0$$

$$dv \neq 0$$

$$\text{div } B = 0$$

$$\boxed{\vec{\nabla} \cdot \vec{\mathbf{B}} = \mathbf{0}}$$

This is Maxwell II equation.

Self-Assessment Question (SAQ) 2: Monopole in magnet does not exist. Explain why?

Self-Assessment Question (SAQ) 3: What is physical significance of magnetic Maxwell second equation?

4.5.3 MAXWELL THIRD EQUATION

The third equation contains the result of Faraday's experiments with "electromagnetic induction"—a changing magnetic field creates an electric field, and that electric field circulates around the area experiencing the change in total magnetic flux. (Remember that the curl operator measures the extent to which a vector field runs in circles.) We won't go into the details of his experiments, except to note that he discovered that moving a coil of wire (a loop to pick up a circulating electric field) through a magnetic field (for example, by putting it on a shaft and turning it) led to the invention of electric generators, and hence made a major contribution to the industrialization of the world.

A changing magnetic field induces an electromotive force (e.m.f.) and, hence, an electric field. The direction of the e.m.f. opposes the change. This third of Maxwell's equations, Equation, is Faraday's law of induction and includes Lenz's law. The electric field from a changing magnetic field has field lines that form closed loops, without any beginning or end. This is obtained from Faraday's law of electromagnetic induction.

A/c to Faraday's law the change in magnetic flux induces an EMF.

$$e = -\frac{d\phi B}{dt}$$

$$\phi_B = \int \vec{B} \cdot \overrightarrow{ds}$$

Now,

$$e = -\frac{d}{dt}\int \vec{B} \cdot \overrightarrow{ds}$$

$$e = -\int \frac{dB}{dt}\overrightarrow{ds} \qquad \text{- - - - - ①}$$

By definition of potential gradient, we have –

$$E = \frac{dv}{dl}$$

$$dv = Edl$$

$$dv = \vec{E} \cdot \overrightarrow{dl}$$

$$\left\{ \begin{array}{c} \vec{E} \parallel \overrightarrow{dl} \\ \theta = 0 \\ \vec{E} \cdot \overrightarrow{dl} = Edl\cos\theta \\ = Edl \end{array} \right\}$$

Integrating, we get –

$$\int dv = \int E \cdot \overrightarrow{dl}$$

$$v = \int \vec{E} \cdot \overrightarrow{dl}$$

Potential v = emf e

$$e = \int \vec{E} \cdot \overrightarrow{dl} \qquad \text{- - - - - ②}$$

Now equating ① and ②

$$\int \vec{E} \cdot \vec{dl} = -\int \frac{dB}{dt} \vec{ds}$$

Using Stoke's theorem

$$\int \vec{E} \cdot \vec{dl} = \int \text{curl}\,\vec{E} \cdot \vec{ds}$$

$$\int \text{curl}\,\vec{E} \cdot \vec{ds} = -\int \frac{dB}{dt} \vec{ds}$$

$$\int \left(\text{curl}\vec{E} + \frac{dB}{dt}\right) \vec{ds} = 0$$

ds is arbitrary take

$$ds \neq 0$$

$$\text{curl}\vec{E} + \frac{dB}{dt} = 0$$

$$\text{curl}\vec{E} = -\frac{dB}{dt}$$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}}$$

Self Assessment Question (SAQ) 4: What is the difference between magnetic fields present in Maxwell second and third equations?

4.5.4 MAXWELL FOURTH EQUATION

Magnetic fields are generated by moving charges or by changing electric fields. This fourth of Maxwell's equations, Equation, encompasses Ampere's law and adds another source of magnetic fields, namely changing electric fields.

Maxwell's equations and the Lorentz force law together encompass all the laws of electricity and magnetism. The symmetry that Maxwell introduced into his mathematical framework may not be immediately apparent. Faraday's law describes how changing magnetic fields produce electric fields. The displacement current introduced by Maxwell results instead from a changing electric field and accounts for a changing electric field producing a magnetic field. The equations for the effects of both changing electric fields and changing magnetic fields differ in form only where the absence of magnetic monopoles leads to missing terms. This symmetry between the effects of changing magnetic and electric fields is essential in explaining the nature of electromagnetic waves.

Later application of Einstein's theory of relativity to Maxwell's complete and symmetric theory showed that electric and magnetic forces are not separate but are different manifestations of the same thing—the electromagnetic force. The electromagnetic force and weak nuclear force are similarly unified as the electroweak force. This unification of forces has been one motivation for attempts to unify all of the four basic forces in nature—the gravitational, electrical, strong, and weak nuclear forces.

AmpereCircuital law

$$\int \vec{B} \cdot dl = \mu_0 I$$

$$J = \int \vec{J} \cdot \overrightarrow{ds}$$

Now,

$$\int \vec{B} \cdot dl = \mu_0 \int \vec{J} \cdot dl$$

$$\therefore \vec{B} = \mu_0 \vec{H}$$

$$\int \mu_0 \vec{H} \cdot \overrightarrow{dl} = \mu_0 \int \vec{J} \cdot \overrightarrow{ds}$$

$$\int \vec{H} \cdot \overrightarrow{dl} = \int \vec{J} \cdot \overrightarrow{ds}$$

Stokes theorem

$$\int \vec{H} \cdot \overrightarrow{dl} = \int curl\vec{H} \cdot \overrightarrow{ds}$$

Now,

$$\int curl\vec{H} \cdot \overrightarrow{ds} = \int \vec{J} \cdot \overrightarrow{ds}$$

$$\int \left(curl\vec{H} - \vec{J}\right) \overrightarrow{ds} = 0$$

$$curl\vec{H} - \vec{J} = 0$$

$$curl\ H = J$$

From equation of continuity

$$div\ J + \frac{d\rho}{dt} = 0$$

$$div(curl\vec{H}) + \frac{d\rho}{dt} = 0$$

$$\therefore div(curl\vec{H}) = 0 \qquad \text{...... 1}$$

$$\frac{d\rho}{dt} = 0$$

Equation – (1) hold for steady condition only in which charge density is not changing, that is not function of time. Therefore, for time dependent electric fields (changing) equation – 1 should be modified. Maxwell added an extra term J'

$$curl\vec{H} = \vec{J} + \vec{J}'$$

Taking divergence both side

$$div\ (curl\vec{H}) = div\vec{J} + divJ'$$

$0 = \text{div}\vec{J} + \text{div}J'$

$$\text{div}J' = -\text{div}J$$

$= -\left(-\frac{d\rho}{dt}\right)$ $\qquad \left\{\begin{matrix} \therefore \text{div}J + \frac{d\rho}{dt} = 0 \\ \text{div}J = -\frac{d\rho}{dt} \end{matrix}\right\}$

$$\text{div}J' = \frac{d\rho}{dt}$$

From Maxwell's first equation, we have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$$

$$\epsilon_0 \vec{E} = D$$

$$\vec{\nabla} \cdot \vec{D} = \rho$$

Now,

$$\text{div}\,\vec{J} = \frac{d\rho}{dt}$$

$= \frac{d}{dt}(\vec{\nabla} \cdot \vec{D})$

$$\text{div}\,\vec{J} = \text{div}\frac{d\vec{D}}{dt}$$

Note:

$$q = 0, \;\; \rho = 0, \;\; I = 0, \;\; J = 0$$

$$\vec{J'} = \frac{d\vec{D}}{dt}$$

Now,

$\text{curl}\,\vec{H} = \vec{J} + \vec{J'}$

$$\boxed{\mathbf{curl\,H = J + \frac{dD}{dt}}}$$

When the electric displacement D is changing with time and is, therefore, termed as Displacement current density.

$$\vec{\nabla} \times E = -\frac{dB}{dt}$$

4.6MAXWELL'S EQUATION IN VARIOUS MEDIUM

4.6.1. MAXWELL'S EQUATION IN FREE SPACE

In the special case of free space, where the current density J and volume charge density ρ are zero, Maxwell equation reduces to

$$\vec{\nabla} \cdot \mathrm{D} = 0$$

$$\vec{\nabla} \cdot \mathrm{B} = 0$$

$$\vec{\nabla} \times \mathrm{E} = -\frac{dB}{dt}$$

$$\vec{\nabla} \times \mathrm{H} = -\frac{dD}{dt}$$

4.6.2. MAXWELL EQUATION IN LINEAR ISOTROPIC MEDIA

In linear isotropic media

$$D = \varepsilon E$$

$$H = \frac{B}{\mu}$$

Where e is dielectric constant, μ is permeability of the medium.

The Maxwell's equation becomes –

$$\vec{\nabla} \cdot \mathrm{E} = \frac{\rho}{\varepsilon}$$

$$\vec{\nabla} \cdot \mathrm{H} = 0$$

$$\vec{\nabla} \times \mathrm{E} + \mu\frac{dH}{dt} = 0$$

$$\vec{\nabla} \times \mathrm{H} - \varepsilon\frac{dE}{dt} = J$$

4.6.3. MAXWELL EQUATION FOR HARMONICALLY VARYING FIELD

If we assume that the field vary harmonically with the time, Maxwell equation can be written expressed in another special form.

$$D = D_0 e^{jwt}$$

$$\frac{dD}{dt} = jwD_o e^{jwt}$$

$$= jwD$$

Similarly, we can write

$$\frac{dD}{dt} = jwB$$

The Maxwell's equation becomes

$$\vec{\nabla} \cdot \mathrm{D} = \rho$$

$$\vec{\nabla} \cdot \mathrm{B} = 0$$

$$\vec{\nabla} \times \mathrm{E} + \mathrm{jwB} = 0$$

$$\vec{\nabla} \cdot \mathrm{H} - \mathrm{jwD} = \mathrm{J}$$

1. Using Maxwell equation curlE $= -\frac{dB}{dt}$

$$\mathbf{curl\,H = J + \frac{dD}{dt}}$$

Show that

$$\mathbf{divB = 0 \text{ and } divD = \rho}$$

Or

Are Maxwell equations independent? If no, explain.

Solution: Maxwell equations are not independent because we can derive Maxwell second equation from third and first equation from fourth equation.

$$\mathrm{curlE} = -\frac{dB}{dt}$$

Taking diversion on both side,

$$\mathrm{div\,(\,curl\,E)} = -\mathrm{div}\left(\frac{dB}{dt}\right)$$

$$\mathrm{div\,curlE} = 0$$

$$0 = -\vec{\nabla}.\frac{dB}{dt}$$

$$0 = -\frac{d}{dt}\left(\vec{\nabla} \cdot \mathrm{B}\right)$$

or

$$-\frac{d}{dt}\left(\mathrm{div}\vec{B}\right) = 0$$

If for each point in space divergence divB becomes zero at any time in past or future, then the above equation becomes

$$\boxed{\mathbf{div\vec{B} = 0}}$$

$$\mathrm{curlH} = \mathrm{J} + \frac{dD}{dt}$$

Taking div both side

$$\text{div}(\text{curl}H) = \text{div}J + \text{div}\frac{dD}{dt}$$

$$\text{div curl}H = 0$$

$0 = J + \nabla\left(\frac{dD}{dt}\right)$

-div$J = \frac{d}{dt}\cdot(\vec{\nabla}\vec{D})$

From continuity equation,

$$\text{div}\vec{J} + \frac{d\rho}{dt} = 0$$

$$\text{div}\vec{J} = -\frac{d\rho}{dt}$$

$= -\left(-\frac{d\rho}{dt}\right) = \frac{d}{dt}(\vec{\nabla}\vec{D})$

$\frac{d}{dt}\rho = \frac{d}{dt}(\vec{\nabla}\cdot\vec{D})$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho}$$

2. Using Maxwell relation $\text{div}D = \rho$ and curl $H = J + \frac{dD}{dt}$

Prove that i) $\vec{E} = \frac{1}{\epsilon_0}\frac{qq_0}{r^3}\vec{\mu}$

Coulomb's Law

ii) $\text{div}J + \frac{d\rho}{dt} = 0$Equation of continuity.

Solution:

$$\text{div}D = \rho$$

Integrating w.r to v

$$\int \text{div}D dv = \int \rho dv$$

Volume charge density

$$\rho = \frac{dq}{dv}$$

$$\Rightarrow dq = \rho dv$$

Integrating

$$\int dq = \int \rho dq$$

$$q = \int \rho dv$$

Now,

$\int \text{div } D dv = q$ - - - - - ①

Using Gauss divergence theorem

$$\int \text{div } D dv = \int \vec{D} \cdot \overrightarrow{ds}$$

$= \int \vec{D} \cdot \overrightarrow{ds} = q$

$$\vec{D} = \epsilon_0 \vec{E}$$

$$\int \epsilon_0 \vec{E} \cdot \overrightarrow{ds} = q$$

$$\int \vec{E} \cdot \overrightarrow{ds} = \frac{q}{\epsilon_0}$$

$\int E ds \cos 0 = \frac{q}{\epsilon_0}$ $\left\{ \begin{matrix} \vec{E} \parallel \overrightarrow{ds} \\ \theta = 0^0 \end{matrix} \right\}$

$$E \int ds = \frac{q}{\epsilon_0}$$

$$E \times 4\pi r^2 = \frac{q}{\epsilon_0}$$

$\int ds$ = Area of sphere of radius

$$r = 4\pi r^2$$

$$E = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2}$$

$E = \frac{F}{q_0}$ $\{q_0 \text{ test charge}\}$

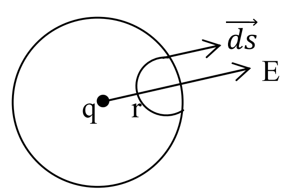
$$F = q_0 E$$

$F = \frac{1}{4\pi\epsilon_0} \frac{q q_0}{r^2}$

$$\vec{F} = F\hat{r}$$

$$\hat{r} = r\hat{r}$$

$$\hat{r} = \frac{\vec{r}}{r}$$



$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q q_0}{r^2} \times \frac{\vec{r}}{r}$

$$\boxed{\vec{\mathbf{F}} = \frac{\mathbf{1}}{\mathbf{4\pi \epsilon_0}} \frac{\mathbf{q q_0}}{\mathbf{r^3}} \vec{\mathbf{r}}}$$

ii. $\text{curl}\vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$

takingdiv

$$\text{div curl}\vec{H} = \text{div}J + \text{div}\,\frac{d\vec{D}}{dt}$$

$$O = \text{div}\vec{J} + \vec{\nabla}\cdot\frac{d\vec{D}}{dt}$$

$$O = \text{div}\vec{J} + \frac{d}{dt}\left(\vec{\nabla}\cdot\vec{D}\right)$$

or

$$\text{div}\vec{J} + \frac{d}{dt}\left(\text{div}\vec{D}\right) = 0$$

$$\boxed{\mathbf{div}\vec{\mathbf{J}} + \frac{d}{dt}\boldsymbol{\rho} = \mathbf{0}}$$ {from using Maxwell first equation $\text{div}D = \rho$}

Example: 2

Starting from the equation of continuity show, for a conducting medium obeying ohm's law J = σE and using Gauss law that

Solution: a. $\frac{d\rho}{dt} + \frac{\sigma}{E}$

Equation of continuity

$$\vec{\nabla}\,.\,\vec{J} + \frac{d\rho}{dt} = 0$$

$$\vec{J} = \sigma\vec{E}$$

$$\vec{\nabla}\,.\,\sigma\vec{E} + \frac{d\rho}{dt} = 0$$

$$\sigma\left(\vec{\nabla}\,.\,\vec{E}\right) + \frac{d\rho}{dt} = 0$$

$\therefore\ \vec{\nabla}\,.\,\vec{E} = \frac{\rho}{\epsilon}$ { maxwell 1^{st}eqtn}

$$\sigma \times \frac{\rho}{\epsilon} + \frac{d\rho}{dt} = 0$$

b. $$\frac{d\rho}{dt} + \sigma\frac{\rho}{\epsilon} = 0$$

$$\frac{d\rho}{dt} = \sigma\frac{\rho}{\epsilon}$$

$$\frac{d\rho}{dt} = \frac{\sigma}{\epsilon}dt$$

Initially $t = 0 \quad \rho = \rho$

$t = t \quad \rho = \rho^0$

Integrating,

$$\int_{\rho\,=\,\rho^0}^{\rho\,=\,\rho^0}\frac{d\rho}{\rho} = -\,\frac{\sigma}{\epsilon}\int_{t\,=\,0}^{t\,=\,t}dt$$

$$[\log \rho]_{\rho_0}^{\rho} = -\frac{\sigma}{\epsilon} t'$$

$$\log \rho - \log \rho_0 = -\frac{\sigma}{\epsilon} t$$

$$\log\left(\frac{\rho}{\rho_0}\right) = -\frac{\sigma}{\epsilon} t$$

Taking antilog:

$$\frac{\rho}{\rho_0} = e^{-\frac{\sigma t}{\epsilon}}$$

$$\boxed{\boldsymbol{\rho_0 = \rho_0\, e^{-\frac{\sigma t}{\epsilon}}}}$$

4.7 SUMMARY

In this unit, you have studied introduction to electrodynamics. To present the clear understanding of electrodynamics, Maxwell equations have been discussed in details. You have studied how Maxwell equations are derived and basic understanding electrodynamics. You have also studied interrelations of Maxwell equations, continuity equation and its physical significance, Coulomb's law from Maxwell equation, continuity equation from Maxwell equation, independence of Maxwell equations, concept of displacement current, Maxwell equations in free space.

Maxwell first equation is Gauss law in electrostatics, Maxwell second equation is Gauss law in magneto statics, Maxwell third equation is Farade's law of electromagnetic induction, Maxwell fourth equation is Ampere – Maxwell law which is modification of ampere circuital law. Many solved examples are given in the unit to make the concepts clear. To check your progress, self-assessment questions (SAQs) are given place to place.

4.8 GLOSSARY

Divergence- a deviation from standards

Continuity-the fact of continuing without stopping or of staying the same

Permeability - is the quality or state of being permeable—able to be penetrated or passed through

4.9 REFERENCES

1.Introduction to Electrodymics David J. Griffiths

2. Classical Electrodynamics John David Jackson

3. Principle of electrodynamics by Melvin Schwartz

4. Electrodynamics by Gupta Kumar

4.10 SUGGESTED READINGS

1. YOU TUBE

2. National Programme on Technology Enhanced Learning (**NPTEL**) Lectures

4.11 TERMINAL QUESTIONS

(Should be divided into Short Answer type, Long Answer type, Numerical, Objective type)

4.11.1 Short Answer type

1. Derive Gauss law in differential form?

2. What is physical significance of Maxwell first equation?

3. Derive coulomb's law from Maxwell first equation.

4. Derive continuity equation from Maxwell fourth equation.

4.11.2 Long Answer type

1. Derive equation of continuity. What are physical significances of it?

2. What is meant by Displacement current? How Maxwell modified Ampere's circuital law.

3. Derive Maxwell equations. Write physical significance of these equations.

4.11.3Numerical Answer type

1. Find the total electric flux through a closed cylinder containing a line charge along its axis with linear charge density $\lambda = \lambda_0(1 - \frac{x}{\lambda})\frac{c}{m}$ if the cylinder and the line charge extend from x=0 to x=h.

2. What is the flux through any closed surface surrounding a charged sphere of radius a0 with volume charge density of $\rho = \rho_0 \left(\frac{r}{a_0}\right)$where r is the distance from center of the sphere?

3.Given the vector field

$$\vec{A} = \cos(\pi y - \frac{\pi}{\alpha})\hat{\imath} + \sin(\pi x)\,\hat{\jmath}$$

Sketch the field lines and divergence of the field.

4. Find the charge density in a region for which electric field in spherical coordinates is given by $\vec{E} = ar^2\hat{r} + \frac{bcos\theta}{r}\hat{\theta} + c\hat{\emptyset}$.

5. A square containing loop with sides of length L rotates so that the angle between the normal to the plane of loop and a fixed magnetic field **B** varies as $\theta = \theta(t) + \theta_0\left(\frac{t}{t_0}\right)$ find the emf induced in the loop.

4.12 OBJECTIVE TYPE QUESTIONS

1. Maxwell's equations, were arrived at mostly through various experiments carried out by different investigators, but they were put in their final form by

A. James Clerk Maxwell
B. Thomas Clerk Maxwell
C. Charles Drawin
D. Stephen Hawking

2. Maxwell's equations can be written in

A. integral form
B. differential form
C. logical form
D. either in integral or differential form

3. Relationship of electric and magnetic field is governed by physical laws, which are known as

A. Kirchhoff's Equations
B. Millman's Equations
C. Maxwell's Equations
D. Arithmetic Equations

4. In Maxwell's equation, ∇·B=0, which term is basically zero

A. ρ_{ev} (electrical)
B. ρ_{mv} (magnetic)
C. σ_{ev} (electrical)
D. σ_{mv} (magnetic)

5. Dielectric constitutive relationship is

A. D=∊
B. D=∊HE
C. D=∊E
D. D=∊H

6. $\nabla \cdot D = \rho_v$, s termed as

A. Faraday's Law of Induction
B. Ampere's Law
C. Gauss's Law (electrical)
D. Gauss's Law for magnetism

7. In Maxwell's equation, $\nabla \cdot D = \rho_{ev}$; D is

A. electric flux density
B. magnetic flux density
C. magnetic field intensity
D. electric field intensity

8. In harmonic plane general equation, $A(x,t)=A_o\cos(kx-wt+\varphi)$, k is

A. magnitude of wave disturbance
B. amplitude
C. wave number
D. angular frequency

9. Law which governs interaction of electromagnetic field with charged matter.

A. Gauss's Law
B. Faraday's Law
C. Ampere's Law
D. Lorentz force Law

10. In harmonic plane general equation, $A(x,t)=A_o\cos(kx-wt+\varphi)$, '$A_o$' is

A. magnitude of wave disturbance
B. amplitude
C. wave number
D. angular frequency

11. $\nabla x H = J + \partial D/\partial t$ is modification of

A. Faraday's Law of Induction

B. Ampere's Law
C. Gauss's Law (electrical)
D. Gauss's Law for magnetism

12. In harmonic plane general equation, $A(x,t)=A_o\cos(kx-wt+\varphi)$, φ represents

A. magnitude of wave disturbance
B. amplitude
C. phase shift
D. angular frequency

13. In Maxwell's equation $\nabla xH=J+\partial D/\partial t$, J is

A. electric flux density
B. magnetic flux density
C. surface current density
D. No physical quantity

1.A 2.D 3.C 4.D 5C. 6.C 7.D 8.C 9.D 10.B 11.B 12.C 13.C

UNIT 5: ELECTROMAGNETIC WAVES

STRUCTURE

5.1 INTRODUCTION

In this chapter, we will discuss Maxwell equation in free space condition, in non-conducting medium, conducting medium. With the help of Maxwell equation, we will derive Electromagnetic wave equation in free space, conducting medium, in non-conducting medium. Electromagnetic waves carry energy with it, the amount of energy carried by it will be discussed in terms of Pointing theorem.

5.2OBJECTIVES

- To learn aboutwave equation for free space condition
- To learn aboutPlane Electromagnetic Waves in Free Space
- To learn aboutEnergy flow due to a plane EM wave
- To discuss plane Electromagnetic Waves in conducting media
- To discuss Pointing theorem.

5.3 WAVE EQUATION FOR FREE SPACE CONDITION

We have Maxwell equations

$$\left.\begin{aligned} &\nabla\cdot\vec{D} = \rho &&\text{-----(a)}\\ &\vec{\nabla}\cdot\vec{B} = 0 &&\text{-----(b)}\\ &\vec{\nabla}\cdot\vec{E} = -\frac{dB}{dt} &&\text{-----(c)}\\ &\vec{\nabla}\cdot\vec{H} = J + \frac{dD}{dt} &&\text{-----(d)}\end{aligned}\right\} \quad \text{- - - (5.1)}$$

In free space there is no charge and no current $q = 0$

$$I = 0 \quad \rho = 0 \quad J = 0$$

Now Maxwell equation in free space conditions can be written as –

$$\left.\begin{array}{ll} \vec{\nabla}\cdot\vec{D} = 0 & \text{- - - - - (a)} \\ \vec{\nabla}\cdot\vec{B} = 0 & \text{- - - - - (b)} \\ \vec{\nabla}\times\vec{E} = -\frac{dB}{dt} & \text{- - - - - (c)} \\ \vec{\nabla}\times\vec{H} = \frac{dD}{dt} & \text{- - - - - (d)} \end{array}\right\} \quad \text{- - -(5.2)}$$

Now taking curl of equation – (5.2c)

$$\vec{\nabla}\times\vec{E} = -\frac{dB}{dt}$$

$$\text{curl}\left(\vec{\nabla}\times\vec{E}\right) = \text{curl}\left(-\frac{d\vec{B}}{dt}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\times\left(-\frac{dB}{dt}\right)$$

Using formula

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}$$

$$\left(\vec{\nabla}\cdot\vec{E}\right)\vec{\nabla} - \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{E} = -\vec{\nabla}\times\frac{d\vec{B}}{dt}$$

$$O - \nabla^2 E = -\frac{d}{dt}\left(\vec{\nabla}\times\vec{B}\right) \qquad \{\text{from equation (5.2a)}\}$$

$$-\nabla^2 E = -\frac{d}{dt}\left(\vec{\nabla}\times\mu_0\vec{H}\right)$$

$$\vec{\nabla}^2 E = -\frac{d}{dt}\left(\vec{\nabla}\times\mu_0\vec{H}\right)\left\{\begin{array}{c} \vec{\nabla}\cdot\vec{D} = 0 \\ \vec{\nabla}\cdot\epsilon_0\,\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{\nabla} = \nabla^2 \end{array}\right\}$$

$$\vec{\nabla}^2 E = \mu_0\frac{d}{dt}\left(\vec{\nabla}\times\vec{H}\right)$$

$$\vec{\nabla}^2 E = \mu_0\frac{d}{dt}\left(\frac{dD}{dt}\right) \qquad \text{from free space}\left\{\vec{\nabla}\times\vec{H} - \frac{dD}{dt}\right\}$$

$$\mu_0\frac{d}{dt}\left(\frac{d}{dt}\epsilon_0\,E\right) \qquad \{O = \epsilon_0\,E\}$$

$$\vec{\nabla}^2 \boldsymbol{E} = \boldsymbol{\mu_0} \in_0 \frac{\boldsymbol{d^2E}}{\boldsymbol{dt^2}}$$

or

$$\vec{\nabla}^2 \boldsymbol{E} = \boldsymbol{\mu_0} \in_0 \frac{\boldsymbol{d^2E}}{\boldsymbol{dt^2}} = \boldsymbol{0} \qquad \text{- - - -(5.3)}$$

This is the wave equation in free space for electric component in EM wave.

Similarly, by taking curl of equation (5.2a), we can show that$\vec{\nabla}^2 H = \mu_0 \in_0 \frac{d^2H}{dt^2}$

$$\vec{\nabla}^2 \boldsymbol{H} = \boldsymbol{\mu_0} \in_0 \frac{\boldsymbol{d^2H}}{\boldsymbol{dt^2}} = \boldsymbol{0} \qquad \text{- - - -(5.4)}$$

This is the wave equation in free space for magnetic component in EM wave.

Now comparing equation (5.3) and (5.4) with wave equation,

$$\frac{d^2y}{dz^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dz^2} - \frac{1}{v^2}\frac{d^2y}{dt^2} = 0$$

or

$$\nabla^2 y - \frac{1}{v^2}\frac{d^2y}{dt^2} = 0$$

$$\frac{1}{v^2} = \mu_0 \in_0$$

$$v = \frac{1}{\sqrt{\mu_0 \in_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}}$$

$= 2.99 \times 10^8 m/s \cong 3 \times 10^8 m/s = \mathrm{C}$ (Speed of light)

Now from equation (5.3) and (5.4), subst.

$\nabla^2 E - \frac{1}{C^2}\frac{d^2E}{dt^2} = 0$and

$$\nabla^2 H - \frac{1}{C^2}\frac{d^2H}{dt^2} = 0$$

SAQ: 1 Show that there exists electromagnetic wave in space and they travel in free space with velocity of light?

SAQ: 2 Show that the wave equation can be written in the form

$$(\nabla^2 + k^2)E = 0$$

where k = $\frac{2\pi}{\lambda}$ is called wave vector or propagation constant?

5.3.1. PLANE ELECTROMAGNETIC WAVES IN FREE SPACE

An electromagnetic wave, whose field vectors are the functions of only one space coordinate and the time coordinate is said to be a uniform plane wave. Let us consider such waves that are propagating in z-directions, so that $\vec{E}$ and $\vec{H}$ will vary only in the direction of z-axis, i.e.

$$\frac{dx}{dy} = \frac{d}{dy} = 0, \qquad \frac{d}{dz} \neq 0$$

and we write

$$\left.\begin{array}{l} \vec{E} = \vec{E}(x,t) \\ \vec{H} = \vec{H}(x,t) \end{array}\right\} \qquad \text{- - - -(5.5)}$$

then now from equation

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \epsilon_0 \, \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right) \cdot \left(Ex\hat{\imath} + Ey\hat{\jmath} + Ez\hat{k}\right) = 0$$

$$\frac{d}{dx}E_x + \frac{d}{dy}E_y + \frac{d}{dz}E_z = 0$$

$$0 + 0 + \frac{dE_z}{dz} = 0$$

$$\frac{dE_z}{dz} = 0$$

(E_Z = const. in space)

or

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \mu_0 \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right) \cdot \left(Hx\hat{\imath} + Hy\hat{\jmath} + Hz\hat{k}\right) = 0$$

$$\frac{d}{dx}H_x + \frac{d}{dy}H_y + \frac{d}{dz}H_z = 0$$

$\frac{d\mathrm{H}_z}{dz} = 0$

[Hz = constant in space]

Also,

$\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}$

$$\left(\vec{\nabla} \times \vec{E}\right)z = -\mu_0 \frac{dHz}{dt}$$

$$\therefore \vec{\nabla} \times \vec{E} = \begin{bmatrix} ij\hat{k} \\ \frac{d}{dx}\frac{d}{dy}\frac{d}{dz} \\ ExEyEz \end{bmatrix}$$

$= \hat{\imath}\left(\frac{d}{dy}Ez - \frac{d}{dz}Ey\right) - j\left(\frac{d}{dx}Ex - \frac{d}{dx}Ex\right) + \hat{k}\left(\frac{d}{dx}Ey - \frac{d}{dy}Ex\right)$

$\left(\vec{\nabla} \times \vec{E}\right)z = \hat{k}\left(\frac{d}{dx}Ey - \frac{d}{dy}Ex\right)$

$$\left(\vec{\nabla} \times \vec{E}\right)z = 0$$

$0 = -\mu_0 \frac{dHz}{dt}$

$\frac{dHz}{dt} = 0$ [Hz = const. in time]

or

Similarly

[Ez = const. in time]

Thus we conclude that Ez and Hz remains constant as regards to time and space. Thus they represent static components and consequently no part of wave motion. We can therefore write –

Ez = Hz = 0

So that

$$\left.\begin{aligned} E &= iEx + jEy \\ H &= iHx + jHy \end{aligned}\right\} \qquad \text{- - - -(5.6)}$$

Since the electric vector $\vec{E}$ and the magnetic vector $\vec{H}$ do not have any z component, the z-direction being the direction of propagation, both these vector are perpendicular to the direction of propagation Maxwell's electromagnetic wave are, therefore purely transverse.E and H vectors for a plane electromagnetic wave travelling in positive Z direction, E and H are in phase but perpendicular to each other.

5.4 **PLANE EM WAVE IN MATTER:**

The propagation of plane EM wave in homogenous, isotropic linear source free and stationery media, whether conducting or non-conducting.

If the constitutive relations:

$$\left.\begin{aligned} \mathrm{D} &= \in \mathrm{E} \\ \mathrm{B} &= \mu\mathrm{H} \\ \mathrm{J} &= \sigma\mathrm{E} \end{aligned}\right\} \qquad \text{- - - (5.7)}$$

Hold good, then the medium, is said to be isotropic and linear further a medium is stationary if it is at rest w.r.t. the coordinate system used.

$$\left.\begin{aligned} &\vec{\nabla}\cdot D = \rho \\ &\vec{\nabla}\cdot B = 0 \\ &\vec{\nabla}\times\vec{E} = -\frac{dB}{dt} \\ &\vec{\nabla}\times\vec{H} = J + \frac{d\vec{D}}{dt} \end{aligned}\right\} \qquad \text{- - - (5.8)}$$

Now taking curl of equation - (5.2c)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) + \vec{\nabla}\left(-\frac{\mathrm{d}\vec{B}}{\mathrm{dt}}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \frac{d}{dt}(\vec{\nabla} \times \vec{B}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \frac{d}{dt}(\vec{\nabla} \times \mu\vec{H}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \mu\frac{d}{dt}\left(\vec{J} + \frac{d\vec{D}}{dt}\right) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \mu\frac{d}{dt}\left(\sigma\vec{E} + E\frac{d\vec{E}}{dt}\right) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + E\mu\frac{d^2\vec{E}}{dt^2} + \sigma\mu\frac{d\vec{E}}{dt} = 0 \qquad \text{- - - (5.9)}$$

We know

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})$$

$$= \vec{\nabla} \times (\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}\left(\frac{\rho}{\in}\right) - \overrightarrow{\nabla^2}\vec{E}$$

Now

$$\vec{\nabla}\left(\frac{\rho}{\in}\right) - \overrightarrow{\nabla^2}\vec{E} + \in\mu\frac{d^2\vec{E}}{dt^2} + \sigma H\frac{d\vec{E}}{dt} = 0$$

$$\boldsymbol{\nabla^2\vec{E} = \in\mu\frac{d^2\vec{E}}{dt^2} + \sigma\mu\frac{d\vec{E}}{dt} + \vec{\nabla}\left(\frac{\rho}{\in}\right)} \qquad \text{- - - (5.10)}$$

Similarly, we can obtain an equation for the Magnetic field $\vec{H}$ by taking the curl of equation (5.2d) and using equation (5.2c) and (5.1). The equation will be:

$$\boldsymbol{\nabla^2 H = \in\mu\frac{d^2\vec{H}}{dt^2} + \sigma\mu\frac{d\vec{H}}{dt}} \qquad \text{- - - -(5.11)}$$

Let us consider a wave propagating in z-direction such that all derivative of E and H with respect to x and y are zero.

$$\frac{d}{dx} = \frac{d}{dy} = 0$$

$$\frac{d}{dz} \neq 0$$

We have, $\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$

$$\vec{\nabla} = \hat{k}\frac{d}{dz}$$

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

$$\nabla^2 = \frac{d^2}{dz^2}$$

Substituting in equation – (5.10)

$$\nabla^2\vec{E} = \in \mu\frac{d^2\vec{E}}{dt^2} + \sigma\mu\frac{d\vec{E}}{dt} + \vec{\nabla}\left(\frac{\rho}{\in}\right)$$

$$\frac{d^2\vec{E}}{dz^2} = \in \mu\frac{d^2\vec{E}}{dt^2} + \sigma\mu\frac{d\vec{E}}{dt} + \hat{k}\frac{d}{dz}\left(\frac{\rho}{\in}\right)$$

or

$$\frac{d^2\vec{E}}{dz^2} = \left(\in \mu\frac{d^2}{dt^2} + \sigma\mu\frac{d}{dt}\right)\vec{E} + \hat{k}\frac{d}{dz}\left(\frac{\rho}{\in}\right)$$

$$\frac{d^2}{dz^2}\left(\hat{\imath}Ex + jEy + \hat{k}Ez\right) = \left(\in \mu\frac{d^2}{dt^2} + \sigma\mu\frac{d^2}{dt}\right)(iEx + \hat{\jmath}Ey + kEz) + \hat{k}\frac{d}{dz}\left(\frac{\rho}{\in}\right)$$

Now field equation –

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\in}$$

$$\left(\hat{\imath}\frac{d}{dx} + j\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\cdot\left(\hat{\imath}Ex + \hat{\jmath}Ey + \hat{k}Ez\right) = \frac{\rho}{\in}$$

$$\frac{d}{dz}(Ez) = \frac{\rho}{\in}$$

But,

$$\hat{k}\frac{d}{dz}\left(\frac{\rho}{\in}\right) = \hat{k}\frac{d}{dz}\left(\frac{d}{dz}Ez\right)$$

$$\hat{k}\frac{d}{dz}\left(\frac{\rho}{\in}\right) = \frac{d^2}{dz^2}\left(\hat{k}Ez\right)$$

Substituting these value in equation – (5.11)

$$\frac{d^2}{dz^2}(iEx + jEy) + \frac{d^2}{dz^2}\left(\hat{k}Ez\right) = \left(\in \mu\frac{d^2}{dt^2} + \sigma\mu\frac{d^2}{dt}\right)\left(\hat{\imath}Ex + \hat{\jmath}Ey + \hat{k}Ez\right) + \frac{d^2}{dz^2}\left(\hat{k}Ez\right)$$

$$\frac{d^2}{dz^2}(iEx + jEy) = \mu\left(\in\frac{d^2}{dt^2} + \frac{\sigma d}{dt}\right)(\hat{\imath}Ex + \hat{\jmath}Ey + \hat{k}Ez)$$

Writing the above equation only for example we have

$$\mu \in \frac{d^2Ez}{dt^2} + \mu\sigma\frac{dEz}{dt} = 0$$

$\boldsymbol{\in \frac{d^2Ez}{dt^2} + \sigma\frac{dEz}{dt} = 0}$.

5.4.1. RELATIVE ORIENTATION OF E AND H VECTORS IN A PLANE WAVE:

From equation

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\frac{d}{dx} = 0 \quad \frac{d}{dy} - 0 \quad \frac{d}{dz} \neq 0$$

$$Ez = 0$$

$$Hz = 0$$

So, $Bz = \mu Hz = 0$

$$Bz = 0$$

Now,

$$\begin{vmatrix} \hat{\imath}\hat{\jmath}\hat{k} \\ \frac{d}{dx}\frac{d}{dy}\frac{d}{dz} \\ ExEyEz \end{vmatrix}$$

$=\frac{-d}{dt}(\hat{\imath}Bx + \hat{\jmath}By + \hat{k}Bz)$

$$\begin{vmatrix} & \hat{\imath}\hat{\jmath}\hat{k} & \\ 0 & 0 & \frac{d}{dz} \\ ExEy & & 0 \end{vmatrix}$$

$= \frac{-d}{dt}(\hat{\imath}Bx + \hat{\jmath}By)$

Let us suppose that the sinusoidal wave of angular frequency w is propagating in z-direction, so that we can write

$E = \in_0 e^{j(\omega t - kx)}$ - - - - (5.12)

$H = H_0 e^{j(\omega t - kx)}$ - - - - (5.13)

Differential equation (5.12) war to z

$$\frac{dE}{dz} = -kj \in_0 e^{j(\omega t - kx)}$$

$$\frac{dE}{dz} = -kjE$$

$\boldsymbol{\frac{d}{dz} = -kj}$ - - - - - (5.14)

Now diff. equation (5.12) war to t

$$\frac{dE}{dt} = \omega j \in_0 ej^{(\omega t - kx)}$$

$\boldsymbol{\frac{d}{dt} = \omega j}$ - - - - - (5.15)

Now

$\begin{vmatrix} & \hat{\imath}\hat{\jmath}\hat{k} & \\ 0 & 0 & -jk \\ ExEy & & 0 \end{vmatrix} = -j\omega(\hat{\imath}Bx + jBy)$

$$\hat{\imath}(0 + jkEy) - \hat{\jmath}(jkEx) + \hat{k}(0)$$

$= j\omega(\hat{\imath}Bx + \hat{\jmath}By)$

Comparing coefficient of $\hat{\imath}$ and $\hat{\jmath}$

$$jkEy = -j\omega Bx$$

$$jkEy = -j\omega\mu Hx$$

$$-jkEx = -j\omega By$$

$$\Rightarrow jkEx = jw\mu Hy$$

$$-\frac{Ey}{Hx} = \frac{Ex}{Hy} = \frac{\omega\mu}{k} \qquad \text{- - - (5.16)}$$

Now we take the dot product of E and H

$$\vec{E}\cdot\vec{H} = ExHx + EyHy$$

$$= \frac{\omega\mu}{k}HyHx + \left(-\frac{\omega\mu}{k}HxHy\right)$$

$$\vec{E}\cdot\vec{H} = 0$$

Hence $\vec{E}$ and $\vec{H}$ are mutually perpendicular.

5.5. PLANE EM WAVE PROPAGATION ISOTROPIC DIELECTRIC (NON CONDUCTING MEDIA)

Maxwell field equations: -

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$with\, J = \sigma E$$
$$B = \mu H$$
$$D = \epsilon E$$

$$\vec{\nabla}\times\vec{H} = J + \frac{d\vec{D}}{dt}$$

For non conducting media, we note that

(i) conductivity σ would be zero, so that

$$J = \sigma E = 0$$

(ii) medium is taken as isotropic due to which there would be no volume distribution of charge. Therefore
$\rho = 0$.

Apply these conditions.

Maxwell equations for non-conducting medium:

$$\vec{\nabla}\cdot\vec{D}=0 \qquad \vec{\nabla}\cdot\vec{E}=0 \qquad \text{----- (5.17)}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\cdot\vec{H}=0 \qquad \text{----- (5.18)}$$

or

$$\vec{\nabla}\times\vec{E}=-\frac{d\vec{B}}{dt}\vec{\nabla}\times\vec{E}=-\mu\frac{d\vec{H}}{dt} \qquad \text{----- (5.19)}$$

$$\vec{\nabla}\times\vec{H}=\frac{d\vec{D}}{dt}\vec{\nabla}\times\vec{H}=\in\frac{d\vec{E}}{dt} \qquad \text{----- (5.20)}$$

a. Equation of propagation of magnetic vector H –

Taking curl of equation - (5.20)

$$\vec{\nabla}\times(\vec{\nabla}\cdot\vec{H})=\vec{\nabla}\times\left(\in\frac{d\vec{E}}{dt}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{H})-(\vec{\nabla}\cdot\vec{\nabla})\vec{H}=\in\frac{d}{dr}(\vec{\nabla}\times\vec{E})$$

$$0-\nabla^2\vec{H}=\in\frac{d}{dt}\left(-\frac{d\vec{B}}{dt}\right)$$

$$\nabla^2\vec{H}=\in\frac{d^2\vec{B}}{dt^2}$$

$$\nabla^2\vec{H}=\mu\in\frac{d^2\vec{H}}{dt^2}$$

$$\boldsymbol{\nabla^2\vec{H}=\frac{1}{v^2}\frac{d^2\vec{H}}{dt^2}} \qquad \text{- - - -5.21}$$

$$\therefore\ \frac{1}{v}=\sqrt{\mu\in}$$

$$\frac{1}{v^2}=\mu\in$$

Equation – (5.21) is the wave equation for magnetic vector $\vec{H}$.

i. Equation of propagation electric vector E:

Taking curl of equation (2c)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}\times\left(-\mu\frac{d\vec{H}}{dt}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-(\vec{\nabla}\cdot\vec{\nabla})\vec{E}=-\mu\frac{d}{dt}(\vec{\nabla}\times\vec{H})$$

$$0-\nabla^2\vec{E}=-\mu\frac{d}{dt}\left(\in\frac{d\vec{E}}{dt}\right)$$

$$\nabla^2\vec{E} = \mu \in \frac{d^2\vec{E}}{dt^2}$$

or

$$\nabla^2\vec{E} = \frac{1}{v^2}\frac{d^2\vec{E}}{dt^2} \qquad \text{- - - - 5.22}$$

Which is the wave equation for electric vector $\vec{E}$

Suppose the EM wave is propagating in z-direction, then

$\frac{d}{dx} = \frac{d}{xy} = 0$so that $\vec{\nabla} = k\frac{d}{dz}$ and $\nabla^2 = \frac{d^2}{dz^2}$

We get from equation (5.21) and (5.22)

$$\left.\begin{aligned} \frac{d^2H}{dz^2} - \frac{1}{v^2}\frac{d^2H}{dt^2} = 0 \qquad \text{- - - - - (a)} \\ \frac{d^2E}{dz^2} - \frac{1}{v^2}\frac{d^2E}{dt^2} = 0 \qquad \text{- - - - - (b)} \end{aligned}\right\} \qquad \text{- - - - - (5.23)}$$

Now solution of these equations are

$$\left.\begin{aligned} E = E_0e^{j(\omega t-kz)} \qquad \text{- - - - - (a)} \\ H = H_0e^{j(\omega t-kz)} \qquad \text{- - - - - (b)} \end{aligned}\right\} \qquad \left\{where\ k = \frac{2\pi}{\lambda}\right\}$$

Differential equation (a) w.r.t. z and t, we get

$$\frac{dE}{dz} = E_0e^{j(\omega t-kz)}(-jk) = -Ejk$$

$$\frac{dE}{dz} = -Ejk$$

and$\frac{dE}{dt} = E_0e^{j(\omega t-kz)}(j\omega) = Ej\omega$

$$\Rightarrow \frac{d}{dz} = -jk$$

$$\frac{d}{dt} = j\omega$$

5.6. PROPAGATION OF PLANE EM WAVE IN CONDUCTING MEDIA

The material do have net conductivity σ which accounts for the loss of energy in the media. It is further assumed that the materials are homogeneous, linear, isotropic, unbounded and stationary. If such a material contains no free volume charge and no other current except the one determined by Ohm's law, then Maxwell's equation in such a material are given by –

$\vec{\nabla} \cdot D = \rho$ ----- (a)

$\vec{\nabla} \cdot B = \rho$ ----- (b)

$\vec{\nabla} \times H = J + \frac{dD}{dt}$ ----- (c)

$\vec{\nabla} \times E = -\frac{dB}{dt}$ ----- (d)

$$with \begin{cases} J = \sigma E \\ B = \mu H \\ D = \in E \end{cases}$$

From these relations, we get:

$$\vec{\nabla} \times (\vec{\nabla} \times E) + \left(\vec{\nabla} \times \frac{dB}{dt}\right) = 0$$

$$\vec{\nabla}(\vec{\nabla} \cdot E) - \overrightarrow{\nabla^2} E + \frac{d}{dt}(\vec{\nabla} \times \mu H) = 0$$

$$\vec{\nabla}\left(\frac{\rho}{\in}\right) - \nabla^2 E + \mu \frac{d}{dt}\left(\sigma E + \in \frac{dE}{dt}\right) = 0$$

$$\vec{\nabla}\left(\frac{\rho}{\in}\right) - \nabla^2 E + E\mu \frac{d^2 E}{dt^2} + \sigma\mu \frac{dE}{dt} = 0$$

$$\nabla^2 E = E\mu \frac{d^2 E}{dt^2} + \sigma\mu \frac{dE}{dt} + \vec{\nabla}\left(\frac{\rho}{\in}\right)$$

$$\vec{\nabla}\left(\frac{\rho}{\in}\right) = 0$$

$$\nabla^2 \vec{E} = \mu\sigma \frac{d\vec{E}}{dt} + \mu \in \frac{d^2 \vec{E}}{dt^2}$$

$$\nabla^2 \vec{E} = \mu\sigma \frac{d\vec{E}}{dt} + \mu \in \frac{d^2 \vec{E}}{dt^2} \quad ---- (5.24)$$

Similarly, we get obtain:

$$\nabla^2 \vec{H} = \mu\sigma \frac{d\vec{H}}{dt} - \mu \in \frac{d^2 \vec{H}}{dt^2} \quad ---- (5.25)$$

5.7. EQUATION OF MOTION (WAVE EQUATION)

Let us consider the plane polarized EM wave is propagating in z-direction with E in x-direction and H in y-direction

So $\frac{d}{dx} = \frac{d}{dy} = 0, \quad \frac{d}{dz} \neq 0$

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

$$\vec{\nabla} = \hat{k}\frac{d}{dz}$$

$$\vec{\nabla}\cdot\vec{\nabla} = \hat{k}\frac{d}{dz}\cdot\hat{k}\frac{d}{dz}$$

$$\nabla^2 = \frac{d^2}{dz^2}$$

Substituting these values in equation (5.24)

$$\frac{d^2\vec{E}}{dz^2} - \mu\sigma\frac{d\vec{E}}{dt} + \mu \in \frac{d^2\vec{E}}{dt^2} = 0$$

or

$$\frac{d^2\vec{E}}{dz^2} - \mu \in \frac{d^2\vec{E}}{dt^2} - \mu\sigma\frac{d\vec{E}}{dt} = 0$$

or since

$$\vec{E} = Ex\hat{\imath}$$

$$\vec{H} = Hy\hat{\jmath}$$

Now,

$$\frac{d^2Ex}{dz^2} - \mu \in \frac{d^2Ex}{dt^2} - \mu\sigma\frac{dEx}{dt} = 0 \qquad \text{- - - - (5.26)}$$

Similarly, for magnetic field

$$\frac{d^2Hy}{dz^2} - \mu \in \frac{d^2Hy}{dt^2} - \mu\sigma\frac{dHy}{dt} = 0 \qquad \text{- - - - (5.27)}$$

Let electric component Ex is function of time only. Let it varies with time as

$$Ex = E_0 e^{j\omega t}$$

$$\frac{dEx}{dt} = E_0 e^{jwt}\frac{d}{dt}(j\omega t)$$

$$\frac{dEx}{dt} = j\omega E_0 e^{j\omega t}$$

$= j\omega Ex$

$$\frac{d^2Ex}{dt^2} = j\omega E_0 e^{j\omega t} \times j\omega$$

$= j^2\omega^2 E_0 e^{j\omega t}$

$\frac{d^2Ex}{dt^2} = -\omega^2 Ex \qquad \{j^2 = -1\}$

Now,

$$\frac{d^2Ex}{dz^2} - \mu \in \frac{d^2Ex}{dt^2} - \mu\sigma\frac{dEx}{dt} = 0$$

$$\frac{d^2Ex}{dz^2} + \mu \in \omega^2 Ex - j\mu\sigma\omega Ex = 0$$

$$\frac{d^2Ex}{dz^2} - (j\mu\sigma\omega - \mu \in^2)Ex \ = 0$$

or

$$\boldsymbol{\frac{d^2Ex}{dz^2} - \gamma^2 Ex \ = 0} \qquad \text{- - - -(5.28)}$$

Wave equation for electric component

where, $j\mu\sigma\omega - \mu \in \omega^2 = \gamma^2$ is called propagation constant.

5.7.1. PHASE VELOCITY AND REFRECTIVE INDEX

The solution of equation (5.28) is

$$Eu = Eoe^{-\gamma z}$$

$$\gamma^2 = j\mu\sigma\omega - \mu \in \omega^2$$

$$\gamma = \sqrt{j\mu\sigma\omega - \mu \in \omega^2}$$

$$= \sqrt{j\mu\sigma\omega + \in^2 \in \omega^2} \qquad \{j^2 = -1\}$$

$$\gamma = \sqrt{j\mu\sigma\omega\left(1 + \frac{j\omega \in}{\sigma}\right)}$$

For conductor

$$\frac{\sigma}{\mu \in} \ggg 1$$

or

$$\frac{\sigma}{\mu \in} \lll 1$$

Neglect term contain $\frac{\mu\in}{\sigma}$

Here,

$$1 + \frac{j\omega \in}{\sigma} \cong 1$$

$$\gamma = \sqrt{j\mu\sigma\omega}$$

or

$$\gamma = (1+j)\sqrt{\frac{\mu\sigma\omega}{2}}$$

$$\lambda = \sqrt{\frac{\mu\sigma\omega}{2}} + j\sqrt{\frac{\mu\sigma\omega}{2}}$$

λhas a real as well as an imaginary part, we can then put

$\gamma = \alpha + j\beta$whereα is real part associated with attenuation and is called attenuation constant, while β is the imaginary part called phase. Therefore, on comparing above equations, we get

$$\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}}$$

$$Eu = Eoe^{-\gamma^2} \qquad \text{- - - - (5.29)}$$

$$Ex = Eoe^{\left(-\sqrt{\frac{\mu\sigma\omega}{2}}+j\sqrt{\frac{\mu\sigma\omega}{2}}\right)Z}$$

$$= Eoe^{-\sqrt{\frac{\mu\sigma\omega}{2}}Z - j\sqrt{\frac{\mu\sigma\omega}{2}}Z}$$

$$= Eoe^{-\sqrt{\frac{\mu\sigma\omega}{2}}Z} e^{-j\sqrt{\frac{\mu\sigma\omega}{2}}Z}$$

The term $e^{-\sqrt{\frac{\mu\sigma\omega}{2}}Z}$ is called attenuation factor, while $e^{-j\sqrt{\frac{\mu\sigma\omega}{2}}Z}$ is called phase factor.

5.7.2. PHASE VELOCITY –

$$y = a\sin(\omega t - kx)$$

$$\omega t - kx = constant$$

$$\omega - k\frac{d\mu}{dt} = 0$$

$$\frac{d\mu}{dt} = \frac{\omega}{k}$$

Here $k = \beta$

$$V = \frac{\omega}{\beta}$$

$$= \frac{\omega}{\frac{\sqrt{\mu\sigma\omega}}{2}}$$

$$= \frac{\omega}{\frac{\sqrt{\mu\sigma\omega}}{2}}$$

$$= \omega\sqrt{\frac{2}{\mu\sigma\omega}}$$

$$= \sqrt{\frac{2\omega^2}{\mu\sigma\omega}}$$

$$V = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Reflective index $\eta = \frac{C}{V}$

$$= \frac{C}{\sqrt{2\omega/\mu\sigma}}$$

$$\boldsymbol{\eta = C\sqrt{\frac{\mu\sigma}{2\omega}}}$$

5.7.3. DEPTH OF PENETRATION (SKIN DEPTH):

In a medium, which has conductivity, the wave suffers attenuation as it progresses. At radio frequencies in a good conductor, the factor $\omega\sigma$ will have high value. Consequently α, the attenuation will be quite high and the wave may penetrate only a very short distance before being reduced to a negligibly small percentage of its original strength. This means that an existing wave in a conducting medium is rapidly attenuated. Therefore, we find the penetration of the wave into the conducting medium, called depth of penetration or skin depth.

The depth of penetration δ is defined as that depth in which the wave attenuates to $\frac{1}{e}$ times of its strength before attenuation constant.

$$Eu = Eoe^{-\alpha z}$$

$$\alpha z = 1$$

$$\frac{Eu}{Eo} = \frac{1}{e}$$

i.e. on traversing distance

$$z = \frac{1}{\alpha}$$

the amplitude of the wave falls $\frac{1}{e}$ times its value at z=0. By definition, such a distance is equal to depth of penetration δ so that for a good conductor,

$$\begin{bmatrix} \delta = \frac{1}{\alpha} \\ \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \end{bmatrix}$$

i) for copper at one mega cycle

$$\omega = 2\pi 10^6 c/s \qquad \begin{Bmatrix} \omega = 2\pi F \\ F = 10^6 c/s \end{Bmatrix}$$

$\mu = \mu o$equal to that of free space, being a non-magnetic material

$= 4\pi 10^{-7}$ Henry/meter

$\sigma = 5.8 \times 10^7$mho/meter, so that depth of penetration is

$$\delta = \sqrt{\left(\frac{2 \times 10^7}{2\pi \times 10^6 \times 4\pi \times 5.8 \times 10^7}\right)}$$

$= 0.0667 \times 10^3$meter

= 0.0667 mm.

ii) For silver at one mega cycle

$\omega = 2\pi \cdot 10^6 c/s$ {equal to that of a free space, as silver is non-

$\mu = 4\pi \times 10^{-7}$ H/m magnetic material}

$$\sigma = 6.2 \times 10^{-7}$$

so that depth of penetration is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi\mu\sigma}} = 0.064\text{mm.}$$

which is very small for copper it is 0.066 mm. Consequently, performance of a pure silver component and a silver plated brass component is expected to be indistinguishable of microwave frequencies.

iii) For sea water: at 10 kc/s

$\omega = 2\pi \cdot 10^4$c/s

$\mu = \mu o = 4\pi \times 10^{-7}$ H/m

$\sigma = 4.0$mho/m

so that depth of penetration

$$\delta = \frac{1}{\sqrt{\pi F \mu \sigma}} = 2.5 \text{ m}$$

and the attenuation constant

$$\alpha = \frac{1}{\delta} = 0.4 \text{neper/m}$$

This is the attenuation for $\delta = 1$ is 8.686db. For lesser δ, there will be more α, the attenuation. At 10 kc/s

$$\omega = 2\pi \cdot 10^4 \text{c/s}$$

$$\mu = \mu o = 4\pi \times 10^{-7} \text{ H/m}$$

$$\sigma = 4.0 \text{mho/m}$$

So the depth of penetration

$$\delta = \frac{1}{\sqrt{\pi F \mu \sigma}} = 2.5 \text{m}$$

and the attenuation constant

$$\alpha = \frac{1}{\delta} = 0.4 \text{neper/m}$$

This is the attenuation for $\delta = 1$ is 8.686 db for lesser δ, there will be more α, the attenuation. At 10kc/s the attenuation is 87 db for a distance of only 25 m. So a communication path cannot be much skin depth long. Sea water is thus a poor medium for communication. If a sufficiently high frequency is used, so that displacement current greatly exceeds, the conduction current, the attenuation constant α is

$$\alpha = \frac{1}{\delta} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

Putting$\sigma = 4$ mho/m,

$$\varepsilon = 8/\varepsilon_0$$

$$\mu = \mu o = 4\pi \times 10^{-7} \text{ H/m}$$

The equivalent skin depth δ is about 1.19 cm. only and the penetration at such frequency is, for all practical purposes, non-existent. For this reason, frequency, which are low enough, so that sea water acts as conducting medium, must be used in any attempt at communication.

iv) Skin depth decreases if either the conductivity σ the permeability μ or the frequency F, increases. Good conductors are, therefore, always highly opaque to light, except when in the form of extremely thin films. A thin sheet of conducting material can act as a low pass filter EM waves.

5.8. ENERGY IN ELECTROMAGNETIC FIELD: POINTING VECTOR (POINTING THEOREM)

Energy may be transported through space by means of electromagnetic waves let the material inside surface be isotropic, homogenous, and characterized by permeability μ, permittivity V∈ and conductivity σ. To derive relation, consider a volume V banded by a closed surface S. Maxwell third and fourth relation are

$$\text{Curl E} = -\frac{dB}{dt}$$

$$\text{Curl H} = J + \frac{dD}{dt}$$

Taking scaler product of above equation with H and E

$$\text{H . Curl E} = -\text{ H .}\frac{dB}{dt}$$

$$\text{E . Curl H} = \text{J.E} + \text{E .}\frac{dD}{dt}$$

No subtracting

$$\text{E . Curl H} - \text{H . Curl E} = \text{J.E} + \text{E .}\frac{dD}{dt} + \text{H .}\frac{dD}{dt} \qquad \text{- - - -(5.30)}$$

We have the following formula

div A×B = B. Curl A - A . Crul B

Now,

$$\text{div}\vec{E}\times\vec{H} = \vec{H}.\text{ Curl }\vec{E} - \vec{E}.\text{ Curl }\vec{H}$$

Multiplying by:

$$-\text{ div}\vec{E}\times\vec{H} = \vec{E}.\text{ Curl }\vec{H} - \vec{H}.\text{ Curl }\vec{E}$$

Substituting these value in eg - 1

$$-\text{ div}\vec{E}\times\vec{H} = \vec{J}.\vec{E} + \vec{E}.\frac{d\vec{D}}{dt} + \vec{H}.\frac{d\vec{B}}{dt}$$

$$\vec{J}.\vec{E} + \left(\vec{E}.\frac{d\vec{D}}{dt} + \vec{H}.\frac{d\vec{B}}{dt}\right) + \text{div }\vec{E}\times\vec{H} = 0$$

We have $\vec{D} = \in \vec{E}$ and

$\vec{B} = \mu_0 \vec{H}$

$\vec{J}.\vec{E} + \left(\vec{E}.\frac{d}{dt} \in \vec{E} + \vec{H}.\frac{d}{dt}\mu\vec{H}\right) + \text{div}\, \vec{E} \times \vec{H}$

$\vec{J}.\vec{E} + \frac{1}{2}\frac{d}{dt}(\in E^2) + \frac{1}{2}\frac{d}{dt}(\mu H^2) + \text{div}\,(\vec{E} \times \vec{H}) = 0$

$\frac{1}{2}\frac{d}{dt} \in E^2 = \frac{1}{2} \times 2 \in E \frac{dE}{dt}$

$= \in E \frac{dE}{dt}$

$= \in \vec{E}.\frac{d\vec{E}}{dt}$

$= \vec{E}.\frac{d}{dt}(\in \vec{E})$

$\vec{E} \parallel \frac{d\vec{E}}{dt}, \ \theta = 0$ $\vec{E}\frac{d\vec{E}}{dt} = \theta$ $\vec{E}\frac{d\vec{E}}{dt}\cos\theta$ $= E \frac{dE}{dt}$

$\vec{J}.\vec{E} + \left(\frac{1}{2}\frac{d}{dt}(\vec{E}. \in \vec{E}) + \frac{1}{2}\frac{d}{dt}(\vec{H}.\mu\vec{H})\right) + \text{div}\,(\vec{E} \times \vec{H}) = 0$

$\vec{J}.\vec{E} + \frac{1}{2}\frac{d}{dt}(\vec{E}.\vec{D}) + \frac{1}{2}\frac{d}{dt}(\vec{H}.\vec{B}) + \text{div}\,(\vec{E} \times \vec{H}) = 0$

$\vec{J}.\vec{E} + \frac{1}{2}\frac{d}{dt}\left[(\vec{E}.\vec{D}) + (\vec{H}.\vec{B})\right] + \text{div}\,(\vec{E} \times \vec{H}) = 0$

$E. \in E = C \in .E$ $\in E\cos\theta$ $= \in E^2$

Integrating over the volume V bounded by the surface S, we get

$$\int (\vec{J}.\vec{E})dv + \int \frac{1}{2}.\frac{d}{dt}(\vec{E}.\vec{D} + \vec{H}.\vec{B})dv + \int \text{div}\,(\vec{E} \times \vec{H})dv = 0$$

$$\int \vec{J}.\vec{E}\, dv + \frac{1}{2}\int \frac{d}{dt}\left[(\vec{E}.\vec{D}) + (\vec{H}.\vec{B})\right]dv = -\int \text{div}\,(\vec{E} \times \vec{H})dv$$

Using Gauss divergence theorem

$$\int \text{div}\,(\vec{E} \times \vec{H})dv = \int \vec{E} \times \vec{H}\, .ds$$

$$\int \vec{J}.\vec{E}\, dv + \frac{1}{2}\int \frac{1}{2}\frac{d}{dt}\left[(\vec{E}.\vec{D}) + (\vec{H}.\vec{B})\right]dv$$

$$= -\int (\vec{E} \times \vec{H}).ds \qquad -(5.31)$$

Interpretations of terms:

1st term:

$$\int (\vec{J}.\vec{E})dv = \int (\vec{J}dv).\vec{E}$$

$$= \int I\, dl.\vec{E} \left\{ J = \frac{I}{A} \right\}$$

$$= \int \frac{d}{dt}\ dl.\vec{E}$$

$$= \int dq\ \left(\frac{dl}{dt}\right).\vec{E}$$

$$= \int dq\ (\vec{V}.\vec{E}) \left\{ V = \frac{d}{t} \right\}$$

We have Lorentz force

$$\vec{F} = q\vec{E} + qVB \sin\theta$$

$$= q\vec{E} + q\,(\vec{V} \times \vec{B})$$

$$\vec{F} = q\,(\vec{E} \times \vec{V} \times \vec{B})$$

Work done for small displacement 'dl'

$$dw = \vec{F} . \overrightarrow{dl} = q\,[\vec{E} + (\vec{V} \times \vec{B})] . \overrightarrow{dl}$$

$$= q\vec{E}.\overrightarrow{dl} + q(\vec{V} \times \vec{B})] . \overrightarrow{dl}$$

$$dw = q\vec{E}.\overrightarrow{dl} + q(\vec{V} \times \vec{B})] . \overrightarrow{dl}$$

Fate of work done

$$\frac{dw}{dt} = q\vec{E}.\frac{\overrightarrow{dl}}{dt} + q(\vec{V} \times \vec{B}) . \frac{\overrightarrow{dl}}{dt}$$

$$\frac{dw}{dt} = q\vec{E}.\vec{V} + (\vec{V} \times \vec{B}) . \vec{V}$$

$$\frac{dw}{dt} = q\,(\vec{V}.\vec{E}) \qquad \{(\vec{V} \times \vec{B}) . \vec{V} = 0 \text{ for scaler triple product}\}$$

Including all charges we can write the above equation:

$$A . (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$A . (\vec{A} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = 0$$

$$\frac{dw}{dt} = \Sigma\, qj\,(\vec{V}j . \overrightarrow{E}j)$$

Now,

$$\int (\vec{J}.\vec{E})\, dv = \Sigma\, qj\, (\vec{V}j\,.\vec{E}j) = \frac{dw}{dt}$$

The first term of equation (5.31) represent the rate at which the work is done by the field on the charges i.e., power second term: Interpretation of

$$\int \frac{1}{2}\cdot\frac{d}{dt}(E\,.\,D + H.\,B)\, dv$$

$$= \int \frac{1}{2}\cdot\frac{d}{dt}(E\,.\in E + H.\,\mu H)\, dv$$

$$= \int \frac{d}{dt}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) dv$$

The first and second term on right hand side represent the time rate of charge of electromagnetic energy stored in the electric and magnetic field respectively within the volume V.

Rate at which work is done by the field on the charges i.e., the power spent by the fields due to motion of charges. And the time rate of change of electromagnetic energy stored in electric and magmatic fields within the Volume V.

We therefore conclude that the R.H.S.

$$\int (E \times H).\, ds \quad \text{of equation}(5.31)$$

Must represent flowing per second (in power) into the volume V through surface S or the power flowing out of volume V through the surface S we denote

$E \times V = P$

Where P, called pointing vector which is define as -

The amount of field energy passing through unit area of the surface is unit time which is normal to the direction of energy flow. It unit is watt/met^2.

5.8.1. POYNTING THEOREM:

Law of conservation of energy for electromagnet is fields. The energy can be written as:

$$J.E + \left[\frac{1}{2}\frac{d}{dt}(E.\,D + H.\,B)\right] + \text{div}\,(E \times H) = 0$$

$= J.E = \frac{d}{dt}\left(\frac{1}{2} \in E^2 + \frac{1}{2}\mu H^2\right) + \vec{\nabla} . P$

$= \frac{d\mu}{dt} + \vec{\nabla} . P$

Where:

$\mu = \frac{1}{2} \in E^2 + \frac{1}{2}\mu H^2$

Which represent energy density of electromagnetic field.

For non – conducting medium J = 0

So that:

$\frac{d\mu}{dt} + \vec{\nabla} . P = 0$

5.8.2. POINTING VECTOR

Pointing vector is define as the amount of field energy (Electric and Magnetic) passing through unit area of the surface perpendicular to a plane containg$\vec{E}$ and $\vec{H}$ per unit time it's as:

$\vec{P} = \vec{E} \times \vec{H}$

Since in an EM wave

$E \perp H$

$P = \vec{E} \times \vec{H}$

$= EH \sin 90\ \hat{n}$

P = EH

About pointing vector, we can further state.

5.8.3. POINTING VECTOR VARIES INVERSELY AS THE SQUARE OF THE DISTANCE FROM THE POINT SOURCE OF RADIATION-PROOF'

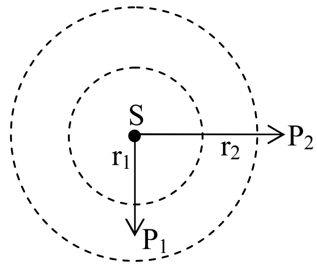
Let as consider a source & of electromagnetic radiation enclosed by two spherical surfaces of radios r_1 of r_2 with source (s) centre.

Let source be emitting radius at the rate of W (Watt) that pointing vector equal to energy flowing up

$$\frac{\text{energy flowing}}{\text{time} \times \text{Area}} = \frac{W}{t \times \text{Area}} = \frac{P}{\text{Area}}$$

Pointing Vector = $\frac{W}{\text{Area}}$

$P_1 = \frac{W}{4\pi r_1^2}$



$P_2 = \frac{W}{4\pi r_2^2}$

$\frac{P_1}{P_2} = \frac{\frac{W}{4\pi r_1^2}}{\frac{W}{4\pi r_2^2}} = \frac{r_1^2}{r_2^2}$

$\frac{P_1}{P_2} = \left(\frac{r_2}{r_1}\right)^2$ or

$$\mathbf{P} \propto \frac{\mathbf{1}}{\mathbf{r^2}}$$

In case of time varying fields P = E × H. Gives the instate nous value of the pointing vector and therefore we calculate average value for one complete period.

Suppose the fields are given by

$\vec{E} = \vec{E}_0 \sin wt$ &

$\vec{H} = \vec{H}_0 \sin wt$

$$\frac{\int_0^T \overrightarrow{P dt}}{\int_0^T \overrightarrow{dt}} = \frac{\int \vec{E} \times \vec{H}\, dt}{[T]_0^T}$$

$$= \int \frac{\vec{E}_0 \sin wt \times \vec{H}_0 \sin wt \ dt}{T}$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{T} \int_0^T \sin^2 wt\, dt$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{T} \int_0^T \frac{1 - \cos^2 wt}{2} \ dt$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{2T} \int_0^T 1 - \cos^2 wt \ dt$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{2T} [T]_0^T \left[\frac{\sin^2 wt}{2w}\right]_0^T$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{2T}\left[T - \frac{1}{2w}\left(\frac{\sin 2\pi\, T - \sin 0}{T}\right)\right]$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{2T}\left[T - \frac{1}{2w}\sin 4\pi\right]$$

$$= \frac{\vec{E}_0 \times \vec{H}_0}{2T}\, T$$

$$\vec{P} = \frac{\vec{E}_0 \times \vec{H}_0}{2}$$

5.9. SUMMARY

In this unit, you have studied the application of Maxwell equations. You have studied Maxwell equations in varies conditions like free space, in conducting medium, non-conducting medium. You have derived wave equations for free space condition, plane electromagnetic wave in free space, plane electromagnetic wave in matter. In this unit you have also learnt energy flow due to a plane EM wave (pointing vector for free space). In this unit you have also learnt relative orientation of E and H vectors in a plane wave and found both mutually perpendicular to each other. In this unit you have also learnt about propagation constant, phase velocity, reflective index, depth of penetration (skin depth). In this unit you have also learnt energy in electromagnetic field pointing vector (pointing theorem). Many solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are given place to place.

5.10 GLOSSARY

Penetration depth- electromagnetic radiation can penetrate

Permeability - is the quality or state of being permeable—able to be penetrated or passed through

5.11 REFERENCES

1. Introduction to Electrodymics David J. Griffiths

2. Classical Electrodynamics John David Jackson

3. Principle of electrodynamics by Melvin Schwartz

4. Electrodynamics by Gupta Kumar

5.12 SUGGESTED READINGS

1. YOU TUBE

2. National Programme on Technology Enhanced Learning (**NPTEL**) Lectures

5.13 TERMINAL QUESTIONS

(Should be divided into Short Answer type, Long Answer type, Numerical, Objective type)

5.13.1 Short Answer type

1. Write short note on following-

(a) Skin depth

(b) Propagation constant

2. Show that electric and magnetic component in an electromagnetic wave are mutually perpendicular.

3. Show that in free space $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

Where c is speed of light,μ_0 permeability of free space, ϵ_0 permittivity of free space.

4. Show that the wave equation can be written in the form

$$(\nabla^2 + k^2)E = 0$$

where k = $\frac{2\pi}{\lambda}$ is called wave vector or propagation constant

5.13.2Long Answer type

1. Derive electromagnetic equation in conducting medium. Explain depth of penetration and propagation constant.

2. Explain pointing vector. What are its physical significance?

UNIT 6: INTERACTION OF ELECTROMAGNETIC WAVE WITH MATTER

Structure

6.1 INTRODUCTION

This chapter explains about the boundary conditions and the macroscopic behavior for a plane electromagnetic wave when it is incident on an infinite plane boundary between two semi-infinite media of different indices of refraction, for instance as free space and a metallic conductor. We shall be led to the familiar laws of reflection and refraction, laws that were established, before the advent of electromagnetic theory merely from general wave theory. In order to discuss the behavior of electromagnetic waves at the boundary, we must find the boundary conditions holding at a surface of discontinuity between two media. Boundary conditions can be worked out for electric displacement vector (D), electric field (E), magnetic induction (B) and magnetic field strength (H).

6.2 ELECTROSTATIC BOUNDARY CONDITIONS

If the electric field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called electrostatic boundary conditions. These conditions are helpful in determining the field on one side of the boundary when the field on other side is known.

6.2.1Boundary Conditions at an interface of two Dielectrics

For determining boundary conditions, we will use following Maxwell's equations

$\oint D.ds = Q_{enc}$ -------------------------- (1)

and

$\oint E.dl = 0$ -------------------------- (2)

Also we need to decompose E to the interface of the interest-

$E = E_t + E_n$

Where $E_t$andE_n are the normal and tangential components of Electric field (E) to the interface of the medium. Let us consider the E field exists in a region that consists of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0\varepsilon_{r1}and\varepsilon_2 = \varepsilon_0\varepsilon_{r2}$as shown below in fig. 1-

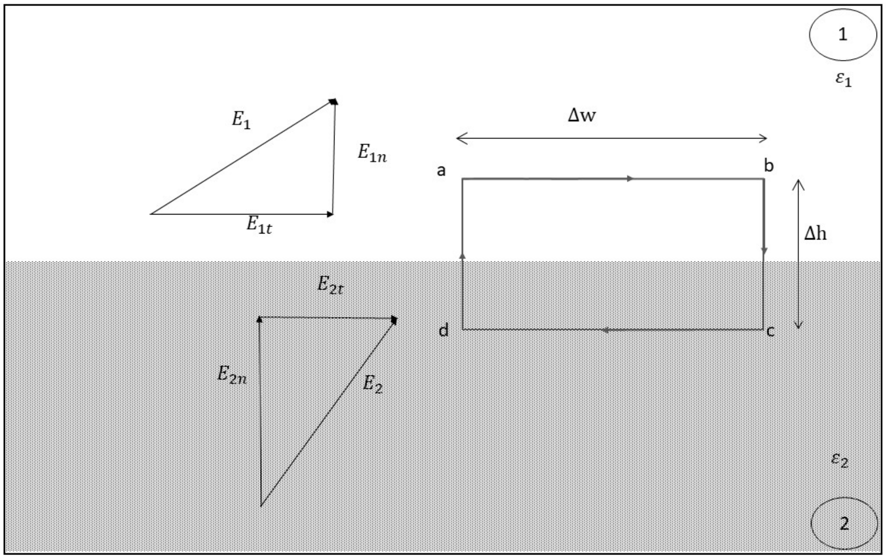


Figure 1

In media 1 and 2, Electric field can be decomposed as-

$$E_1 = E_{1t} + E_{1n}$$

$$E_2 = E_{2t} + E_{2n}$$

Using Maxwell Equation $\oint E.dl = 0$ along the closed path abcda-

$$0 = E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

As Δh→0, above equation becomes

$E_{1t} = E_{2t}$ -------------------------- (3)

Thus the tangential components are the same on the two sides of the boundary, i.e.$E_t$is continuous across the boundary.

Since $D = \varepsilon E = D_n + D_t$

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

or

$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$ -------------------------- (4)

Thus, $D_t$undergoes some change, hence it is said to be discontinuous across the interface.We can also apply first Maxwell equation, to the Cylindrical Gaussian Surfaceof the fig 2.

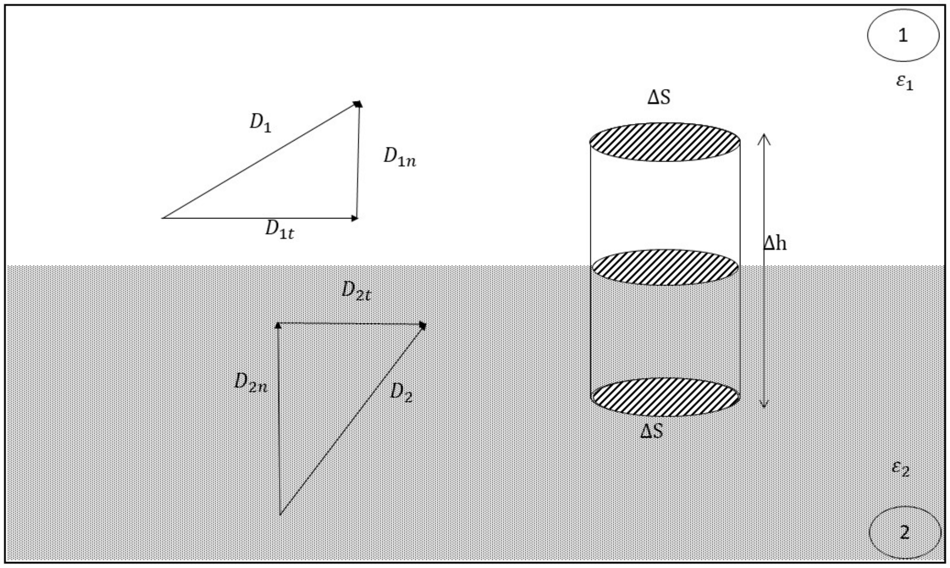


Figure 2

The contribution due to curved sides vanishes. Allowing $\Delta h \to 0$ gives

$$\Delta Q = \rho_s \Delta S = D_{1n}\Delta S - D_{2n}\Delta S$$

Or

$$D_{1n} - D_{2n} = \rho_s$$

hereρ_s is the free charge density at boundary. Generally$\rho_s = O$ (until and unless, we are not putting free charge at the interface deliberately), the above equation become-

$D_{1n} = D_{2n}$ -------------------------- (5)

Thus the normal component of D is continuous across the interface. Further since $D = \varepsilon E$, we can write-

$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$ -------------------------- (6)

Showing that the normal component of the E is discontinuous at the boundary.Equations (3-6) are known as the boundary conditions for adielectric- dielectric boundary.

6.2.2 Conductor-Dielectric Boundary Conditions

We follow the same procedure as we did in case of dielectric-dielectric boundary conditions, as the electric field inside a conductor is always zero.

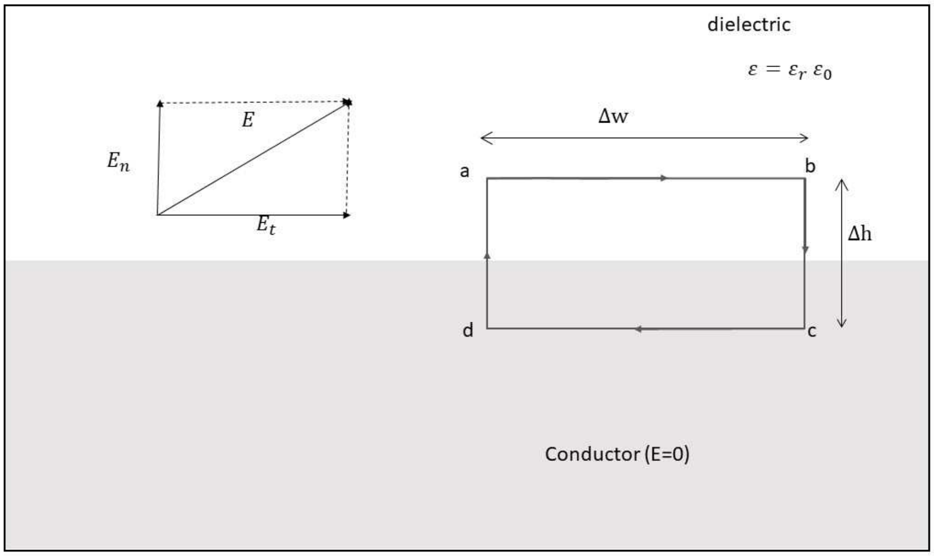


Figure 3

Working out the *E.dl*for loop *abcda*-

$$0 = 0.\Delta w + 0.\frac{\Delta h}{2} + E_n\frac{\Delta h}{2} - E_t\Delta w - E_n\frac{\Delta h}{2} - \quad 0.\frac{\Delta h}{2}$$

Which further can be written as-

$E_t = 0$ -------------------------- (7)

Using above formula, we can write-

$D_t = \varepsilon E_t = 0$ -------------------------- (8)

Working out the first Maxwell equation$\oint D.ds = Q_{enc}$for cylindrical Gaussian surface-

$$\Delta Q = \rho_s \Delta S = D_n \Delta S - 0$$

Or

$$D_n = \rho_s$$

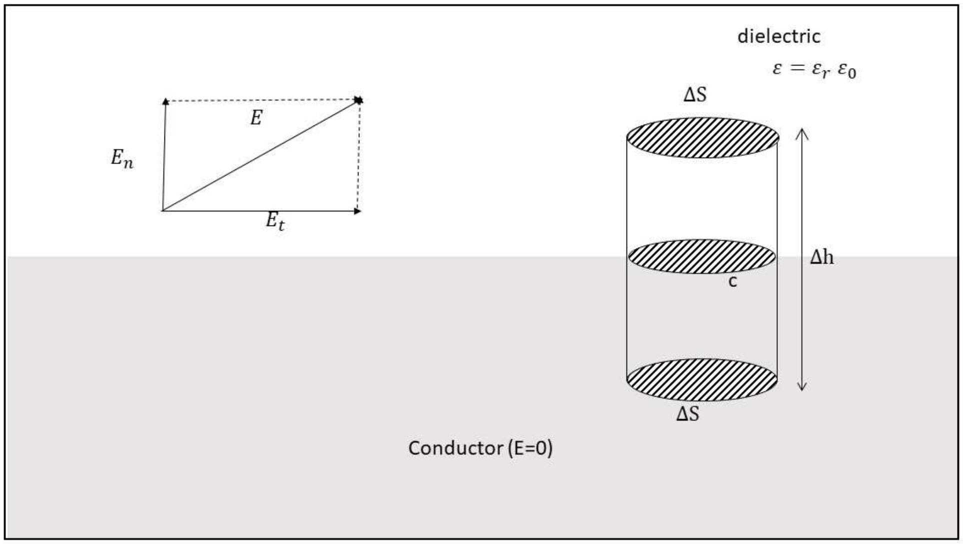


Figure 4

Above formula can also be written in terms of E_n-

$$D_n = \varepsilon E_n = \rho_s \qquad \text{-------------------------- (9)}$$

$$\therefore E_n = \frac{\rho_s}{\varepsilon} \qquad \text{-------------------------- (10)}$$

Eqns. (7- 10) give the conductor-dielectric boundary conditions. Similarly,if we incorporate

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1$$

then the eqns. (7- 10) would give the boundary conditions, for free space.

6.3 MAGNETIC BOUNDARY CONDITIONS

The magnetic boundary conditions are the conditions that H (or B) field must satisfy at the boundary between two different media. This derivation make use of Gauss's law for magnetic fields $\oint B.ds = 0$ and Ampere's circuit law $\oint H.dl = I$

Consider the boundary between two magnetic media 1 and 2, characterized, respectively, by μ_1 and μ_2 as in Figure 5. Applying Gauss law of magnetostatics to the pillbox (Gaussian surface) of Figure 5(a) and allowing $\Delta h \rightarrow 0$, we obtain -

$$B_{1n}\Delta s - B_{2n}\Delta s = 0$$

Thus $B_{1n} = B_{2n}$

or$\mu_1 H_{1n} = \mu_2 H_{2n}$ -------------------------- (11)

since $B = \mu H$, Equation (11) shows that the normal component of B is continuous at the boundary. It also shows that the normal component of H is discontinuous at the boundary; H undergoes some change at the interface.

Similarly, we apply Ampere's law to the closed path abcda of Figure 5(b) where surface current K on the boundary is assumed normal to the path. We obtain

$K.\Delta s = H_{1t}.\Delta w + H_{1n}\frac{\Delta h}{2} + H_{2n}\frac{\Delta h}{2} - H_{2t}.\Delta w - H_{2n}\frac{\Delta h}{2}$ ----------------------(12)

As $\Delta h \to 0$, above equation leads to-

$H_{1t} - H_{2t} = K$ -------------------------- (13)

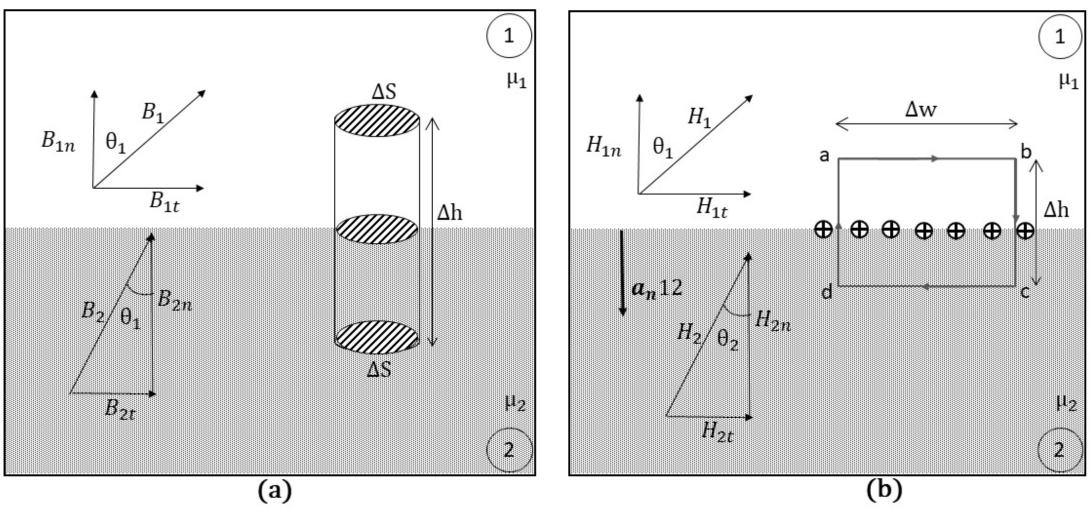


Figure5 Boundary condition between two magnetic media: (a) for B, (b) for H

This shows that the tangential component of H is also discontinuous while that of B is discontinuous at the boundary. Equation (13) may be written in terms of B as-

$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} = K$ -------------------------- (14)

In the general case, eq. (13) becomes-

$$(H_1 - H_2) \times a_{n12} = K$$

wherea_{n12} is a unit vector normal to the interface and is directed from medium 1 to medium 2. If the boundary is free of current or the media are not conductors (for K is free current density), K = 0 and eq. (13) becomes-

$H_{1t} = H_{2t}$ -------------------------- (15)

or

$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$ -------------------------- (16)

Thus the tangential component of H is continuous while that of B is discontinuous at the boundary. If the field make an angle θ with the normal to the interface-

$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2$ -------------------------- (17)

And eq. (16) produces-

$\frac{B_1}{\mu_1}\sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2}\sin\theta_2$ -------------------------- (18)

Dividing eq. (18) by (17) gives-

$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$ -------------------------- (19)

Eqn (11), (16) and (19) give the boundary conditions for B, H and θ respectively.

6.4 REFLECTION AND REFRACTION OF PLANE ELECTROMAGNETIC WAVES

Previous section describes the change in the electromagnetic components acroos the interface of two different media. We shall now investigate the behaviour of EM waves in different combinations of different types of media, i.e. dielectrics, conductors (including good conductors) and combinations of these. We remind ourselves that the dielectrics are non-conductors, and they may be either magnetic or non-magnetic; we shall restrict ourselves to non-magnetic dielectrics.

We assume an ideally thin, infinitely plane interface between the two Linear, Isotropic and Homogeneous (LIH) media. An incident wave along n_i would. in general, give rise to a reflected wave along n_0 and a transmitted wave along n_t(Figure 6). These three waves, combined together, satisfy the continuity conditions for thc tangential Components of E and H, and for the normal components of D and B at the interface. Initially, we exclude the total reflection from the dielectric, and assume the media to extend to infinity on the sides of the interface.

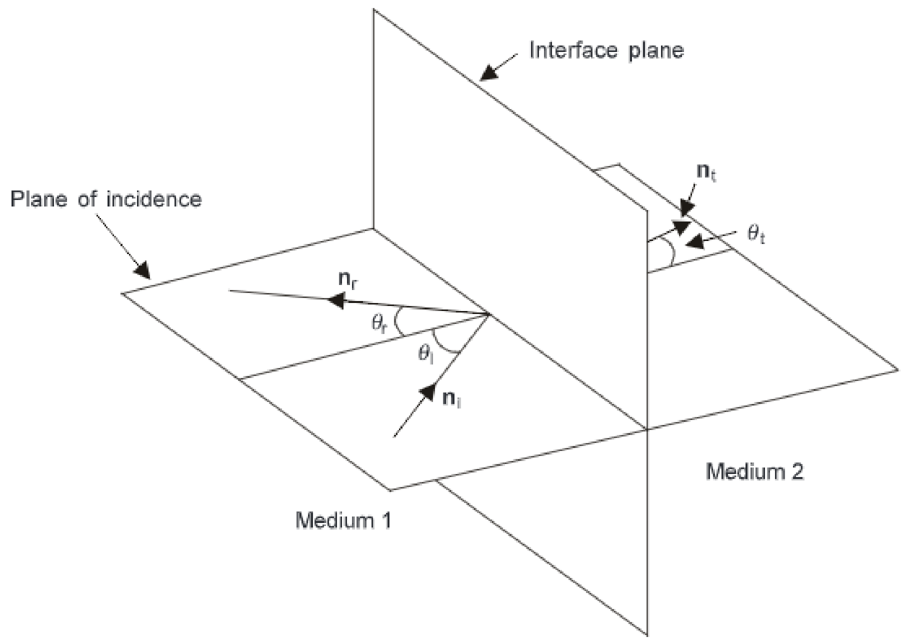


Figure 6 An electromagnetic wave in medium 1, incident on the interface between media 1 and 2, giving rise to both the reflected wave in the medium 1 and the transmitted wave in the medium 2.

The whole analysis is classified by the method through which the incident wave hits the interface surface of discontinuity, i.e. (l) the incident wave meeting the interface normally; and (2) the incident wave meeting the interface obliquely, ir which case, the interference pattern produced in the first medium, by the combination of the incident and the reflected waves would be stationary in one direction and travelling in an orthogonal direction. The properties of such patterns are of great practical importance in design and study of waveguides.

Therefore, it is to be mentioned that when a wave meets a discontinuity in the media of propagation, then, in general, it (the incident wave) produces a transmitted wave in the medium and a reflected wave which travels back in the first medium, the exact directions of propagation being decided upon by the interface continuity conditions. We shall now consider waves incident normally on different types of interface surfaces, startirg first with a perfectly conducting surface.

6.4.1 Laws of Reflection and Snell's Law of Refraction

Referring to Figure 6, we consider the electromagnetic wave incident on the interface, to be both plane and plane-polarized, so that its electric field intensity is of the form

$$E_{ti} = E_{0i}\exp\left[j\omega_r\left(t - n_i.\frac{r}{u_1}\right)\right] \quad \text{-------------------------- (20)}$$

where u_1 is the phase velocity Of the wave in the medium 1. The time t = 0, and the origin r=0 can be chosen arbitrarily. We, however, choose the origin at a point on the interface plane. The above equation defines a plane wave for all values of t and r, but we shall use it only for medium 1, and when we consider normal incidence, then $n_i \cdot r = z$ (+ve Z-direction). The reflected and the refracted waves from the plane interface are also plane

$$E_r = E_{0r} \exp\left[j\omega_r\left(t - n_r.\frac{r}{u_1}\right)\right] \quad \text{-------------------------- (21)}$$

$$E_t = E_{0t} \exp\left[j\omega_t\left(t - n_t.\frac{r}{u_2}\right)\right] \quad \text{-------------------------- (22)}$$

where u_2 is the phase velocity of the wave in the medium 2. It should be noted that so far no assumptions have made about the amplitudes, phases, frequencies, and the directions of the reflected and the refracted waves. The amplitudes E_{0r}, E_{0t} can be complex if required.

The characteristics of the reflected and the transmitted waves are obtained from the interface continuity conditions that the tangential components of E and H must be continuous across the interface, i.e. the sum of the tangential components of E_i and E_r must equal that of E_t on the interface (z = 0). Similar conditions must holds for H. These conditions must hold for all instants of time and at all points on the interface (z = 0)

$$\therefore \omega_i = \omega_r = \omega_t$$

i.e. all the three waves must be of same frequency. Also, since these conditions hold at all the points on the interface,

$$\therefore \frac{n_i.r_i}{u_1} = \frac{n_r.r_i}{u_1} = \frac{n_t.r_i}{u_2} \quad \text{-------------------------- (23)}$$

where r_i is any point on the interface. From the first two terms of the above equation

$$(n_i - n_r).r_i = 0$$

Since r_i lies on the interface, the $(n_i - n_r)$ must normal to the interface plane z = 0, i.e. referring to the figure 6.

$$\theta_i = \theta_r$$

which means that the angle of reflection equals the angle of incidence. Since $(n_i - n_r)$ is parallel to n (the normal to the interface), the three vectors n_i, n_r and n are coplanar. These are the laws of reflection of the waves. The plane containing these three vectors is the 'plane of incidence'. Going back to Eq. (23), we get

$$\left(\frac{n_i}{u_i} - \frac{n_t}{u_2}\right).r_i = 0 \quad \text{-------------------------- (24)}$$

Therefore, the vector in the brackets must be normal to the interface plane z = 0, and hence n_i, n_t, and n are coplanar, and hence all the four normal vectors n_i, n_t, $n_r$and n must lie in the plane ofincidence. Furthermore, the tangential components of $(n_i/u_i,)$ and (n_t/u_2) must be equal, i.e.

$$\frac{\sin\theta_i}{u_1} = \frac{\sin\theta_t}{u_2} \quad \text{-------------------------- (25)}$$

or since the wave number β= ω/u, the above equality can be expressed as-

$$\beta_1(sin\theta_i) = \beta_2(sin\theta_t)$$

or

$$\frac{sin\theta_t}{sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2} = \frac{\sqrt{\varepsilon_{r1}\mu_{r1}}}{\sqrt{\varepsilon_{r2}\mu_{r2}}} \quad \text{-------------------------- (26)}$$

where n is the index of refraction. This is the 'Snell's law of refraction'. Thus the quantity'βsinθ' is conserved across the interface. The above laws are general, and apply to any two media. They hold true for total reflection as well.

6.5 POLARIZATION

An important property of an electromagnetic wave is polarization which describes the orientation of thc electric field E. It is defined as the property of a radiated electromagnetic wave describing the time-varying direction and relative magnitude of the electric field vector; specifically, the figure traced as a function of time by the extremity of the vector at a fixed location in space, and the same in which it is traced, as observed along the direction of propagation, i.e. polarization is rhe curve traced out by the end-point of the arrows representing the instantaneous electric field. The field must be observed along and towards the direction of propagation. For the simplicity of understanding we consider the simplest case of a uniform plane wave, trasverse in nature, propagating along the z-axis of the referred co-ordinate system, travelling in the +ve z-direction. In this case, E and H vectors have to lie in a plane orthogonal to z-axis, but there is no constraint that E and H vectors have to be constant as function of space and/or time. The only requirements are that E and H should be to each other, and the ratio of their magritudes should equal to η (= intrinsic impedance of the medium). Hence both the vectors can rotate in the transverse plane by the sane angle and scale in the same proportion, without affecting the nature of the wave. Thus a knowledge of the E wave is sufficient to specify the H field unambiguously. Hence for completeunderstanding, it would be sufficient to discuss the behaviour of E field only (see Figur**e 7**).

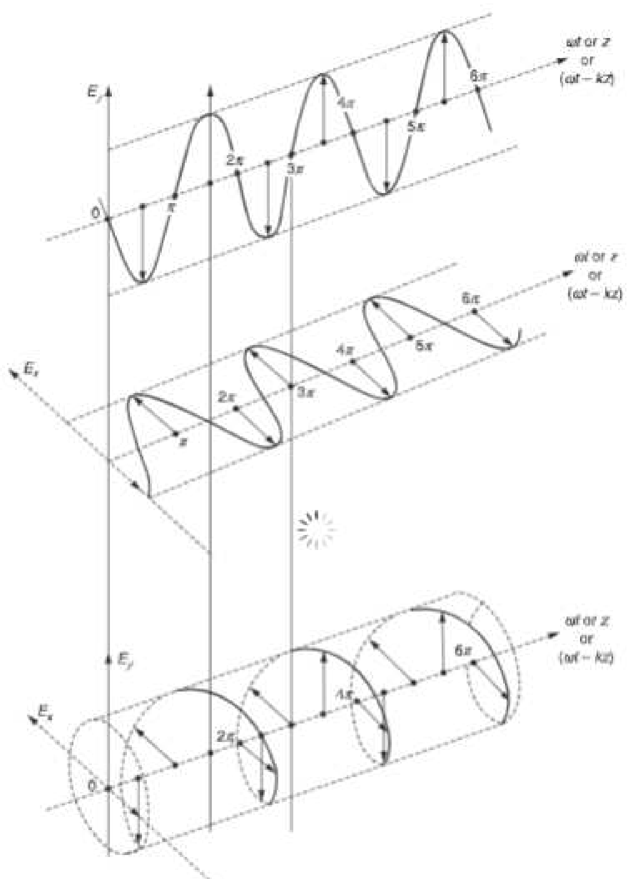


Figure 7- Rotation a plane electromagnetic wave, at a fixed space point (say z = 0 asa function of time. The figure shows that the two orthogonal components (x- and y-) of space phase displacement ofx/2 and also the time phase difference of these two components. If the axis is replaced by z-axis, then the above 3 diagrams would the space rotation of the wave at a given instant of time $t=t_o$. Furthermore. If ωt is is replaced by kz, then the diagrams show the behaviour of a travelling wave propagating in z-direction, showing both time and space phase variatons of the orthogonal cornponents.

Polarization can be classified into three categories: linear, circular and elliptic. If the vecor which describes thc electric field at a point in space as a function of time is always directcd along astraight line which is normal to the direction of propagation of the wave, then thc field is said to be linearly polarized. In general, if the figure that the tip of the electric field vector traces is an ellipse, then the field is said to be elliptically polarized. Linear and circular polarizations arc special cases of the elliptic polarization, and they can be obtained when the ellipse degenerates to a straight line or a circle. The figure of the electric field (i.e. the closed loop) would be traced either in clockwise (CW) or counterclockwise (CCW) sense, looking at it along the direction of propagation as specified earlier. It is to be noted that thc clockwise polarization is also known as right-handed (i.e. CW = RH) polarization and the counterclockwise polarization is known as left-handed (CCW = LH) polarization for all three-types of polarizations. It is being reminded that this sense of rotation (right-handed or left-handed) is obtained by viewing the progressive wave from its ’rear' in the direction of propagation. In the present example, the wave is travelling in the +ve z-direction (which has been taken as into the page), so that the rotation is being examined from an observation point looking into the page and perpendicular to it. (**Figure 8**).To study and analyse the different states of polarization of a wave, we consider two waves of same frequency propagating along the z-axis in the z-direction and having electric fields oriented along the x- and the y-directions respectively. To maintain the generality, let the amplitudes of these two waves be unequal and an arbitrary phase difference between them. We assume the time-variations of both the waves to be of time-harmonic and hence we can express the fields as-

$$E_{xc} = E_1 = Rc\{E_x e^{j(\omega t-\beta z)}\} = E_{x0} cos(\omega t - \beta z + \phi_x) \quad \text{-------------------------- (27a)}$$

$$E_{yc} = E_2 = Rc\{E_y e^{j(\omega t-\beta z)}\} = E_{y0} cos(\omega t - \beta z + \phi_y) \quad \text{-------------------------- (27b)}$$

where ω is the angular frequency of the waves and β is their propagation constant. To keep the analysis completely general, E_{xc} , E_{yc} , E_x , E_y are all complex with E_{x0} , E_{y0} being real and $\phi_y - \phi_x \neq 0$ (at this stage).

Next, in the above expression, if $\phi_y - \phi_x$ is +ve, E_y leads E_x, and if $\phi_y - \phi_x$, is -ve, $E_y$lags E_x. Since at present, the investigation is of the behaviour of the two fields as a function of time at any specified point in space, we can assume that point to be located at z = 0 (for principality of analysis) without any loss of generality. Thus the equations for the two fields become:

$$E_1 = E_x = E_{x0} cos(\omega t + \phi_x) \quad \text{-------------------------- (28a)}$$

$$E_2 = E_y = E_{y0} cos(\omega t + \phi_y) = E_{y0} cos\{(cot\ \omega t + \phi_x) + (\phi_y - \phi_x)\} \quad \text{-------------- (28b)}$$

At any instant of time, the resultant E field would be the vector sum of these two instantaneous fields.

∴ At some instant of time t, the resultant field is:

$$E = i_x E_{x0} cos(\omega t + \phi_x) + i_y E_{y0} cos(\omega t + \phi_y)$$

$$= i_{\phi yx} \sqrt{\{E_{x0}{}^2 cos^2(\omega t + \phi_x) + E_{y0}{}^2 cos^2(\omega t + \phi_x) + (\phi_y - \phi_x)\}}$$

$$\angle \tan^{-1}\left\{\frac{E_{y0} cos\{(cot\ \omega t + \phi_x) + (\phi_y - \phi_x)\}}{E_{x0} cos(\omega t + \phi_x)}\right\} \quad \text{-------------------------- (29)}$$

where $i_{\phi yx}$ is the unit vector in the direction of E (see Fig. 8).

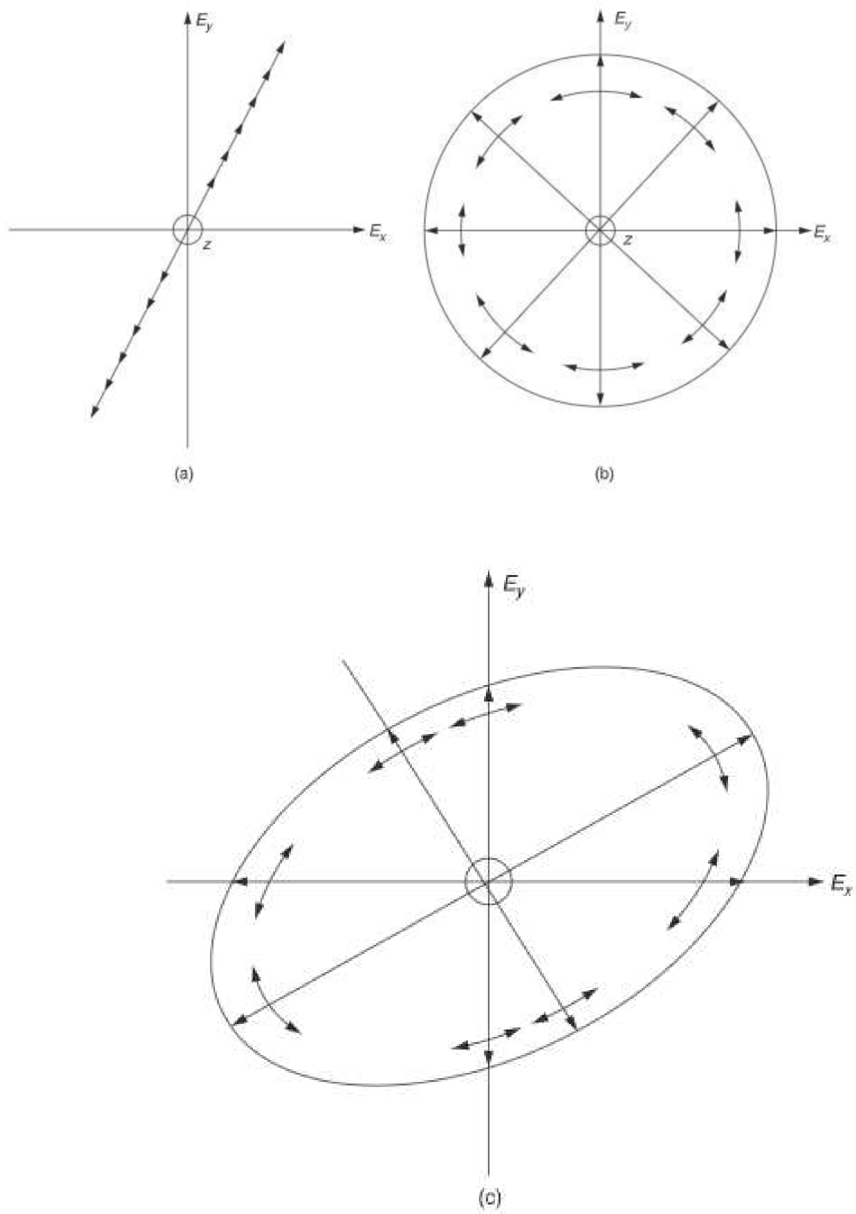


Figure8: Polarization figure traces of an electric fields vector tip as a function of time for given spece position: (a) linear (b) circular (c) elliptical

It is Obvious from the above equation that the magnitude and the direction of the resultant Electric field are functions of time and hence change with time. To obtain the locus of the tip of the E vector (by eliminating from these equations):-

$cos(\omega t + \Phi_x) = \frac{E_x}{E_{x0}}$ and $\sin(\omega t + \Phi_x) = \sqrt{\left\{1 - \frac{{E_x}^2}{{E_{x0}}^2}\right\}}$ -------------------------- (30)

in the expression for E_y–

$$\frac{E_y}{E_{y0}} = cos(\omega t + \Phi_y) = cos(\omega t + \Phi_x) + (\Phi_y - \Phi_x)$$

$$= cos(\omega t + \Phi_x).cos(\Phi_y - \Phi_x) - sin(\omega t + \Phi_x).sin(\Phi_y - \Phi_x)$$

$$= \frac{E_x}{E_{x0}} cos(\Phi_y - \Phi_x) - \sqrt{\left\{1 - \frac{{E_x}^2}{{E_{x0}}^2}\right\} sin(\Phi_y - \Phi_x)} \text{------------------------} (31)$$

Rearranging and squaring Sides,

$$\left\{\frac{E_x}{E_{x0}} cos(\Phi_y - \Phi_x) - \frac{E_y}{E_{y0}}\right\}^2 = \left\{1 - \frac{{E_x}^2}{{E_{x0}}^2}\right\} sin^2(\Phi_y - \Phi_x)$$

Further rearranging-

$$\frac{{E_x}^2}{E^2{}_{x0}} - \frac{2E_xE_y cos(\Phi_y - \Phi_x)}{E_{x0}E_{y0}} + \frac{{E_y}^2}{{E_{y0}}^2} = sin^2(\Phi_y - \Phi_x) \qquad \text{--------------------------} (32)$$

This is an equation of an ellipse. Hence the tip of the E-vector, in general, for a time-harmonic plane wave. traces an ellipse. as a function of time. Once in every time-period. So this ellipse will be traced ω/2π times every second. Thus this wave is called an elliptically polarized wave.

The equation of the ellipse and hence its orientation {i.e. its tilt to the x or y-axis) would change with any change in E_{x0}, E_{y0} and/or $(\Phi_y - \Phi_x)$. Thus it is the phase of the ellipse, and not its absolute size which is of interest in the study of the polarization of waves. The shape of the ellipse can be characterized by two of parameters, each set consisting of two parameters or their equivalent angles.

The first set of parameters consists of (i) the axial ratio (AR), i.e. the ratio of thc major to the minor axes of the ellipse (which can also be expressed as angle ε) and the tilt angle (τ), i.e. the orientation of the major axis of the ellipse with respect to the x-direction. The second set of parameters. denoted by (τ,δ) are also related to AR and the phase difference $(\Phi_y - \Phi_x)$.

LINEAR POLARIZATION-

The two components of the waveE_x andE_y may or may not have the same magnitude, let their

phase difference$(\Phi_y - \Phi_x)$ is zero. Then Eq. (32) becomes:

$$\left\{\frac{E_y}{E_{x0}} - \frac{E_y}{E_{x0}}\right\}^2 = 0 \qquad \text{-------------------------} (33)$$

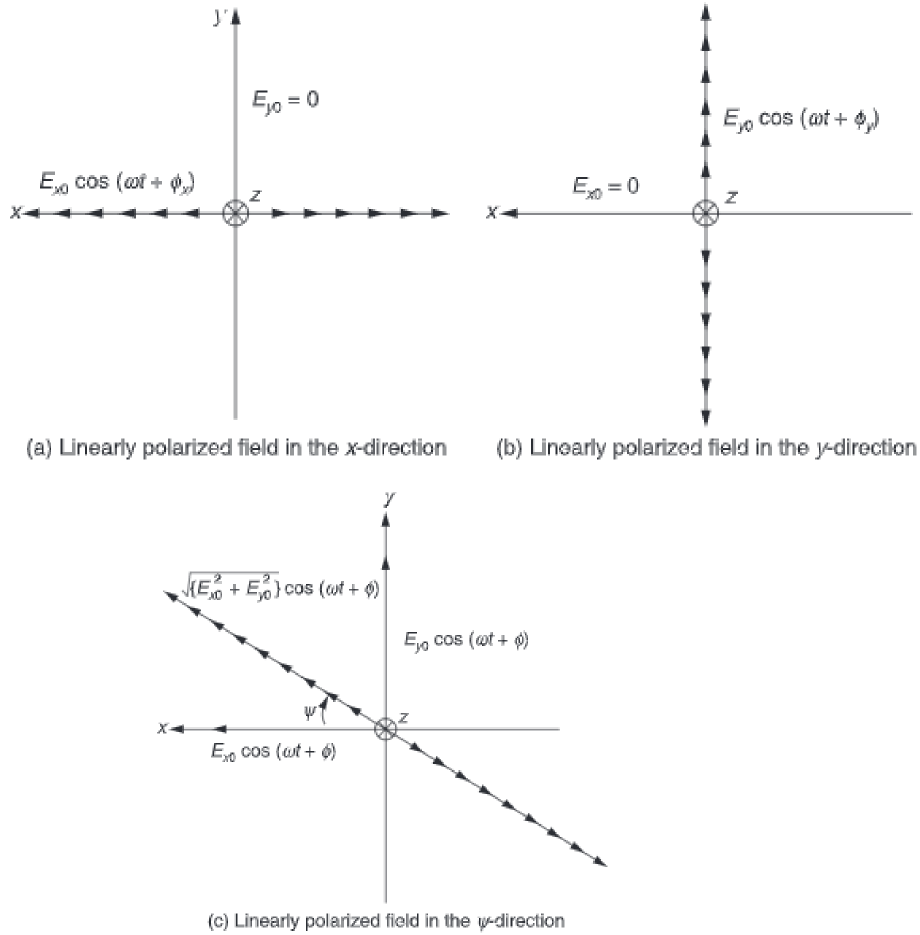


Fig.8 A linearly polarized wave (the co-ordinate system is right handedwith the +ve z directionin to the plane of the paper and normal to the paper plane)

This gives
$E_y = \left\{\frac{E_{y0}}{E_{x0}}\right\} E_x$ -------------------------- (34)

This is the equation of a straight line whose slope is $\frac{E_{y0}}{E_{x0}}$, and so the tip of the electric field vector draws a straight line when $(\Phi_y - \Phi_x) = 0((\Phi_y = \Phi_x = \Phi = 0))$, independent of the relative amplitudes of $E_x$and E_y This Polarization is known as 'linear polarization', and the wave is saidto be linearly polarized.

The instantaneous value of thc resultant E vector will vary from zero to$\sqrt{\{E_{xo}{}^2 - E_{yo}{}^2\}}cos(\omega t + \Phi)$ when the two components are $E_{x0}cos(\omega t + \Phi)$and $E_{y0}cos(\omega t + \Phi)$ in the first half-cycle and then to -ve maximum and back to zero in the next half-cycle, and this pattern along the straight line given by Eq. (34) will keep on repeating itself. The slope of this line would depend on the relative magnitudes of $E_{x0}$and E_{y0}
Hence summarizing:
l. If $E_{x0} = 0$, this line becomes vertical and the wave is called 'vertically polarized wave'.
2. if $E_{y0=0}$, the line becomes horizontal giving rise to a 'horizontally polarized wave'.

3. If E_{x0} =E_{y0} the wave is said to be linearly polarized with 45^0 polarization angle.
4. In general, when both components E_x and $E_y$are at the same phase, the wave is said to be linearly polarized along a line that makes an angle V with the x-axis where-

$$\psi = tan^{-1}\left\{\frac{E_y}{E_x}\right\} = tan^{-1}\left\{\frac{E_{y0}}{E_{x0}}\right\} \quad \text{-------------------------- (35)}$$

Hence, a time-harmonic field is linearly polarized at a given point in space if the electric field (or the magnetic field) vector at that point is always oriented along the same straight line at all instants of time.

6.6 FRESNEL'S EQUATIONS (DYNAMIC PROPERTIES OF REFLECTION AND REFRACTION)

The equations relating the amplitudes of the reflected and transmitted waves with that of incident wave are known as Fresnel's equations (or Fresnel's formulae). We know that field vectors E and B in a plane electromagnetic wave are mutually perpendicular and also they are perpendicular to the direction of propagation of the wave. Though the field vector E is always at right angles to the direction of propagation, but that does not fix the direction uniquely, and is said that the wave is polarised in a particular direction of its electric (or according to an alternative convention, its magnetic vector) points in that direction.

It is convenient to consider two extreme cases (i) in which incident wave is polarişed such that electric field vector E is normal to the plane of incidence, and (ii) in which field vector E is parallel to the plane of incidence. The general result may be obtained by a suitable linear combination of these two extreme results.

Case (i) When E-vector is perpendicular to the plane of incidence: The electric and magnetic field vector E and B of the incident wave are Z perpendicular to direction of propagation k as shown B, or H in fig. 9. The electric field vector E is perpendicular to the plane of incidence. Since the media are isotropic, the electric field vectors of the reflected and refracted waves are also perpendicular to the plane of incidence. All the electric field vectors are shown directed away from the observer. The orientations of B, or H, the B-vectors are chosen to give a positive flow of energy in the direction of wave vectors.

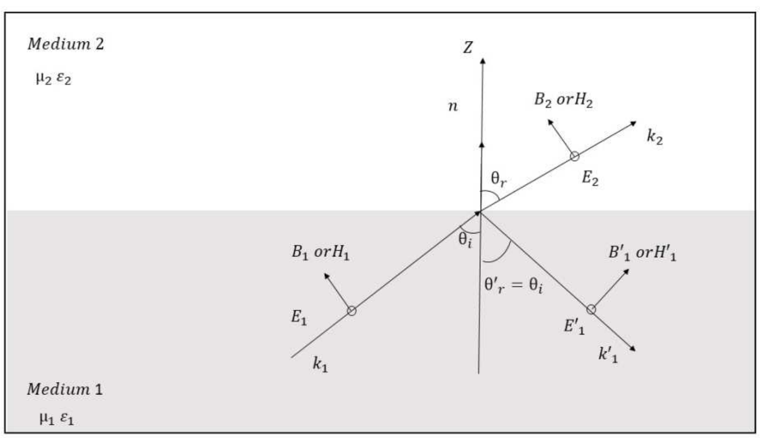


Fig. 9. Reflection and Refraction with polarization perpendicular to the plane of incidence

Since electric field vectors are parallel to theboundary surface, the continuity of the tangential Fig. 9)Reflection and refraction with polarizationcomponent of electric field vectors E at the interface perpendicular to plane of incidence,requires that

$E_{01} + E'_{01} = E_{02}$ -------------------------- (36)

Similarly, the continuity of the tangential component of the magnetic field at the interface requires that

$H_{1t} + H'_{1} = H_{2t}$ -------------------------- (37)

i.e. $-H_{01}cos\theta_i + H'_{01}cos\theta'_r = -H_{02}cos\theta_r$

i.e. $(H_{01} - H'_{01})cos\theta_i = (H_{02})cos\theta_r$ -------------------------- (38)

(Since from law of reflection ;θ_i=θ'_r)

But $B_1 = \frac{k_1 \times E_1}{\omega_1}$; therefore $H_1 = \frac{k_1 \times E_1}{\mu_1 \omega_1}$ (since $B_1 = \mu_1 H_1$)

Or $H_1 = \frac{k_1 n_1 \times E_1}{\mu_1 \omega_1}$

$= \frac{\sqrt{(\mu_1 \epsilon_1)} n_1 \times E_1}{\mu_1}$

(since$k_1 = \frac{\omega_1}{v_1} = \omega_1 \sqrt{(\mu_1 \epsilon_1)}$ i.e. $\frac{k_1}{\omega_1} = \mu_1 - 1$)

This gives-

$H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_{01}$ -------------- (39a)

$H'_{01} = \sqrt{\frac{\epsilon_1}{\mu_1}} E'_{01}$ -------------- (39b)

$H_{02} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{02}$------------- (39c)

Using these results, eqn (38) gives-

$\sqrt{\frac{\epsilon_1}{\mu_1}} (E_{01} - E'_{01})cos\theta_i = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{02}cos\theta_r$ -------------------------- (40)

Eliminating E_{02}, we get-

$$\sqrt{\frac{\epsilon_1}{\mu_1}} (E_{01} - E'_{01})cos\theta_i = \sqrt{\frac{\epsilon_2}{\mu_2}} (E_{01} + E'_{01})cos\theta_r$$

On simplification we get-

$$\left(\frac{(E'_{01}}{(E_{01}}\right)_{\perp} = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}}cos\theta_i - \sqrt{\frac{\epsilon_2}{\mu_2}}cos\theta_r}{\sqrt{\frac{\epsilon_1}{\mu_1}}cos\theta_i + \sqrt{\frac{\epsilon_2}{\mu_2}}cos\theta_r} \qquad \text{-------------------------- (41)}$$

Where the symbol$\perp$ denotes that in the case under consideration electric field is perpendicular to the plane of incidence. Equation (41) gives the ratio of amplitude of electric field vectors in reflected and incident waves, i.e. relative electric field amplitude of reflected wave with respect to incident wave.

Similarly, eliminatingE'_{01}, we get-

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\{E_{01} - (E_{02} - E_{01})\}cos\theta_i = \sqrt{\frac{\epsilon_2}{\mu_2}}E_{02}cos\theta_r$$

Or $$\sqrt{\frac{\epsilon_1}{\mu_1}}\{2E_{01} - E_{02}\}cos\theta_i = \sqrt{\frac{\epsilon_2}{\mu_2}}E_{02}cos\theta_r$$

On simplification, we get-

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\sqrt{\frac{\epsilon_1}{\mu_1}}cos\theta_r}{\sqrt{\frac{\epsilon_1}{\mu_1}}cos\theta_i + \sqrt{\frac{\epsilon_2}{\mu_2}}cos\theta_r} \qquad \text{-------------------------- (42)}$$

This equation gives relative electric field amplitude of refracted wave with respect to that of incidentwave. Equations (41), and (42) are-known as Fresnel's equations. Now let us find the form of these equationin the particular case of non-conducting medium, where$\mu_1 = \mu_2 = \mu_0$, so that

$$n_1 = \sqrt{(K_{1m}K_{1e})} = \sqrt{\frac{\mu_1\epsilon_1}{\mu_0\epsilon_0}} = \sqrt{\frac{\epsilon_1}{\epsilon_0}}$$

$$n_2 = \sqrt{(K_{2m}K_{2e})} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_0\epsilon_0}} = \sqrt{\frac{\epsilon_2}{\epsilon_0}}$$

So that-

$$\frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2/\epsilon_0}{\epsilon_1/\epsilon_0}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad \text{-------------------------- (43)}$$

In this case, equation (41) becomes

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \frac{\sqrt{\frac{\epsilon_1}{\mu_0}}cos\theta_i - \sqrt{\frac{\epsilon_2}{\mu_0}}cos\theta_r}{\sqrt{\frac{\epsilon_1}{\mu_0}}cos\theta_i + \sqrt{\frac{\epsilon_2}{\mu_0}}cos\theta_r}$$

$$= \frac{cos\theta_i - \sqrt{\frac{\epsilon_2}{\epsilon_1}}cos\theta_r}{cos\theta_i + \sqrt{\frac{\epsilon_2}{\epsilon_1}}cos\theta_r}$$

$$= \frac{cos\theta_i - \sqrt{\frac{n_2}{n_1}}cos\theta_r}{cos\theta_i + \sqrt{\frac{n_2}{n_1}}cos\theta_r} \quad \text{-------------------------- (44a)}$$

(Since $\frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$)

Now using Snell's law $\frac{n_2}{n_1} = \frac{sin\ \theta_i}{sin\ \theta_r}$, we obtain

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \frac{cos\theta_i - \frac{sin\ \theta_i}{sin\ \theta_r}cos\theta_r}{cos\theta_i + \frac{sin\ \theta_i}{sin\ \theta_r}cos\theta_r}$$

$$= \frac{sin\ \theta_r cos\theta_i - sin\ \theta_i cos\theta_r}{sin\ \theta_r cos\theta_i + sin\ \theta_i cos\theta_r}$$

i.e. $$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \frac{sin\ (\theta_i - \theta_r)}{sin\ (\theta_i + \theta_r)} \quad \text{-------------------------- (44b)}$$

Similarly, equation (42) becomes

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\sqrt{\frac{\epsilon_1}{\mu_0}}cos\theta_i}{\sqrt{\frac{\epsilon_1}{\mu_0}}cos\theta_i + \sqrt{\frac{\epsilon_2}{\mu_0}}cos\theta_r}$$

$$= \frac{2cos\theta_i}{cos\theta_i + \sqrt{\frac{\epsilon_2}{\epsilon_1}}cos\theta_r}$$

$$= \frac{2cos\theta_i}{cos\theta_i + \sqrt{\frac{n_2}{n_1}}cos\theta_r} \quad \text{-------------------------- (45a)}$$

Using Snell's law

$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2cos\theta_i}{cos\theta_i + \frac{sin\,\theta_i}{sin\,\theta_r}cos\theta_r}$

$$= \frac{2sin\,\theta_r cos\theta_i}{sin\,\theta_r cos\theta_i + sin\,\theta_i cos\theta_r}$$

$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2cos\theta_i sin\,\theta_r}{\sin(\theta_i + \theta_r)}$ -------------------------- (45b)

Equations (44) and (45) represents Fresnel's equations for non-conducting media when electric field vectorEis perpendicular to the plane of incidence.Fresnel's equations lead to following results

(a) When (n1/n2) < 1 i.e. n1<n2 or in words when an electromagnetic wave in incident on the interfaceof two dielectrics from a medium, then from Snell's law sin θ_i/sin θ_r = n2/n1> 1; sin θ_i> sin θ_r, or θ_i>θ_r; i.e. refracted rays are deviated towards the normal hence from equation (44b) is negative thereby indicating that the reflected and the incident waves are in a opposite phases atthe interface. In other words, when an electromagnetic wave is reflected from a denser medium; it suffers a phase change of π radians

(b) When (n1/n2) >1 i.e. n1>n2 or in words, when an electromagnetic wave is incident on the interface of two dielectric from a denser medium, then from Snell's law

$$\frac{\sin\theta_i}{\sin r} = \frac{n_2}{n_1} < \ 1 \; or \; \theta_i < \theta_r$$

i.e. refracted ray is deviated away from the normal and hence from equation (44a), $\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}$ thereby indicating that reflected and incident waves are in same phase.

(c) In both the above cases (a) and (b), $\left(\frac{E_{02}}{E_{01}}\right)_{\perp}$ in equation (45b) is always positive; there by indicatingthat the refracted or transmitted wave docs not suffer any phase change.

6.7 COHERENCE

Coherence has been explained in thc context of wave polarization, i.e. if the two components of
the E wave are expressed as -

$$E_x = E_1(t)\sin\omega t$$

$$E_y = E_2(t)\sin\omega t + \ \delta t$$

where all the three time functions are independent, then the resultant wave is said to completely unpolarized or incoherent (which implies that a completely polarized wave is a coherent wave), it being assumed that –

$$\langle {E_1}^2(t) \rangle = \langle {E_2}^2(t) \rangle$$

Coherence will now be discussed in a different sense. Let us consider a plane E wave travelling to the right as shown in Figure 10 . As the wave travels from point 1 to point 2, the phase at point 2 will be retarded from that at point 1 by an amount equal to βd. However, on the wave-front, points 1 and 3 will be in phase (i.e. zero phase difference). Similarly, the phase difference points 2 and 4 {both on the moved position of the wave-front) will also be zero. The condition described herecharacterize the providing of wave-front coherence as distinct from the situation regarding polarization. In homogeneous media, wave-front coherence exists, but the waves not necessarily polarization coherent.

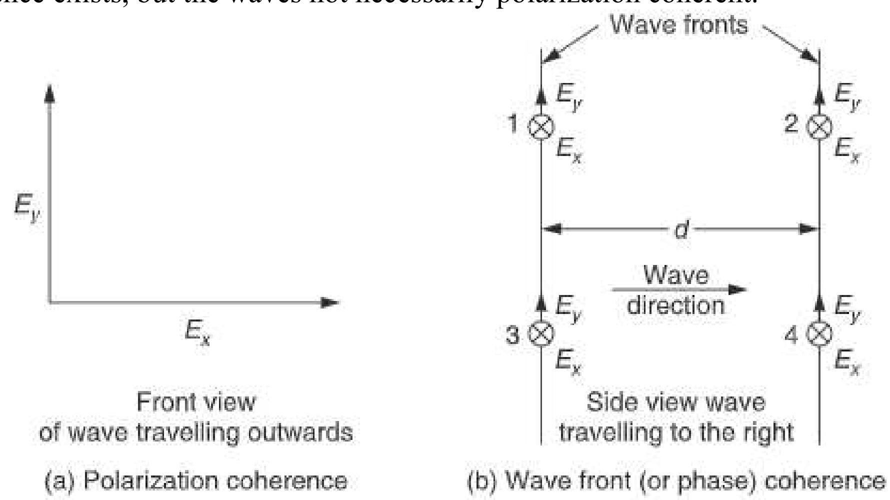


Figure 10 Two types of coherence in waves

In non-homogeneous media there can exist wave-front incoherence, in which case it is not possible to define a wave-front, i.e. an equiphase surface. In such a medium, referring to Figure 10, the E_x component of the field at points 1 and 3 will not necessarily in phase and be time-independent. Wave-front incoherence can exist for wavelengths of 10 - 20 m for waves coming through earth' s ionosphere from celestial radio source.

6.8 SUMMARY

Present chapter discusses the change in electromagnetic quantities across different interfaces. Boundary conditions for electric field, electric displacement, magnetic field and magnetic induction has been obtained. These relationships later have been used to derive the expressions for several optical phenomena like reflection, refraction and polarization etc.

6.9 Glossary

Interface – Boundary
Homogeneous- uniform
Specified- particular
Limited- restricted

6.10 References

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6.11 Suggested Readings

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2. Feynman Lectures Vol.2, R.P.Feynman, R.B.Leighton, M. Sands, 2008, Pearson Education
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6.12 Terminal Questions

1. A homogeneous dielectric ($\varepsilon_r = 2.5$) fills region 1 ($x \leq 0$) while region 2 ($x \geq 0$) is free space. If $D_1 = 12a_x - 10a_y + 4a_z\, nC/m^2$, find D_2

2. It is found that $E = 60a_x + 20a_y - 30a_z\, mV/m$ at a particular point on the interface between air and a conducting surface. Find D and ρ_s at that point.

3. Given $H_1 = -2a_x + 6a_y + 4a_z\, A/m$ in the region $y - x - 2 \leq 0$ where$\mu_1 = 5\mu_0$, calculate –

(a) M_1 and B_1
(b) H_2 and B_2 in region $y - x - 2 \geq 0$ where $\mu_2 = 2\mu_0$, calculate –

6.12 Answer

1. $D_2 = 12a_x - 4a_y + 1.6\, a_z\, nC/m^2$

2. 0.531 $a_x + 0.177a_y \pm 0.265a_z\, pC/m^2$, 0.619 pC/m^2

3. (a) $M_1 = -8a_x + 24a_y + 16a_z\, A/m,\ B_1 = -12.57a_x + 37.7a_y + 25.13a_z\, \mu Wb/m^2$

(b) $H_2 = -8a_x + 12a_y + 4a_z\, A/m,\ B_2 = -20.11a_x + 30.16a_y + 10.05a_z\, \mu Wb/m^2$

UNIT 7: Electromagnetic radiations

Structure

7.1 Introduction

In this unit, you will study about the ultimate sources of all type electromagnetic radiation. A moving charge produce electromagnetic radiation and the field vectors for such conditions will be calculated. The behavior of such electromagnetic radiations will be discussed. An understanding of special radiations, viz Bremsstrulung and Cronokov will also be conferred.

7.2 Objectives

After studying this unit, you should be able to understand-

- the concept of radiation emission due to a moving charge.
- dipole radiation
- different potential forms of electromagnetic radiations.
- power radiation from a moving charge
- Bremsstrahlung and Cronokov types of radiations.

7.3 Radiation from an accelerated charge particle

Radiation is a process of transporting energy from the electromagnetic field of an accelerating charge. Radiation is an irreversible flow of electromagnetic energy from the source (charges) to infinity. This is possible only because the electromagnetic fields associated with accelerating charges fall off as 1/r instead of $1/r^2$ as in the case for charges at rest or moving uniformly. So the total energy flux obtained from the Poynting flux is finite at infinity.

To understand this, let us assume a source which is placed at origin. The energy radiating by the source at time t_0 is $P(t_0)$. When the time passes to t, the radiated energy covered a distance r with the speed of light. Therefore, we can express $P(t_0)$ as below:

$$P\ (t_0) = \lim_{r\to\infty} P\ \left(r, t_0 + \frac{r}{c}\right)$$

Where power is integral over the area (of sphere with radius r) of Poynting vector:

$$P = \oint \boldsymbol{S}.d\boldsymbol{a} = \frac{1}{\mu_0}\oint(\boldsymbol{E}\times\boldsymbol{B}).d\boldsymbol{a}$$

The above equations represent the energy per unit time (radiation) that is carried away and never come back. Thus the study radiation involves the electric and magnetic fields which construct Poynting vector results in the radiation (in terms of power) on integrating over a large spherical surface.

7.4 Radiation from dipole

A dipole is defined as the two and equal opposite point charges separated by very small distance say d. Now, if we suppose that these two point charges are connected by a thin metallic wire and can move along it with angular frequency ω

Than at any time t, total charge

$$q(t) = q_0 \cos(\omega t) \qquad (7.1)$$

and electric dipole moment is

$$p(t) = p_0 \cos(\omega t)\hat{z} \qquad (7.2)$$

Where $p_0 = q_0 d$, is the maximum dipole moment.

The retarded potential is

$$V(r,t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0 \cos\left[\omega\left(t - {r_+}/{c}\right)\right]}{r_+} - \frac{q_0 \cos\left[\omega\left(t - {r_-}/{c}\right)\right]}{r_-}\right\} \qquad (7.3)$$

where

$$r_\pm = \sqrt{r^2 \mp rd\cos\theta + \left({d}/{2}\right)^2}$$

As the distance between the charges is very small as compared to r ($d \ll r$).

$$r_\pm \cong \sqrt{r^2 \mp rd\cos\theta}$$

$$= r\sqrt{1 \mp \frac{d}{r}\cos\theta}$$

since $\frac{d}{r} \ll 1$

Hence, we can use here the binomial expansion

$$\frac{1}{r_\pm} = \frac{1}{r}\left(1 \pm \frac{d}{2r}\right)^{-1/2}$$

i.e.

$$\frac{1}{r_+} = \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

And, $$\frac{1}{r_-} = \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$$

Now,

$$\cos\left[\omega\left(t - {}^{r_+}\!/_{c}\right)\right] \cong \cos\left[\omega\left\{t - \frac{r}{c}\left(1 - \frac{d}{2r}\cos\theta\right)\right\}\right]$$

$$\cos\left[\omega\left(t - {}^{r_+}\!/_{c}\right)\right] \cong \cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega d}{2c}\cos\theta\right]$$

Now use the formula $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Hence,

$$= \cos\omega\left(t - \frac{r}{c}\right)\cos\frac{\omega d}{2c}\cos\theta - \sin\omega\left(t - \frac{r}{c}\right)\sin\frac{\omega d}{2c}\cos\theta$$

For a perfect dipole, d is very small as compared to $\frac{c}{\omega}$, where ω is related to a wavelength $\lambda = \frac{2\pi c}{\omega}$ and λ is very large to d.

$$\cos\left[\omega\left(t - {}^{r_+}\!/_{c}\right)\right] \cong \cos\left[\omega\left(t - \frac{r}{c}\right) - \frac{\omega d}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)\right]$$

From the above equations, we can write the potential of an oscillating perfect dipole:

$$V(r,\theta,t) = \frac{\mathrm{p}_0\cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega}{c}\sin\left[\omega\left(t - \frac{r}{c}\right)\right] + \frac{1}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}$$

In static limit ($\omega \to 0$), the second term provides the familiar potential of a stationary dipole:

$$V(r,\theta,t) = \frac{\mathrm{p}_0\cos\theta}{4\pi\varepsilon_0 r^2}$$

As we are concerned about the fields at large distances from the source with the condition ($d \ll \lambda \ll r$) and ($r \gg \frac{c}{\omega}$). For such conditions, potential is now reduced to

$$V(r,\theta,t) = -\frac{\mathrm{p}_0\omega}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - \frac{r}{c}\right)\right] \qquad (7.4)$$

Calculation of vector potential:

The vector potential is determined by the current flowing in the z-direction in the wire:

$\boldsymbol{I}(t) = \frac{dq}{dt}\boldsymbol{k} = -q_0\omega\sin(\omega t)\boldsymbol{k}$

Where $\boldsymbol{k}$ is the unit vectoralong z direction (Fig 7.1).

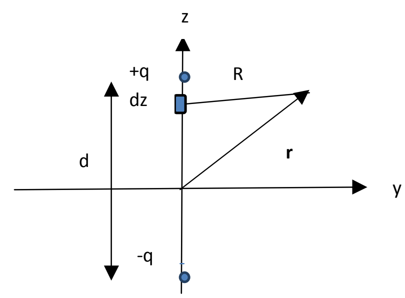


Fig (7.1)

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\boldsymbol{k}}{R}dz$$

After solving the above equation (keeping only first order 1 and 2, we have

$$\boldsymbol{A}(\boldsymbol{r},\theta,t) = \frac{\mu_0\, p_0\omega}{4\pi r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\boldsymbol{k} \qquad (7.5)$$

Calculations of E and B:

The electric field and magnetic field can be calculated from the desired relations:

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

Where, $\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$

$$\nabla V = -\frac{\mathrm{p}_0\omega}{4\pi\varepsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega}{rc}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{r} - \frac{\sin\theta}{r^2}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\}$$

As we know $(r \gg \frac{c}{\omega})$.

$$\nabla V = -\frac{\mathrm{p}_0\omega^2}{4\pi\varepsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\frac{\omega}{rc}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$

Whereas,

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\mu_0\mathrm{p}_0\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right](\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

On substituting the values of ∇V and $\frac{\partial \boldsymbol{A}}{\partial t}$, we get electric field as

$$\mathbf{E} = -\frac{\mu_0\mathrm{p}_0\omega^2}{4\pi r}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta} \qquad (7.6)$$

Also, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

Where

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\emptyset}$$

$$\nabla \times \boldsymbol{A} == -\frac{\mu_0 \mathrm{p}_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\left[\omega\left(t - \frac{r}{c}\right)\right] + \left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}\hat{\emptyset}$$

As we know $(r \gg \frac{c}{\omega})$.

$$\nabla \times \boldsymbol{A} = -\frac{\mu_0 \mathrm{p}_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\emptyset}$$

Hence, the magnetic field is calculated as

$\mathbf{B} = -\frac{\mu_0 \mathrm{p}_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\emptyset}$ (7.7)

The above equations of electric and magnetic fields represents a monochromatic wave with wavelength (λ), frequency(ω) propagating in radial direction. This, in other words, represents electromagnetic wave in free space.

The pointing vector gives the energy radiated by such oscillating dipole:

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}) = \frac{\mu_0}{c}\left\{\frac{\mathrm{p}_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}^2 \hat{r}$$

The intensity is obtained by arranging (in time) over a complete cycle:

$\langle \mathbf{S} \rangle = -\frac{\mu_0 \mathrm{p}_{0_0}^2 \omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$ (7.8)

The total power radiated is found by integrating $\langle \mathbf{S} \rangle$over a sphere of radius r:

$$\langle P \rangle = \int \langle \mathbf{S} \rangle . d\boldsymbol{a} = -\frac{\mu_0 \mathrm{p}_{0_0}^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta d\emptyset.$$

$\langle P \rangle = \frac{\mu_0 \mathrm{p}_{0_0}^2 \omega^4}{32\pi c}$ (7.9)

7.5Retarding potential

Consider Poisson's equation for the static case with scalar potential (V) and vector potential (**A**).

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho\ ,\ \nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

where ρ and $\boldsymbol{J}$ defines charge and current densities respectively. The solution of above equations can be given as:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\mathscr{r}}d\tau' \text{and} \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}d\tau'$$

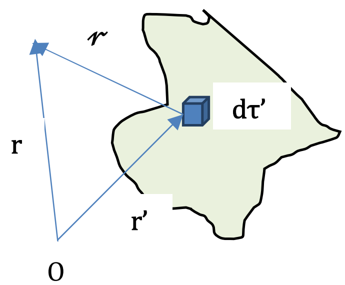


Fig (7.2)

Where $\mathscr{r}$ is the distance from the source point $\boldsymbol{r}'$to the field point **r** (Fig 7.2).Let us consider now that some information is travelling at a speed of electromagnetic wave (c) and travels a distance$\mathscr{r}$ upto a reciever. But, the information received by the receiver is that which was generated at earlier time. Hence, we can say that there is a delay in receiving the information. This delay in time is known as retarded time (t_r), which can be defined mathematically as below:

$$t_r = t - \frac{\mathscr{r}}{c} \qquad (7.10)$$

This earlier time (t_r) is called retarded time when the message left. Since this message covered a distance $\mathscr{r}$, the delay is $\mathscr{r}/c$.

In view of retarded time, the above mentioned potentials can be written in a more general way as:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}',t_r)}{\mathscr{r}}d\tau' \qquad (7.11)$$

And in similar way $$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{\mathscr{r}}d\tau' \qquad (7.12)$$

Where $\rho(\boldsymbol{r}',t_r)$ is the charge density at $\boldsymbol{r}'$ point and retarded timet_r.

The above mentioned potentials are called as retarded potentials.

7.6 Lienard -Wiechert Potential

From the last section, we understand the retarded potential. In this section, we will derive the famous Lienard - Wiechert potential for a moving point charge using the concept of retarded potential. For this purpose, consider a moving charge q on a specified trajectory. It is also learnt from earlier units that Poisson's equation can be described as below:

$\nabla^2 V = -\frac{1}{\epsilon_0}$ and $\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$

The solution of above scale potential can be obtained as below for a travel of electromagnetic wave for a distance ($\mathscr{r}$) from a point source **r'** to a field point **r** unrewarded time t_r. Hence, the scalar and vector (retarded) potentials due to moving charge can be given by -

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}', t_r)}{\mathscr{r}} d\tau' \quad (7.13)$$

and $$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}', t_r)}{\mathscr{r}} d\tau'$$
(7.14)

where, the retarded time can be defined as $t_r = t - \frac{\mathscr{r}}{c}$ and $\rho(r', t_r)$ and $\boldsymbol{J}(\boldsymbol{r}', t_r)$ are the charge and current densitiesthat prevailed at point **r'** at the retarded time t_r. The main difficulty to integrate equation (7.13) and (7.14) because time (t_r) is not fixed and hence the volume of integration. This implies that the volume is not specified and nonzero.

The above-mentioned integrals can be solved by using one dimensional delta function. As we know that

$\delta(u) = 0$ for $u \neq 0$ and $\delta(0) \neq 0$

and $$\int_{-\infty}^{\infty} \delta(u) du = 1$$
(7.15)

Using the above property of delta function, the scalar and vector potential given by equation (7.13 and 7.14) can be re-written as

$$V(r, t) = \frac{1}{4\pi\varepsilon_0}\iint_{-\infty}^{\infty} \frac{\rho(r', t_r)}{\mathscr{r}} \delta\left(t_r - \left|t - \frac{r}{c}\right|\right) d\tau' dt_r$$
$$\boldsymbol{A}(r, t) = \frac{\mu_0}{4\pi}\iint_{-\infty}^{\infty} \frac{\boldsymbol{J}(r', t_r)}{\mathscr{r}} \delta\left(t_r - \left|t - \frac{r}{c}\right|\right) d\tau' dt_r \quad (7.16)$$

However, $\rho(r', t_r) d\tau' = dq$ is the charge in the volume element $d\tau'$ and $\boldsymbol{J}(r', t_r) = \rho(r', t_r)\boldsymbol{v}(t_r)$ is the current density produced by an element of charge density $\rho(r', t_r)$ moving with velocity $\boldsymbol{v}(t_r)$, so

$$V(r, t) = \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\int_{-\infty}^{\infty} \frac{dq}{\mathscr{r}} \delta\left(t_r - t + \frac{r}{c}\right) d\tau'$$
$$\boldsymbol{A}(r, t) = \frac{\mu_0}{4\pi}\int_{\tau'}\int_{-\infty}^{\infty} \frac{\boldsymbol{v}(t_r)}{\mathscr{r}} \delta\left(t_r - t + \frac{r}{c}\right) d\tau' \quad (7.17)$$

If the charge distribution limits to point charge q, we can rewrite the equation (7.17) as

$$V(r,t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r$$
$$\boldsymbol{A}(r,t) = \frac{\mu_0 q}{4\pi}\int_{-\infty}^{\infty}\frac{v(t_r)}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r \qquad (7.18)$$

To solve above equation, we substitute

$$\zeta = t_r - t + \frac{r}{c} \qquad (7.19)$$

On differentiating with respect to t_r

$$\frac{d\zeta}{dt_r} = 1 + \frac{1}{c}\frac{dr}{dt_r} \qquad (7.20)$$

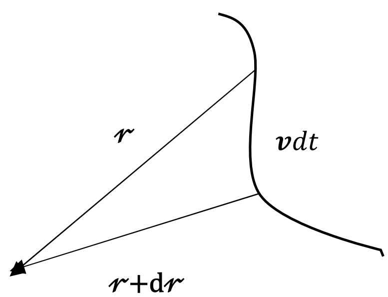


Fig (7.3)

Now consider

$$r^2 = \boldsymbol{r}.\boldsymbol{r}$$

$$2r\frac{d\boldsymbol{r}}{d\tau_r} = 2\boldsymbol{r}.\frac{d\boldsymbol{r}}{d\tau_r}$$

$$\frac{dr}{d\tau_r} = \frac{\boldsymbol{r}}{r}.\frac{d\boldsymbol{r}}{d\tau_r} \qquad (7.21)$$

For a charge moving from position vector $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$ in time interval dt_r with velocity$\boldsymbol{v}$, $\boldsymbol{r}$can be written as:

$$\boldsymbol{r} = (\boldsymbol{r} + d\boldsymbol{r}) + \boldsymbol{v}dt_r$$

$$\frac{d\boldsymbol{r}}{d\tau_r} = -\boldsymbol{v}$$

The equation (7.21) is then becomes

$$\frac{dr}{d\tau_r} = -v.\frac{\boldsymbol{r}}{r}$$

From equation (7.20)

$$\frac{d\zeta}{dt_r} = 1 - \frac{\boldsymbol{v}}{c}.\frac{\boldsymbol{r}}{r}$$

or

$$\frac{d\zeta}{dt_r} = \frac{cr - \boldsymbol{v}.\boldsymbol{r}}{cr}$$

$dt_r = \frac{cr\, d\zeta}{cr - \boldsymbol{v}.\boldsymbol{r}}$(7.22)

This value of dt_rcan be substituted in equation (7.18), the integrals of scalar and vector potentials reduced to the following form:

$$V = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\delta(\zeta)}{r}\frac{cr d\zeta}{cr - \boldsymbol{v}.\boldsymbol{r}}$$
$$\boldsymbol{A} = \frac{\mu_0 q}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{v}\delta(\zeta)}{r}\frac{cr d\zeta}{cr - \boldsymbol{v}.\boldsymbol{r}} \qquad (7.23)$$

From the property of delta function, we have

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{c}{cr - \boldsymbol{v}.\boldsymbol{r}}\right]_{\zeta=0}$$
$$\boldsymbol{A} = \frac{\mu_0 q}{4\pi}\left[\frac{c\boldsymbol{v}}{cr - \boldsymbol{v}.\boldsymbol{r}}\right]_{\zeta=0} \qquad (7.24)$$

But $\zeta = 0$ implies$t_r - t + \frac{r}{c} = 0$ or $t_r = t + \frac{r}{c}$, which is retarded time

Therefore,

$$V(r,t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{c}{cr - \boldsymbol{v}.\boldsymbol{r}}\right]_{ret}$$
$$\boldsymbol{A}(r,t) = \frac{\mu_0 q}{4\pi}\left[\frac{c\boldsymbol{v}}{cr - \boldsymbol{v}.\boldsymbol{r}}\right]_{ret} \qquad (7.25)$$

This form of potentials which exhibit the dependence of the potentials on the velocity of the particle was first of all given by A. Lienard and E. Wiechert. On their name, these scalar and vector potential forms are known as Liennard-Wiechert potentials.

7.7 Larmor's radiation power formula and interpretation

This Larmor's formula describes the power radiated by an accelerated charge particle. A moving charge can have a uniform velocity of acceleration. As in earlier section we have developed the scalar and vector potential forms, famously known as Liennard-Wiechert potentials, for a moving charge. These potentials will be useful to calculate the associated electric and magnetic fields. Finally, poynting vector will be calculated and represents the energy flow per unit area per unit time *i.e.* power radiated per unit area by moving charge.

From equation (6), according to Liennard-Wiechert potentials:

$$V(r,t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r$$
$$\boldsymbol{A}(r,t) = \frac{\mu_0 q}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{v}(t_r)}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r \qquad (7.26)$$

We know that the fields of a charged particles are related to scalar and vector forms as below:

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

By substituting the values of V and A, we have

$$\boldsymbol{E}(r.t) = -\left[\frac{q}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{-\infty}^{\infty}\frac{1}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r + \frac{\mu_0 q}{4\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{\boldsymbol{v}(t_r)}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r\right]$$
(7.27)

and $\boldsymbol{B}(r,t) = \boldsymbol{\nabla} \times \frac{\mu_0 q}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{v}(t_r)}{r}\delta\left(t_r - t + \frac{r}{c}\right)dt_r$ (7.28)

On solving equation (7.27) and (7.28) (Ref 3) using delta functions, the simplified forms of electric and magnetic fields are

$$\boldsymbol{E}(r.t) = -\frac{q}{4\pi\varepsilon_0}\left[\frac{(\boldsymbol{r} - r\boldsymbol{\beta})(1-\beta^2)}{(r - \boldsymbol{r}.\boldsymbol{\beta})^3} + \frac{\boldsymbol{r} \times (\boldsymbol{r} - r\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}}{c(r - \boldsymbol{r}.\boldsymbol{\beta})^3}\right]$$

And

$$\boldsymbol{B}(r,t) = \frac{\hat{\boldsymbol{r}} \times \boldsymbol{E}}{c} = \frac{\boldsymbol{r} \times \boldsymbol{E}}{rc}$$

or

$$\boldsymbol{B}(r,t) = \frac{\mu_0 cq}{4\pi}\left[\frac{(\boldsymbol{\beta} \times \boldsymbol{r})(1-\beta)^2}{(r - \boldsymbol{r}.\boldsymbol{\beta})^3} + \frac{\boldsymbol{r} \times \{\boldsymbol{r} \times (\boldsymbol{r} - r\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{rc(r - \boldsymbol{r}.\boldsymbol{\beta})^3}\right]$$

Where $\boldsymbol{\beta}$is a time derivative which has the values as $\boldsymbol{\beta} = \frac{\boldsymbol{v}(t_r)}{c}$ and$\dot{\boldsymbol{\beta}} = \frac{\dot{\boldsymbol{v}}(t_r)}{c} = \frac{\boldsymbol{a}(t_r)}{c}$.

The above fields (**E** and **B**) divide themselves naturally into velocity fields (independent of acceleration) and acceleration field (independent of velocity).

$$\boldsymbol{E} = \boldsymbol{E}_v + \boldsymbol{E}_a \text{ and } \boldsymbol{B} = \boldsymbol{B}_v + \boldsymbol{B}_a$$

The *velocity field* falls off as $\frac{1}{r^2}$ and is just the generalisation of Coulomb's Law to uniformly moving charges. The second term (*acceleration field* contribution) falls off as $\frac{1}{r}$, is proportional to the particle's acceleration and is perpendicular to$\boldsymbol{r}$. In order to calculate the energy radiated by a particle, we must integrate the normal component of S over the surface of a sphere. Hence only the acceleration field will contribute to produce radiation due to a moving charge.

Hence, the electric field and magnetic field due to acceleration constitute the radiation field of the moving charge:

$$\boldsymbol{E_{rad}}(r.t) = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{r}\times(\boldsymbol{r}-r\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}}{c(r-\boldsymbol{r}.\boldsymbol{\beta})^3}\right] \quad (7.29)$$

and

$$\boldsymbol{B_{rad}}(r,t) = \frac{\hat{\boldsymbol{r}}\times\boldsymbol{E_{rad}}(r,t)}{c} \quad (7.30)$$

When $(\beta \ll 1)$, the radiation fields simplify to

$$\boldsymbol{E_{rad}}(r,t) = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}}\times\frac{1}{c^2}(\hat{\boldsymbol{r}}\times\boldsymbol{a})\right]$$

Where $\dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}$.

$\boldsymbol{E_{rad}}$lies in the plain containing $\hat{\boldsymbol{r}}$and $\boldsymbol{a}$and $\boldsymbol{B_{rad}}$is perpendicular to this plane. If θ is the angle between $\hat{\boldsymbol{r}}$and $\boldsymbol{a}$**,** then

$$\boldsymbol{E_{rad}} = \boldsymbol{c}B_{\boldsymbol{rad}} = \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{a}}{r c^2}\sin\theta$$

The pointing vector gives the energy radiated along:

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$$

$$\boldsymbol{S} = \frac{\mu_0}{16\pi^2 c}\frac{q^2a^2}{r^2}\sin^2\theta\,\hat{\boldsymbol{r}} \quad (7.31)$$

This is the required expression and represent the energy flow per unit area per unit time i.e. power radiated per unit area.

We may express the angular distribution of the radiation as the power radiated per unit solid angle, i.e.

$$\frac{dP}{d\Omega} = \frac{(4\pi r^2\hat{\boldsymbol{r}}).\boldsymbol{S}}{4\pi}$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\sin^2\theta$$

The total power radiated is obtained by integrating the above equation over the entire sphere

$$P = \int dP = \int_{4\pi}\frac{\mu_0 q^2 a^2}{16\pi^2 c}\sin^2\theta\, d\Omega$$

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int_0^{\pi}\int_0^{2\pi}(\sin^2\theta)\sin\theta \; d\theta d\emptyset$$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3} \qquad (7.32)$$

This is the famous Larmor's formula for the power radiated by an accelerated (non relativistic) charge particle.

Radiation by an accelerated (relativistic) charge particle:

To find the appropriate generalization of Larmor formula for arbitrary velocities of charge, Lorentz transformations will be used. The power (P=dE/dt) is a Lorentz invariant quantity. The main aim is to find a Lorentz invariant which reduces to Larmor formula for $\beta \ll 1$.

From Larmor formula where ($m\boldsymbol{a} = \frac{d\boldsymbol{p}}{dt}$):

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{d\boldsymbol{p}}{dt}\cdot\frac{d\boldsymbol{p}}{dt}\right)$$

Lorentz invariant generalization will be

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp_\mu}{dt_0}\cdot\frac{dp_\mu}{dt_0}\right)$$

Where p_μ is a four vector given by $p_\mu = \left(\boldsymbol{p}, \frac{iE}{c}\right)$ **and** t_0 is the proper time interval given by $t = \gamma t_0$, where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

The above equation will be reduced as $\beta \to 0$,

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{m^2c^3}\left(\left[\frac{dp_1}{dt_0}\right]^2 + \left[\frac{dp_2}{dt_0}\right]^2 + \left[\frac{dp_3}{dt_0}\right]^2 + \left[\frac{dp_4}{dt_0}\right]^2\right)$$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{m^2c^3}\left(\left[\frac{d\boldsymbol{p}}{dt_0}\right]^2 + \left\{\frac{d}{dt_0}\left(\frac{i}{c}E\right)\right\}^2\right)$$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2\gamma^2}{m^2c^3}\left(\left[\frac{d\boldsymbol{p}}{dt}\right]^2 + \frac{1}{c^2}\left\{\frac{dE}{dt}\right\}^2\right)$$

Now from $E = \frac{mc^2}{\sqrt{1-\beta^2}}$ and $= \frac{mv}{\sqrt{1-\beta^2}}$, substitute the values of $\frac{dE}{dt}$ and $\frac{d\boldsymbol{p}}{dt}$ in the above equation and after simplifying we get,

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2q^2\gamma^6}{3c}\left(\dot{\beta}^2 - \beta^2\dot{\beta}^2\sin^2\theta\right) \qquad (7.33)$$

Where θ is the angle between velocity ($\beta = \frac{v}{c}$) and acceleration ($\dot{\beta} = \frac{a}{c}$). The above expression is also known as Lienard radiation formula. There will be three cases

Case I: If the moving particle has non-relativistic motion, then as $\beta \to 0, \gamma \to 1$. Therefore, above formula reduces to Larmor formula

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}$$

Case II: If the moving particle has relativistic motion and velocity and accelerations are linear to each other this implies $\theta = 0\ degree$ and hence

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}\gamma^6$$

Example of such type of radiation is Bremsstrahlung.

Case III: If the moving particle has relativistic motion and velocity and accelerations are perpendicular to each other this implies $\theta = 90\ degree$ and hence

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}\gamma^4$$

Examples of such type of radiation are found in circular acceleration.

7.8 Bremsstrahlung radiation

Bremsstrahlung is a German originated word, which means 'breaking the radiation'. Bremsstrahlung can be observed when a free electron is accelerated in the field of an ion. The English name of Bremsstrahlung is "free-free," represents the transition between two unbound electronic states. Electrons with energies smaller than 1 MeV lose their energies mostly by excitation or ionization of the atoms through which they pass. While, electrons with energies greater than 1 MeV will be radiating a considerable fraction of energy in the form of X-rays during inelastic collision with atomic nuclei.

When high velocity electron passes close to nucleus of an atom, it experiences strong coulomb's force of attraction due to the nucleus and is suddenly slowdown. Therefore, the electron suffers a deflection in its path. During its retardation, the electrons emits electromagnetic radiation, which falls in X-ray region of the electromagnetic spectrum. Due to different losses in velocity of the incident electrons, e.m. radiations of all possible wavelength with in a certain range are emitted and, therefore, a continuous spectrum of X-ray is produced. These radiations having continuous spectrum is called Bremsstrahlung. It has been found that rate of loss of energy per unit path length of the incident electron by this type of radiative interaction, is proportional to square of atomic number Z of the absorbing material i.e.

$$-\frac{dE}{dx} \propto Z^2$$

The energy loss by radiative collisions depends on Z^2 in contrast to Z dependence in case of ionization losses due to atomic electrons of absorber material. The loss of energy due to radiative collision is, therefore more prominent in heavy material. It has also been found that the loss of energy also increases with the energy E of the incident electrons.

7.9 Cerenkov radiation

As of now, we know that a moving charge radiates electromagnetic waves. If the velocity of a charged particle becomes greater than the phase velocity of light in a dielectric medium, the emitted electromagnetic radiation is called Cerenkov radiation. Electromagnetic shock wave (light) emitted by a high-speed charged particle when the particle passes through a transparent, non-conducting, solid material at a speed greater than the speed of light in the material is known as Cerenkov radiation The discovery of Cerenkov radiation was made in 1934 by Soviet physicist Pavel Alekseyevich Cherenkov. He was awarded Nobel prize in 1958 for this discovery. On his name, it is called Cerenkov radiation.

Cerenkov radiation is an observable effect. A brief flash of light is observed when a charge particle travels faster than the speed of light in a transparent medium like water or air. This has been firstly observed when a beta electron, a light weight charged particle, is accelerated in water to the energy more than or 175 KeV energy or velocity 2 x 10^8 m/s (speed of light in water). Cerenkov radiation is not possible with heavy and too slow alpha particles. Hence, Cerenkov radiation is found very useful to detect high-energy beta (or other lighter and faster) charged particles in nuclear reactors or in a spent nuclear fuel pool where such highly energetic beta particles are released as the fission fragments decay. The glow is visible also after the chain reaction stops (in the reactor). Cerenkov radiation are also being used to check the remaining nuclear fuel in controlled nuclear reactions. Therefore, it can be very helpful for measuring of fuel burn up.

Self Assessment Question (SAQ) 1: Accelerated **c**harge particles ____________energy.

Self Assessment Question (SAQ) 2: Larmor's formula used to calculate the radiated________by an accelerated charge particle.

Self Assessment Question (SAQ) 3:___________________is an example of power radiated by a moving charge particle which has co-linear velocity and acceleration.

Self Assessment Question (SAQ) 4: What is the speed of electromagnetic potentials **A** and V propagated in space.

7.10 Summary

In this unit, we have analyzed some mathematical and physical problems which arise due to moving charge and due to electric dipole during its oscillation along the vector of the dipole moment. The concept of retarded potential explains that the present potentials at a given position are due to the past positions of the charge. This means the charges does not instantaneously establish the potential but the potentials are related casually to motion of charge. The Liénard–Wiechert potentials explains the electrodynamics of a moving charge in terms of a vector potential and a scalar potential. Electromagnetic radiation in the form of waves can be obtained from these potentials. It was also concluded that charge particles with uniform motion does not radiate energy, while accelerated charge particles produce radiation. The power radiated by such accelerated charged particles can be calculated with the help of Larmor's formula. Bremsstrahlung radiations are special type of radiation occurs when electrons are slowed down in a target and gives rise to continuous spectrum.

7.11Glossary

Retarding potential –the potentials that can provide earlier values based on retarded time.

Bremsstrahlung- an example of power radiated by a moving charge particle which has co-linear velocity and acceleration

Charge density- the charge per unit volume

Dipole- two equal and opposite charges separated by a finite distance

Current density- current flowing per unit area

7.12 References

1. Electrodynamics, IGNOU, New Delhi

2. Introduction to Electrodynamics, D.J. Griffiths, Pearson Publication, Chennai

3. Classical Electrodynamics, J.D. Jackson, Willey Eastern Ltd., New Delhi

7.13 Suggested Readings

1. Electrodynamics, Gupta Kumar, Pragati Publication

2. Foundation of Electromagnetic Theory, J.R. Reitz, F.J. Milford and R.W. Christy, Narosa publications, New Delhi

3. Foundation of Electrodynamics, Parry Moor, Dover publication, New York.

7.14 Terminal Questions

(Should be divided into Short Answer type, Long Answer type, Numerical, Objective type)

7.14.1 Objective type

1. Power radiated by a point charge is proportional to
(a) position
(b) acceleration
(c) square of acceleration
(d) none of these

2. A combination of equal and opposite charges separated by a finite distance is called
(a) dipole
(b) magnet
(c) antenna
(a) none of these

3. In synchrotron radiation, a charge moves with relativistic velocity in
(a) linear path
(b) parabolic path
(c) circular path
(d) none of these

4. Electric field (**E**) is calculated using scalar field (V) and vector field (**A**) as
(a) $\boldsymbol{E} = -\boldsymbol{\nabla}.\boldsymbol{A} - \frac{\partial V}{\partial t}$
(b) $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$
(c) $\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$
(d) $\boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{A}$

5. Electric field (**E**) is calculated using scalar field (V) and vector field (**A**) as
(a) $\boldsymbol{E} = -\boldsymbol{\nabla}.\boldsymbol{A} - \frac{\partial V}{\partial t}$
(b) $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$
(c) $\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$
(d) $\boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{A}$

6. Poynting vector (**S**) is calculated from electric field (**E**) and magnetic field (**B**) as
(a) $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{B} \times \boldsymbol{E})$
(b) $\boldsymbol{S} = \frac{1}{\varepsilon_0}(\boldsymbol{B} \times \boldsymbol{E})$
(c) $\boldsymbol{S} = \frac{1}{\varepsilon_0}(\boldsymbol{E} \times \boldsymbol{B})$
(d) $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$

7. If the kinetic energy of incident particles passing through matter is comparable to rest mass energy, then the energy loss by the emission of electromagnetic radiations is called as
   (e) Compton effect
   (f) Bremsstrahlung
   (g) Cerinkov radiation
   (h) Straggling

7.14.2 Short Answer type

1. Why only accelerated charge particles radiate energy.

2. Explain why it is not possible to have a consistent model for a stable atom based on the laws of classical mechanics and electrodynamics.

3. What is the power radiated by a non relativistically accelerated charge particle?

4. What will be the power radiation for a relativistically accelerated charge particle with velocity and acceleration in perpendicular direction. Give one example for such radiation.

5. Why linear accelerators are more efficient to circular one?

7.14.3 Long Answer type

1. Explain retarded potentials. Derive an expression for the retarded potentials and show that they are become the solution of Poisson's equation for stationary charges and steady currents.

2. What are Lienard Wiechert potentials? Calculate the electric and magnetic field vectors for a uniformly moving charged particle using Lienard Wiechert potentials.

3. Using Lienard Wiechert potential derive an expression for the total energy radiated by a uniformly accelerated point charge.

4. Derive an expression for the radiated energy from a high velocity electron moving with an acceleration parallel to the velocity.

5. Show that the radiative energy loss per unit revolution in a circular orbit of radius R meter by an electron (rest mass energy E_0) of total energy E MeV in the relativistic limit is given by

$$\Delta E = \frac{e^2}{3\varepsilon_0 R}\left(\frac{E}{E_0}\right)^4$$

6. Write short notes on-

   (a) Larmor's formula

   (b) Bremsstrahlung and Cerenkov radiation.

7.14.4 Numerical Answer type

1) An early model of hydrogen atom pictured an electron moving in a stationary circular orbit around a proton. If the radius of the orbit is 0.53 Å (Bohr first orbit), Show that on the basis of classical theory the electron would radiate energy at a rate of approximately 0.46 ergs per second?

2) In a betatron accelerator, an electron is revolving in an orbit of radius 50 meter with total energy 20 MeV. Calculate the energy loss due to radiation.

3) If an electron is accelerated in synchrotron with energy 5 BeV in a radis of 10 meter, calculate the power radiated by the electron and energy loss per unit turn.

4) Find the radiation resistance of the wire joining the two ends of the dipole. The wires are in ordinary radio (d=5cm and $\lambda=10^3$ meter)

7.15 Answers

7.15.1 Self Assessment Questions (SAQs):

1. radiates

2. power

3. Bremsstrahlung

4. speed of light

7.15.2 Terminal Questions: Objective type

1. (c), 2. (a), 3. (c), 4. (c), 5. (b) , 6. (d) , 7. (b)

7.15.3 Terminal Questions: Short Answer type

1. The contribution due to accelerated moving charge is perpendicular to its motion. In order to calculate the energy radiated by a particle, we must integrate the normal component of S over the surface of a sphere. Hence only the acceleration field will contribute to produce radiation due to a moving charge.

2. The classical theory represents a non radiative motion of electron in an orbit while according to electrodynamics the orbital motion is an accelerated motion of electron and it should radiate.

3. $P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}$

4. $P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}\gamma^4$, circular accelerators

5. For a given magnitude of applied force the power radiation emitted with transverse acceleration is γ^2 higher than parallel acceleration.

7.15.4 Numerical type questions

1. $P = \frac{1}{4\pi\varepsilon_0}\frac{2e^2a^2}{c^3}$, $a = \frac{v^2}{R}$, $\frac{mv^2}{R} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2}$; On solving P=0.46 x$10^{-7}$watts=0.46 ergs/sec

2. $\Delta E = \frac{e^2}{3\varepsilon_0 R}\left(\frac{E}{E_0}\right)^4 = 0.03\ eV$, where$E_0 = 0.51\ MeV$ for electron

3. $\Delta E = \frac{e^2}{3\varepsilon_0 R}\left(\frac{E}{E_0}\right)^4 = 5.5\ MeV$

4. $P = I^2R = q^2\omega^2\,\mathrm{R}\ \sin^2\omega t \Rightarrow \langle P\rangle = \frac{1}{2}q^2\omega^2R = \frac{\mu_0 q^2 d^2\omega^2}{12\,\pi c} \Rightarrow R = \frac{4\mu_0\pi^2 d^2 c^2}{6\,\pi c\lambda^2} = 2\times10^{-6}\Omega$

UNIT 8: Four Vector Formalism of Maxwell's equation

Structure

8.1 Introduction

The relativistic mechanics indicates that space and time coordinates depend on each other. The measurement of time coordinate in an inertial frame includes both the space and time coordinates of another inertial frame where the event takes place. Therefore, it is natural that space and time coordinators can be handled together as similar as to three space coordinator. This procedure was first developed by H. Minkowski and known as four dimensional formulations or Minkowski space. A vector in four dimensional Minkowski space is called a four vector. These four vectors hold the properties similar to those of ordinary vectors and their components transform from one frame to another follows Lorentz transformation. These four vectors are generally belong to tensors of the first rank.

8.2 Objectives

After studying this unit, you should be able to -

- understand operations of four vector forms
- write quantities in electrodynamics in term of four vector
- write and understand Maxwell's equations in four vector
- derive four vector form of continuity equation and Lorentz force.

8.3 D'Alembertian operator

In the earlier chapter, it has been cleared that gradient is known as the rate of a change of a scalar function. In three dimensions, gradient is represented as:

$$\boldsymbol{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Or

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

This definition of three dimensional gradient operator can be extended to four dimensional gradient operator, which is given by

$$\boldsymbol{\nabla}_\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{i}{c}\frac{\partial}{\partial t}\right)$$

The above operator behaves as a four-operator. Similar to Laplacian operator in three dimension ($\nabla^2 = \boldsymbol{\nabla}.\boldsymbol{\nabla}$), a new operator can also be defined as D'Alembertian operator $\left(\ {}^2 = \boldsymbol{\nabla}_\mu . \boldsymbol{\nabla}_\mu\right)$, which can be defined as below:

$${}^2 = \boldsymbol{\nabla}_\mu . \boldsymbol{\nabla}_\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{i}{c}\frac{\partial}{\partial t}\right) . \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{i}{c}\frac{\partial}{\partial t}\right)$$

$$^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad (1)$$

8.4 The Electromagnetic field tensor

The electromagnetic field vectors **E** and **B** can be expressed in terms of electromagnetic potential **A** and ∅ as

$$\boldsymbol{E} = -\nabla\emptyset - \frac{\partial \boldsymbol{A}}{\partial t}\text{and}\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}(2)$$

The vector **B**, **E** and **A** has three components (B_x, B_y, B_z),(E_x, E_y, E_z) and (A_1, A_2, A_3) respectively.

From equation (1),

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ **,** where$\boldsymbol{\nabla}$ is an Laplacian operator, Hence

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{bmatrix}$$

Therefore,

$$B_x = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = F_{23}$$

$$B_y = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = F_{31}$$

$$B_z = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = F_{12}$$

Also from $= -\nabla\emptyset - \frac{\partial \boldsymbol{A}}{\partial t}$,

$$E_x = \frac{\partial A_1}{\partial t} - \frac{\partial \emptyset}{\partial x}$$

$$\frac{iE_x}{c} = -\frac{i\,\partial A_1}{c\,\partial t} - \frac{i\,\partial \emptyset}{c\,\partial x}$$

$$\frac{iE_x}{c} = -\frac{\partial A_1}{\partial (ict)} - \frac{\partial (\frac{i\emptyset}{c})}{\partial x_1}$$

Therefore, let us consider

$$\frac{iE_x}{c} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} = F_{41}$$

Similarly

$$\frac{iE_y}{c} = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2} = F_{42}$$

and

$$\frac{iE_z}{c} = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3} = F_{43}$$

In general, from above equations one can write

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \qquad (3)$$

With $F_{\mu\nu} = -F_{\nu\mu}$ and $F_{\mu\mu} = F_{\nu\nu} = 0$

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} F_{13} & F_{14} \\ F_{21} & F_{22} F_{23} & F_{24} \\ F_{31} & F_{32} F_{33} & F_{34} \\ F_{41} & F_{42} F_{43} & F_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_z - B_y \frac{-iE_x}{c} \\ -B_z & 0 \quad B_x \frac{-iE_y}{c} \\ B_y - B_x & 0 \quad \frac{-iE_z}{c} \\ \frac{-iE_x}{c} \frac{-iE_y}{c} \frac{-iE_z}{c} & 0 \end{bmatrix} \qquad (4)$$

This is the electromagnetic field tensor of rank 2.

8.5 Lorentz invariance in field tensor

If the field equations are to be covariant with respect to Lorentz transformations it is of course necessary that the field tensor component $F_{\mu\nu}$ have the same form in all Lorentz reference frames. Using above equation (3), let us consider electromagnetic field tensor is in S-frame as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \qquad (5)$$

If any S’ frame is moving with constant velocity with respect to S frame, the above electromagnetic tensor can be described as:

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} \qquad (6)$$

The Lorentz transformation for x_μ and A_μ are

$$x'_\mu = \sum_\lambda a_{\mu\lambda} x_\lambda$$

And

$$A'_\mu = \sum_\phi a_{\mu\phi} x_\phi$$

The inverse Lorentz transformation of x_μ are

$$x_\lambda = \sum_\mu a_{\mu\lambda} x'_\mu$$

On differentiating with respect to x'_μ ,

$$\frac{\partial x_\lambda}{\partial x'_\mu} = a_{\mu\lambda}$$

Now from equation (8.6),

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} = \frac{\partial}{\partial x'_\mu}\left(\sum_\phi a_{\nu\phi} A_\phi\right) - \frac{\partial}{\partial x'_\nu}\left(\sum_\lambda a_{\mu\lambda} A_\lambda\right)$$

$$= \sum_\phi a_{\mu\phi} \frac{\partial A_\phi}{\partial x'_\mu} - \frac{\partial A_\lambda}{\partial x'_\nu} = \sum_{\phi,\lambda} a_{\nu\phi} \frac{\partial A_\phi}{\partial x_\lambda} . \frac{\partial A_\lambda}{\partial x'_\mu} - \sum_{\lambda,\phi} a_{\mu\lambda} \frac{\partial A_\lambda}{\partial x_\phi} . \frac{\partial A_\phi}{\partial x'_\nu}$$

$$= \sum_{\phi,\lambda} a_{\nu\phi} a_{\mu\lambda} \frac{\partial A_\phi}{\partial x_\lambda} - \sum_{\lambda,\phi} a_{\mu\lambda} a_{\nu\phi} \frac{\partial A_\lambda}{\partial x_\phi} = \sum_{\phi,\lambda} a_{\nu\phi} a_{\mu\lambda}\left(\frac{\partial A_\phi}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\phi}\right)$$

$$= \sum_{\phi,\lambda} a_{\nu\phi} a_{\mu\lambda} F_{\lambda\phi} \qquad (7)$$

Where,

$$\begin{bmatrix} a_{11} & a_{12} a_{13} & a_{14} \\ a_{21} & a_{22} a_{23} & a_{24} \\ a_{31} & a_{32} a_{33} & a_{34} \\ a_{41} & a_{42} a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

Using equation (7) we can calculate the field components very easily, for example

$$F'_{12} = \sum_{\lambda,\phi} a_{1\lambda} a_{2\phi} F_{\lambda\phi} = \sum_\lambda a_{1\lambda}[a_{21}F_{\lambda 1} + a_{22}F_{\lambda 2} + a_{23}F_{\lambda 3} + a_{24}F_{\lambda 4}]$$

$$F'_{12} = a_{11}[a_{21}F_{11} + a_{22}F_{12} + a_{23}F_{13} + a_{24}F_{14}]$$

$$+a_{12}[a_{21}F_{21} + a_{22}F_{22} + a_{23}F_{23} + a_{24}F_{24}]$$

$$+a_{13}[a_{21}F_{31} + a_{22}F_{32} + a_{23}F_{33} + a_{24}F_{34}]$$

$$+a_{14}[a_{21}F_{41} + a_{22}F_{42} + a_{23}F_{43} + a_{24}F_{44}]$$

As we know $a_{11} = 1$ and $a_{12} = a_{13} = a_{14} = 0$

$$F'_{12} = a_{11}[a_{21}F_{11} + a_{22}F_{12} + a_{23}F_{13} + a_{24}F_{14}]$$

$$F'_{12} = F_{12}$$

Similarly, $F'_{13} = F_{13}$, $F'_{14} = F_{14}$ and others

Hence,

$$F'_{\mu\nu} = F_{\mu\nu} \qquad (8)$$

This is the desired result. Hence electromagnetic field tensor is invariant under Lorentz transformation. This is also known as Lorentz invariance of field tensor.

8.6 Covariant form of Maxwell's equation in four-vector

We start with Maxwell's equation in free space

$$\nabla.\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad (9)$$

$$\nabla.\boldsymbol{B} = 0 \qquad (10)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (11)$$

$$\nabla \times \boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0\boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \qquad (12)$$

From considering the non-homogeneous pair of equations i.e. equation (9) and (12)-

$$\nabla.\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \Longrightarrow \nabla.\frac{i\boldsymbol{E}}{\boldsymbol{c}} = \frac{i\rho}{c\varepsilon_0} \Longrightarrow \nabla.\frac{i\boldsymbol{E}}{\boldsymbol{c}} = \mu_0 ic\rho \Longrightarrow \nabla.\frac{i\boldsymbol{E}}{\boldsymbol{c}} = \mu_0 j_4 \qquad (13)$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \Longrightarrow \nabla \times \boldsymbol{B} = \mu_0\boldsymbol{J} + \frac{\partial(i\boldsymbol{E}/\boldsymbol{c})}{\partial(ict)} \Longrightarrow \nabla \times \boldsymbol{B} - \frac{\partial(i\boldsymbol{E}/\boldsymbol{c})}{\partial(ict)} = \mu_0\boldsymbol{J} \qquad (14)$$

In terms of four co-ordinates, we introduce $(x, y, z, ict) = (x_1, x_2, x_3, x_4)$. The above equations can be written as

$$0 + \frac{\partial B_z}{\partial x_2} - \frac{\partial B_y}{\partial x_3} + \frac{\partial(iE_x/c)}{\partial x_4} = \mu_0 J_1 \qquad (15)$$

$$-\frac{\partial B_z}{\partial x_1} + 0 + \frac{\partial B_x}{\partial x_3} + \frac{\partial(-iE_y/c)}{\partial x_4} = \mu_0 J_2 \qquad (16)$$

$$-\frac{\partial B_y}{\partial x_1} + \frac{\partial B_x}{\partial x_2} + 0 + \frac{\partial(-iE_z/c)}{\partial x_4} = \mu_0 J_3 \qquad (17)$$

$$\frac{\partial(iE_x/c)}{\partial x_1} + \frac{\partial(iE_y/c)}{\partial x_2} + \frac{\partial(iE_z/c)}{\partial x_3} + 0 = \mu_0 J_4 \qquad (18)$$

Treating the right hand numbers of this system as the components of a four current density and introducing in the LHS as a set of independent variables defined as:

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} F_{13} & F_{14} \\ F_{21} & F_{22} F_{23} & F_{24} \\ F_{31} & F_{32} F_{33} & F_{34} \\ F_{41} & F_{42} F_{43} & F_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_z - B_y \frac{-iE_x}{c} \\ -B_z & 0 \quad B_x \frac{-iE_y}{c} \\ B_y - B_x & 0 \quad \frac{-iE_z}{c} \\ \frac{-iE_x}{c} \frac{-iE_y}{c} \frac{-iE_z}{c} & 0 \end{bmatrix}$$

Hence equations (15) to (18) are

$$\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_2} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4} = \mu_0 J_1$$

$$\frac{\partial F_{21}}{\partial x_1} + \frac{\partial F_{22}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_3} + \frac{\partial F_{24}}{\partial x_4} = \mu_0 J_2$$

$$\frac{\partial F_{31}}{\partial x_1} + \frac{\partial F_{32}}{\partial x_2} + \frac{\partial F_{33}}{\partial x_3} + \frac{\partial F_{34}}{\partial x_4} = \mu_0 J_3$$

$$\frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3} + \frac{\partial F_{44}}{\partial x_4} = \mu_0 J_4$$

From the above equations, Maxwell electromagnetic field equations (9) and (12) are obtained in compact form as below:

$$\sum_{v=1}^{4} \frac{\partial F_{\mu v}}{\partial x_v} = \mu_0 J_v \qquad (19)$$

Now consider equations (10) and (11)

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0 \qquad (20)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Longrightarrow \boldsymbol{\nabla} \times \boldsymbol{E} + ic\frac{\partial \boldsymbol{B}}{\partial x_4} = 0 \Longrightarrow \boldsymbol{\nabla} \times \frac{-i\boldsymbol{E}}{\boldsymbol{c}} + \frac{\partial \boldsymbol{B}}{\partial x_4} = 0 \qquad (21)$$

Thus equation (20) and (21) in terms of components can be written as:

$$0 + \frac{\partial(-iE_z/c)}{\partial x_2} + \frac{\partial(iE_y/c)}{\partial x_3} + \frac{\partial B_x}{\partial x_4} = 0 \qquad (22)$$

$$\frac{\partial(iE_z/c)}{\partial x_1} + 0 + \frac{\partial(iE_x/c)}{\partial x_3} + \frac{\partial B_y}{\partial x_4} = 0 \qquad (23)$$

$$\frac{\partial(-iE_y/c)}{\partial x_1} + \frac{\partial(iE_x/c)}{\partial x_2} + 0 + \frac{\partial B_z}{\partial x_4} = 0 \qquad (24)$$

$$\frac{\partial(B_x)}{\partial x_1} + \frac{\partial(B_x)}{\partial x_2} + \frac{\partial(B_x)}{\partial x_3} + 0 = 0 \qquad (25)$$

The above equations can be rewritten as:

$$0+\frac{\partial F_{34}}{\partial x_2}+\frac{\partial F_{42}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_4}=0$$

$$\frac{\partial F_{43}}{\partial x_1}+0+\frac{\partial F_{14}}{\partial x_3}+\frac{\partial F_{31}}{\partial x_4}=0$$

$$\frac{\partial F_{24}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_2}+0+\frac{\partial F_{21}}{\partial x_4}=0$$

$$\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}+\frac{\partial F_{12}}{\partial x_3}+0=0$$

All the above equations and hence Maxwell electromagnetic field equations (10) and (11) can be written more compactly in a single equation

$$\frac{\partial F_{\lambda\mu}}{\partial x_v}+\frac{\partial F_{\mu v}}{\partial x_\lambda}+\frac{\partial F_{v\lambda}}{\partial x_\mu}=0 \qquad (26)$$

As tensor equations are invariant under coordinate transformation. Hence, equations (19) and (26) express the covariant form of Maxwell's equations in terms of electromagnetic field tensor ($F_{\mu v}$) .

8.7 Four vectorpotential

The four vector potential form can be obtained in two steps. In the first step, we will reduce Maxwell four equations in two equations in terms of scalar and vector potentials. In the next step four vector potential will be obtained using reduced Maxwell two equations.

The four Maxwell's field equations are:

$$(i)\nabla.\boldsymbol{E}=\frac{\rho}{\varepsilon_0}\,(ii)\nabla.\boldsymbol{B}=0\ (iii)\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\,(iv)\nabla\times\boldsymbol{B}=\mu_0\left(\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}+\boldsymbol{J}\right)$$

From (ii) and (iii) Maxwell equations, the electromagnetic field vectors **E** and **B**are expressed as

$$\boldsymbol{B}=\nabla\times\boldsymbol{A} \qquad (27)$$

$$\boldsymbol{E}=-\nabla\mathrm{V}-\frac{\partial\boldsymbol{A}}{\partial t} \qquad (28)$$

Now take dot product with ∇in equation (28) andcross product with ∇in equation (27) on both the side, we have

$$\nabla.\boldsymbol{E}=\nabla.\left(-\nabla\mathrm{V}-\frac{\partial\boldsymbol{A}}{\partial t}\right)=-\nabla^2\phi-\frac{\partial}{\partial t}(\nabla.\boldsymbol{A}) \qquad (29)$$

$$\nabla\times\boldsymbol{B}=\nabla\times(\nabla\times\boldsymbol{A})=\nabla(\nabla.\boldsymbol{A})-\nabla^2\boldsymbol{A} (30)$$

Substituting the div E and curl B in Maxwell's equation (i) and (iv), we obtain

$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla.\boldsymbol{A}) = \frac{\rho}{\varepsilon_0}$ (31)

$\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \nabla^2\boldsymbol{A} = \mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}$(32)

But from equation (28) and (32), we have

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \nabla^2\boldsymbol{A} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\mathrm{V} - \frac{\partial \boldsymbol{A}}{\partial t}\right) + \mu_0\boldsymbol{J}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \nabla^2\boldsymbol{A} = -\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \frac{1}{c^2}\boldsymbol{\nabla}\left(\frac{\partial V}{\partial t}\right) + \mu_0\boldsymbol{J}$$

Hence, equation (31) and (32) can be rewritten as

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(\boldsymbol{\nabla}.\boldsymbol{A}) = -\frac{\rho}{\varepsilon_0}$$

And

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla\left(\boldsymbol{\nabla}.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0\boldsymbol{J}$$

From the relation $\ ^{\mathbf{2}} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, we have

$^{\mathbf{2}}\boldsymbol{A} - -\nabla\left(\boldsymbol{\nabla}.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0\boldsymbol{J}$ (33)

and $^{\mathbf{2}}V + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\frac{\rho}{\varepsilon_0}$(34)

Equation number (33) and (34) are the two reduced equations of Maxwell four field equations and represents the Maxwell's equations in terms of the potential $\boldsymbol{A}$ and V.

The coupled equations in (33) and (34) and these equations become in much simplified form, if we choose following gauge (Lorentz condition) such that

$\nabla.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$ (35)

When the Lorentz condition is impressed on equation (33) and (34), the Maxwell's field equations assume

$^{2}\boldsymbol{A} = -\mu_0 j$ (36)

$^{2}V = -\frac{\rho}{\varepsilon_0}$ (37)

Equation (36) and (37) are uncoupled second order differential equations and are known as D' Alembertian equations for electromagnetic potentials.

For defining four vector potential, equation (37) can be written as:

$$^2\frac{iV}{c} = -\frac{i\mu_0\rho}{c\mu_0\varepsilon_0} = -i\mu_0(ic\rho)$$

Where $c^2 = {}^1\!/_{\mu_0\varepsilon_0}$

Thus we may write equations (36) and (37) as below:

$$^2\boldsymbol{A} = -\mu_0 j$$

$$^2\frac{iV}{c} = -i\mu_0 J_4$$

From above two equations, we can write

$$A_\mu = \left(\boldsymbol{A}, \frac{iV}{c}\right) = (A_1, A_2, A_3, A_4)\text{where}A_4 = \frac{iV}{c}.$$

Where $A_\mu = \left(\boldsymbol{A}, \frac{iV}{c}\right)$ is a four vector and caller four vector potential.

8.8 Four vectorcurrent and continuity equation

Charges on a moving body is independent of the observer's motion. In other words, total charge in an isolated system remain invariant. Let us consider dq charge is distributed in dV volume. Therefore charge density (ρ) is defined as dq/dV for the considered system. We can write

$$dq = \rho dV = \rho dx_1 dx_2 dx_3$$

Multiplying by four vector dx_μ both sides in the above expression, we get

$$dq\ dx_\mu = \rho\ dx_\mu dx_1 dx_2 dx_3$$

$$dq\ dx_\mu = \rho\ \frac{dx_\mu}{dt} dx_1 dx_2 dx_3 dt \qquad (38)$$

Now as we know that $x_4 = ict$ and on differentiating both sides we have $dx_4 = ic\ dt$. Therefore,

$$dx_1 dx_2 dx_3 dt = dx_1 dx_2 dx_3 \frac{dx_4}{ic}$$

Which is invariant. Also the left hand side of equation (38) is invariant and represent a four-vector. On right hand side $dx_1 dx_2 dx_3 dt$ represents a scalar (invariant) quantity. Therfore the remaining quantity $\left(\rho\ \frac{dx_\mu}{dt}\right)$on right hand side will be a four –vector. This four vector is represented by j_μ.

$$j_\mu = \left(\rho\ \frac{dx_\mu}{dt}\right) \qquad (39)$$

Where, the four vector $j_\mu = (j_1, j_2, j_3, j_4) = \left(\rho \frac{dx_1}{dt}, \rho \frac{dx_2}{dt}, \rho \frac{dx_3}{dt}, \rho \frac{dx_4}{dt}\right) = (\rho u_1, \rho u_2, \rho u_3, ic\rho)$.

The four vector $j_\mu = (j_1, j_2, j_3, ic\rho) = (\boldsymbol{j}, ic\rho)$ is known as four-vector current.

Continuity equation in four- vector form

, the continuity equation,in electrodynamics, represents conservation of charge and mathematically expressed as

$$\nabla.\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \qquad (40)$$

In relativity theory it is clear that charge density and current densities are completely separable quantities. Charge distribution in static reference frame will appear as a current distribution in other moving reference frame.

The continuity equation (40) can also be written as

$$\nabla.\boldsymbol{J} + \frac{\partial(ic\rho)}{\partial(ict)} = 0$$

from the definition of current density above $J_4 = ic\rho$ and $x_4 = ict$

Hence, the above equation can be rewritten as

$$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0$$

$$\frac{\partial J_\mu}{\partial x_\mu} = 0 \Longrightarrow \quad {}^2.\boldsymbol{J}_\mu = 0 \qquad (41)$$

Where ^{2} is D'Alembertian operator and equation (41) represents the continuity equation in four vector form.

8.9 Gauge invariance of Maxwell equation

Maxwell equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. Maxwell equation in free space can be written as below

$$(i)\nabla.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}(ii)\nabla.\boldsymbol{B} = 0\ (iii)\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}(iv)\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$$

Where ρ is the charge density, $\boldsymbol{J}$is the current density. μ_0 and ε_0 are the permeability anf permittivity in free space. We know that electric field **($\boldsymbol{E}$)** and magnetic field **($\boldsymbol{B}$)** canbe written in form of scalar (V) and vector potential **(A)** as follows:

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \qquad (42)$$

and, $\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$ (43)

The equations (42) and (43) can also be considered as the solutions of Maxwell's equations. However these solutions are not unique, because in former case, an addition of the gradient of scalar function Λ does not change the magnetic field vector. Thus

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda \qquad (44)$$

Then from equation (42)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{A} + \boldsymbol{\nabla}\Lambda) = \boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\Lambda$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{0}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

Which is similar toequation (42). Therefore, in a similar way, a new electric field vector (**$\boldsymbol{E}'$**) does not change if$\boldsymbol{A}$is replaced by$\boldsymbol{A}'$ in equation (43).

$$\boldsymbol{E}' = -\boldsymbol{\nabla} V' - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{E}' = -\boldsymbol{\nabla} V' - \frac{\partial}{\partial t}(\boldsymbol{A} + \boldsymbol{\nabla}\Lambda)$$

$$\boldsymbol{E}' = -\boldsymbol{\nabla}\left(V' + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \boldsymbol{A}}{\partial t}$$

Now we define, $V' + \frac{\partial \Lambda}{\partial t} = V$ or

$V' = V - \frac{\partial \Lambda}{\partial t}$ (45)

thenfrom the above equation

$$\boldsymbol{E}' = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{E}$$

The equations (44) and (45) are called gauge transformations. Therefore, Maxwell equations remain unchanged (or invariant) under gauge transformationgiven in equation (44) and (45).

8.10 Electromagnetic energy-momentum tensor

It is now well understood that position four vector are expressed as

$$x_\mu = (x_1, x_2, x_3, x_4) = (\boldsymbol{r}, ict)$$

From the above the component of velocity four-vector $\left[u_\mu = (u_1, u_2, u_3, u_4)\right]$ can be obtained as

$$u_1 = \frac{dx_1}{d\tau} = \frac{dx_1}{dt}\frac{dt}{d\tau} = \frac{dx}{dt}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_2 = \frac{dx_2}{d\tau} = \frac{dx_2}{dt}\frac{dt}{d\tau} = \frac{dy}{dt}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_3 = \frac{dx_3}{d\tau} = \frac{dx_3}{dt}\frac{dt}{d\tau} = \frac{dz}{dt}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_4 = \frac{dx_4}{d\tau} = \frac{d(ict)}{dt}\frac{dt}{d\tau} = \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}$$

where, $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma$

Hence, $u_\mu = (u_1, u_2, u_3, u_4) = u_\mu = (\gamma u_x, \gamma u_y, \gamma u_z, i\gamma c) = (\gamma \boldsymbol{u}, i\gamma c)$

These are known as four vector velocity or velocity four vector. From the velocity four vector, one can easily derive momentum four vector as follows

$$p_1 = m_0 u_1 = \frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}} = m u_x = p_x$$

Similarly,

$$p_2 = m_0 u_2 = \frac{m_0 u_y}{\sqrt{1-\frac{u^2}{c^2}}} = m u_y = p_y$$

$$p_3 = m_0 u_3 = \frac{m_0 u_z}{\sqrt{1-\frac{u^2}{c^2}}} = m u_z = p_z$$

$$p_4 = m_0 u_4 = \frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}} = imc = i\frac{E}{c}$$

Hence, $p_\mu = (p_1, p_2, p_3, p_4) = (p_x, p_y, p_z, imc) = (\boldsymbol{p}, i\frac{E}{c})$ with $\boldsymbol{p} = m\boldsymbol{u}$

This p_μ is also called energy momentum four vector which is a tensor of rank one.

8.11 Motion of a charge in electromagnetic field : Lorentz force

The Lorentz force is the force experienced by a charged particle moving in an electric and magnetic field. Let us consider a charge q is moving with velocity **u** through magnetic induction **B** and electric field **E**. Then the Lorentz force applicable on the charged particle is

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

If in a given volume (τ), there are n charge carriers then the net force per unit volume is given by

$$\boldsymbol{f} = \frac{d\boldsymbol{F}}{d\tau} = \rho\boldsymbol{E} + J(\boldsymbol{u} \times \boldsymbol{B})$$

or

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$$

where$\rho = \frac{dq}{d\tau}$ and $\boldsymbol{J} = n(dq)\boldsymbol{u}$

In terms of components, the above equation can be written as

$$f_1 = \rho E_x + J_2 B_z - J_3 B_y$$

$$f_2 = \rho E_y + J_3 B_x - J_1 B_z$$

$$f_3 = \rho E_z + J_{1B_y} - J_2 B_x$$

In terms of electromagnetic field tensor ($F_{\mu\nu}$) , the above equations are obtained as

$$f_1 = F_{11}J_1 + F_{12}J_2 + F_{13}J_3 + F_{14}J_4$$

$$f_2 = F_{21}J_1 + F_{22}J_2 + F_{23}J_3 + F_{24}J_4$$

$$f_3 = F_{31}J_1 + F_{32}J_2 + F_{33}J_3 + F_{34}J_4$$

The above equations can also be written in compact form as

$$f_\alpha = \sum_{\nu=1}^{4} F_{\alpha\nu} J_\nu \text{ , where } \alpha = 1,2,3$$

Where, $J_\nu = (\boldsymbol{J}, ic\rho)$ is the corrent four-vector. The right hand side of this equation is the space component of a four vector f_μ. Therefore above equation can be expressed as

$$f_\mu = \sum_{\nu=1}^{4} F_{\mu\nu} J_\nu$$

This f_μ is called **force-density four vector**. To see the meaning of the fourth component of force density four vector, we substitute$\mu = 4$ in above equation

$$f_4 = F_{41}J_1 + F_{42}J_2 + F_{43}J_3 + F_{44}J_4$$

$$f_4 = \frac{i}{c}(E_1J_1 + E_2J_2 + E_3J_3)$$

$$f_4 = \frac{i}{c}(\boldsymbol{E}.\boldsymbol{J})$$

The above equation implies that the fourth component of f_μ is imaginary. Hence in the light of above equation, force-density four vector f_μ can be expressed as

$$f_\mu = \frac{1}{\mu_0}\sum_{\nu=1}^{4} F_{\mu\nu}\sum_{\lambda=1}^{4}\frac{\partial F_{\nu\lambda}}{\partial x_\lambda}, \text{ where } \sum_{\nu=1}^{4}\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu$$

$$or \quad f_\mu = \frac{1}{\mu_0}\sum_{\nu,\lambda=1}^{4} F_{\mu\nu}\frac{\partial F_{\nu\lambda}}{\partial x_\lambda}$$

This tensor form of force density four vector represents the covariant form of the Lorentz force.

Self Assessment Question (SAQ) 1: The current four vector is _____

Self Assessment Question (SAQ) 2: The fourth component of the force-density four vector is ____________

Self Assessment Question (SAQ) 3: What is the form of continuity equation in four vector.

Self Assessment Question (SAQ) 4: What is Lorentz condition?

8.12 Summary

In this unit, you have studied about the four vector formalism to simplify the complex behaviors of four coordinates including time. Maxwell equations and other quantities in electrodynamics are represented in terms of four vector. The operations of four vectors are similar to ordinary vectors and hence are easy to understand mathematically.

8.13 Glossary

Transformation equations- A set of equation to covert from one reference frame to another.

Tensor- a mathematical object analogous to but more general than a vector

Covariance- no change in any equation due to change in frame.

8.14 References

1. Electrodynamics, IGNOU, New Delhi

2. Introduction to Electrodynamics, D.J.Griffiths, Pearson Publication, Chennai

3. Classical Electrodynamics, J.D. Jackson, Willey Eastern Ltd., New Delhi

4. Classical Mechanics, J.C. Upadhyay, Himgiri publishing house, Mumbai

8.15 Suggested Readings

1. Electrodynamics, Gupta Kumar, Pragati Publication

2. Foundation of Electromagnetic Theory, J.R. Reitz, F.J. Milford and R.W. Christy, Narosa publications, New Delhi

3. Foundation of Electrodynamics, Parry Moor, Dover publication, New York.

8.16 Terminal Questions

(Should be divided intoShort Answer type, Long Answer type, Numerical, Objective type)

8.16.1 Objective type

8. For gauge transformation
- (e) The electric and magnetic field vector do not change
- (f) The electric field vector change but not the magnetic field vector
- (g) The magnetic field vector changes but not electric field vector.
- (h) Both the electric and magnetic field vector change

9. Which of the following remains under Lorentz transformations:
- (d) $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} - \frac{1}{c^2}\frac{\partial}{\partial t}$
- (e) $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + \frac{1}{c^2}\frac{\partial}{\partial t}$
- (f) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$
- (g) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

10. Choose the correct statement/s
- (i) $\sum_\mu \frac{\partial A_\mu'}{\partial \mu'} = \sum_\mu \frac{\partial A_\mu}{\partial \mu}$
- (j) $\sum_\mu \frac{\partial A_\mu'}{\partial \mu'} \neq \sum_\mu \frac{\partial A_\mu}{\partial \mu}$
- (k) Lorentz condition is $\;^2A_\mu = -\mu_0 J_\mu$

(l) None of these

11. Covariant form of Maxwell's field equations $\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J}$ and $\nabla . \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ in terms of electromagnetic field tensor is

(e) $^2A_\mu = -\mu_0 J_\mu$

(f) $\sum_{v=1}^{4} \frac{\partial F_{\mu v}}{\partial x_v} = \mu_0 J_\mu$

(g) $\sum_{v=1}^{4} \frac{\partial F_{\mu v}}{\partial x_v} = \frac{\rho}{\varepsilon_0}$

(h) none of these

12. Lorentz force equation in the covariant form gives

(a) The rate of change of linear momentum per unit volume as the space part
(b) The rate of change of linear momentum per unit volume as the time part
(c) The rate of change of mechanical energy per unit volume as the space part
(d) The rate of change of mechanical energy per unit volume as the time part

13. The Lorentz condition is

(e) $\nabla . \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

(f) $\nabla . \boldsymbol{A} - \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

(g) $\nabla \times \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

(h) $\nabla \times \boldsymbol{A} - \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

8.16.2 Short Answer type

1. What is Lorentz condition?

2. What is electromagnetic field tensor?

3. What is D'Alembertian operator?

4. Write down the current and potential four vectors.

5. Express the covariant form of Maxwell's equations.

8.16.3 Long Answer type

1. Show that the D'Alembertian operator is invariant under Lorentz transformation.

2. Establish the covariant for of Maxwell's electromagnetic field equations by four vectors. Does it represent the covariant formulation of electrodynamics.

3. What is four vector potential? Show that the Maxwell's field equation can be written in one single equation, given by $^2A_\mu = -\mu_0 J_\mu$, where A_μ is the four vector potential and J_μ is the current four vector. .

4. Derive an expression for the Lorentz force on a charged particle in an electromagnetic field in terms of four vector.

5. Show that the four vector potential of electrodynamics can be expressed as Lienard-Wiechert potential. Define the electromagnetic field tensor $F_{\mu v}$.

6. Write the Maxwell's equations in terms of scaler and vector potentials. Show that these equations are invariant under gauge transformation. Discuss the significance of the transformation.

7. Discuss the Lorentz invariance of Maxwell's field equations.

8.16.4 Numerical Answer type

5) Show that the self product of electromagnetic field tensor is given by

$$F_{\mu v}^2 = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

6) Prove that the quantities $\boldsymbol{E}.\boldsymbol{B}$ **and** $E^2 - c^2B^2$ are invariant under Lorentz transformation.

7) Find the maximum electric field produced at a stationary atom by a 10 BeV proton which passes it at a distance of 10^{-7}cm., if the rest mass of the proton is equivalent to 1 BeV energy.

8) Find the force in the laboratory frame as well as in the proper frame for two electrons, moving along the axis of a linear accelerator in parallel paths separated by $5 \times 10^{-9}\ m$ at speed $v = 0.999c$. The line connecting the two charges is perpendicular to the direction of their motion.

8.17 Answers

8.17.1 Self Assessment Questions (SAQs):

1. $(j_1,\ j_2, j_3, ic\rho) = (\boldsymbol{j}, ic\rho)$

2. $\frac{i}{c}\rho(\boldsymbol{E}.\boldsymbol{u})$

3. $\ ^2.\boldsymbol{J}_\mu = 0$

4. $\nabla.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

8.17.2 Terminal Questions: Objective type

1. (a), 2. (c), 3. (a), 4. (b), 5. (a), (c), 6. (c),

8.17.3 Terminal Questions: Short Answer type

1. $\nabla.\boldsymbol{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}= \ 0$
2. The electromagnetic field tensor is

$$F_{\mu\nu}=\begin{bmatrix} F_{11} & F_{12} F_{13} & F_{14} \\ F_{21} & F_{22} F_{23} & F_{24} \\ F_{31} & F_{32} F_{33} & F_{34} \\ F_{41} & F_{42} F_{43} & F_{44} \end{bmatrix}=\begin{bmatrix} 0 & B_z - B_y & \frac{-iE_x}{c} \\ -B_z & 0 & B_x \ \frac{-iE_y}{c} \\ B_y - B_x & 0 & \frac{-iE_z}{c} \\ \frac{-iE_x}{c}\frac{-iE_y}{c}\frac{-iE_z}{c} & & 0 \end{bmatrix}$$

3. $\quad ^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

4. current four vector is $j_\mu=(j_1,\ j_2, j_3, ic\rho)=(\boldsymbol{j}, ic\rho)$. vector four vector is $p_\mu=(p_1,p_2,p_3,p_4)=\left(p_x,p_y,p_z,imc\right)=(\boldsymbol{p}, i\frac{E}{c})$ with $\boldsymbol{p}=m\boldsymbol{u}$

5. $\sum_{v=1}^{4}\frac{\partial F_{\mu v}}{\partial x_v}=\mu_0 J_v$ and $\frac{\partial F_{\lambda\mu}}{\partial x_v}+\frac{\partial F_{\mu v}}{\partial x_\lambda}+\frac{\partial F_{v\lambda}}{\partial x_\mu}=0$

8.17.4 Numerical type questions

3. $E=\frac{q}{4\pi\varepsilon_0 r^2}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}=1.5\times 10^{10}$ V/m

4. $F'_y=9.22\ \times 10^{-12}N, 4.12\ \times 10^{-13}\ N$

UNIT 9 WAVE GUIDE

9.1 INTRODUCTION

The process of communication defined as the transmission of information from one place to another place. In this case modulation is used to encode the information onto a carrier wave (a wave of higher frequency), and may involve analog or digital methods. The signal will propagate over any significant distance will determined only by the characteristics of the carrier wave.

Generally, if the frequency of a signal or a particular band of signals is high, the bandwidth utilization is high as the signal provides more space for other signals to get accumulated. However, high frequency signals can't travel longer distances without getting attenuated. We have studied that transmission lines help the signals to travel longer distances.

Microwaves propagate through microwave circuits, components and devices, which act as a part of microwave transmission lines, broadly called as waveguides. A hollow metallic tube of uniform cross-section for transmitting electromagnetic waves by successive reflections from the inner walls of the tube is called as a waveguide. A waveguide is generally preferred in microwave communications. Waveguide is a special form of transmission line, which is a hollow metal tube. Unlike a transmission line, a waveguide has no center conductor.

Now in following sections, we will discuss the transmission line and waveguide in detail:

9.2 OBJECTIVES

After studying this unit, you should be able to-

- To introduce the various types of transmission lines and to discuss the losses associated with it.
- To give through understanding about impedance transformation and matching.
- To impart knowledge on wave guides theories
- To give understanding about characteristics and advantages of waveguide.
- To know about various modes of wave guide.

9.3 GENERAL THEORY OF TRANSMISSION LINES

Transmission line

Transmission line directed electrical energy from one point to another. It is normally used to transfer the output radio frequency energy of a transmitter to an antenna. This energy will not travel through normal electrical wire without great losses. Although the antenna can be connected directly to the transmitter, the antenna is usually located some distance away from the transmitter. The transmitter and antenna is connected by using transmission line. The transmission line has a single purpose for both the transmitter and the antenna.

Theory of Transmission Line:

The electrical characteristics of a two-wire transmission line depend primarily on the construction of the line. The two-wire line acts like a long capacitor. The change of its capacitive reactance is noticeable as the frequency applied to it is changed. Since the long conductors have a magnetic field about them when electrical energy is being passed through them, they also exhibit the properties of inductance.

The values of inductance and capacitance presented depend on the various physical factors are:

For example, the type of line used, the dielectric in the line, and the length of the line must be considered. The effects of the inductive and capacitive reactance of the line depend on the frequency applied. Since no dielectric is perfect, electrons manage to move from one conductor to the other through the dielectric.

Each type of two-wire transmission line also has a conductance value. This conductance value represents the value of the current flow that may be expected through the insulation, If the line is uniform (all values equal at each unit length), then one small section of the line may represent several feet. This illustration of a two-wire transmission line will be used throughout the discussion of transmission lines; but, keep in mind that the principles presented apply to all transmission lines.

A transmission line has the properties of inductance, capacitance, and resistance just as the more conventional circuits have. Usually, however, the constants in conventional circuits are lumped into a single device or component. For example, a coil of wire has the property of inductance. When a certain amount of inductance is needed in a circuit, a coil of the proper dimensions is inserted.

The inductance of the circuit is lumped into the one component. Two metal plates separated by a small space, can be used to supply the required capacitance for a circuit. In such a case, most of the capacitance of the circuit is lumped into this one component. Similarly, a fixed resistor can be used to supply a certain value of circuit resistance as a lumped sum.

Ideally, a transmission line would also have its constants of inductance, capacitance, and resistance lumped together. Unfortunately, this is not the case. Transmission line constants are as described in the following paragraphs.

9.3.1 Distributed Constants

Transmission line constants, called distributed constants, are spread along the entire length of the transmission line and cannot be distinguished separately. The amount of inductance, capacitance, and resistance depends on the length of the line, the size of the conducting wires, the spacing between the wires, and the dielectric (air or insulating medium) between the wires.

The electrical lines which are used to transmit the electrical waves along them are represented as transmission lines. Thedifferent line parameters of a transmission line are: Resistance (R),Inductance (L), capacitance (C), and conductance (G). The line parameters R, L, C and G are distributed over the entire length of the transmission line. Hence, transmission line is called distributed network. They are also called primary constants.

- Resistance (R) is defined as the loop resistance per unit length of the wire. Its unit is ohms/km.
- Inductance (L) is defined as the loop inductance per unit length of the wire. Its unit is Henries/km.

- Capacitance (C) is defined as the loop capacitance per unit length of the wire. Its unit is Farad/km.
- Conductance (G) is defined as the loop conductance per unit length of the wire. Its unit is mhos/km.

9.3.2 Application of Transmission Lines

1.They are used to transmit signal i.e. EM waves from one point to another.
2.They can be used for impedance matching purpose.
3.They can be used as circuit elements like inductors, capacitors.
4.They can be used as stubs by properly adjusting their lengths.

Wavelength of a line is the distance the wave travels along the line while the phase angle is changing through 2π radians is a wavelength.

Characteristic impedance is the impedance measured at the sending end of the line. It is given by Z_0 = Z/Y,where Z = R + jωL is the series impedance Y = G + jωC is the shunt admittance.

The secondary constants of a line are:

(i) Characteristic Impedance
(ii) Propagation Constant

Since the line constants R, L, C and G are distributed through the entire length of the line, they are called as distributed elements. They are also called as primary constants.

9.4 THE TRANSMISSION LINE, GENERAL SOLUTIONS & THE INFINITE LINE

A finite line is a line having a finite length on the line. It is a line, which is terminated, in its characteristic impedance (Z_R=Z_0), so the input impedance of the finite line is equal to the characteristic impedance (Zs=Z_0).

An infinite line is a line in which the length of the transmission line is infinite. A finite line, which is terminated in its characteristic impedance, is termed as infinite line. Therefore, for an infinite line, the input impedance is equivalent to the characteristic impedance.

9.4.1 Symmetrical T Network

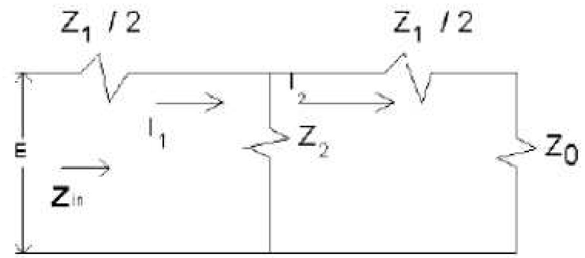


Figure 9.1: Symmetrical T Network.

The value of Z_0 (image impedance) for a symmetrical network can be easily determined. For the symmetrical T network of Figure 9.1, terminated in its image impedance Z_0, and if $Z_1 = Z_2 = Z_T$.

9.4.2 General Solution of the Transmission Line

It is used to find the voltage and current at any points on the transmission line. Transmission lines behave very oddly at high frequencies. In traditional (low-frequency) circuit theory, wires connect devices, but have zero resistance. There is no phase delay across wires; and a short-circuited line always yields zero resistance.

For high-frequency transmission lines, things behave quite differently. For instance, short-circuits can actually have an infinite impedance; open-circuits can behave like short-circuited wires. The impedance of some load (Z_L=X_L+jYL) can be transformed at the terminals of the transmission line to an impedance much different than Z_L.

Let's start by examining a diagram. A sinusoidal voltage source with associated impedance Z_S is attached to a load Z_L (which could be an antenna or some other device – in the circuit diagram we simply view it as an impedance called a load). The load and the source are connected via a transmission line of length L:

Since antennas are often high-frequency devices, transmission line effects are often very important. That is, if the length L of the transmission line significantly alters Z_{in}, then the current into the antenna from the source will be very small. Consequently, we will not be delivering power properly to the antenna.

The same problems hold true in the receiving mode: a transmission line can skew impedance of the receiver sufficiently that almost no power is transferred from the antenna. Hence, a thorough understanding of antenna theory requires an understanding of transmission lines. A great antenna can be hooked up to a great receiver, but if it is done with a length of transmission line at high frequencies, the system will not work properly.

Examples of common transmission lines include the coaxial cable, the microstrip line, which commonly feeds patch/microstrip antennas, and the two wire line (Figure 9.2):

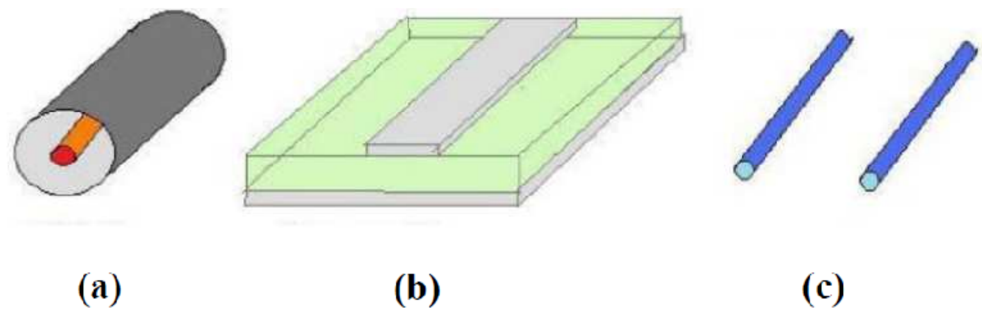


Figure 9.2 (a): Coaxial Cable (b): Microstrip Line (c): Two-Wire Line

To understand transmission lines, we will set up an equivalent circuit to model and analyze them. To start, we will take the basic symbol for a transmission line of length L and divide it into small segments:

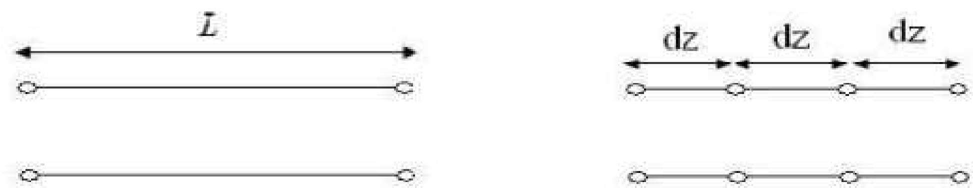


Figure 9.3(a): Parallel-wire representation (b): Divided into small sections of length dz.

Then we will model each small segment with a small series resistance, series inductance, shunt conductance, and shunt capacitance:

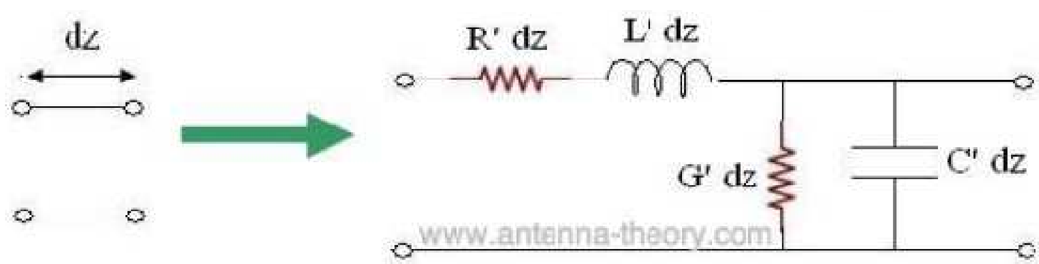


Figure 9.4

The parameters in the above figure are defined as follows: R' - resistance per unit length for the transmission line (Ohms/meter) L' - inductance per unit length for the tx line (Henries/meter) G' - conductance per unit length for the tx line (Siemans/meter) C' - capacitance per unit length for the tx line (Farads/meter) We will use this model to understand the transmission line. All transmission lines will be represented via the above circuit diagram. For instance, the model for coaxial cables will differ from microstrip transmission lines only by their parameters R', L', G' and C'.

To get an idea of the parameters, R' would represent the d.c. resistance of one meter of the transmission line. The parameter G' represents the isolation between the two conductors of the transmission line. C' represents the capacitance between the two conductors that make up the tx line; L' represents the inductance for one meter of the tx line. These parameters can be derived for each transmission line.

General solutions:

A circuit with distributed parameter requires a method of analysis somewhat different from that employed in circuits of lumped constants. Since a voltage drop occurs across each series increment of a line, the voltage applied to each increment of shunt admittance is a variable and thus the shunted current is a variable along the line.

Hence the line current around the loop is not a constant, as is assumed in lumped constant circuits, but varies from point to point along the line. Differential circuit equations that describes that action will be written for the steady state, from which general circuit equation will be defined as follows.

R= series resistance, ohms per unit length of line(includes both wires)
L= series inductance, henrys per unit length of line
C= capacitance between conductors, faradays per unit length of line
G= shunt leakage conductance between conductors, mhos per unit length of line
ωL = series reactance, ohms per unit length of line
Z = R+jωL
ωL = series susceptance, mhos per unit length of line
Y = G+jωC
S = distance to the point of observation, measured from the receiving end of the line
I = Current in the line at any point
E= voltage between conductors at any point
l = length of line

The below figure illustrates a line that in the limit may be considered as made up of cascaded infinitesimal T sections, one of which is shown.

This incremental section is of length of ds and carries a current I. The series line impedance being Z ohms and the voltage drop in the length ds is

$$dE = IZds \quad (9.1)$$

$$\frac{dE}{ds} = IZ \quad (9.2)$$

The shunt admittance per unit length of line is Y mhos, so that

The admittance of the line is Yds mhos. The current dI that follows across the line or from one conductor to the other is

$$dI = EYds \quad (9.3)$$

$$\frac{dI}{ds} = EY \quad (9.4)$$

The equation 9.2 and 94 may be differentiated with respect to s

$$\frac{d^2E}{dS^2} = Z\frac{dI}{dS}$$

$$\frac{d^2I}{dS^2} = Y\frac{dE}{dS}$$

$$\frac{d^2E}{dS^2} = ZYE \quad (9.5)$$

$$\frac{d^2I}{dS^2} = ZYI \quad (9.6)$$

These are the differential equations of the transmission line, fundamental to circuits of distributed constants.

$$E = Ae^{\sqrt{AY_z}} + Be^{-\sqrt{ZY_z}} \quad (9.7)$$

$$I = Ce^{\sqrt{ZY_z}} + De^{-\sqrt{ZYz}} \quad (9.8)$$

Where A,B,C,D are arbitrary constants of integration.

Since the distance is measured from the receiving end of the line, it is possible to assign conditions such that at

s=0, I=I_R, E=E_R

The equation 9.7 and becomes

$$ER=A+B$$

$$I = C+D$$

(9.9)

A second set of boundary condition is not available, but the same set may be used over again if a new set of equations are formed by differentiation of equation 9.7 and 9.8. Thus

$$\frac{dE}{dS} = A\sqrt{ZY}e^{\sqrt{ZY_s}} - \mathrm{B}\sqrt{ZY}e^{-\sqrt{ZY_s}} \quad (9.10)$$

$$IZ = A\sqrt{ZY}e^{\sqrt{ZY_s}} - \mathrm{B}\sqrt{ZY}e^{-\sqrt{ZY_s}} \quad (9.11)$$

$$I = A\sqrt{\frac{Y}{Z}}e^{\sqrt{ZY_s}} - \mathrm{B}\sqrt{\frac{Y}{Z}}e^{-\sqrt{ZY_s}}$$

$$\frac{dI}{dS} = C\sqrt{ZY}e^{\sqrt{ZY_s}} - \sqrt{ZY}e^{-\sqrt{ZY_s}} \quad (9.12)$$

$$E = C\sqrt{\frac{Z}{Y}}e^{\sqrt{ZY_s}} - \mathrm{D}\sqrt{\frac{Y}{Z}}e^{-\sqrt{ZY_s}} \quad (9.13)$$

$$I_R = A\sqrt{\frac{Y}{Z}} - B\sqrt{\frac{Y}{Z}}$$

$$E_R = C\sqrt{\frac{Z}{Y}} - D\sqrt{\frac{Z}{Y}}$$

Simultaneous solution of equation 9.9, 9.12 and 9.13, along with the fact that $E_R = I_R Z R$ and that $Z\ Y$ has been identified as the Z_0 of the line,leads to solution for the constants of the above equations as

$$A = \frac{E_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2} + \frac{Z_0}{Z_R} \quad (9.14\ a)$$

$$B = \frac{E_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2} - \frac{Z_0}{Z_R} \quad (9.14\ b)$$

$$C = \frac{I_R}{2} + \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2} + \frac{Z_R}{Z_0} \quad (9.14\ c)$$

$$D = \frac{I_R}{2} - \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2} - \frac{Z_R}{Z_0} \quad (9.14\ d)$$

The solution of the differential equations of the transmission line may be written

$$E = \frac{E_R}{2} - \frac{Z_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2} - \frac{Z_R}{Z_0} \quad (9.15)$$

$$I = \frac{I_R}{2} - \frac{Z_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2} - \frac{Z_R}{Z_0} \quad (9.16)$$

The above equations are very useful form for the voltage and current at any point on a transmission line. After simplifying the above equations, we get the final and very useful form of equations for voltage and current at any point on a k=line, and are solutions to the wave equation.

$$E = E_R cosh\sqrt{ZY_S} + I_R Z_0\, sinh\sqrt{ZY_S} \quad (9.17)$$

$$I = I_R cosh\sqrt{ZY_S} + \frac{E_R}{Z} sinh\sqrt{ZY_S} \quad (9.18)$$

This result indicates two solutions, one for the plus sign and the other for the minus sign before the radical.

9.5 WAVELENGTH, VELOCITY OF PROPAGATION

Wave propagation is any of the ways in which waves travel. With respect to the direction of the oscillation relative to the propagation direction, we can distinguish between longitudinal wave and transverse waves. For electromagnetic waves, propagation may occur in a vacuum as well as in a material medium. Other wave types cannot propagate through a vacuum and need a transmission medium to exist.

Wavelength

The distance the wave travels along the line while the phase angle is changed through 2Π radians is called wavelength. $\lambda = 2\pi/ \beta$

The change of 2π in phase angle represents one cycle in time and occurs in a distance of one wavelength, $\lambda = v/f$

Velocity

$$V = f\lambda$$
$$V = \omega/ \beta$$

This is the velocity of propagation along the line based on the observation of the change in the phase angle along the line. It is measured in miles/second if ß is in radians per meter.
We know that

$$Z = R + j\,\omega L$$
$$Y = G + j\,\omega C$$

Then

$$\gamma = \alpha + j\beta = \sqrt{ZY}$$

$$\alpha + j\beta = \sqrt{RG - \omega^2 LC + J\omega(LG + CR)}$$

On squaring above equation both side

$$\alpha + 2j\alpha\beta - \beta = RG - \omega^2 LC + J\omega(LG + CR)$$

By equating real and imaginary part:

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)}}{2}} \qquad (9.19)$$

Therefore, the equation for theβ is:

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC) + \omega^2 (LG + CR)}}{2}} \qquad (9.20)$$

For the perfect line condition, R and G both are 0. Therefore, the above equation would be:

$$\beta = \omega\sqrt{LC}$$

The velocity of propagation of such ideal line is given by:

$$\vartheta = \frac{\omega}{\beta}$$

Thus the above equation showing that the line parameter values fix the velocity of propagation.

9.6 THE DISTORTION LESS LINE

It is desirable, however to know the condition on the line parameters that allows propagation without distortion. The line having parameters satisfy this condition is termed as a distortion less line.

The condition for a distortion less line was first investigated by Oliver Heaviside. Distortion less condition can help in designing new lines or modifying old ones to minimize distortion.

A line, which has neither frequency distortion nor phase distortion is called a distortion less line.

Condition for a distortion less line

The condition for a distortion less line is RC=LG. Also,

a)The attenuation constant should be made independent of frequency. $\alpha = RG$

b) The phase constant should be made dependent of frequency. $\beta = \omega LC$

c) The velocity of propagation is independent of frequency.

$$V=1 / LC$$

For the telephone cable to be distortion less line, the inductance value should be increased by placing lumped inductors along the line.

For a perfect line, the resistance and the leakage conductance value were neglected. The conditions for a perfect line are R=G=0. Smooth line is one in which the load is terminated by its characteristic impedance and no reflections occur in such a line. It is also called as flat line.

The distortion Less Line

If a line is to have neither frequency nor delay distortion, then attenuation constant and velocity of propagation cannot be function of frequency.
Then the phase constant be a direct function of frequency

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC) + \omega^2 (LG + CR)}}{2}} \tag{9.21}$$

The above equation shows that if the term under the second radical be reduced to equal

$$(RG + \omega^2 LC)^2 \quad (9.22)$$

Then the required condition for β is obtained. Expanding the term under the internal radical and forcing the equality gives

$$R^2G^2 - 2\omega^2 LCRG + \omega^4 L^2C^2 + \omega^2 L^2G^2 + 2\omega^2 LCRG + \omega^2 CR^2 = (RG + \omega^2 LC)^2 \quad (9.23)$$

This reduces to

$$2\omega^2 LCRG + \omega^2 L^2G^2 + \omega^2 CR^2 = 0$$

$$(LG - CR)^2 = 0 \quad (9.24)$$

Therefore the condition that will make phase constant a direct form is

$$LG = CR$$

A hypothetical line might be built to fulfill this condition. The line would then have a value of ß obtained by use of the above equation.

Already we know that the formula for the phase constant

$$\beta = \omega LC \quad (9.25)$$

Then the velocity of propagation will be $v = 1/LC$

This is the same for the all frequencies, thus eliminating the delay distortion.

May be made independent of frequency if the term under the internal radical is forced to reduce to $(RG + \omega LC)^2$

Analysis shows that the condition for the distortion less line $LG = CR$, will produce the desired result, so that it is possible to make attenuation constant and velocity independent of frequency simultaneously. Applying the condition $LG = RC$ to the expression for the attenuation gives $\alpha = RG$

This is the independent of frequency, thus eliminating frequency distortion on a line. To achieve

$LG = CR$

Require a very large value of L, since G is small. If G is intentionally increased, attenuation is increased, resulting in poor line efficiency.

To reduce R raises the size and cost of the conductors above economic limits, so that the hypothetical results cannot be achieved.

Propagation constant is as the natural logarithm of the ratio of the sending end current or voltage to the receiving end current or voltage of the line. It gives the manner in the wave is propagated along a line and specifies the variation of voltage and current in the line as a function of distance. Propagation constant is a complex quantity and is expressed as $\gamma = \alpha + j\beta$.

The real part is called the attenuation constant, whereas the imaginary part of propagation constant is called the phase constant.

9.7 LOADING AND DIFFERENT METHODS OF LOADING

In ordinary telephone cables, the wires are insulated with paper and twisted in pairs, therefore there will not be flux linkage between the wires, which results in negligible inductance, and conductance. If this is the case, there occurs frequency and phase distortion in the line.

Quarter wave length

For the case where the length of the line is one quarter wavelength long, or an odd multiple of a quarter wavelength long, the input impedance becomes

$$Z_{in} = \frac{Z_0^2}{Z_L} \quad (9.26)$$

Matched load

Another special case is when the load impedance is equal to the characteristic impedance of the line (i.e. the line is matched), in which case the impedance reduces to the characteristic impedance of the line so that

$$Z_{in} = Z_L = Z_0 \quad (9.27)$$

For all l and all λ.

Short: For the case of a shorted load (i.e. $ZL = 0$), the input impedance is purely imaginary and a periodic function of position and wavelength (frequency)

$$Z_{in}(l) = jZ_0 \tan(\beta l) \quad (9.28)$$

Open: For open load case ($Z_L = \infty$), input impedance is once again imaginary and periodic

$$Z_{in}(l) = -jZ_0 \cot(\beta l) \quad (9.29)$$

9.8 LINE NOT TERMINATED IN Z_O

The insertion loss of a line or network is defined as the number of decibels by which the current in the load is changed by the insertion. Insertion loss=Current flowing in the load without insertion of the Network Current flowing in the load with insertion of the network.

Coaxial cable

Coaxial lines confine the electromagnetic wave to the area inside the cable, between the center conductor and the shield. The transmission of energy in the line occurs totally through the dielectric inside the cable between the conductors. Coaxial lines can therefore be bent and twisted (subject to limits) without negative effects, and they can be strapped to conductive supports without inducing unwanted currents in them.

In radio-frequency applications up to a few gigahertz, the wave propagates in the transverse electric and magnetic mode (TEM) only, which means that the electric and magnetic fields are both perpendicular to the direction of propagation (the electric field is radial, and the magnetic field is circumferential). However, at frequencies for which the wavelength (in the dielectric) is significantly shorter than the circumference of the cable, transverse electric (TE) and transverse magnetic (TM) waveguide modes can also propagate.

When more than one mode can exist, bends and other irregularities in the cable geometry can cause power to be transferred from one mode to another.

The most common use for coaxial cables is for television and other signals with bandwidth of multiple megahertz. In the middle 20th century they carried long distance telephone connections.

Microstrip

A microstrip circuit uses a thin flat conductor, which is parallel to a ground plane. Microstrip can be made by having a strip of copper on one side of a printed circuit board (PCB) or ceramic substrate while the other side is a continuous ground plane. The width of the strip, the thickness of the insulating layer (PCB or ceramic) and the dielectric constant of the insulating layer determine the characteristic impedance. Microstrip is an open structure whereas coaxial cable is a closed structure.

Stripline

A stripline circuit uses a flat strip of metal which is sandwiched between two parallel ground planes. The insulating material of the substrate forms a dielectric. The width of the strip, the thickness of the substrate and the relative permittivity of the substrate determine the characteristic impedance of the strip which is a transmission line.

Balanced lines

A balanced line is a transmission line consisting of two conductors of the same type, and equal impedance to ground and other circuits. There are many formats of balanced lines, amongst the most common are twisted pair, star quad and twin-lead.

Twisted pair

Twisted pairs are commonly used for terrestrial telephone communications. In such cables, many pairs are grouped together in a single cable, from two to several thousand.
The format is also used for data network distribution inside buildings, but in this case the cable used is more expensive with much tighter controlled parameters and either two or four pairs per cable.

Single-wire line

Unbalanced lines were formerly much used for telegraph transmission, but this form of communication has now fallen into disuse. Cables are similar to twisted pair in that many cores are bundled into the same cable but only one conductor is provided per circuit and there is no twisting. All the circuits on the same route use a common path for the return current (earth return). There is a power transmission version of single-wire earth return in use in many locations.

Waveguide

Waveguides are rectangular or circular metallic tubes inside which an electromagnetic wave is propagated and is confined by the tube. Waveguides are not capable of transmitting the transverse electromagnetic mode found in copper lines and must use some other mode. Consequently, they cannot be directly connected to cable and a mechanism for launching the waveguide mode must be provided at the interface.

9.9 REFLECTION COEFFICIENT

The reflection coefficient is used in physics and electrical engineering when wave propagation in a medium containing discontinuities is considered. A reflection coefficient describes either the amplitude or the intensity of a reflected wave relative to an incident wave. The reflection coefficient is closely related to the transmission coefficient.

Reflection occurs because of the following cases:

1) When the load end is open circuited

2) When the load end is short-circuited

3) When the line is not terminated in its characteristic impedance.

When the line is either open or short circuited, then there is not resistance at the receiving end to absorb all the power transmitted from the source end. Hence all the power incident on the load gets completely reflected back to the source causing reflections in the line. When the line is terminated in its characteristic impedance, the load will absorb some power and some will be reflected back thus producing reflections.

Reflection Coefficient can be defined as the ratio of the reflected voltage to the incident voltage at the receiving end of the line Reflection Coefficient K=Reflected Voltage at load /Incident voltage at the load.

$$K = \frac{V_r}{V_i}$$

In telecommunications, the reflection coefficient is the ratio of the amplitude of the reflected wave to the amplitude of the incident wave. In particular, at a discontinuity in a transmission line, it is the complex ratio of the electric field strength of the reflected wave (E^-) to that of the incident wave (E^+). This is typically represented with a Γ (capital gamma) and can be written as

$$\Gamma = \frac{E^-}{E^+}$$

The reflection coefficient may also be established using other field or circuit quantities.

The reflection coefficient can be given by the equations below, where ZS is the impedance toward the source, ZL is the impedance toward the load:

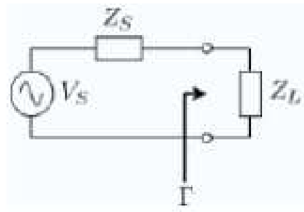


Figure 9.5

Simple circuit configuration showing measurement location of reflection coefficient.

$$\Gamma = \frac{Z_L - Z_S}{Z_L + Z_S} \quad (9.30)$$

Notice that a negative reflection coefficient means that the reflected wave receives a 180°, or π, phase shift.

The absolute magnitude (designated by vertical bars) of the reflection coefficient can be calculated from the standing wave ratio,

SWR:

$$|\Gamma| = \frac{SWR - 1}{SWR + 1} \quad (9.31)$$

9.10 CALCULATION OF CURRENT, VOLTAGE, POWER DELIVERED AND EFFICIENCY OF TRANSMISSION

Voltage and Current ratios

In order to simplify calculations, sinusoidal voltage and current waves are commonly represented as complex-valued functions of time denoted as v and I

$$V = V_0 e^{j(\omega t + \phi V)}$$
$$I = I_0 e^{j(\omega t + \phi I)}$$

Impedance is defined as the ratio of these quantities

$$Z = \frac{V}{I}$$

Substituting these into Ohm's law we have

$$V e^{j(\omega t + \phi V)} = I_0 e^{j(\omega t + \phi I)} Z e^{j\theta}$$
$$= I_0 Z e^{j(\omega t + \phi I + \theta)}$$

Noting that this must hold for all t, we may equate the magnitudes and phases to obtain

$$V_0 = I_0 Z$$
$$\phi_V = \phi_V + \theta$$

The magnitude equation is the familiar Ohm's law applied to the voltage and current amplitudes, while the second equation defines the phase relationship.

Validity of complex representation

This representation using complex exponentials may be justified by noting that (by Euler's formula):

$$cos(\omega t + \phi) = \frac{1}{2}\,[e^{j(\omega t+\phi)} + \; e^{-j(\omega t+\phi)}]$$

i.e. a real-valued sinusoidal function (which may represent our voltage or current waveform) may be broken into two complex-valued functions.

By the principle of superposition, we may analyze the behavior of the sinusoid on the left-hand side by analyzing the behavior of the two complex terms on the right-hand side. Given the symmetry, we only need to perform the analysis for one right-hand term; the results will be identical for the other. At the end of any calculation, we may return to real-valued sinusoids by further noting that

$$cos(\omega t + \phi) = R[e^{j(\omega t+\phi)}]$$

Phasors

A phasor is a constant complex number, usually expressed in exponential form, representing the complex amplitude (magnitude and phase) of a sinusoidal function of time. Phasors are used by electrical engineers to simplify computations involving sinusoids, where they can often reduce a differential equation problem to an algebraic one.

The impedance of a circuit element can be defined as the ratio of the phasor voltage across the element to the phasor current through the element, as determined by the relative amplitudes and phases of the voltage and current. This is identical to the definition from Ohm's law given above, recognizing that the factors of cancel.

Power quantities

When referring to measurements of power or intensity, a ratio can be expressed in decibels by evaluating ten times the base-10 logarithm of the ratio of the measured quantity to the reference level. Thus, if L represents the ratio of a power value P1 to another power value P0, then LdB represents that ratio expressed in decibels and is calculated using the formula:

$$L_{dB} = 10log_{10}\left(\frac{P_1}{P_0}\right)$$

P_1 and P_0 must have the same dimension, i.e. they must measure the same type of quantity, and the same units before calculating the ratio: however, the choice of scale for this common unit is irrelevant, as it changes both quantities by the same factor, and thus cancels in the ratio—the ratio of two quantities is scale-invariant. Note that if P1 = P0 in the above equation, then LdB = 0. If P1 is greater than P0 then LdB is positive; if P1 is less than P0 then LdB is negative.

Rearranging the above equation gives the following formula for P1 in terms of P0 and LdB:

$$P_1 = 10^{\frac{L_{dB}}{10}} P_0$$

Since a bel is equal to ten decibels, the corresponding formulae for measurement in bels (LB) are

$$L_B = log_{10}\left(\frac{P_1}{P_0}\right)$$

$$P_1 = 10^{L_B} P_0$$

9.11 WAVE PROPAGATION IN FREE SPACE

Free space is a region where these are nothing - the vacuum of outer space is a fair approximation for most purposes. There are no obstacles to get in the way, no gases to absorb energy, nothing to scatter the radio waves. Unless you are into space communications, free space is not something you are likely to encounter, but it is important to understand what happens to a radio wave when there is nothing to disturb it.

In free space, a radio wave launched from a point in any given direction will propagate outwards from that point at the speed of light. The energy, carried by photons, will travel in a straight line, as there is nothing to prevent them doing so. For all practical purposes, a radio wave when launched carries on in a straight line forever traveling at the speed of light.

Free space loss is not really a loss at all. It relates to the intensity of the wave at a distance from the source measured by some standard collector, like an antenna or a telescope. As the wave spreads out, the intensity becomes lower.

Consider a radio wave source that radiates in all directions with equal intensity from a single point (like a light bulb).

All the points at a given radius **r** from a single point form the surface of a sphere and the total energy is uniformly spread out over the area of this sphere (remember our source is radiating equally in all directions). So the amount of energy that can be collected over the section of the total area represented by our collector is proportional to the ratio of the "capture area" of our collector to the total area.

The area of this sphere is proportional to the radius: Area = $4\pi r^2$

The power per unit area is simply the total power divided by the total area. If the power is measured in watts this is: Watts / m2 = Total watts / total area in m2

This power is usually referred to as the power flux density: Power flux density = $\frac{P}{4\Pi r^2}$

The amount of power collected by an ideal antenna is simply the power flux density multiplied by the effective capture area of the antenna Ae.

$$P_{rX} = A_e \times Power\ Flux\ Density\ (watts) P_{rX} = A_e P_{rX}/4\pi r^2\ watts \quad (9.32)$$

The effective capture area of an antenna is related to the gain of the antenna. If the Gain of the receiving antenna is G_{rx} the following holds:

$$A_e = (G_{rX})\lambda^2/4\pi$$
$$P_{rx} = P_{tx}(\lambda/4\pi r)^2 \quad watts \quad (9.33)$$

Normalizing this to a receiver antenna of unity gain so G_{rx} =1, the ratio of the received power to the transmitted power which is the proportion we "lose" on the path is called the free space loss represented by:

$$Free\ space\ loss = \frac{P_{rx}}{P_{tx}} = (\lambda/4\pi r)^2 (9.34)$$

The electromagnetic waves that are guided along or over conducting or dielectric surface are called guided waves.

Examples: Parallel wire, transmission lines

Cut off frequency is the wavelength below which there is wave propagation and above which there is no wave propagation.

9.12 POYNTING'S THEOREM

Poynting's theorem is a hugely important mathematical statement in electromagnetic that concerns the flow of power through space. **Poynting's theorem** is a statement of conservation of energy for the electromagnetic field, in the form of a partial differential equation. Poynting's theorem is analogous to the work-energy theorem in classical mechanics, and mathematically similar to the continuity equation, because it relates the energy stored in the electromagnetic field to the work done on a charge distribution (i.e. an electrically charged object), through energy flux.

$$\nabla.(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left|\vec{H}\right|^2 + \frac{1}{2}\varepsilon\left|\vec{E}\right|^2\right) - \sigma\left|\vec{E}\right|^2 \quad (9.35)$$

The wave polarization is defined by the time behavior of the electric field of a TEM wave at a given point in space. In other words, the state of polarization of a wave is describe d by the geometricalshape which the tip of the electric field vector draws as a function of time at a given point in space. Polarization is a fundamental characteristic of a wave, and every wave has a definite state of polarization.

9.13 ELECTROMAGNETIC POWER FLUX

Consider the propagation of an electromagnetic wave through a conducting medium which obeys Oh m's law:

$$j = \sigma E$$

Here, σ is the *conductivity* of the medium. Maxwell's equations for the wave take the form:

$$\nabla.E = 0,$$
$$\nabla.B = 0,$$
$$\nabla\times E = \frac{\partial B}{\partial t},$$
$$\nabla\times B = \mu_0 j + \varepsilon\varepsilon_0\mu_0\frac{\partial E}{\partial t}$$

It can be seen that the skin-depth for a good conductor *decreases* with increasing wave

$$\nabla\times\nabla\times E = -\nabla^2\mathrm{E} = -\frac{\partial\nabla\times B}{\partial t} = \frac{\partial}{\partial t}\left[\mu_0\sigma E + \varepsilon\varepsilon_0\mu_0\frac{\partial E}{\partial t}\right]$$

Looking for a wave like solution of the form

$$E = E_0 e^{i(kz-\omega t)},$$

We obtained the dispersion relation

$$k^2 = \mu_0\omega(\varepsilon\varepsilon_0\omega + i\sigma)$$

Let us consider the case for poor conductor for which $\sigma \ll \epsilon\varepsilon_0\omega$. Applying this the dispersion relation becomes

$$k \cong n\frac{\omega}{c} + i\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon}}$$

Where n is the refractive index having value$n = \sqrt{\epsilon}$. Substituting this in equation results.

$$E = E_0 e^{-\frac{z}{d}} e^{i(k_r z - \omega t)},$$

Where $d = \frac{2}{\sigma}\sqrt{\frac{\varepsilon_0\varepsilon}{\mu_0}}$ and $k_r = \frac{n\omega}{c}$

From the above equation, we can conclude that the propagation of electromagnetic wave through a conductor decays exponentially on the length scale of "d", which is known as "skin depth".

Let us consider the case for good conductor for which $\sigma \gg \epsilon\varepsilon_0\omega$. Applying this the dispersion relation becomes

$$k \cong \sqrt{i\mu_0\sigma\omega}$$

And $d = \frac{1}{k_r} = \sqrt{\frac{2}{\mu_0\sigma\omega}}$

It can be seen that the skin-depth for a good conductor decreases with increasing wave frequency. The fact that k,d=1 indicates that the wave only penetrates a few wave-lengths into the conductor before decaying away. Now the power per unit volume dissipate d via ohmic heating in a conducting medium takes the form

$$P = j.E = \sigma E^2$$

Consider an electromagnetic wave of the form. The mean power dissipated per unit area in the region z> 0 is written

$$\langle P \rangle = \frac{1}{2}\int_0^\infty \sigma E_0^2\, e^{-\frac{2z}{d}} dz = \frac{d\sigma}{4} E_0^2 = \sqrt{\frac{\sigma}{8\mu_0\omega}} E_0^2,$$

For a good conductor. Now, the mean electromagnetic power flux into the region z>0 takes the form

$$\langle u \rangle = \langle \frac{E \times B.\hat{z}}{\mu_0} \rangle_{z=0} = \frac{1}{2}\frac{E_0^2 k_r}{\mu_0\omega} = \sqrt{\frac{\sigma}{8\mu_0\omega}} E_0^2$$

The amplitude of an electromagnetic wave propagating through a conductor decays exponentially on some length-scale, which is termed the skin-depth. The skin-depth for a poor conductor is independent of the frequency of the wave. For a poor conductor, indicating that the wave penetrates many wave-lengths into the conductor before decaying away.

9.14 WAVEGUIDE

Wave guide is normally an enclosed conductor and may be rectangular, circular, etc. in shape. It was invented by "George C Southworth".Waveguide played an important role in Radar Systems during World War II. To honor his work in Wave-guide, he was bestowed with the Morris N. Liebmann award of the IRE in 1938 and the Stuart Ballantine Medal of the Franklin Institute, in 1947. He also received the Louis Levy Medal of the Franklin Institute for his work on Microwave Radiation from the sun.
Waveguides are used to direct and propagate Electromagnetic waves from one point to another. They are generally used to transmit high frequency waves such as Microwaves, Radio waves, Infrared waves etc. For low frequency waves which are less than 1 MHz, parallel transmission lines or co-axial cables are used.
Wave-guide is represented by its dispersion characteristics that has a certain cut-off frequency. The signals having frequencies above this cut-off frequency are allowed to propagate through the Wave-guide and the signals having frequencies below this frequency will face a high reflection. A Waveguide acts like a high pass filter due to this characteristic. The dispersion characteristics can be altered by loading the Wave-guide with metal or di-electric medium.

Some of the components of waveguide are:

- Load – This is responsible for absorbing microwave energy from the system
- Tuner – This component is used to match the load impedance with the source impedance.
- Isolator – They are circulators which have three ports and are responsible for transmitting microwave energy between the different ports while directing and reflecting energy to a port with an attached load.
- Power Measuring (couplers)– These are the devices made to couple power taken from the waveguide system which helps in measuring power, frequency, and other parameters to ensure proper flow of energy.

Classification of Waveguide

Waveguides are classified into two types:
- Metal Waveguides

- Dielectric Waveguides

Metal Waveguide

Metal Waveguides consists of an enclosed conducting metal pipe and the wave guiding principle works on the total internal reflection from the conducting walls. They are of two types:

I. Rectangular Waveguides
II. Circular Waveguides

Dielectric Waveguide

Dielectric Waveguides consists of dielectrics and the reflection from dielectric interfaces helps in the propagation of electromagnetic waves along the Waveguide. They are of two types:

I. Dielectric Slab Waveguides
II. Optical Fiber

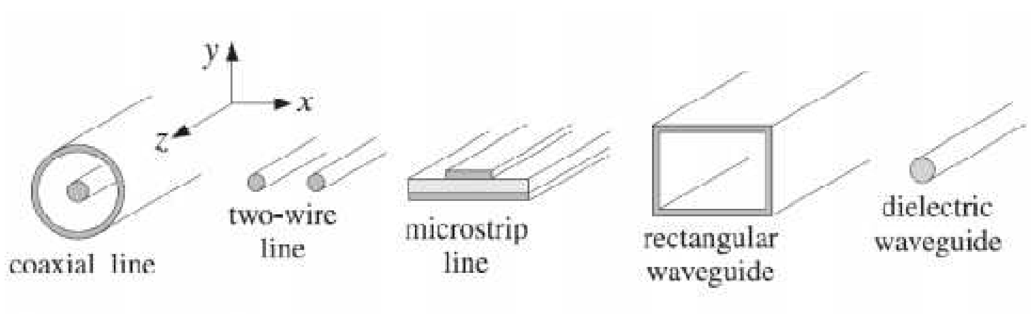


Figure 9.6

- Coaxial cables are widely used to connect RF components. Their operation is practical for frequencies below 3 GHz. Above that the losses are too excessive. For example, the attenuation might be 3 dB per 100 m at 100 MHz, but 10 dB/100 m at 1GHz, and50 dB/100 m at 10 GHz.

- Their power rating is typically of the order of one kilowatt at100 MHz, but only 200 W at 2 GHz, being limited primarily because of the heating of the coaxial conductors and of the dielectric between the conductors (dielectric voltage breakdown is usually a secondary factor.) However, special short-length coaxial cables do exist that operate in the 40 GHz range.

• Another issue is the single-mode operation of the line. At higher frequencies, in order to prevent higher modes from being launched, the diameters of the coaxial conductors must be reduced, diminishing the amount of power that can be transmitted. Two-wire lines are not used at microwave frequencies because they are not shielded and can radiate. One typical use is for connecting indoor antennas to TV sets. Micro strip lines are used widely in microwave integrated circuits.

• Rectangular waveguides next, we discuss in detail the case of a rectangular hollow waveguide with conducting walls, as shown in Fig. Without loss of generality, we may assume that the lengths a, b of the inner sides satisfy $b \leq a$. The guide is typically filled with air, but any other dielectric material ε, μ may be assumed.

The main characteristics of a Waveguide are:

- The tube wall provides distributed inductance.
- The empty space between the tubes walls provide distributed capacitance.
- These are bulky and expensive.

Advantages of Waveguides

Some advantages of waveguides are mentioned below:

- Waveguides are easy to manufacture.
- They can handle very large power in kilowatts.
- Power loss is very negligible in waveguides.
- They offer very low loss
- Low value of attenuation
- When microwave energy travels through waveguide, it experiences lower losses than a coaxial cable.

Waveguide Modes

The two types of Wave-guide Modes that is necessary for propagation of Electromagnetic waves in the Waveguides are:

- TE (Transverse Electric) Mode
- TM (Transverse Magnetic) Mode

TE (Transverse Electric) Mode

In TE (Transverse Electric) Mode, Electric Field vector is transverse or perpendicular to the Waveguide's axis.

TM (Transverse Magnetic) Mode

In TM (Transverse Magnetic) Mode, Magnetic Field (E) vector is transverse or perpendicular to the Waveguide's axis.

9.15 WAVEGUIDE'S EQUATIONS

According to Ampere's law and Faraday's law, we obtain

$$H_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial H_z^0}{\partial x} - j\omega\varepsilon\frac{\partial E_z^0}{\partial y}\right), \qquad H_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial H_z^0}{\partial y} + j\omega\varepsilon\frac{\partial E_z^0}{\partial x}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial E_z^0}{\partial x} + j\omega\mu\frac{\partial H_z^0}{\partial y}\right), \qquad H_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial E_z^0}{\partial y} - j\omega\mu\frac{\partial H_z^0}{\partial x}\right)$$

Where $h^2 = \gamma^2 + k^2$

$$\text{and } \nabla^2\vec{E} + k^2\vec{E} = 0$$
$$\nabla^2\vec{H} + k^2\vec{H} = 0$$

$$\nabla_t^2\vec{E} + (\gamma^2 + k^2)\vec{E} = 0 = [\nabla_t^2 + h^2]\vec{E}$$
$$\nabla_t^2\vec{H} + (\gamma^2 + k^2)\vec{H} = 0 = [\nabla_t^2 + h^2]\vec{H}$$

Case I TEM mode: $E_z = 0, H_z = 0$

$h^2 = 0 = {\gamma_{TEM}}^2 + k^2 \rightarrow \gamma_{TEM} = jk = j\omega\sqrt{\mu\epsilon}\,, v_p = \frac{1}{\sqrt{\mu\epsilon}}$

$$Z_{TEM} = \frac{E_x^0}{H_y^0} = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{\gamma_{TEM}}{j\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta \text{ and } \vec{H} = \frac{1}{Z_{TEM}}\hat{z}\times\hat{E}$$

It is important to note that all the frequencies make γ_{TEM} is pure imaginary, which shows that TEM wave can propagate at any frequency, there is no cut off.

Case I TM mode: $E_z = 0, E_z \neq 0$ and $\nabla_t^2 E_z + h^2 E_z = 0$

$$H_x^0 = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z^0}{\partial y},\ H_y^0 = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z^0}{\partial x},\ E_x^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial x},\ E_y^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial y}$$

$$\Rightarrow Z_{TM} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{\gamma}{j\omega\varepsilon}\left(\neq\frac{j\omega\mu}{\gamma}\right) \text{ and } \bar{H} = \frac{1}{Z_{TM}}(\hat{z}\times\bar{E})$$

$$\gamma = \sqrt{h^2 - k^2} = h\sqrt{1-(\frac{f}{f_c})^2}\text{, where } f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}$$

Case 3TE mode: E_z=0, H_z≠0 and $\nabla_t^2 H_z + h^2 H_z = 0$

$$H_x^0 = -\frac{\gamma}{h^2}\frac{\partial H_z^0}{\partial x},\ H_y^0 = -\frac{\gamma}{h^2}\frac{\partial H_z^0}{\partial y},\ E_x^0 = -\frac{j\omega\mu}{h^2}\frac{\partial H_z^0}{\partial y},\ E_y^0 = \frac{j\omega\mu}{h^2}\frac{\partial H_z^0}{\partial x}$$

$$\Rightarrow Z_{TE} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma}\left(\neq \frac{\gamma}{j\omega\varepsilon}\right)\text{and } \bar{E} = -Z_{TE}(\hat{z}\times\bar{H})$$

$$\text{If } f > f_c,\ \gamma = j\beta = jk\sqrt{1-(\frac{f_c}{f})^2} = j\omega\sqrt{\mu\varepsilon}\sqrt{1-(\frac{f_c}{f})^2}$$

$$\Rightarrow Z_{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}},\ v_p = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}}$$

Case 3 TE mode: $E_z = 0, H_z \neq 0$ and $\nabla_t^2 \mathrm{H_z} + \mathrm{h^2 H_z} = 0$

$H_x^0 = -\frac{\gamma}{h^2}\frac{\partial H_z^0}{\partial x},\ H_y^0 = -\frac{\gamma}{h^2}\frac{\partial H_z^0}{\partial y}, E_x^0 = -\frac{j\mu\omega}{h^2}\frac{\partial H_z^0}{\partial y}, E_y^0 = -\frac{j\mu\omega}{h^2}\frac{\partial H_z^0}{\partial x}$

So,

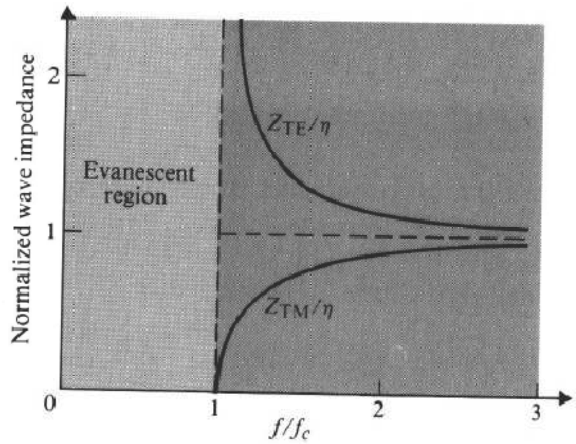
$Z_{TE} = \frac{E_x^0}{H_x^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma}\left(\neq \frac{\gamma}{j\omega\varepsilon}\right)$ and $\vec{E} = -Z_{TE}(\hat{z}\times\widehat{H})$

<u>Difference between TE and TM waves:</u>

TE	TM
Electric field strength E is entirely transverse.	Magnetic field strength is entirely transverse.
It has z component of magnetic field (Hz).	It has z component of electric field (Ez).
It has no z component of electric field (Ez).	It has no z component of magnetic field (Hz).

Mode	Wave impedance, Z	Guide Wavelength
TEM	$\eta = \sqrt{\frac{\mu}{\varepsilon}}$	$\lambda = \frac{1}{f\sqrt{\mu\varepsilon}}$
TM	$\eta\sqrt{1-(f_c/f)^2}$	$\frac{\lambda}{f\sqrt{1-(f_c/f)^2}}$
TE	$\frac{\eta}{\sqrt{1-(f_c/f)^2}}$	$\frac{\lambda}{\sqrt{1-(f_c/f)^2}}$

Figure: Impedances and Guide wavelengths for $f > f_c$.



General Guided Wave Solutions:

-
General solutions to the fields associated with the waves that propagate on a guiding structure using Maxwell's equations.

Assume the following about the guiding structure:

(1) Guiding structure is infinitely long, oriented along the *z*axis, and uniform along its length.

(2) Guiding structure is constructed from ideal materials (conductors are PEC and insulators are lossless).

(3) Fields are time-harmonic. The fields of the guiding structure must satisfy the source free Maxwell's equations given by

$$\nabla \times E = -j\omega\mu H$$
$$\nabla \times H = j\omega\epsilon E$$

For a wave propagating along the guiding structure in the *z*-direction, the associated electric and magnetic fields may be written as

$$\mathrm{E(x,y,z) = [e(x,y) + e_Z\,(x,y)a_z\,]e^{-j\beta Z}}$$
$$\mathrm{H(x,y,z) = [h(x,y) + h_Z\,(x,y)a_z\,]e^{-j\beta Z}}$$

The vectors ***e***(*x,y*) and ***h***(*x,y*) represent the transverse field components of the wave while vectors *ez*(*x,y*)***a****z* and *hz*(*x,y*)***a****z* are the longitudinal components of the wave. By expanding the curl operator in rectangular coordinates, and noting that the derivatives of the transverse components with respect to *z* can be evaluated as

$$\frac{\partial E_x}{\partial z} = j\beta E_x$$

$$\frac{\partial E_y}{\partial z} = j\beta E_y$$
$$\frac{\partial H_x}{\partial z} = j\beta H_x$$
$$\frac{\partial H_y}{\partial z} = j\beta H_y$$

Equate the vector components on each side of the equation to write the six components of the electric and magnetic field as

$$j\omega\varepsilon E_x = \frac{\partial H_x}{\partial y} + j\beta H_y \quad (9.36\ a)$$

$$j\omega\varepsilon E_y = -j\beta H_x - \frac{\partial H_z}{\partial y} \quad (9.36\ b)$$

$$j\omega\varepsilon E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \quad (9.36\ c)$$

$$-j\omega\mu H_x = \frac{\partial E_x}{\partial y} + j\beta E_y \quad (9.37\ a)$$

$$-j\omega\mu H_y = -j\beta E_x - \frac{\partial E_z}{\partial x} \quad (9.37\ b)$$

$$-j\omega\mu H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \quad (9.37\ c)$$

Equations (9.36) and (9.37) are valid for any wave (guided or unguided) propagating in the *z*-direction in a source-free region with a propagation constant of jâ. We may use Equations (9.36) and (9.37) to solve for the longitudinal field components in terms of the transverse field components.

By solving equation (9.36 a) and (9.37 b) for $H_y E_x = \frac{-j}{k_c^2}(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y})$

By solving equation (9.36 b) and (9.37 a) for $H_x E_y = \frac{j}{k_c^2}(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x})$

By solving equation (9.36 b) and (9.37 a) for $E_y H_x = \frac{j}{k_c^2}(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x})$

By solving equation (9.36 a) and (9.37 b) for $E_x H_y = \frac{-j}{k_c^2}(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y})$

Where k_c is the cut off wave number and defined as:

$$k_c^2 = k^2 - \beta^2$$

The cutoff wavenumber for the wave guiding structure is determined by the wavenumber of the insulating medium through which the wave propagates and the propagation constant for the structure. The equations for the transverse components of the fields are valid for all of the modes defined previously. These transverse field component equations can be specialized for each one of these guided structure modes.

TEM Mode:

Using the general equations for the transverse fields of guided waves [Equation (9.38)], we see that the transverse fields of a TEM mode (defined by $E_z = H_z = 0$) are non-zero only when $kc = 0$. When the cutoff wavenumber of the TEM mode is zero, an indeterminant form of (0/0) results for each of the transverse field equations

$$E_x = \frac{-j}{k_c^2}(\beta \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}) \quad (9.38\ a)$$

$$E_y = \frac{j}{k_c^2}\left(-\beta \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x}\right) \quad (9.38\ b)$$

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x}\right) \quad (9.38\ c)$$

$$H_y = \frac{-j}{k_c^2}(\omega\varepsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y}) \quad (9.38\ d)$$

Zero valued cutoff wave number yields the following:

$$k_c^2 = k^2 - \beta^2 = 0$$

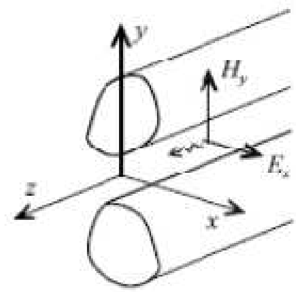
$$\beta = k = \omega\sqrt{\mu\varepsilon}$$

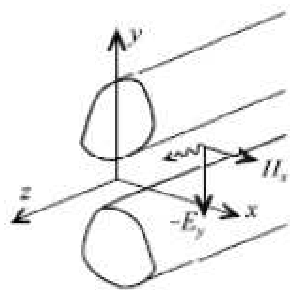
$$k_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon} = 0$$

$$f_c = 0$$

The first equation above shows that the phase constant â of the TEM mode on a guiding structure is equivalent to the phase constant of a plane wave propagating in a region characterized by the same medium between the conductors of the guiding structure. The second equation shows that the cutoff frequency of a TEM mode is 0 Hz. This means that TEM modes can be propagated at any non-zero frequency assuming the guiding structure can support a TEM mode.

Relationships between the transverse fields of the TEM mode can be determined by returning to the source-free Maxwell's equation results for guided waves [Equations (9.36) and (9.37)] and setting $E_z = H_z = 0$ and â = k.





$$E_x = \frac{k}{\omega\epsilon} H_y$$
$$E_y = -\frac{k}{\omega\epsilon} H_x$$
$$\frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y}$$
$$H_x = -\frac{k}{\omega\mu} E_y$$
$$H_y = \frac{k}{\omega\mu} E_x$$
$$\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}$$

Note that the ratios of the TEM electric and magnetic field components define wave impedances which are equal to those of equivalent plane waves.

$$\frac{E_x}{H_Y} = \frac{k}{\omega\epsilon} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \eta = Z_{TEM} \quad \rightarrow \quad E_x = Z_{TEM} H_y$$
$$\frac{-E_y}{H_x} = \frac{k}{\omega\epsilon} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \eta = Z_{TEM} \quad \rightarrow \quad E_y = -Z_{TEM} H_x$$

The previous results can combine to yield

$$h(x,y) = \frac{1}{Z_{TEM}} a_z \times e(x,y)$$

The fields of the TEM mode must also satisfy the respective wave equation:

$$\nabla^2 E + k^2 E = 0$$
$$\nabla^2 H + k^2 H = 0$$

Where

$$E(x,y,z) = e(x,y)e^{-j\beta z}$$
$$H(x,y,z) = h(x,y)e^{-j\beta z}$$

In rectangular coordinates, the vector Laplacian operator is

$$\begin{aligned}\nabla^2 F &= \nabla^2 F_x a_x + \nabla^2 F_y a_y + \nabla^2 F_z a_z \\ &= (\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}) a_x \\ &+ (\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2}) a_y \\ &+ (\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}) a_z\end{aligned}$$

By separating the rectangular coordinate components in the wave equation, we find that each of the field components F_0 (E_x, Ey, H_x, H_y) must then satisfy the same equation [Helmholtz equation].

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + k^2 F = 0$$
$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \beta^2 F + k^2 F = 0$$
$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + k_c^2 F = 0$$
$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

So that the TEM field components must satisfy:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} = 0$$
$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = 0$$
$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = 0$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} = 0$$

This result can be written in compact form as

$$\nabla_t^2 e(x, y) = 0$$
$$\nabla_t^2 h(x, y) = 0$$

where L*t* defines the transverse Laplacian operator which in rectangular 2 coordinates is

$$\nabla_t^2 F = \nabla^2 F_x a_x + \nabla^2 F_y a_y$$

According to the previous result, the transverse fields of the TEM mode must satisfy Laplace's equation with boundary conditions defined by the conductor geometry of the guiding structure, just like the static fields which would exist on the guiding structure for $f = 0$. Thus, the TEM transverse field vectors $\boldsymbol{e}(x,y)$ and $\boldsymbol{h}(x,y)$ are identical to the static fields for the transmission line. This allows us to solve for the static fields of a given guiding structure geometry (Laplace's equation) to determine the fields of the TEM mode.

<u>TE Modes</u>:

The transverse fields of TE modes are found by simplifying the general guided wave equations in (9.38) with $Ez = 0$. The resulting transverse fields for TE modes are

$$E_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \quad (9.39\text{ a})$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \quad (9.39\text{ b})$$

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x} \quad (9.39\text{ c})$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y} \quad (9.39\text{ d})$$

The cutoff wavenumber k_c must be non-zero to yield bounded solutions for the transverse field components of TE modes. This means that we must operate the guiding structure above the corresponding cutoff frequency for the particular TE mode to propagate. Note that all of the transverse field components of the TE modes can be determined once the single longitudinal component (H_z) is found. The longitudinal field component H_z must satisfy the wave equation so that

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \beta^2 H_z + k^2 H_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_c^2 H_z = 0$$

Given the basic form of the guided wave magnetic field

$$H(x,y,z) = [h(x,y) + h_z(x,y)a_z]e^{-j\beta z}$$
$$H_z(x,y,z) = h_z(x,y)e^{-j\beta z}$$

we may write

$$\frac{\partial^2 h_Z(x,y)}{\partial x^2} + \frac{\partial^2 h_Z(x,y)}{\partial y^2} + k_c^2 h_z(x,y) = 0$$

The equation above represents a reduced Helmholtz equation which can be solved for $h_z(x,y)$ based on the boundary conditions of the guiding structure geometry. Once $h_z(x,y)$ is found, the longitudinal magnetic field is known, and all of the transverse field components are found by evaluating the derivatives in Equation (9.39).

The wave impedance for TE modes is found from Equation (9.39):

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$$

Note that the TE wave impedance is a function of frequency.

TM Modes

The transverse fields of TM modes are found by simplifying the general guided wave equations in (9.38) with $Hz = 0$. The resulting transverse fields for TM modes are

$$E_x = -\frac{j\beta}{k_c^2}\frac{\partial E_x}{\partial x} \qquad (9.40\text{ a})$$

$$E_y = -\frac{j\beta}{k_c^2}\frac{\partial E_z}{\partial y} \qquad (9.40\text{ b})$$

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} \qquad (9.40\text{ c})$$

$$H_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} \qquad (9.40\text{ d})$$

The cutoff wavenumber k_c must also be non-zero to yield bounded solutions for the transverse field components of TM modes so that we must operate the guiding structure above the corresponding cutoff frequency for the particular TMmode to propagate. Note that all of the transverse field components of the TMmodes can be determined once the single longitudinal component (E_z) is found. The longitudinal field component E_z must satisfy the wave equation so that

$$\frac{\partial^2 E_Z}{\partial x^2} + \frac{\partial^2 E_Z}{\partial y^2} + \frac{\partial^2 E_Z}{\partial z^2} + k^2 E_z = 0$$
$$\frac{\partial^2 E_Z}{\partial x^2} + \frac{\partial^2 E_Z}{\partial y^2} - \beta^2 E_z + k^2 E_z = 0$$
$$\frac{\partial^2 E_Z}{\partial x^2} + \frac{\partial^2 E_Z}{\partial y^2} + k_c^2 E_z = 0$$

Given the basic form of the guided wave electric field

$$E(x,y,z) = [e(x,y) + e_z(x,y)a_z]e^{-j\beta z}$$

$$E_z(x,y,z) = e_z(x,y)e^{-j\beta z}$$

We may write it as:

$$\frac{\partial^2 e_Z(x,y)}{\partial x^2} + \frac{\partial^2 e_Z(x,y)}{\partial y^2} + k_c^2 e_z(x,y) = 0$$

The equation above represents a reduced Helmholtz equation which can be solved for $e_z(x,y)$ based on the boundary conditions of the guiding structure geometry. Once $e_z(x,y)$ is found, the longitudinal magnetic field is known, and all of the transverse field components are found by evaluating the derivatives in Equation (9.40).

The wave impedance for TM modes is found from Equation (9.40):

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k}$$

Note that the TM wave impedance is also a function of frequency.

9.16 SUMMARY

In the present unit, you have studied about transmission line and waveguide. Along this you have also studied, the distributed constants and application of transmission lines. You have seenthe Transmission line, general solution & the infinite line. Wavelength, velocity of propagation and the distortion less line, reflection coefficienthave been discussed in detail. Waveguide and mode of propagation have been also discussed in detail. To present the clear understanding and to make the concepts of the unit clear, many solved examples are given in the unit. To check your progress, terminal questions are given place to place.

9.17 GLOSSARY

Waveguide	A metal tube or other device confining and conveying microwaves
Mode	When light travels in a waveguide, we can have electromagnetic field pattern that does not change as a function of distance.
Propagation loss	The loss in power as an optical wave travels in space.
Waveguide loss	The optical propagation loss in an optical waveguide. Typically expressed in dB/cm
Transmission line	Conductor for transmitting electrical or optical signals or electric power.
Decibel	Logarithmic measure of power.

Attenuation constant (α)	The magnitude of ratio between input and output quantities of the network.
Cutoff frequency	Frequency at which the network changes from a pass band region to a stop band region or vice versa

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9.20 TERMINAL QUESTIONS

Short Answer type

1. What do you mean by Characteristic Impedance?

2.Define the line parameters of a transmission line.

3. Why the line parameters are called distributed elements?

4. What do you mean by propagation constant?

5. What is a distortion less line?

6. What is the condition for the distortion less line?

7. What do you understand by loading?

8. Define Reflection coefficient?

9. What is called dominant mode?

10. What is called cutoff frequency?
11. Distinguish between TE and TM waves.
12. Define resonant cavity?
13. Define the quality factor of the cavity resonator?
14. What is dominant mode? Name the dominant mode in TE and TM waves?

Long Answer type

1. Derive the general transmission line equations for voltage and current at any point on a line.
2. Derive an expression for the attenuation constant and phase constant of a transmission line in terms of the line constants R, L, G and C.
3. Derive the conditions required for a distortion less line.
4. Derive the expressions for the field components of TEM waves between parallel planes. Discuss the properties of TEM waves.
5. Distinguish between the characteristics of TE and TM waves.
6. What is meant by cavity resonator? Derive the expression for the resonant frequency of the rectangular cavity resonator?
7. Derive the expression for the resonant frequency of the circular cavity resonator?

Numerical type

1. A line has the following primary constants: R = 100 Ω/km, L = 0.001 H/km, G = 1.5 μmho/km, C = 0.062 μF/km.
Find the characteristic impedance, propagation constant, velocity of propagation and wavelength.

2. A generator of 1V, 1 kHz supplies power to a 100 km open wire line terminated in 200 Ω resistance. The line parameters are R = 10 Ω/km, L = 3.8 mH/km, G = 1x10-6 mho/km, C = 0.0085 μF/km. Calculate the input impedance, reflection coefficient, the input power, and the output power.

3. For a frequency of 15 GHz and plane separation of 8 cm in air, find the following for TM1 mode.

a) Cut-off wavelength

b) Characteristic impedance

c) Phase constant

4. For a frequency of 20 GHz and plane separation of 5cm in air, find the following:

a) Cu-off wavelength

b) Phase velocity

c) Group velocity

Multiple choice type questions

1. The waveguide is employed in the transmission lines, when operated at the range of:
a) Hz
b) KHz
c) MHz
d) GHz

2. The cut off frequency for a waveguide to operate is:
a. 3 MHz
b. 3 GHz
c. 6 MHz
d. 6 GHz

3. In rectangular waveguides, the dimensions a and b represent the:
a. Broad wall dimensions
b. Broad wall and side wall dimension respectively
c. Side wall and broad wall dimension respectively
d. Side wall dimensions

4. When electromagnetic waves are propagated in a waveguide:
a. they travel along a broader walls of the guide
b. they are reflected from the walls but do not travel along them
c. they travel through the dielectric without touching the walls
d. they travel along all four walls of the waveguide

5. Waveguides are used mainly for microwave signals because:
a. they depend on straight-line propagation which applies to microwaves only
b. losses would be too heavy at lower frequencies
c. there are no generators powerful enough to excite them at lower frequencies
d. they would be too bulky at lower frequencies

6. The wavelength of a wave in a waveguide

a. is greater than of free space

b. depends only on the waveguide dimensions and the free-space wavelength

c. is inversely proportional to the phase velocity

d. is directly proportional to the group velocity

7. The main difference between the operation of transmission lines and waveguides is that

a. the latter are not distributed, like transmission lines

b. the former can use stubs and quarter-wave transformers, unlike the latter

c. transmission lines use the principal mode of propagation, and therefore do not suffer from low-frequency cut-off

d. terms such as impedance matching and standing-wave ratio cannot be applied to waveguides

8. When a particular mode is excited in a waveguide, three appears an extra electric component, in the direction of propagation. The resulting mode is

a. transverse-electric

b. transverse-magnetic

c. longitudinal

d. transverse-electromagnetic

9. A signal propagation in a waveguide has a full wave of electric intensity change between the two further walls, and no component of the electric field in the direction of propagation. The mode is

a. TE1,1

b. TE1,0

c. TM2,2

d. TE2,0

Answer: 1: d**, 2:** d**, 3:** b, **4:** b, **5:** d, **6:** a, **7:** c, **8:** b, **9:** d