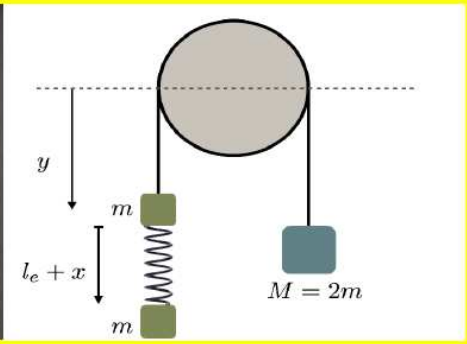
MSCPH502

M. Sc. Ist Semester

CLASSICAL MECHANICS



DEPARTMENT OF PHYSICS

SCHOOL OF SCIENCES

UTTARAKHAND OPEN UNIVERSITY

Board of Studies

Department of Physics (School of Sciences)

Unit writing and Editing

Course Title and Code : Classical Mechanics (MSCPH502)
ISBN :
Copyright : Uttarakhand Open University
Edition : 2021
Published By : Uttarakhand Open University, Haldwani, Nainital- 263139
Printed By :

Classical Mechanics

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Credit: 3

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UNIT 1 MECHANICS OF A SYSTEM OF PARTICLES

Structure

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1.1 INTRODUCTION

We have study that the applications of Newton's laws of motion require the specification of all the forces acting on the body at all instants of time. But in practical situations, when the constraint forces are present, the applications of the Newtonian approach may be a difficult task. You will see that the greatest drawback with the Newtonian procedure is that the mechanical problems are always tried to resolve geometrically rather than analytically. When there is constrained motion, the determination of all the insignificant reaction forces is a great bother in the Newtonian approach. In order to resolve these problems, methods have been developed by D'Alembert, Lagrange and others. The techniques of Lagrange use the generalized coordinates which will be discussed and used in this unit. You will see that in the Lagrangian formulation, the generalized coordinates used are position and velocity, resulting in the second order linear differential equations.

1.2 OBJECTIVES

After studying this unit, you should be able to-

- understand degrees of freedom, constraints and generalized coordinates
- understand and use D'Alembert principle
- understand and use Lagrange's equation of motion
- solve the problems based on D'Alembert principle and Lagrange's equation of motion

1.3 SYSTEM OF PARTICLES

We know that a system of particles means a group of particles inter-related. The equations for a system of particles can be readily used to develop those for a rigid body. An object of ordinary size known as a "macroscopic" system — contains a huge number of atoms or molecules. A very important concept introduced with a system of particles is the center of mass.

Let us consider a system consisting of N point particles, each labeled by a value of the index i which runs from 1 to N. Each particle has its own mass m_i and (at a particular time) is located at its particular place r_i. The center of mass (CM) of the system is defined by the following position vector-

Center of mass $\overrightarrow{r_{CM}} = (1/M)\sum_i m_i \vec{r}_i$ .....(1)

where M is the total mass of all the particles. This vector locates a point in space which may or may not be the position of any of the particles. It is the mass-weighted average position of the particles, being nearer to the more massive particles. Let us consider a simple example to understand this. Let us consider a system of only two particles, of masses m and 2m, separated by a distance l. Let us choose the coordinate system so that the less massive particle is at the origin and the other is at x= l, as shown in Figure (1). Then we have m_1 =

m, m_2 =2m, x_1 =0, x_2 = l. (The y and z coordinates are zero of course.) By the definition x_{CM} = (1/ 3m) (m×0+2m×l)= (2/ 3)l

With time, the position vectors of the particles $\vec{r_i}$ generally change, therefore, in general the center of mass (CM) moves.

The velocity of its motion is the time derivative of its position-

$\overrightarrow{v_{CM}} = (1/M) \sum_{i=1}^{N} m_i \vec{v}_i$ .....(2)

But the sum on the right side is just the total linear momentum of all the particles. Solving for this, we find an important result-

Total linear momentum of a system $\overrightarrow{p_{tot}} = M\overrightarrow{v_{CM}}$ .....(3)

This is just like the formula for the linear momentum of a single particle. Therefore, we see that the total linear momentum of a system is the same as if all the particles were located together at the CM and moving with its velocity.

The total force on the ith particle consists of the (net) external force on it, plus the net force due to the interactions with other particles in the system, which we call the internal forces. We can write this out as follows-

$$\vec{F}_i = \overrightarrow{F_i^{ext}} + \sum_j \vec{F}_{ij} \quad .....(4)$$

where $\vec{F}_{ij}$ denotes the force exerted on the ith particle by the jth particle.

To get the total force on the whole system, we simply add up all these forces-

$$\overrightarrow{F_{tot}} = \sum_i \vec{F}_i^{ext} + \sum_i \sum_{j \neq 1} \vec{F}_{ij} \quad .....(5)$$

In the double sum on the right the terms cancel in pairs by Newton's 3rd law (for example, $\overrightarrow{F_{12}} + \overrightarrow{F_{21}}$ =0). The double sum thus gives zero, therefore-

$$\overrightarrow{F_{tot}} = \sum_i \vec{F}_i^{ext} \quad .....(6)$$

The total force on the system is the sum of only the external forces. Since there are very many internal forces, the fact that they give no net contribution to the total force is why it is possible for many applications to treat an object of macroscopic size as a single particle. The internal forces do play important roles in determining some aspects of the system, such as its energy.

For each particle individually, we have Newton's 2nd law-

$$\vec{F}_i = \frac{\overrightarrow{dp_i}}{dt} \quad .....(7)$$

Adding these for all the particles, and using the above result for $\overrightarrow{p_{tot}}$, we find two forms of the 2nd law as it applies to systems of particles-

Newton's 2nd law for systems-

$$\overrightarrow{F_{tot}^{ext}} = \frac{\overrightarrow{dp_{tot}}}{dt} = M\,\overrightarrow{a_{CM}} \qquad .....(8)$$

Here $\overrightarrow{a_{CM}} = \frac{\overrightarrow{dv_{CM}}}{dt}$ is the acceleration of the CM. We see that the total external force produces an acceleration of the center of mass, as though all the particles were located there.

This is not the only possible effect of the external forces. They can also cause rotational motion about the CM. But the internal forces do not change the motion of the CM. From the first formula above we find one of the most important laws of mechanics i.e. law of conservation of momentum. If the total external force on a system is zero, the total momentum of the system is conserved. In physics, the term "is conserved" means "remains constant in time

The total kinetic energy of all the particles is

$$\frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i(\vec{v}_i + \vec{v}_{CM})^2 = \frac{1}{2}\sum_i m_i[v_i^2 + v_{CM}^2 + 2\vec{v}_i.\vec{v}_{CM}]$$

$$= \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\sum_i m_i v_i^2 + \vec{v}_{CM}.\sum_i m_i\vec{v}_i$$

But the value of last term is zero, therefore we have-

$$\frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\sum_i m_i v_i^2$$

We can interpret the terms on the right as-

• The first term is what the kinetic energy would be if all the particles really were at the CM and moving with its speed. We often call this the kinetic energy of the CM motion.

• The second term is the total kinetic energy as it would be measured by an observer in the CM reference frame. We call this the kinetic energy relative to the CM, or sometimes the internal kinetic energy.

The total kinetic energy is the sum of these two terms-

Kinetic energy of a system, $K = \frac{1}{2}Mv_{CM}^2 + K(rel.\,to\ CM)$

This breakup of the kinetic energy into that of the CM plus that relative to the CM is an example of the general property. We will see that this property also holds for angular momentum. It holds for linear momentum too, but the second part, the total linear momentum relative to the CM, is always zero.

Newton's second law of motion says that the total (external) force is equal to the rate of change of total (linear) momentum. It follows that

$$d\vec{p}_{tot} = \vec{F}_{tot}\,dt$$

Integrating both sides, we find for the net change in momentum-

$$\Delta\vec{p}_{tot} = \int_{t_0}^{t_1} \vec{F}_{tot}\,dt$$

This integral (over time) of the force is called the impulse. We have shown a theorem:

We know that according to Impulse-momentum theorem, "The impulse of the total force is equal to the change of the total linear momentum."

One use of this relation is to define the average force that acts during a specified time interval. Let the force act for time Δt, producing a net change $\Delta\vec{p}$ in the total momentum. The average force is given by-

Average force $\vec{F}_{av} = \frac{\Delta\vec{p}_{tot}}{\Delta t}$

This is useful in cases where the force is an unknown function of time and we would like to describe its average effect over some specific time interval without having to investigate the detailed behaviour.

1.4 DEGREE OF FREEDOM

"The minimum number of independent variables or coordinates required to specify (or define) the position of a dynamical system, consisting of one or more particles, is called the number of degrees of freedom of the system."

Let us consider the example of the motion of a particle, moving freely in space. This motion can be described by a set of three coordinates (x, y, z) and hence the number of degrees of freedom possessed by the particle is three. Similarly, a system of two particles moving freely in space needs two sets of three coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) i.e. six coordinates to specify its position. Thus, the system has six degrees of freedom. If a system consists of N particles, moving freely in space, we require 3N independent coordinates to describe its position. Hence, the number of degrees of freedom of the system is 3N.

The configuration of the system of N particles, moving freely in space, may be represented by the position of a single in 3N dimensional space which is known as configuration space of the system. The configuration space for a system of one freely moving particle is 3-dimensional and for a system of two freely moving particles, it is six dimensional.

The Number of coordinates required to define (or specify) a dynamical system, becomes smaller, when the constraints are present in the system. Therefore, the degrees of freedom of a dynamical system is defined as the minimum number of independent coordinates or variables required to specify the system well-matched with the constraints.

If there are n independent variables, say q_1, q_2 ,..... q_n and n constants C_1, C_2, ..., C_n such that $\sum_{i=1}^{n} C_i dq_i = 0$ at any position of the system, then we must have-

$C_1 = C_2 = ...C_n = 0$

1.5 CONSTRAINTS

Generally, the motion of a particle or system of particles is restricted by one or more conditions. The restrictions on the motion of a system are called constraints and the motion is said to be constrained motion. A constrained motion cannot proceed arbitrarily in any manner. For example, a particle motion is restricted to occur only along some specified path, or on a surface (plane or curved) arbitrarily oriented in space. The motion along a specified path is the simplest example of a constrained motion.

1.5.1 Holonomic Constraints and Nonholonomic Constraints

The constraints that can be expressed in the form $f(x_1, y_1, z_1: x_2, y_2, z_2; x_n, y_n, z_n; t) = 0$, where time t may occur in case of constraints which may vary with time, are called holonomic and the constraints not expressible in this way are termed as non-holonomic.

The motion of the particle placed on the surface of sphere under the action of gravitational force is bound by non-holonomic constraint $(r^2 - a^2) \geq 0$

1.5.2 Scleronomic Constraint and Rheonomic Constraints

If the constraints are independent of time, they are called as scleronomic but if they contain time explicity, they are termed as rheonomic. A bead sliding on a moving wire is an example of rheonomic constraint.

1.6 FORCES OF CONSTRAINT

You should know that the constraints are always related to forces which restrict the motion of the system. These forces are known as “forces of constraint”. For example, the reaction force on a sliding particle on the surface of a sphere is the force of constraint. If we consider the case of a rigid body, the inertial forces of action and reaction between any two particles are the forces of constraint. In a simple pendulum, the force of constraint is the tension in the string. Similarly, in the case of a bead sliding on the wire is the reaction by the wire exerted on the bead at each point. These forces of constraint are elastic in nature and generally appear at the surface of contact because the motion due to external applied forces is slowed down by the contact. Newton has not given any direction to calculate these forces of constraint. Generally, the forces of constraint act in a direction perpendicular to the surface of constraints while the motion of the object is parallel to the surface. In such cases, the work done by the forces of constraint is zero. These constraints are known as workless and may be called as ideal constraints.

1.7 GENERALIZED COORDINATES

The smallest possible number of variables to describe the configuration of a system are called "Generalized coordinates". Thus the name generalized coordinates is given to a set of independent coordinates sufficient in number to describe completely the state of configuration of a dynamical system. The coordinates are represented as q_1, q_2, q_3,.....q_k, ...q_n; where n is the total number of generalized coordinates. Thus, we should know that these are the minimum number of coordinates required to describe the motion of the system.

Let us consider some examples of generalized coordinates. For a particle constrained to move on the circumference of a circle, only one generalized coordinate $q_1 = \theta$ is enough and two generalized coordinates $q_1 = \theta$ and $q_2 = \phi$ for a particle moving on the surface of a sphere. The number of generalized coordinates for a system of N particles, constrained by m equations, are n = 3N- m. It is not necessary that these coordinates should be rectangular, spherical or cylindrical.

Let us see the generalized notations for displacement, velocity, acceleration, momentum, force, potential energy in terms of generalized co-ordinates as follows-

(a) **Generalized Displacement:** δq_j are called generalized displacement or arbitrary displacements. If q_j is an angle coordinate, δq_j is an angular displacement.

(b) **Generalized Velocity:** Generalized velocity may be described in terms of time derivative $\dot{q}_j$ of the generalized coordinate q_j.

(c) **Generalized Acceleration:** The double derivative of q_j i.e. $\ddot{q}_j$ is known as generalized acceleration.

(d) **Generalized Momentum:** The momentum associated with generalized coordinate q_k is called the generalized momentum p_k associated with a coordinate q_k. Generalized momentum $p_k = \frac{\partial T}{\partial \dot{q}_k}$, where T is the kinetic energy of a system

(e) **Generalized Force:** $Qj = \sum_{i=1}^{i=N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$, here Q_j is called the generalized force associated with a coordinate q_j and N , the number of free particles in the system.

1.8 PRINCIPLE OF VIRTUAL WORK

Let us consider a system of N particles. We know that an infinitesimal virtual displacement of ith particle of the system is represented by $\delta\vec{r}_i$. This is the displacement of position coordinates only and does not involve variation of time i.e.

$$\delta\vec{r}_i = \delta\vec{r}_i\,(q_1, q_2, q_3, ...q_n) \qquad .....(9)$$

Let us assume that the system is in equilibrium, then we know that the total force on any particle is zero i.e.

$$\vec{F}_i = 0, \qquad i= 1, 2, 3, ...N$$

The virtual work of the force $\vec{F}_i$ in the virtual displacement $\delta\vec{r}_i$ will also be zero i.e.

$$\delta W_i = \vec{F}_i . \delta\vec{r}_i = 0$$

Similarly, the sum of virtual work done for all the particles should be zero i.e.

$$\delta W = \sum_{i=1}^{i=N} \vec{F}_i . \delta\vec{r}_i = 0 \qquad .....(10)$$

The equation (10) represents the "Principle of Virtual Work". It states that the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium.

The total force $\vec{F}_i$ on the ith particle can be expressed in the following way-

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i \qquad .....(11)$$

Here, $\vec{F}_i^a$ is the applied force and $\vec{f}_i$ is the force of constraint. Therefore, equation (10) takes the form as-

$$\delta W = \sum_{i=1}^{i=N} (\vec{F}_i^a + \vec{f}_i) . \delta\vec{r}_i = 0$$

or

$$\sum_{i=1}^{i=N} \vec{F}_i^a . \delta\vec{r}_i + \sum_{i=1}^{i=N} \vec{f}_i . \delta\vec{r}_i = 0 \qquad .....(12)$$

Now we limit ourselves to the systems where the virtual work of the forces of constraints is zero (i.e. in the case of rigid body), then from equation (12), we have-

$$\sum_{i=1}^{i=N} \vec{F}_i^a . \delta\vec{r}_i = 0 \qquad .....(13)$$

i.e. for the equilibrium of a system, the virtual work of applied forces is zero. It is obvious that the principle of virtual work deals with the statics of a system of particles.

1.9 D'ALEMBERT'S PRINCIPLE

D'Alembert's principle, alternative form of Newton's second law of motion, stated by the 18th-century French polymath Jean le Rond d'Alembert. In effect, the principle reduces a problem in dynamics to a problem in statics. The Newton's second law states that the force *F* acting on a body is equal to the product of the mass *m* and acceleration *a* of the body, or $F = ma$; in D'Alembert's form, the force *F* plus the negative of the mass *m* times acceleration *a* of the body is equal to zero: $F - ma = 0$. In other words, the body is in equilibrium under the action of the real force *F* and the fictitious force $-ma$. The fictitious force is also called an inertial force and a reversed effective force.

This method is based on the principle of virtual work. We know that according to Newton's second law of motion, the force acting on the ith particle is given by-

$$\vec{F}_i = \frac{\overrightarrow{dp_i}}{dt} = \dot{\vec{p}}_i$$

Or $$\vec{F}_i - \dot{\vec{p}}_i = 0$$ where i = 1, 2, 3.....N

It is clear from the above equation that any particle in the system is in equilibrium under a force which is equal to the actual force $\vec{F}_i$ plus a reversed effective force $\dot{\vec{p}}_i$. Therefore, for virtual displacement $\delta\vec{r}_i$, we can write-

$$\sum_{i=1}^{i=N}(\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \qquad .....(14)$$

But $\vec{F}_i = \vec{F}_i^a + \vec{f}_i$, therefore from equation (14), we have-

$$\sum_{i=1}^{i=N}(F_i^a - \dot{\vec{p}}_i).\delta\vec{r}_i + \sum_{i=1}^{i=N} f_i.\delta\vec{r}_i = 0 \qquad .....(15)$$

Again, the system is restricted for which the virtual work of the constraints is zero, i.e. $\sum_{i=1}^{i=N} \vec{f}_i.\delta\vec{r}_i = 0$. Therefore, above equation (15) reduces as-

$$\sum_{i=1}^{i=N}(F_i^a - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \qquad .....(16)$$

This is known as D'Alembert's principle.

In general, we can write-

$$\sum_{i=1}^{i=N}(\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \qquad .....(17)$$

1.10 LAGRANGE'S EQUATION OF MOTION

Let us consider a system of N particles. The coordinate transformation equations are-

$\vec{r}_i = \vec{r}_i\,(q_1, q_2, q_3, ..., q_k ....., q_n, t)$ .....(18)

where 't' is the time and q_k (k = 1, 2, 3, ...n) are the generalized coordinates.

Differentiating equation (18) with respect to t, we get-

$$\frac{d\vec{r}_i}{dt} = \frac{\partial\vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots\cdots + \frac{\partial\vec{r}_i}{\partial q_k}\frac{dq_k}{dt} + \cdots\cdots + \frac{\partial\vec{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

Or $$\vec{v}_i = \dot{\vec{r}}_i = \sum_{k=1}^{k=n}\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \qquad .....(19)$$

where $\dot{q}_k$ are the generalized velocities.

The virtual displacement is given by-

$$\delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial\vec{r}_i}{\partial q_2}\delta q_2 + \ldots\ldots\ldots\ldots + \frac{\partial\vec{r}_i}{\partial q_k}\delta q_k + \ldots\ldots + \frac{\partial\vec{r}_i}{\partial q_n}\delta q_n$$

Or $$\delta\vec{r}_i = \sum_{k=1}^{k=n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k \qquad .....(20)$$

(since by definition, the virtual displacements do not depend on time)

According to D'Alembert's principle-

$$\sum_{i=1}^{i=N}(\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \qquad .....(21)$$

Here $\sum_{i=1}^{i=N}(\vec{F}_i).\delta\vec{r}_i = \sum_{i=1}^{i=N}(\vec{F}_i).\sum_{k=1}^{k=n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k$

$$= \sum_{k=1}^{k=n}\sum_{i=1}^{i=N}\left[F_i.\frac{\partial\vec{r}_i}{\partial q_k}\right]\delta q_k = \sum_{k=1}^{k=n}G_k\,\delta q_k \qquad .....(22)$$

where $G_k = \sum_{i=1}^{i=N}\vec{F}_i.\frac{\partial\vec{r}_i}{\partial q_k} = \sum_{i=1}^{i=N}\left\{F_{xi}\frac{\partial x_i}{\partial q_k} + F_{yi}\frac{\partial y_i}{\partial q_k} + F_{zi}\frac{\partial z_i}{\partial q_k}\right\}$ .....(23)

are known as the components of generalized force associated with the generalized coordinates q_k.

Now, $$\sum_{i=1}^{i=N}(\dot{\vec{p}}_i).\delta\vec{r}_i = \sum_{i=1}^{i=N}m_i\ddot{\vec{r}}_i.\sum_{k=1}^{k=n}\frac{\partial r_i}{\partial q_k}\delta q_k$$

$$= \sum_{k=1}^{k=n}\left\{\sum_{i=1}^{i=N}m_i\ddot{\vec{r}}_i.\frac{\partial r_i}{\partial q_k}\right\}\delta q_k \qquad .....(24)$$

and $$\sum_{i=1}^{i=N}m_i\ddot{\vec{r}}_i.\frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{i=N}\left\{\frac{d}{dt}\left(m_i\dot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_k}\right\} - m_i\dot{\vec{r}}_i.\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_k}\right)\right. \qquad .....(25)$$

We can write the above equation (25) as-

$$\sum_{i=1}^{i=N}m_i\ddot{\vec{r}}_i.\frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{i=N}\left\{\frac{d}{dt}\left(m_i\vec{v}_i.\frac{\partial\vec{v}_i}{\partial q_k}\right\} - m_i\vec{v}_i.\left(\frac{\partial\vec{v}_i}{\partial q_k}\right)\right. \qquad .....(26)$$

Putting equating (26) in equation (24), we get-

$$\sum_{i=1}^{i=N}(\dot{\vec{p}}_i).\delta\vec{r}_i = \sum_{k=1}^{k=n}\left\{\sum_{i=1}^{i=N}\left\{\frac{d}{dt}\left(m_i\vec{v}_i.\frac{\partial\vec{v}_i}{\partial q_k}\right\} - m_i\vec{v}_i.\left(\frac{\partial\vec{v}_i}{\partial q_k}\right)\right\}\delta q_k\right.$$

$$= \sum_{k=1}^{k=n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{q}_k}\left(\sum_{i=1}^{i=N}\frac{1}{2}m_i(v_i.v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{i=N}\frac{1}{2}m_i(v_i.v_i)\right\}\right]\delta q_k$$

$$= \sum_{k=1}^{k=n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k \qquad .....(27)$$

Using equations (22) and (27) in equation (21), we get-

$$\sum_{k=1}^{k=n}G_k\,\delta q_k - \sum_{k=1}^{k=n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k = 0$$

Or $$\sum_{k=1}^{k=n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}\right]\delta q_k-\sum_{k=1}^{k=n} G_k\,\delta q_k = 0$$

Or $$\sum_{k=1}^{k=n}\left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}\right\}-G_k\right]\delta q_k = 0$$

The constraints are holonic, i.e. any virtual displacement δq_k is independent of δq_j.

Therefore, $$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}\right\}-G_k = 0$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}= G_k \qquad .....(28)$$

The above equation represents the general form of Lagrange's equations.

For a conservative system-

$$\vec{F}_i = \nabla_i V = -\hat{i}\frac{\partial V}{\partial x_i}-\hat{j}\frac{\partial V}{\partial y_i}-\hat{k}\frac{\partial V}{\partial z_i} \qquad .....(29)$$

Comparing equations (23) and (29), we get-

$$G_k = -\frac{\partial V}{\partial q_k} \qquad .....(30)$$

From equation (28)-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}= -\frac{\partial V}{\partial q_k}$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k} = 0$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial (T-V)}{\partial q_k} = 0 \qquad .....(31)$$

Or we can write –

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_k}-\frac{\partial (T-V)}{\partial q_k} = 0 \qquad .....(32)$$

(since the scalar potential V is the function of generalized coordinates q_k only not depending on generalized velocities.)

We define a new function as-

$$L = T - V \qquad .....(33)$$

which is called the Lagrangian of the system i.e. the difference of kinetic energy and potential energy is the Lagrangian.

Now we can write the above equation as-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{.....(34)}$$

These equations are known as Lagrange's Equations for conservative system

1.11 DISSIPATION FUNCTION

In physics, the **Rayleigh dissipation function**, named for Lord Rayleigh, is a function used to handle the effects of velocity-proportional frictional forces in Lagrangian mechanics. It is defined for a system of N particles as-

$$F = \frac{1}{2}\sum_{i=1}^{i=N}(k_x v_{i,x}^2 + k_y v_{i,y}^2 + k_z v_{i,z}^2) \qquad \text{.....(35)}$$

The force of friction is negative the velocity gradient of the dissipation function, $\vec{F}_f = -\nabla_v \mathrm{F}$. The function is half the rate at which energy is being dissipated by the system through friction.

As friction is not conservative, it is included in the Q_j term of Lagrange's equations.

It can be shown that if a system involves frictional forces or in general dissipative forces, then in suitable circumstance, such a system can also be described in terms of extended Lagrangian formulation.

1.12 CYCLIC COORDINATES

Cyclic Coordinate is the coordinate , on which the physical parameter (like momentum) doesn't depend or moreover , one can conclude that this physical parameter will remain conserve when the motion is being taken in that coordinate . These coordinates are also known as ignorable coordinates.

This is quite useful to determine the conservative nature of motion . Let us take ,

L = m/2 (x'² + y'²) + a(2x)

So , here the motion is in xy plain but our potential term (2ax) is y coordinate independent , hence by the use of cyclic coordinate , one can conclude that the momentum in y direction will be conserve , however this may not be the only conserve quantity.

Example 1: Consider the motion of a particle of mass m. Using Cartesian coordinates as generalized coordinates, deduce Newton's equation of motion from Lagrange's equations.

Solution: We know the general form of the Lagrange's equations-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k$$

Here $q_1 = x$, $q_2 = y$, $q_3 = z$ and generalized force components $G_1 = F_x$, $G_2 = F_y$, $G_3 = F_z$

The kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

For x-coordinate, we can write the Lagrange's equation as-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x$$

But $\frac{\partial T}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial T}{\partial x} = 0$

Therefore, $\frac{d}{dt}(m\dot{x}) - 0 = F_x$ or $F_x = \frac{d}{dt}(m\dot{x}) = \frac{dp_x}{dt}$

Where p_x is the x-component of the momentum.

Similarly for y and z- components-

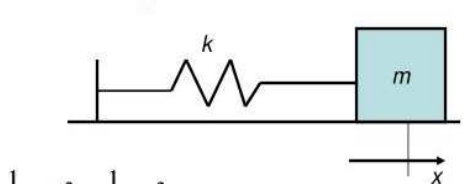
$$F_y = \frac{dy}{dt}, \qquad F_z = \frac{dp_z}{dt}$$

Thus, $$\vec{F} = \frac{d\vec{p}}{dt}$$

This is Newton's equation of motion.

Example 2: A mass M is attached a spring of force constant k. The other end of the spring is fixed in a wall. The entire system is placed on a frictional less surface. If the mass is slightly displaced by linear displacement x, deduce the equation of motion and hence find out the expression for time period of the system.

Solution:



We know the Lagrangian's equation of motion-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{.....(1)}$$

In this case, only one generalized coordinate i.e. x is required, thus $q_k = x$.

The above equation can be written as-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \text{.....(2)}$$

The kinetic energy of the system, $T = (1/2)\, M\, \dot{x}^2$

The potential energy of the system, $V = (1/2)\, k\, x^2$

Thus, we can write Lagrangian as-

$$L = T - V$$

$$= (1/2)\, M \dot{x}^2 - (1/2)\, k\, x^2 \qquad .....(3)$$

$\frac{\partial L}{\partial \dot{x}} = M \dot{x}$ and $\frac{\partial L}{\partial x} = -kx$

From equation (2), we have-

$$\frac{d}{dt}(M \dot{x}) - (-kx) = 0$$

Or $$M \ddot{x} + kx = 0 \qquad .....(4)$$

This is required equation.

From above equation, $\ddot{x} = -(k/M)\, x$

Standard equation of Simple Harmonic Motion, $\alpha = -\omega^2 x$

Comparing the above equation with standard equation of motion of linear mass spring system (Simple Harmonic Motion), we get-

$$\omega^2 = (k/M) \quad \text{or} \quad \omega = \sqrt{\frac{k}{M}}$$

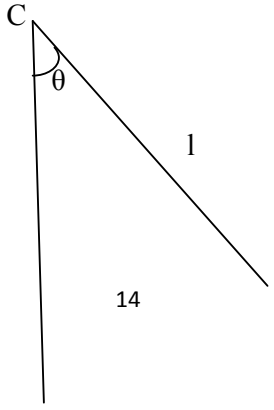
Or $$\frac{2\pi}{T} = \sqrt{\frac{k}{M}}$$

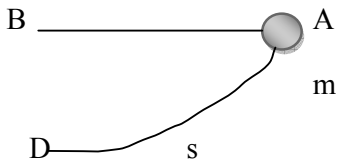
Or $$T = 2\pi\sqrt{\frac{M}{k}}$$

This is the expression for time period of linear mass spring system.

Example 3: Obtain the equation of motion of a simple pendulum by using Lagrange's equation of motion and hence deduce the formula for its time period for small amplitude oscillations.

Solution:





Let point of suspension C is taken as reference point. The angle θ between rest position and deflected position is chosen as generalized coordinate.

The kinetic energy of the system, $T = (1/2)\, mv^2 = (1/2)\, m\, s^2 = (1/2)\, m(l\dot{\theta})^2 = (1/2)\, ml^2\dot{\theta}^2$

Here m is the mass of the bob and l, the effective length of pendulum.

As the bob comes from point A to point D, it falls freely through a vertical distance BD. Thus, potential energy of the system, V = mg(BD) = mg (CD-CB) = mg (l- l cos θ)

$$= mgl\,(1- \cos\theta)$$

The Lagrangian can be written as-

$$L = T - V$$

Or $$L = (1/2)\, ml^2\dot{\theta}^2 - mgl\,(1- \cos\theta) \qquad .....(1)$$

Now, $\frac{\partial L}{\partial \dot{\theta}} = m\, l^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = - mgl \sin\theta$

We can write Lagrange's equation of motion –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Or $$\frac{d}{dt}\left(m\, l^2\dot{\theta}\right) - (- mgl \sin\theta) = 0$$

Or $$ml^2\,\ddot{\theta} + mgl \sin\theta = 0$$

Or $$\ddot{\theta} + (g/l) \sin\theta = 0$$

This is the equation of motion of simple pendulum.

Since oscillations are of small amplitude, therefore $\sin\theta \approx \theta$. Hence equation of motion of simple pendulum becomes-

$$\ddot{\theta} + (g/l)\ \theta = 0$$

Or $$\ddot{\theta} = -(g/l)\ \theta$$

Comparing the above equation with standard equation of motion of SHM , , $\alpha = -\omega^2 x$

We get-

$$\omega^2 = (g/l) \quad \text{or} \quad \omega = \sqrt{\frac{g}{l}}$$

Or $$\frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

Or $$T = 2\pi\sqrt{\frac{l}{g}}$$

This is the expression for time period of simple pendulum.

Example 4: Atwood's Machine: Obtain the equation of motion of a system of two masses, connected by an inextensible string passing over a small smooth pulley.

Solution: Atwood's machine is an example of a conservative system with holonomic constraint. The pulley is small, mass less and frictionless.

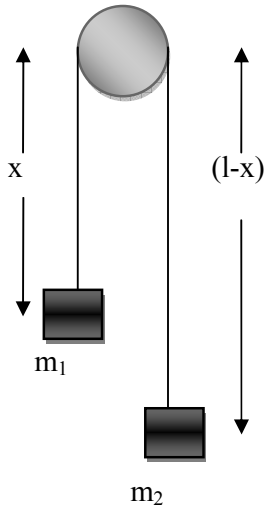
Let us consider that two masses m_1 and m_2 are connected by an inextensible string of length l. Let us suppose that x be the variable vertical distance from the pulley to the mass m_1, then the mass m_2 will be at a distance (l-x) from the pulley.

Obviously, here is only one independent coordinate i.e. generalized coordinate x.

Let us assume the pulley as reference.

The total kinetic energy of the system T = Kinetic energy of mass m_1 + kinetic energy of mass $m_2 = (1/2)\, m_1\dot{x}^2 + (½)\, m_2\dot{x}^2$

$$= (1/2)\,(m_1 + m_2)\,\dot{x}^2$$



Obviously, $m_2 > m_1$

Total potential energy of the system, V = Potential energy of mass m_1 + potential energy of mass m_2

$$= - m_1 gx - m_2 g (l-x)$$

Thus we can write the Lagrangian as –

$$L = T- V$$

$$= (1/2) (m_1 + m_2) \dot{x}^2 - \{ - m_1 gx - m_2 g (l-x) \}$$

$$= (1/2) (m_1 + m_2) \dot{x}^2 + m_1 gx + m_2 g (l-x)$$

Now, $\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2) \dot{x}$ and $\frac{\partial L}{\partial x} = (m_1 - m_2) g$

Lagrange's equation of motion, $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

Putting for $\frac{\partial L}{\partial \dot{x}}$ and $\frac{\partial L}{\partial x}$ in the above equation-

$$\frac{d}{dt} \{ (m_1 + m_2) \dot{x} \} - (m_1 - m_2) g = 0$$

$$(m_1 + m_2) \frac{d}{dt} (\dot{x}) - (m_1 - m_2) g = 0$$

$$(m_1 + m_2) \ddot{x} = (m_1 - m_2) g$$

Or $$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)} g$$

Or $$\ddot{x} = - \frac{(m_2 - m_1)}{(m_1 + m_2)} g$$

This is the required equation of motion.

Self Assessment Question (SAQ) 1: Find Lagrange's equation of motion for an electrical circuit comprising an inductance L and capacitance C. The capacitor is charged to a q coulombs and current flowing in the circuit is i amperes.

Self Assessment Question (SAQ) 2: Obtain the equation of motion of a compound pendulum using Lagrange's equation of motion. Also find out the expression for its time period.

Self Assessment Question (SAQ) 3: Choose the correct option-

(i) A rigid body moving freely in space has degrees of freedom-

(a) 4 (b) 9 (c) 6 (d) 3

(ii) Constraint in a rigid body is-

(a) holonomic (b) rheonomic (c) scleronomic (d) nonholonomic

(iii) Generalized coordinated-

(a) depend on each other (b) may be Cartesian coordinates (c) are necessarily spherical coordinates (d) are independent of each other

1.13 SUMMARY

In this unit, you have studied about System of particles, degree of freedom, constraints and generalized coordinates. You have learnt that the minimum number of independent variables or coordinates required to specify (or define) the position of a dynamical system, consisting of one or more particles, is called the number of degrees of freedom of the system. You have studied that the generalized coordinates are the smallest possible number of variables to describe the configuration of a system. You have known about generalized displacement, generalized velocity, generalized acceleration, generalized momentum and generalized force. In this unit, you have also studied about principle of virtual work and established D'Alembert's principle. We have also established the Lagranges's equation of motion. An glimpse of Rayleigh dissipation function and cyclic coordinates has been given. To present the clear understanding and to make the concepts of the unit clear, some solved examples are given in the unit. To check your progress, self assessment questions (SAQs) are given in the unit

1.14 GLOSSARY

Degree of freedom- The minimum number of independent variables or coordinates required to specify the position of a dynamical system

Constraints- restrictions

Generalized coordinates- the smallest possible number of variables to describe the configuration of a system

Dissipation- rakishness, indulgence

1.15 TERMINAL QUESTIONS

1. What do you understand by a system of particles. Explain.

2. What are constraints? Explain the various types of constraints. Give their significance.

3. Explain the following-

(i) Degrees of freedom (ii) Generalized coordinates

4. Determine the number of degrees of freedom for a massless rod, moving freely in space with a particle which is constrained to move on the rod.

5. Explain the principle of virtual work. Hence deduce D'Alembert's principle.

6. Establish the Lagrange's equation of motion. Give its importance.

7. Write notes on-

(i) Dissipation Function (ii) Cyclic Coordinates

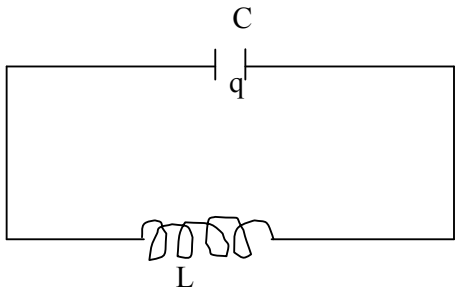
8. Derive Lagrange's equation oif motion from D'Alembert's principle.

9. A bead is sliding on a uniform rotating rod in a force-free space, find its equation of motion.

1.16 ANSWERS

Self Assessment Questions (SAQs):

1.



Here charge q is playing a role of displacement. It is generalized coordinate.

The kinetic energy of the system, $T = (1/2) L i^2$

The potential energy, $V = q^2/(2C)$

The Lagrangian $L = T - V = (1/2) L i^2 - q^2/(2C) = (1/2) L \dot{q}^2 - q^2/(2C)$

$\frac{\partial L}{\partial \dot{q}} = L \dot{q}$ and $\frac{\partial L}{\partial q} = -(q/C)$

Lagrange's equation of motion-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Or $\frac{d}{dt}(L\dot{q}) - \{ -(q/C) \} = 0$

Or $L\ddot{q} + q/C = 0$ or $\ddot{q} = -(1/LC)\, q$

This is the required equation of motion.

2. $\ddot{\theta} = -(mgl/I)\,\theta$

3. (i) (c), (ii) (a) , (c) (iii) (b), (d)

Terminal Questions:

4. 4

9. $\ddot{r} = r\omega^2$

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1.18 SUGGESTED READINGS

1. Classical Mechanics, H. Goldstein, Pearson Education

2. Introduction to Classical Mechanics, David Morin, Cambridge University Press

3. A Course on Classical Mechanics, Madhumangal Pal, Narosa

4. Classical Mechanics and General Properties of Matter, P.D. Raychaudhuri, New Age International

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UNIT 2 HAMILTON'S AND VARIATIONAL PRINCIPLE

Structure

2.1 Introduction

2.2 Objectives

2.3 Hamilton's Principle

2.3.1 Modified Hamilton's Principle

2.3.2 Advantage of Hamilton's Variational principle

2.4 Lagrange's Equations of Motion from Hamilton's Principle

2.5 Calculus of Variations

2.6 Euler- Lagrange's equations

2.7 Conservation Laws

2.7.1 Conservation of Linear Momentum

2.7.2 Conservation of Angular Momentum

2.7.3 Conservation of Energy

2.8 Noether's Theorem

2.9 Symmetries

2.10 Summary

2.11 Glossary

2.12 Terminal Questions

2.13 Answers

2.14 References

2.15 Suggested Readings

2.1 INTRODUCTION

In the previous unit, we have studied about system of particles, degrees of freedom, constraints and their type, generalized coordinates and D'Alembert's principle and used the D'Alembert's principle to deduce Lagranges equations of motion. In that unit, we have learnt about Lagrange's equations of motion and applied these equations to solve the problems. In this unit, we shall study Hamilton's principle and calculus of variation. Lagrange's equations of motion can be derived by an entirely different way, namely Hamilton's variational principle. We shall learn to deduce Lagrange's equations of motion from Hamilton's principle. We shall also discuss various conservation laws in the present unit.

2.2 OBJECTIVES

After studying this unit, you should be able to-

- understand Hamilton's variational principle
- apply Hamilton's principle in various cases
- understand various conservation laws

2.3 HAMILTON'S PRINCIPLE

Let us know about Hamilton's principle. Hamilton's principle is one of the variational principles in classical mechanics. All the laws of mechanics can be derived by using the Hamilton's principle. Hence it is one of the most fundamental and important principles of mechanics and mathematical physics.

We shall first explain the meaning of the motion of the system as it is involved in Hamilton's variational principle. The determination of the motion of the single particle in three dimensional space is a mechanical problem. The mechanical problem involving two particles, every particle being described by a set of three co-ordinates, can be reduced to a single particle problem simply by regarding that the single particle moves in a six dimensional space. Thus in general, a problem N- particles can be treated as one of a single particle moving along a trajectory in 3N- dimensional space. This space is referred to as configuration space and the single particle as system point. The motion of system point in configuration space is called the motion of the system between any two given instants. Configuration space has no necessary connection with the real three dimensional space.

Hamilton's principle states that for a conservative holonomic system, its motion from time t_1 to time t_2 is such that the line integral (known as action or action integral)

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

with L = T - V has stationary (extreme) value for the correct path of the motion. The quantity I is called as Hamilton's principal function. We can express the principle as-

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0$$

or simply $\delta \int_{t_1}^{t_2} L dt = 0$ …..(1)

or where δ is the variation symbol.

2.3.1 Modified Hamilton's Principle

According to Hamilton's principle-

$$\delta \int_{t_1}^{t_2} L dt = 0$$

where L= T-V = L (q_k, $\dot{q}_k$, t)

We can write the above equation in terms of Hamiltonian H which is written as-

$$H(p_k, q_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Or $$L(q_k, \dot{q}_k, t) = \sum_k p_k \dot{q}_k - H(p_k, q_k, t)$$

Hence the Hamilton's principle in the new form can be written as-

$$\delta \int_{t_1}^{t_2} \{ \sum_k p_k \dot{q}_k - H(p_k, q_k, t) \} dt = 0$$

or simply

$$\delta \int_{t_1}^{t_2} \{ \sum_k p_k \dot{q}_k - H \} dt = 0 \quad \text{.....(2)}$$

This is known as modified Hamilton's principle.

2.3.2 Advantage of Hamilton's Variational principle

Hamilton's principle of mechanics has special advantages as the beginning point for approximations. First, it is extremely succinct. Secondly, it easily accommodates moving disconnecting fluid boundaries. Thirdly, approximations - however strong - that maintain the symmetries of the Hamiltonian will automatically preserve the corresponding conservation laws. For example, Hamilton's principle allows useful analytical and numerical approximations to the equations governing the motion of a homogeneous rotating fluid with free boundaries. Hamilton's principle of mechanics governs the motions of classical fluids. As a statement of dynamical law, it has important practical advantages over the more conventional Eulerian formulation of fluid mechanics.

2.4 LAGRANGE'S EQUATIONS OF MOTION FROM HAMILTON'S PRINCIPLE

Here, we shall derive Lagrange's equations of motion from Hamilton's principle. The Lagrangian L is a function of generalized coordinates q_k's and generalized velocities $\dot{q}_k$'s and time t, i.e.

$$L = L(q_1, q_2, q_3, \ldots\ldots\ldots\ldots q_k, \ldots\ldots\ldots q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots\ldots\ldots \dot{q}_k, \ldots\ldots\ldots \dot{q}_n, t)$$

If the Lagrangian does not depend on time t explicitly, then the variation δL can be written as-

$$\delta L = \sum_{k=1}^{k=n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{k=n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \quad \text{.....(3)}$$

We integrate both sides from $t=t_1$ to $t = t_2$ –

$$\int_{t_1}^{t_2} \partial L \, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt \quad \text{.....(4)}$$

But by Hamilton's principle-

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$

Therefore, we have-

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \quad dt = 0$$

…..(5)

Where $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

Integrating by parts, the second term on the left hand side of equation (5), we get-

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \quad dt = \sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_2}^{t_2} - \int_{t_2}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \quad dt$$

…..(6)

At the end points of the path at the times t_1 and t_2, the coordinates must have definite values $q_k(t_1)$ and $q_k(t_2)$ respectively, i.e. $\delta q_k(t_1) = \delta q_k(t_2) = 0$ and hence we have-

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_2}^{t_2} = 0$$

Therefore, equation (5) takes the form-

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt - \int_{t_2}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt = 0$$

$$\sum_k \int_{t_2}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right] \delta q_k \quad dt = 0$$

…..(7)

For holonomic system, the generalized coordinates δq_k are independent of each other. Therefore, the coefficient of each δq_k must vanish, i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

…..(8)

Where k = 1, 2, 3, ……, n are the generalized coordinates.

Equation (8) represent the Lagrange's equations of motion.

2.5 CALCULUS OF VARIATIONS

In this section, we shall discuss the calculus of variations. Calculus of variations is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of functionals: mappings from a set of functions to the real numbers. Functionals are often expressed as definite integrals involving functions and their derivatives. Functions that maximize or minimize functionals may be found using the Euler–Lagrange equation of the calculus of variations.

A simple example of such a problem is to find the curve of shortest length connecting two points. If there are no constraints, the solution is obviously a straight line between the points. However, if the curve is constrained to lie on a surface in space, then the solution is less obvious, and possibly many solutions may exist. Such solutions are known as geodesics. A related problem is posed by Fermat's principle: light follows the path of shortest optical

length connecting two points, where the optical length depends upon the material of the medium. One corresponding concept in mechanics is the principle of least action.

Many important problems involve functions of several variables. Solutions of boundary value problems for the Laplace equation satisfy the Dirichlet principle. Plateau's problem requires finding a surface of minimal area that spans a given contour in space: a solution can often be found by dipping a frame in a solution of soap suds. Although such experiments are relatively easy to perform, their mathematical interpretation is far from simple: there may be more than one locally minimizing surface, and they may have non-trivial topology. The calculus of variations may be said to begin with Newton's minimal resistance problem in 1687, followed by the brachistochrone curve problem raised by Johann Bernoulli (1696). It immediately occupied the attention of Jakob Bernoulli and the Marquis de l'Hôpital, but Leonhard Euler first elaborated the subject, beginning in 1733. Lagrange was influenced by Euler's work to contribute significantly to the theory. After Euler saw the 1755 work of the 19-year-old Lagrange, Euler dropped his own partly geometric approach in favor of Lagrange's purely analytic approach and renamed the subject the calculus of variations in his 1756 lecture Elementa Calculi Variationum.

Legendre (1786) laid down a method, not entirely satisfactory, for the discrimination of maxima and minima. Isaac Newton and Gottfried Leibniz also gave some early attention to the subject. To this discrimination Vincenzo Brunacci (1810), Carl Friedrich Gauss (1829), Siméon Poisson (1831), Mikhail Ostrogradsky (1834), and Carl Jacobi (1837) have been among the contributors. An important general work is that of Sarrus (1842) which was condensed and improved by Cauchy (1844). Other valuable treatises and memoirs have been written by Strauch (1849), Jellett (1850), Otto Hesse (1857), Alfred Clebsch (1858), and Carll (1885), but perhaps the most important work of the century is that of Weierstrass. His celebrated course on the theory is epoch-making, and it may be asserted that he was the first to place it on a firm and unquestionable foundation. The 20th and the 23rd Hilbert problem published in 1900 encouraged further development. In the 20th century David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jacques Hadamard among others made significant contributions. Marston Morse applied calculus of variations in what is now called Morse theory. Lev Pontryagin, Ralph Rockafellar and F. H. Clarke developed new mathematical tools for the calculus of variations in optimal control theory. The dynamic programming of Richard Bellman is an alternative to the calculus of variations.

The calculus of variations is concerned with the maxima or minima (collectively called extrema) of functionals. A functional maps functions to scalars, so functionals have been described as "functions of functions." Functionals have extrema with respect to the elements y of a given function space defined over a given domain. A functional $J[y]$ is said to have an extremum at the function f if $\Delta J = J[y] - J[f]$ has the same sign for all y in an arbitrarily small neighborhood of f. The function f is called an extremal function or extremal. The extremum $J[f]$ is called a local maximum if $\Delta J \leq 0$ everywhere in an arbitrarily small neighborhood of f, and a local minimum if $\Delta J \geq 0$ there. For a function space of continuous functions, extrema of corresponding functionals are called weak extrema or strong extrema, depending on whether the first derivatives of the continuous functions are respectively all continuous or not.

Both strong and weak extrema of functionals are for a space of continuous functions but weak extrema have the additional requirement that the first derivatives of the functions in the space be continuous. Thus a strong extremum is also a weak extremum, but the converse may

not hold. Finding strong extrema is more difficult than finding weak extrema. An example of a necessary condition that is used for finding weak extrema is the Euler–Lagrange equation.

The solution of a dynamical problem means that we want to locate the position of the system e.g. a particle, at a particular instant of time. We are also interested in the path adopted by the system. The piecewise information of this path i.e. where it is maximum or minimum is protected through differential calculus by putting $\dot{y}(x) = 0$ etc. But if we want the

information about the whole path then we shall have to look for integral calculus and will be interested in the arguments like whether the path as a whole is largest or shortest (extremum or having a stationary value). This type of study requires the technique of calculus of variations (Figure 1).

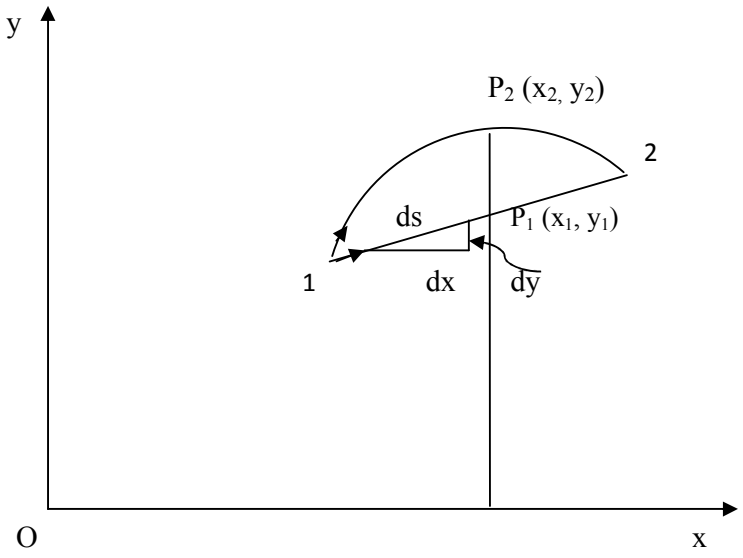


Figure (1)

Two paths which a particle may follow in going from position 1 to position 2 are shown. The straight line path is shortest and can be represented as-

$$y = mx + c$$

or $y = y(x)$ …..(9)

in functional form indicating that y is a function of independent parameter x. For each value of x, there will be fixed value of y.

As we are interested in the length of the path, denoted by say, I , we can write-

$$I = \int ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{dx^2[1 + (\frac{dy}{dx})^2} \quad = (\int \sqrt{1 + \dot{y}^2}) \quad dx$$

$$= \int f(\dot{y}) \quad dx \quad = \quad \int f(y, \dot{y} \quad , \quad x \quad) \quad dx$$

…..(10)

because $\dot{y}$ involves both y and x, we have expressed it in f. Thus, we note that path I is the integral of the function f which itself is the function of y. If we want that the path I be extremum, then δI should be equal to zero, i.e. δI = 0

or $$\delta \int f(y, \dot{y}, x) \; dx = 0$$

or $$= \delta \int f [\, y(x), \; \dot{y}(x), x \,] \, dx = 0$$

This is the formulation of the problem of calculus of variations. It is obvious that variation δ is defined as the variation in the quantity to which it is applied at the fixed value of independent parameter i.e.

$$\delta \quad y \quad = \quad (\quad y_2 \quad - \quad y_1 \quad)_x$$

…..(11)

If we report for both curves then the equation y = y (x) is incapable to represent both e.g. y = mx + c will represent straight line path only. Therefore, to include other paths, we require another parameter, say, α, designating path, to be introduced in y i.e.

$$y = y(x, \alpha)$$

so that $$y = y(x, \alpha_1) \quad \text{and} \quad y = y(x, \alpha_2)$$

may represent the two paths. However, the relationship can be expressed as-

$$y \quad (x, \quad \alpha) \quad = \quad y \quad (x, \quad 0) \quad + \quad \alpha \quad \eta(x)$$

…..(12)

where η(x) is any arbitrary function of x which vanishes at end points. Note α = 0 in y (x, 0) may be taken to represent extremum path. Thus α represents the paths. It means I, the length of path, which is different for different paths, will also be a function of α, i.e. I = I(α) so that we may write the integral-

$$I(\alpha) \quad = \quad \int f[y(x, \alpha), \dot{y} \quad (x, \quad \alpha), \quad x \quad] \quad dx$$

…..(13)

In this way, it may be noted that δ is taken at a fixed value of independent parameter, i.e. x is same for all paths considered i.e. x is not the function of α or we can write

$$\frac{\partial x}{\partial \alpha} = 0$$

Further, at end points, all paths meet and therefore, there is no variation even in y coordinate at end points

Or $\left|\frac{\partial y}{\partial \alpha}\right|_{\text{end points 1 \& 2}} = 0$ …..(14)

The shortest path is taken as extremum path and the other paths with which we compare it are called comparison paths.

δ- variation represents the increase in the quantity to which it is applied as we pass from extremum path to the comparison path at the same value of independent variable. Thus δy represents arbitrary variation of y(x) with respect to an arbitrary parameter, α about its extremum value. In other words, δy corresponds to a virtual displacement. Such a virtual displacement does not always coincide with possible actual displacement occurring in the course of motion i.e. varied path may or may not be an actual path.

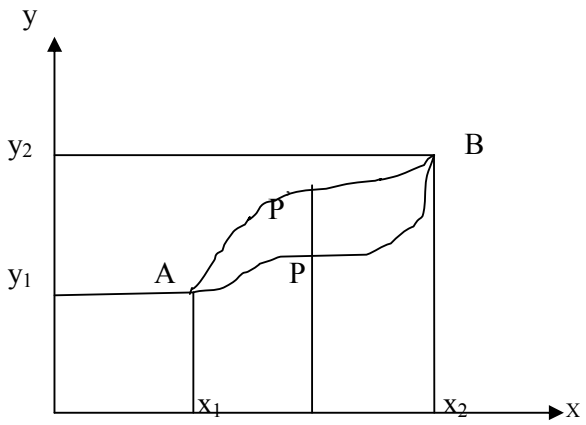


Figure (2)

The two important points in this variation, significant to note are-

(i) end points' coordinates are same for each curve which gives $\left|\frac{\partial y}{\partial \alpha}\right|_{\text{end points}} = 0$. Further, variation is considered at a fixed value of independent coordinate x; $\frac{\partial x}{\partial \alpha} = 0$ for such a variation.

(ii) The system must travel from one end point A to other end point B in the same time for all the paths conceived. It also predicts that end points' time is also fixed for every path. This gives that $\frac{\partial t}{\partial \alpha} = 0$ along the path and also at the end points. For satisfying these conditions, we consider that system particle moves with different velocities along different paths so as to keep end points' time fixed i.e. there is no variation of time along any path i.e. δt = 0.

2.6 EULER- LAGRANGE'S EQUATIONS

In this section, we shall establish Euler-Lagrange's equations. Let us consider a function f (y, $\dot{y}$, x) defined on a curve given by y = y(x) between two points A (x_1, y_1) and B (x_2, y_2). Here, $\dot{y}$ = dy/dx. We shall find a particular curve y(x) for which the line integral I of the function f between the two points I = $\int_{x_1}^{x_2} f\,(y, \dot{y}, x)\,dx$ has a stationary value.

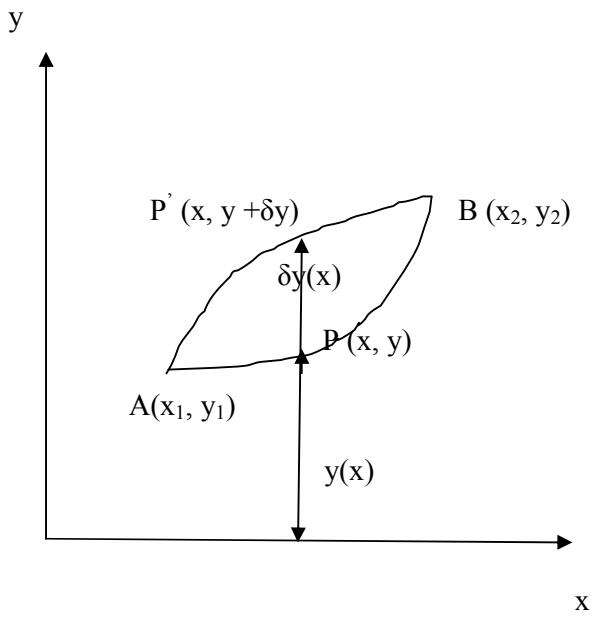


Figure (3)

Let us suppose that APB be the curve for which I is stationary. Now, let us consider a neighbouring curve AP'B with the same end points A and B. The point P (x, y) of the curve APB corresponds to the point P'(x, y + δy) of the curve AP'B, keeping x-coordinate of the points fixed. This defines a δ-variation of the curve. The variation is arbitrary but small and may be represented as-

$$\delta y = \frac{\partial y}{\partial \alpha}\delta\alpha = \eta\,(x)\,\delta\alpha \qquad \text{.....(15)}$$

where α is a parameter common to all points of the path which is independent of x and η (x) is a function of x with the condition that-

$$\delta y_1 = \delta y_2 = \eta\,(x_1) = \eta\,(x_2) = 0 \qquad \text{.....(16)}$$

By choosing different η(x), we may construct different varied paths. The corresponding variation in $\dot{y}$ is

$$\delta\dot{y} = \dot{\eta}\,(x)\,\delta\alpha \qquad \text{.....(17)}$$

Now, the integral on the varied path is-

$$I' = \int_{x_1}^{x_2} f(y + \delta y,\ \dot{y} + \delta\dot{y}, x)\,dx$$

Or

$$I' = \int_{x_1}^{x_2} f(y + \eta\delta\alpha,\ \dot{y} + \dot{\eta}\delta\alpha,\ x)\,dx \qquad \text{.....(18)}$$

Since the variation is small, the integral I' may be obtained by considering only first order terms in the Taylor expansion of the function f i.e.,

$$I' = \int_{x_1}^{x_2} \left[f(y,\dot{y},x) + \frac{\partial f}{\partial y}\eta\delta\alpha + \frac{\partial f}{\partial \dot{y}}\dot{\eta}\,\delta\alpha\right] dx \qquad \text{.....(19)}$$

Hence

$$\delta I = I' - I = \delta\alpha \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial \dot{y}}\,\dot{\eta}\right) dx \qquad \text{.....(20)}$$

But

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\,\dot{\eta}\,dx = \frac{\partial f}{\partial \dot{y}}\eta \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\eta\,dx$$

or

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\,\dot{\eta}\,dx = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\eta\,dx \qquad [\text{as } \eta(x_1) = \eta(x_2) = 0]$$

Therefore,

$$\delta I = \delta\alpha \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\eta\,dx \qquad \text{.....(21)}$$

The condition that the integral I is stationary means that δI = 0, i.e.

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\eta\,dx = 0$$

.....(22)

As η is arbitrary, the integrand of equation (22) must be zero, i.e.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

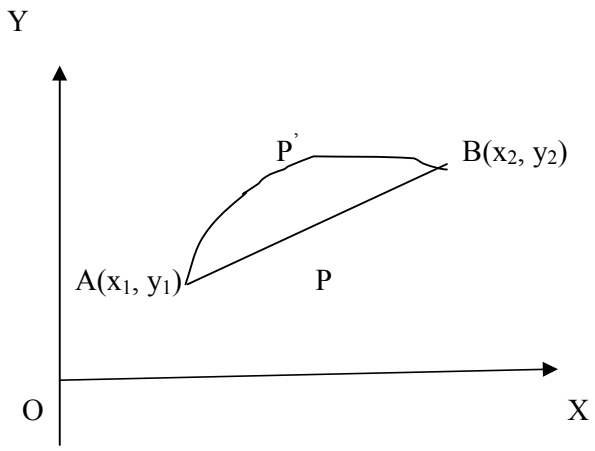
.....(23)

The above equation (23) is known as Euler-Lagrange equation.

This is a relation that should be satisfied by a function f if the integral I is to be extremum. Obviously, Euler- Lagrange equations bear a great similarity with Lagrange's equation of motion.

Example 1: Show that the shortest distance between two points in a plane is a straight line.

Solution: Let us consider two points A (x_1 , y_1) and B (x_2, y_2) in XY- plane. An element of length ds of any curve, say AP'B, passing through A and B points is given by-



$$ds^2 = dx^2 + dy^2$$

or $$ds = \sqrt{1 + \dot{y}^2}\, dx = 0 \qquad \text{.....(i)}$$

Total length of the curve from point A to the point B is given by-

$$s = \int_A^B \sqrt{1 + \dot{y}^2}\, dx = \int_A^B f\, dx \qquad \text{.....(ii)}$$

where $f = \sqrt{1 + \dot{y}^2}$.

The length of the curve s will be minimum if $\delta s = 0$. It means that f should satisfy the Euler-Lagrange's equation i.e.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\delta f}{\delta \dot{y}}\right) = 0$$

.....(iii)

Here, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$

Therefore, from equation (iii), we have-

$$0 - \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0$$

Or $$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0$$

Or $\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right)$ = constant = C (say)

Or $\dot{y}^2 = C\,(1 + \dot{y}^2)$

Or $\dot{y} = \frac{C}{\sqrt{1-C^2}}$ = a constant (K)

Or $\dot{y} = \frac{C}{\sqrt{1-C^2}} = K$

Or $\frac{dy}{dx} = K$

Integrating both sides, we get-

$$y = Kx + J \qquad \text{.....(iv)}$$

where J is again a new constant.

Obviously, the above equation represents a straight line. Therefore, the shortest distance between any two points in a plane is a straight line. The constants K and J can be determined by the condition that the straight line (iv) passes through points A (x_1, y_1) and B (x_2, y_2).

Example 2: Apply variational principle to find the equation of 1-D harmonic oscillator.

Solution: For 1-D harmonic oscillator-

The potential energy $V = (1/2)\,kx^2$ and kinetic energy $T = (1/2)\,m\,\dot{x}^2$

The Lagrangian L for 1-D harmonic oscillator can be written as-

$$L = T\text{-}V = (1/2)\,m\,\dot{x}^2 - (1/2)\,kx^2$$

According to Hamilton's principle or variational principle $\int L\,dt$ or $\int f\,(x, x, t)\,dt$ is extremum.

Euler-Lagrange's equation is-

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

Here, $\frac{\partial f}{\partial x} = -kx$, $\frac{\partial f}{\partial \dot{x}} = m\dot{x}$

Therefore, from the above equation, we have-

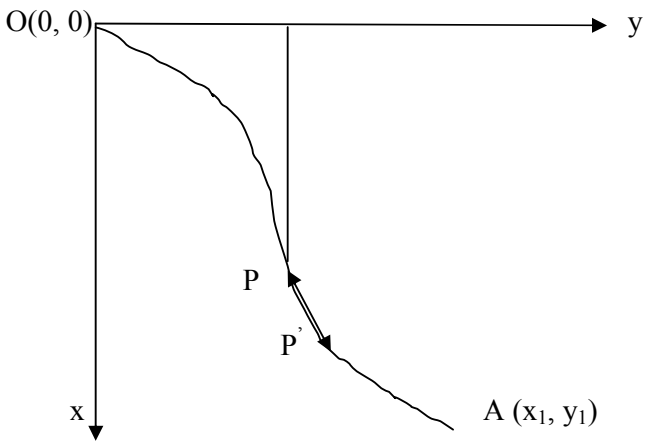
$$m\ddot{x} + kx = 0$$

or $$\ddot{x} = -(k/m)\,x$$

which is the equation of motion for 1-D harmonic oscillator.

Example 3: A particle slides from rest at one point on a frictionless wire in a vertical plane to another point under the influence of the earth's gravitational field. If the particle travels in the shortest time, show that the path followed by it is a cycloid [**Brachistochrone Problem**].

Solution: Let the shape of wire be in the form of a curve OA. The particle starts to travel from O (0, 0) from rest and moves to A (x_1, y_1) under the influence of gravity on the frictionless wire.



Let v be the speed at P.

The time taken in moving PP' = ds element = ds/v

Therefore, the total time taken by the particle in moving from the higher point O to the lower point A is given by-

$$t = \int_O^A ds/v \qquad \text{.....(i)}$$

If the vertical distance of fall from point O to point P be x, then from the particle of conservation of energy, we have-

$$(1/2)\, mv^2 = mgx$$

Or $$v = \sqrt{2gx}$$

Therefore, $t = \int_O^A \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}$ [$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \dot{y}^2}$]

.....(ii)

So that $f = \sqrt{(1 + \dot{y}^2)/2gx}$ and for t to be minimum

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$

…..(iii)

Here, $\frac{\partial f}{\partial y} = 0$ and $\left(\frac{\partial f}{\partial \dot{y}}\right) = \frac{\dot{y}}{\sqrt{2gx\ (1+\dot{y}^2)}}$

Substituting in equation (iii), we get-

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{2gx\ (1+\dot{y}^2)}}\right) = 0$$

Or $\frac{\dot{y}}{\sqrt{x\ (1+\dot{y}^2)}}) =$ constant (C)

Squaring both sides, we get-

$$\frac{\dot{y}^2}{x(1+\dot{y}^2)} = C^2$$

Or $\frac{\dot{y}^2}{C^2} = x(1+\dot{y}^2)$

Or $\dot{y}^2 = C^2 x\ (1+\dot{y}^2)$

Or $\dot{y}^2 = C^2 x + C^2 x\ \dot{y}^2$

Or $\dot{y}^2(1 - C^2x) = C^2x$

Or $\dot{y}^2 = \frac{C^2x}{1-C^2x} = \frac{C^2x}{(\frac{1}{C^2} - x)C^2} = \frac{x}{\frac{1}{C^2} - x}$

Or $\dot{y}^2 = \frac{x}{B-x}$, where $\frac{1}{C^2} = B$, a constant

Or $\frac{dy}{dx} = \sqrt{\frac{x}{B-x}}$

Or $y = \int \sqrt{\frac{x}{B-x}}\, dx + C'$, another constant of integration …..(iv)

Let us consider that-

$x = B \sin^2\theta$, then $dx = 2B \sin\theta \cos\theta\, d\theta$

Therefore, $y = \int \frac{\sin\theta}{\cos\theta} 2B \sin\theta \cos\theta\, d\theta + C'$

$= B \int 2\sin^2\theta d\theta + C'$

$= B \int (1 - \cos 2\theta) d\theta + C'$

$$= B\left[\theta - \frac{\sin 2\theta}{2}\right] + C' = (B/2)\left[2\theta - \sin 2\theta\right] + C'$$

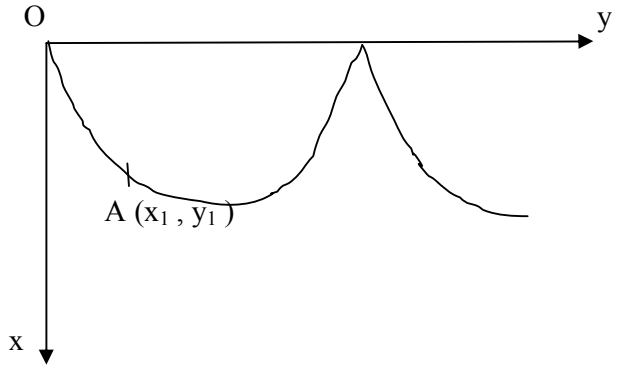
Thus the parametric equations of the curve are-

$X = B \sin^2\theta = (B/2)(1 - \cos 2\theta)$ and $y = (B/2)(2\theta - \sin 2\theta) + C'$

Since the curve passes through point (0, 0), $C = 0$

Therefore-

$x = (B/2)(1 - \cos 2\theta)$ and $y = (B/2)(2\theta - \sin 2\theta)$ …..(v)



Let $2\theta = \phi$ and $B/2 = A$, then the parametric equations of the curve are-

$$x = A(1 - \cos\phi) \quad \text{and} \quad y = A(\phi - \sin\phi) \qquad \text{…..(vi)}$$

The above equation represents a cycloid (the above figure). The constant A can be determined because the curve passes through the point A (x_1, y_1).

Self Assessment Question (SAQ) 1: If we take a curve passing through the fixed points (x_1, y_1) and (x_2, y_2) and revolve it about Y-axis to form a surface of revolution, find the equation of the curve for which the surface area is minimum [Minimum surface of revolution].

Self Assessment Question (SAQ) 2: Choose the correct option-

(i) In δ-variation-

(a) the time is not involved (b) time as well as position coordinates are allowed to vary

(c) the time is involved (d) only time is allowed to vary

(ii) In case of modified Hamilton's principle-

(a) the path refers to phase space (b) the path refers to configuration space

(c) $\delta \int_{t_1}^{t_2}\{\sum_k p_k \dot{q}_k - H\}dt = 0$ (d) $\int_{t_1}^{t_2}\{\sum_k p_k \dot{q}_k + H\}dt = 0$

2.7 CONSERVATION LAWS

The word conservation applies in the sense of constantness when some characteristics of the motion of a system remains constant in time. In this physical world, there exist a number of conservation laws. There are conservation laws relating to linear momentum, angular momentum, energy and various other quantities. We apply Newtonian mechanics to deduce conservation laws for a particle in motion. Conservation laws prove to be very powerful tools in solving mechanical problems in the following respects-

(i) Conservation laws are independent of the details of the trajectory and often, of the details of a particular force. The laws are, therefore, a way of stating very general and significant consequence of equations of motion. Conservation laws assure us many times that some aspects of motion are impossible and must be left out.

(ii) Conservation laws have an intimate relation with invariance. Their failure in certain cases may result in the discovery of new and not yet understood phenomenon.

(iii) Conservation laws tell us a graet deal about the motion even if forces affecting motion are not known in advance.

(iv) Even when the force is known exactly, a conservation law may provide a simple and suitable method of solving the motion of the particle and particle systems.

In this section, we shall discuss some conservation laws.

2.7.1 Conservation of Linear Momentum

Let us consider that a force $\vec{F}$ is acting on a particle of mass m, then according to Newton's second law of motion, we have-

$$\frac{\overrightarrow{dp}}{dt} = \vec{F}$$

$$\vec{F} = \frac{\overrightarrow{dp}}{dt} = \frac{d(m\vec{v})}{dt} \qquad \text{.....(24)}$$

Where $\vec{p} = m\vec{v}$, the linear momentum of the particle.

If the external force acting on the particle is zero, then-

$$\frac{\overrightarrow{dp}}{dt} = \frac{d(m\vec{v})}{dt} = 0$$

Or $$\vec{p} = m\vec{v} = \text{constant} \qquad \text{.....(25)}$$

Thus, in the absence of external force, the linear momentum of a particle is constant (conserved) i.e. if there is no external force applied on a particle, then its linear momentum is conserved. This is known as conservation law of linear momentum.

2.7.2 Conservation of Angular Momentum

We know that the angular momentum of a particle about a fixed point is defined as the moment of its linear momentum i.e. the angular momentum of a particle P of mass m about a point O is defined as-

$$\vec{j} = \vec{r} \times \vec{p} \qquad \text{.....(26)}$$

Where $\vec{r}$ is the position vector of the particle P and $\vec{p} = m\vec{v}$ is its linear momentum.

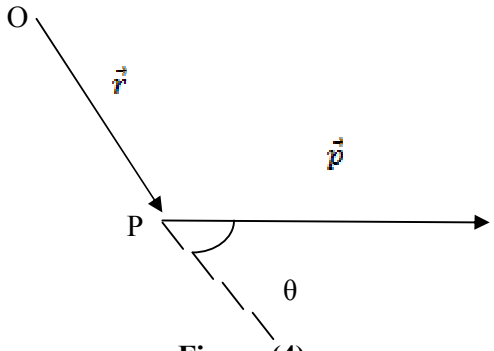


Figure (4)

If force $\vec{F}$ is acting on the particle, then the moment of force i.e. torque about point O is given as-

$$\vec{\tau} = \vec{r} \times \vec{F}$$

.....(27)

Differentiating equation (26) with respect to time t, we get-

$$\frac{d\vec{j}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

Or $$\frac{d\vec{j}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + 0 \qquad [\text{since } \frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = m(\vec{v} \times \vec{v}) = m(0) = 0]$$

$$= \vec{r} \times \vec{F}$$

$$= \vec{\tau} \qquad \text{[from equation (27)]}$$

i.e. $\vec{\tau} = \frac{d\vec{j}}{dt}$ .....(28)

Thus, the time rate of change of angular momentum of a particle is equal to the torque acting on it.

If the torque acting on the particle is zero i.e. $\vec{\tau} = 0$, then from equation (28), we have-

$$\frac{d\vec{j}}{dt} = 0$$

Or $\vec{j} = \text{constant}$ .....(29)

Thus, if there is no torque acting on the particle then the angular momentum of that particle is constant (conserved). This is known as law of conservation of angular momentum.

2.7.3 Conservation of Energy

We know that the work done by an external force $\vec{F}$ acting on the particle in displacing from point 1 to point 2 is given by-

$$W_{12} = \int_1^2 \vec{F}.\overrightarrow{dr}$$ .....(30)

According to Newton's second law-

$$\vec{F} = m\frac{d\vec{v}}{dt}$$ .....(31)

Putting for $\vec{F}$ in equation (30), we get-

$$W_{12} = \int_1^2 m\frac{d\vec{v}}{dt}.\overrightarrow{dr}$$

$$= m\int_1^2 \frac{d\vec{v}}{dt}.\overrightarrow{dr} = m\int_1^2 \frac{d\vec{v}}{dt}.\left(\frac{d\vec{r}}{dt}dt\right) = m\int_1^2 \frac{d\vec{v}}{dt}.(\vec{v}dt)$$

$$= m\int_1^2 \frac{d}{dt}\left[\frac{1}{2}\vec{v}.\vec{v}\right]dt = m\int_1^2 d\left[\frac{1}{2}mv^2\right]$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$ .....(32)

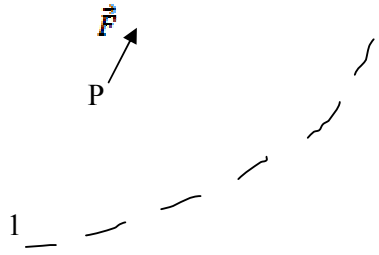


Figure (5)

The scalar quantity $\frac{1}{2}mv^2$ is defined as the kinetic energy and denoted by T. Thus, the work done by the force acting on the particle appears equal to the change in the kinetic energy i.e.

$$W_{12} = \int_1^2 \vec{F}.\overrightarrow{dr} = T_2 - T_1 \qquad \text{……..(33)}$$

This is known as work-energy theorem.

Further, we have read that-

$$\vec{F} = -\nabla V$$

We can write-

$$W_{12} = \int_1^2 \vec{F}.\overrightarrow{dr} = \int_1^2 (-\nabla V).\overrightarrow{dr}$$

$$= \int_1^2 [-\frac{dV}{dr}\vec{dr}] = -\int_1^2 dV = V_1 - V_2 \qquad \text{…..(34)}$$

From equations (33) and (34), we get-

$$T_2 - T_1 = V_1 - V_2$$

Or $T_2 + V_2 = T_1 + V_1$

Or $T_1 + V_1 = T_2 + V_2 = \text{constant}$

Or $T + V = \text{constant} = \text{Total energy}$

i.e. the total energy of the particle is constant (conserved).

Thus the sum of kinetic energy and potential energy (i.e. total mechanical energy) of a particle remains constant in a conservative force field. This is known as the law of conservation of energy.

2.8 NOETHER'S THEOREM

Noether's theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law. The theorem was proven by mathematician Emmy Noether in 1915 and published in 1918, although a special case was proven by E. Cosserat & F. Cosserat in 1909. The action of a physical system is the integral over time of a Lagrangian function (which may or may not be an integral over space of a Lagrangian density function), from which the system's behavior can be determined by the principle of least action.

Noether's theorem is used in theoretical physics and the calculus of variations. A generalization of the formulations on constants of motionin Lagrangian and Hamiltonian mechanics (developed in 1788 and 1833, respectively), it does not apply to systems that cannot be modeled with a Lagrangian alone (e.g. systems with a Rayleigh dissipation function). In particular, dissipative systems with continuous symmetries need not have a corresponding conservation law.

Let us consider an example to understand the thing. If a physical system behaves the same regardless of how it is oriented in space, its Lagrangian is symmetric under continuous rotations: from this symmetry, Noether's theorem dictates that the angular momentum of the system be conserved, as a consequence of its laws of motion. The physical system itself need not be symmetric; a jagged asteroid tumbling in space conserves angular momentum despite its asymmetry. It is the laws of its motion that are symmetric.

Let us consider another example, if a physical process exhibits the same outcomes regardless of place or time, then its Lagrangian is symmetric under continuous translations in space and time respectively: by Noether's theorem, these symmetries account for the conservation laws of linear momentum and energy within this system, respectively.

As a final example, if the behavior of a physical system does not change upon spatial or temporal reflection, then its Lagrangian has reflection symmetry and time reversal symmetry respectively: Noether's theorem says that these symmetries result in the conservation laws of parity and entropy, respectively.

Noether's theorem is important, both because of the insight it gives into conservation laws, and also as a practical calculational tool. It allows investigators to determine the conserved quantities (invariants) from the observed symmetries of a physical system. Conversely, it allows researchers to consider whole classes of hypothetical Lagrangians with given invariants, to describe a physical system. As an illustration, suppose that a physical theory is proposed which conserves a quantity *X*. A researcher can calculate the types of Lagrangians that conserve *X* through a continuous symmetry. Due to Noether's theorem, the properties of these Lagrangians provide further criteria to understand the implications and judge the fitness of the new theory.

There are numerous versions of Noether's theorem, with varying degrees of generality. The original version applied only to ordinary differential equations (used for describing distinct

particles) and not partial differential equations (used for describing fields). The original versions also assume that the Lagrangian depends only upon the first derivative, while later versions generalize the theorem to Lagrangians depending on the n^{th} derivative. There are natural quantum counterparts of this theorem, expressed in the Ward–Takahashi identities. Generalizations of Noether's theorem to super spaces also exist.

2.9 SYMMETRIES

Symmetry in everyday language refers to a sense of harmonious and beautiful proportion and balance. In mathematics, "symmetry" has a more precise definition, that an object is invariant to any of various transformations; including reflection, rotation or scaling. Although these two meanings of "symmetry" can sometimes be told apart, they are related, so in this article they are discussed together. Mathematical symmetry may be observed with respect to the passage of time; as a spatial relationship; through geometric transformations; through other kinds of functional transformations; and as an aspect of abstract objects, theoretic models, language, music and even knowledge itself.

In this section, we shall discuss the symmetry from three perspectives: in mathematics, including geometry, the most familiar type of symmetry for many people; in science and nature; and in the arts, covering architecture, art and music. The opposite of symmetry is asymmetry.

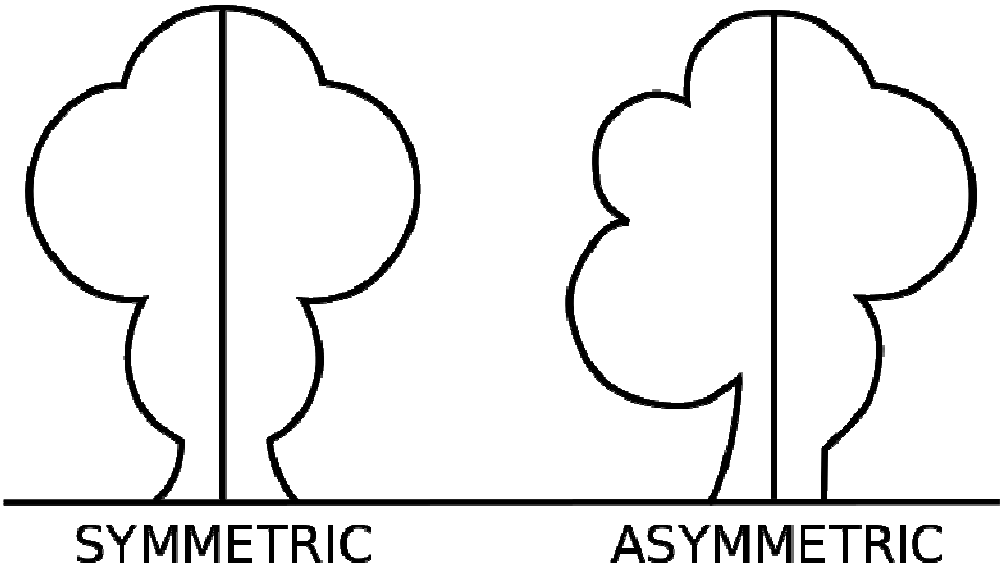


Figure (6)

A geometric shape or object is symmetric if it can be divided into two or more identical pieces that are arranged in an organized fashion. This means that an object is symmetric if there is a transformation that moves individual pieces of the object but doesn't change the overall shape. The type of symmetry is determined by the way the pieces are organized, or by the type of transformation:

- An object has reflectional symmetry (line or mirror symmetry) if there is a line going through it which divides it into two pieces which are mirror images of each other.
- An object has rotational symmetry if the object can be rotated about a fixed point without changing the overall shape.
- An object has translational symmetry if it can be translated without changing its overall shape.
- An object has helical symmetry if it can be simultaneously translated and rotated in three-dimensional space along a line known as a screw axis.
- An object has scale symmetry if it does not change shape when it is expanded or contracted. Fractals also exhibit a form of scale symmetry, where small portions of the fractal are similar in shape to large portions.
- Other symmetries include glide reflection symmetry and roto reflection symmetry.

Symmetry in physics has been generalized to mean invariance—that is, lack of change—under any kind of transformation, for example arbitrary coordinate transformations. This concept has become one of the most powerful tools of theoretical physics, as it has become evident that practically all laws of nature originate in symmetries. In fact, this role inspired the Nobel laureate PW Anderson to write in his widely read 1972 article *More is Different* that "it is only slightly overstating the case to say that physics is the study of symmetry." See Noether's theorem (which, in greatly simplified form, states that for every continuous mathematical symmetry, there is a corresponding conserved quantity such as energy or momentum; a conserved current, in Noether's original language); and also, Wigner's classification, which says that the symmetries of the laws of physics determine the properties of the particles found in nature. Important symmetries in physics include continuous symmetries and discrete symmetries of spacetime; internal symmetries of particles; and supersymmetry of physical theories.

2.10 SUMMARY

In this unit, we have studied Hamilton's principle, modified Hamilton's principle and its advantages. In this unit, we have derived Lagrange's equations of motion from Hamilton's variational principle. Here, we have discussed and learnt the calculus of variations by considering some important examples and problems. We have seen that the calculus of variations is concerned with the maxima or minima (collectively called extrema) of functionals. Euler-Lagrange's equations also have been derived in this unit. We have discussed various conservation laws and their importance. To present the clear understanding

and to make the concepts of the unit clear, some solved examples are given in the unit. To check your progress, self assessment questions (SAQs) are given in the unit.

2.11 GLOSSARY

Fundamental – basic

Configuration- pattern, design

Conservative- traditional, conventional

2.12 TERMINAL QUESTIONS

1. State Hamilton's principle and derive Lagrange's equations of motion from it.

2. Explain modified Hamilton's principle. Discuss advantages of Hamilton's variational principle.

3. What is δ-variation? Establish Euler- Lagrange's equations.

4. What do you mean by variational principle? Explain.

5. What are conservative laws? Prove that if there is no external force acting on a particle, then its linear momentum is conserved.

6. Prove that if external torque acting on the particle is zero, its angular momentum is conserved.

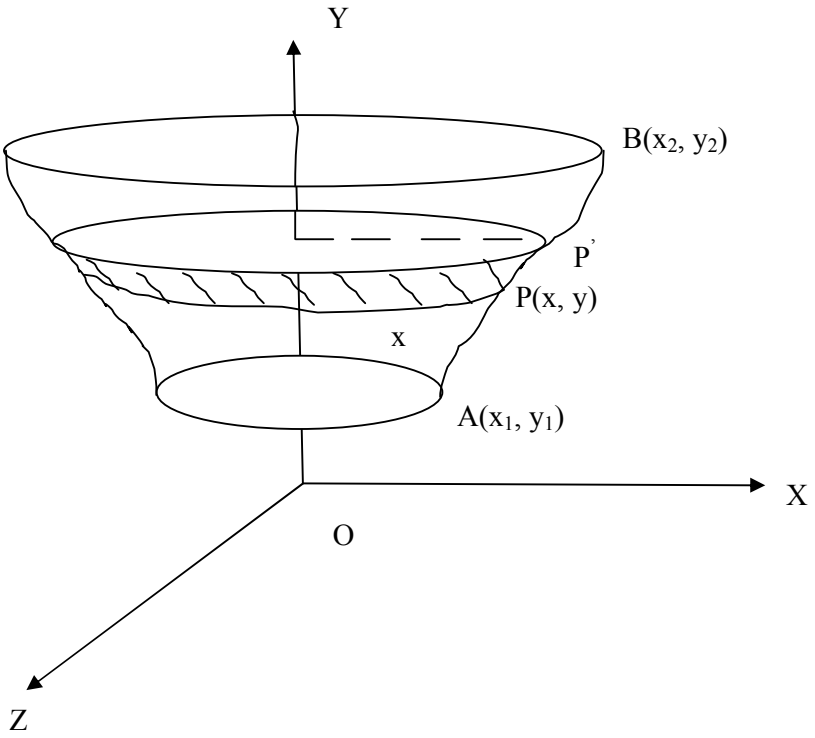
7. Explain and discuss Noether's theorem. What are symmetries? Explain.

8. State and prove energy conservation law.

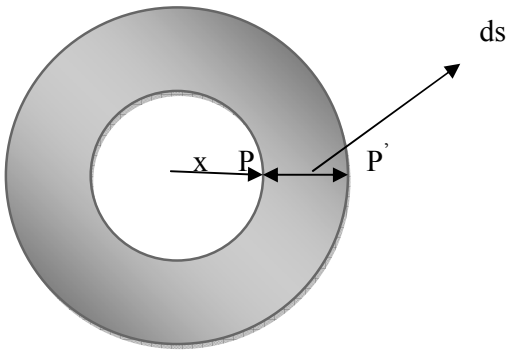
2.13 ANSWERS

Self Assessment Questions (SAQs):

1. Let us consider a curve AB which passes through the fixed points A (x_1, y_1) and B (x_2, y_2). The curve AB has been revolved about Y-axis to generate a surface.
   Let us consider a strip of the surface with radius x and breadth PP' = ds.
   $ds^2 = dx^2 + dy^2$ or $ds = \sqrt{1 + \dot{y}^2}\ dx$
   The area of the strip $ds = 2\pi x\ ds = 2\pi x \sqrt{1 + \dot{y}^2}\ dx$



Minimum surface area of revolution



Circular strip of area 2π x ds

Total area of revolution S = $2\pi \int_A^B x\sqrt{1+\dot{y}^2}$ dx

This area will be minimum (extremum), if δS = 0, for which Euler- Lagrange equation is to be satisfied i.e.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

Where f = $x\sqrt{1+\dot{y}^2}$

Here, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$

From above equation, we get-

$$\frac{d}{dx}\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = 0 \quad \text{or} \quad \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a \text{ (constant of integration)}$$

Squaring both sides, we get-

$$\left[\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right]^2 = a^2$$

Or $$\dot{y} = \frac{a}{\sqrt{x^2-a^2}} \quad \text{or} \quad dy/dx = \frac{a}{\sqrt{x^2-a^2}}$$

Or $$y = \int \frac{a}{\sqrt{x^2-a^2}}\,dx = a\cosh^{-1}(x/a) + b$$

where b is another constant of integration.

From the above equation, we get-

$$\cosh^{-1}(x/a) = \frac{y-b}{a}$$

or $$x = a\cosh\left(\frac{y-b}{a}\right)$$

This is the equation of a catenary. This is the equation of the curve for which the surface of revolution is minimum. The constants a and b can be determined by applying the conditions that the curve passes through points (x_1, y_1) and (x_2, y_2).

2. (i) (a), (b) (ii) (a), (c)

2.14 REFERENCES

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2.15 SUGGESTED READINGS

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2. Introduction to Classical Mechanics, David Morin, Cambridge University Press

3. A Course on Classical Mechanics, Madhumangal Pal, Narosa

4. Classical Mechanics and General Properties of Matter, P.D. Raychaudhuri, New Age International

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UNIT 3 CENTRAL FORCE MOTIONS - I

Structure

3.1 Introduction

3.2 Objectives

3.3 Reduction to One-Body Problem

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3.1 INTRODUCTION

In this unit, we will discuss the motion of two bodies each of which exerts a conservative, central force on the other but which are subject to no other, external forces. There are many examples of this problem: the two stars of a binary star system, a planet orbiting the sun, the moon orbiting the earth, the electron and proton in a hydrogen atom, the two atoms of a diatomic molecule. In most cases the true situation is more complicated. For example, even if we are interested in just one planet orbiting the sun, we cannot completely neglect the effects of all the other planets; likewise, the moon—earth system is subject to the external force of the sun. Nonetheless, in all such cases, it is an acceptable starting approximation to treat the two bodies of interest as being isolated from all outside influences.

Here it is important to also state that the examples of the hydrogen atom and the diatomic molecule do not belong in classical mechanics, since all such atomic-scale systems must really be treated by quantum mechanics. However, many of the ideas that we will develop in this unit (e.g. the important idea of reduced mass) play a crucial role in the quantum mechanical two-body problem.

3.2 OBJECTIVES

After studying this unit, you should be able to

- describe how a two-body problem can be reduced to a mathematically equivalent one-body problem
- understand the idea of reduced mass
- describe the general properties of central force
- define Kepler's second law of planetary motion
- describe what is meant by effective potential
- derive the differential equation for the orbit
- describe the conditions for stability and closure of orbits
- derive the inverse-square force law and describe how it relates to the Kepler's laws of planetary motion

3.3 REDUCTION TO ONE-BODY PROBLEM

Let us consider a conservative isolated system of two point masses m_1 and m_2 with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the origin O of some inertial reference frame as shown in Figure 3.1.

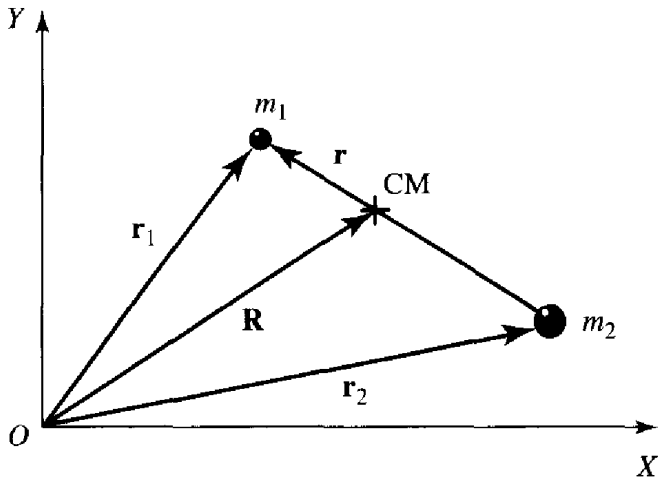


FIGURE 3.1 A conservative system of two point masses.

The two masses interact via central force for which the potential $V(r)$ is a function of the separation distance between them, i.e. $r = |r_1 - r_2|$. Such a system has six degrees of freedom, which means six independent generalized coordinates are required to describe the state of the system. We can conveniently choose the three generalized coordinates to describe the relative position r, since the potential energy is expressed as $V(r)$, and the other three, describing the center-of-mass (CM) position, R, of the two masses. Because of the symmetries in the system, the two-body problem can be reduced to a mathematically equivalent one-body problem.

The total kinetic energy of the system T can be written as the sum of the kinetic energy of the CM, T_{CM} plus the kinetic energy of the motion about the CM, T', i.e.

$$T = T_{CM} + T'$$

$$= \frac{1}{2}M\dot{R}^2 + \left(\frac{1}{2}m_1\dot{r}_1'^2 + \frac{1}{2}m_2\dot{r}_2'^2\right)$$

where $M = m_1 + m_2$ is the total mass of the CM. Here r_1' and r_2' are the position vectors of the two point masses relative the CM,

$$r_1' = (r_1 - R) = \left[r_1 - \left(\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}\right)\right]$$

$$= \frac{m_2(r_1 - r_2)}{m_1 + m_2} = -\frac{m_2 r}{m_1 + m_2}$$

where $r = r_2 - r_1$. Similarly,

$$r_2' = (r_2 - R) = \frac{m_1 r}{m_1 + m_2}$$

Therefore, the kinetic energy for the system is given as

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}m_1\left(\frac{m_2\dot{r}}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1\dot{r}}{m_1+m_2}\right)^2$$

$$= \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{(m_1+m_2)m_1m_2}{(m_1+m_2)^2}\dot{r}^2$$

$$= \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{r}^2$$

This can be thought of as the kinetic energy of two fictitious objects, one of mass M moving with the speed of the CM, and one of mass μ moving with the speed of the relative position r, where μ is called the *reduced mass* of the two-particle system,

$$\mu = \frac{m_1m_2}{m_1+m_2}$$

We may consider two simple limits here:

- If $m_2 \gg m_1$, then the reduced mass, $\mu \to m_1$, $r_2' \to 0$ and $r \to r_1'$. That is, the CM is fixed on the heavier mass and the motion is entirely of the smaller mass.
- If $m_2 = m_1 = m$, then $\mu = m/2$, $r_1' = r/2$ and $r_2' = -r/2$. In this case, the motion of the two particles is completely symmetric about the CM.

The Lagrangian for the system can be written as

$$L = T - V = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{r}^2 - V(r)$$

From the above expression, it is obvious that R is not present, i.e. the three coordinates of R are cyclic or ignorable and hence, the corresponding linear momentum or the velocity of the CM, $\dot{R} = \text{constant}$, which means that the CM does not move or moves with a constant velocity. Therefore, the first term in the above expression is a constant and the Lagrangian takes the form

$$L = \frac{1}{2}\mu\dot{r}^2 - V(r) \qquad (3.1)$$

The rest of the Lagrangian is exactly same as for the one-body problem, i.e. a fixed center of force with a single particle of mass μ at a distance r from it. Thus, the central force motion of two bodies about their CM can always be reduced to an equivalent one-body problem.

3.4 GENERAL PROPERTIES OF CENTRAL FORCE MOTION

From the results of the previous section, we can see that we only need to consider the problem of a single particle of reduced mass μ moving about a fixed center of force, which is taken as the origin of the coordinate system as shown in the Figure 3.2.

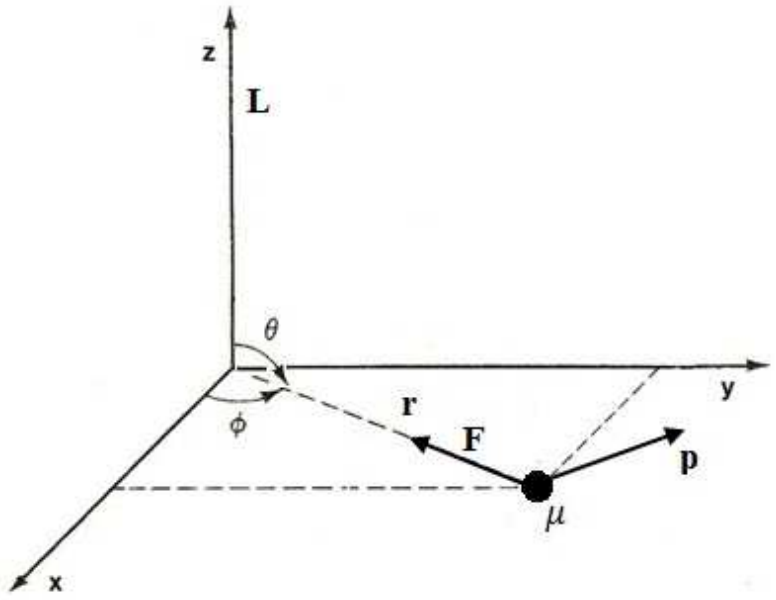


FIGURE 3.2 Motion of a particle subjected to a central force is confined to an x-y plane perpendicular to $\vec{L}$.

3.4.1 Angular Momentum

We know that the torque is given as

$$\boldsymbol{\tau = \dot{L} = r \times F}$$

If $\boldsymbol{F}$ is in the direction of the position vector $\boldsymbol{r}$, then we can see that the net torque should be zero, and therefore,

$$\boldsymbol{L = constant}$$

Since, $\boldsymbol{L = r \times p}$, $\boldsymbol{r}$ is always perpendicular to the fixed direction of $\boldsymbol{L}$ in space. This can be true if and only if $\boldsymbol{r}$ always lies in the plane whose normal is parallel to $\boldsymbol{L}$. When $\boldsymbol{L}$ is directed along the z-axis, then $\boldsymbol{r}$ will lie in the x-y plane, that is, the polar angle $\boldsymbol{\theta = \pi/2}$. Therefore,

$$\dot{\theta} = 0$$

We know that the Lagrangian in the spherical polar coordinate system is given by

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\varphi}^2\right) - V(r,\theta,\varphi)$$

In our case, since $\theta = \pi/2$ and $\dot{\theta} - 0$, the Lagrangian takes the form,

$$L = \frac{1}{2}m(\dot{r}^2 \mid r^2\dot{\varphi}^2) \quad V(r)$$

As it is clear that $\boldsymbol{L}$ does not contain φ, i.e. φ is a cyclic coordinate, whose corresponding generalized momentum is the angular momentum of the system, i.e.

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} = constant$$

One of the two Lagrange's equations of motion is then simply,

$$\dot{p}_\varphi = \frac{d}{dt}(\mu r^2 \dot{\varphi}) = 0 \qquad (3.2)$$

$$or \quad \mu r^2 \dot{\varphi} = l \qquad (3.3)$$

where l is the constant magnitude of the angular momentum.

3.4.2 Law of Equal Areas

From equation (3.2), we also get the following interesting result,

$$\frac{d}{dt}\left(\frac{1}{2} r^2 \dot{\varphi}\right) = 0 \qquad (3.4a)$$

The factor $1/2$ is inserted because $r^2\dot{\varphi}/2$ is nothing but the areal velocity – the area swept out by the radius vector per unit time as shown in Figure 3.3.

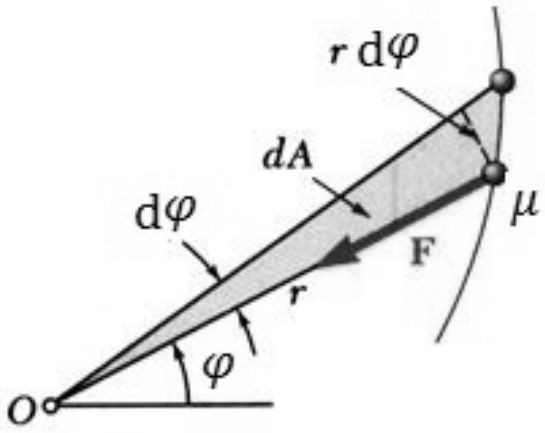


FIGURE 3.3 The area dA swept by the radius vector in time dt.

From the figure we can see that the differential area dA swept out in time dt is

$$dA = \frac{1}{2} r(r d\varphi)$$

And hence, the areal velocity is given as

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\varphi} \qquad (3.4b)$$

From equation (3.4a), we can see that the areal velocity is constant. The conservation of angular momentum is thus, equivalent to saying that the areal velocity is constant, which is also the *Kepler's second law* of planetary motion.

The Lagrange's equation for the coordinate r is given as

$$\frac{d}{dt}(\mu\dot{r}) - \mu r\dot{\varphi}^2 + \frac{\partial V}{\partial r} = 0 \qquad (3.5)$$

Substituting, $\dot{\varphi}$ from equation (3.3) in the above equation, we get

$$\mu\ddot{r} - \mu r\left(\frac{l^2}{\mu^2 r^4}\right) = -\frac{\partial V}{\partial r}$$

$$or \qquad \mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{\partial V}{\partial r} \qquad (3.6)$$

$$or \qquad \mu\ddot{r} = -\frac{\partial}{\partial r}\left[\frac{l^2}{2\mu r^2} + V\right]$$

$$or \qquad \mu\ddot{r}\dot{r} = -\frac{\partial}{\partial r}\left[\frac{l^2}{2\mu r^2} + V\right]\frac{dr}{dt}$$

$$or \qquad \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}\left[\frac{l^2}{2\mu r^2} + V\right]$$

$$or \qquad \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V\right) = 0$$

$$or \qquad \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V = constant, E \qquad (3.7)$$

As we know the total energy E is given as

$$E = T + V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{l^2}{\mu^2 r^4}\right) + V(r)$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r)$$

Therefore, the constant in equation (3.7) is equal to the total energy of the system E, which is conserved. The constants, angular momentum l appearing in equations (3.3) and total energy E in (3.7) are called the integrals of motion (or first integrals of motion).

From equation (3.7), we have

$$\frac{1}{2}\mu\dot{r}^2 = E - V - \frac{l^2}{2\mu r^2}$$

$$or \qquad \dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}$$

$$or \qquad dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}}$$

Integrating both sides, we have

$$\int_0^t dt = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}}$$

$$or \qquad t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}} \qquad (3.8)$$

The above equation gives $t(r)$, but we can rearrange to solve for $r(t)$. Once, we have obtained $r(t)$, we can integrate equation (3.3) to obtain $\varphi(t)$, i.e.

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{l}{\mu r^2}$$

On integrating the above equation, we get

$$\varphi = \varphi_0 + \frac{l}{\mu}\int_0^t \frac{dt}{r^2} \qquad (3.9)$$

Thus, equations (3.8) and (3.9) provide us $r(t)$ and $\varphi(t)$, by which we can locate the position of the particle on the path at any instant of time t and hence, are the solutions of our central force problem. If the nature of force is known, that is, $V(r)$ is known, then the integration of these two equations may be done. The result will contain l, E, r_0 and φ_0.

3.5 DIFFERNTIAL EQUATION FOR THE ORBIT

We want to deduce the nature of an orbit, in case of a central force, so that the solution to the equation of the orbit can give us the radial distance r as a function of φ.

From equation (3.3),

$$\frac{d\varphi}{dt} = \frac{l}{\mu r^2}$$

Therefore,

$$\dot{r} = \frac{dr}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{l}{\mu r^2}\frac{dr}{d\varphi}$$

$$\Rightarrow \ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d\varphi}{dt}\frac{d}{d\varphi}\left(\frac{dr}{dt}\right)$$

$$or \qquad \ddot{r} = \frac{l}{\mu r^2}\frac{d}{d\varphi}\left(\frac{l}{\mu r^2}\frac{dr}{d\varphi}\right)$$

From equation (3.6), the equation of motion for a particle of reduced mass μ, moving under central force $F(r)$, can be written as

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} = F(r)$$

$$\rightarrow \mu\left[\frac{l}{\mu r^2}\frac{d}{d\varphi}\left(\frac{l}{\mu r^2}\frac{dr}{d\varphi}\right)\right] - \frac{l^2}{\mu r^3} - F(r)$$

$$or \qquad \frac{l^2}{\mu r^2}\left[\frac{1}{r^2}\frac{d^2r}{d\varphi^2} - \frac{2}{r^3}\left(\frac{dr}{d\varphi}\right)^2 - \frac{1}{r}\right] = F(r)$$

$$or \qquad \frac{d^2r}{d\varphi^2} - \frac{2}{r}\left(\frac{dr}{d\varphi}\right)^2 - r = \frac{\mu r^4}{l^2}F(r) \qquad (3.10)$$

This is a differential equation for the orbit connecting r and φ. It can be written in more compact form if instead of r, it is written in terms of $u = 1/r$. Therefore,

$$\frac{dr}{d\varphi} = -\frac{1}{u^2}\frac{du}{d\varphi}$$

$$\Rightarrow \frac{d^2r}{d\varphi^2} = \frac{d}{d\varphi}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right) = -\frac{1}{u^2}\frac{d^2u}{d\varphi^2} + \frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2$$

Putting these values in equation (3.10), we get

$$\frac{d^2u}{d\varphi^2} + u = -\frac{\mu}{l^2u^2}F\left(\frac{1}{u}\right) \qquad (3.11)$$

This is the differential equation for the orbit provided the force law $F(r) = F(1/u)$ is known. Conversely, if the equation of orbit is known, i.e. $r = f(\varphi)$, then one can work backwards and obtain the force law. However, one thing that should be noted carefully is that

an orbit is uniquely specified for a central force but the converse is not true, i.e. there might be infinitely large number of force laws or force fields for a specified orbit.

Example 1: Show that the force that will cause a particle to describe a circular orbit, which passes through the center of the force, varies as the inverse fifth power of the distance?

Solution:

Equation of a circle of radius a whose circumference passes through the origin is

$$r = 2a\cos\varphi$$

where r and φ are the usual polar coordinates (Figure 3.4).

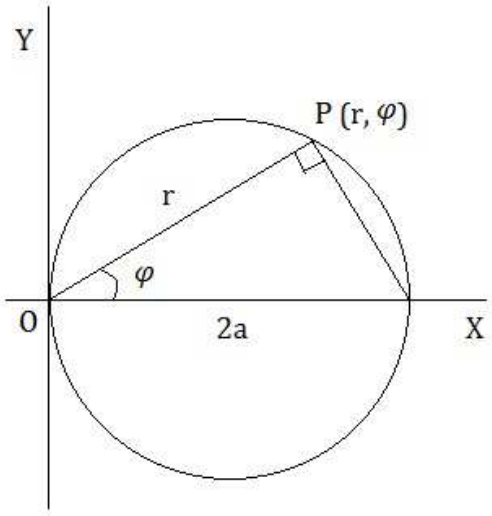


FIGURE 3.4

$$u = \frac{1}{r} = \frac{1}{2a\cos\varphi}$$

$$or \qquad \frac{du}{d\varphi} = \frac{\sin\varphi}{2a\cos^2\varphi},$$

$$or \qquad \frac{d^2u}{d\varphi^2} = \frac{1}{2a\cos\varphi} + \frac{\sin^2\varphi}{a\cos^3\varphi}$$

$$or \qquad \frac{d^2u}{d\varphi^2} + u = \frac{1}{2a\cos\varphi} + \frac{\sin^2\varphi}{a\cos^3\varphi} + \frac{1}{2a\cos\varphi}$$

Substituting the above expression in the differential equation (3.11), we get

$$\frac{\sin^2\varphi}{a\cos^3\varphi}+\frac{1}{a\cos\varphi}=-\frac{\mu}{l^2u^2}F\left(\frac{1}{u}\right)$$

$$or \qquad \frac{1}{a\cos^3\varphi}--\frac{\mu}{l^2u^2}F\left(\frac{1}{u}\right) \qquad [\because \sin^2\varphi+\cos^2\varphi-1]$$

$$or \qquad F(r)=-\frac{l^2}{\mu r^2}.\frac{1}{a\cos^3\varphi}=-\frac{l^2}{\mu r^2}.\left(\frac{1}{2a\cos\varphi}\right)^3.8a^2$$

$$or \qquad F(r)=-\frac{8l^2a^2}{\mu r^5} \qquad (3.12)$$

Example 2: Show that for the orbit described in the previous example, the total energy of the particle is zero.

Solution:

The kinetic energy of the particle moving in a circular orbit is given as

$$T=\frac{1}{2}\mu(\dot{r}^2+r^2\dot{\varphi}^2)$$

$$or \qquad T=\frac{1}{2}\mu(4a^2\sin^2\varphi\,\dot{\varphi}^2+4a^2\cos^2\varphi\,\dot{\varphi}^2)$$

$$or \qquad T=\frac{1}{2}\mu(4a^2\dot{\varphi}^2)=2\mu a^2\dot{\varphi}^2$$

In terms of $l=\mu r^2\dot{\varphi}$ [equation (3.3)], the KE becomes

$$T=\frac{2a^2l^2}{\mu r^4} \qquad (3.13)$$

Potential energy of the particle is given as

$$V=-\int F(r)dr$$

$$or \qquad V=\frac{8l^2a^2}{\mu r^5}\int\frac{1}{r^5}dr$$

$$or \qquad V=-\frac{2a^2l^2}{\mu r^4} \qquad (3.14)$$

Therefore, the total particle energy $E = T + V = 0$.

3.5.1 Integrable Power-Law Potentials

The equation of the orbit can be worked out from the conservation of total energy. We know that

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V$$

$$or \qquad E - V = \frac{1}{2}\mu\left(\frac{l}{\mu r^2}\frac{dr}{d\varphi}\right)^2 + \frac{l^2}{2\mu r^2}$$

$$or \qquad \frac{2\mu(E-V)}{l^2} = \frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2}$$

Substituting $u = 1/r$ in the above equation, we get

$$\frac{2\mu(E-V)}{l^2} = \left(\frac{du}{d\varphi}\right)^2 + u^2 \qquad (3.15)$$

From equation (3.3)

$$dt = \frac{\mu r^2}{l}d\varphi$$

Integrating both sides, we get

$$t = \frac{\mu r^2}{l}(\varphi - \varphi_0)$$

For any particular force law, one can obtain the actual equation of the orbit by eliminating t using the solution (3.8), to get

$$\frac{\mu r^2}{l}(\varphi - \varphi_0) = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}}$$

$$or \qquad \varphi = \varphi_0 + \int_{r_0}^{r} \frac{dr}{\frac{\mu r^2}{l}\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}} \qquad (3.16a)$$

Replacing r by $1/u$ in the above expression, we get

$$\varphi = \varphi_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu V}{l^2} - u^2}} \qquad (3.16b)$$

The most important force laws are the power-law functions of r,

$$V = ar^{n+1} \qquad (3.17)$$

$$or \qquad V = au^{-n-1} \qquad \left[\because u = \frac{1}{r}\right]$$

so that the force varies as the nth power of r. With this potential, equation (3.13) becomes

$$\varphi = \varphi_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu a}{l^2} u^{-n-1} - u^2}} \qquad (3.18)$$

If $-n-1 = 0, 1$ and 2, the denominator in the above expression takes the form $\sqrt{\alpha u^2 + \beta u + \gamma}$ and the integration can directly be expressed in terms of circular functions. Since, $n = -1$ corresponds to a constant potential [equation (3.17)], which is no force at all, we exclude this case. We are therefore, left with $n = -2$ and -3 corresponding to the inverse-square and inverse-cube force laws.

3.6 EFFECTIVE POTENTIAL

While describing the motion of a particle under the influence of a central force, we have seen that the two-dimensional motion can be reduced to a one-dimensional problem by using conservation of angular momentum and energy. Before obtaining any specific force laws, let us look into the motion in the general case, without explicitly arriving at any solutions but just making use of equations of motion and the conservation theorems and making plots of V_{eff} vs. r. The conservation of total energy according to equation (3.7), is given as

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) = T_{rad} + T_{ang}(r) + V(r)$$

where T_{rad} and T_{ang} are the kinetic energies due to the radial and angular motions. Since, T_{ang}, also known as the centrifugal energy, is a function of r and so it V, the two terms are combined to given the effective potential energy, i.e.

$$E - \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r) \qquad (3.19)$$

where $V_{eff}(r) = T_{ang}(r) + V(r)$.

As an illustration, consider a plot of V_{eff} against r for the case of an attractive inverse-square law of force,

$$F = -\frac{k}{r^2}$$

For positive k, the minus sign ensures that the force is attractive and towards the center of the force. The potential energy for this force is given as

$$V = -\int F\,dr = -\frac{k}{r}$$

And the corresponding effective potential is

$$V_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \qquad (3.20)$$

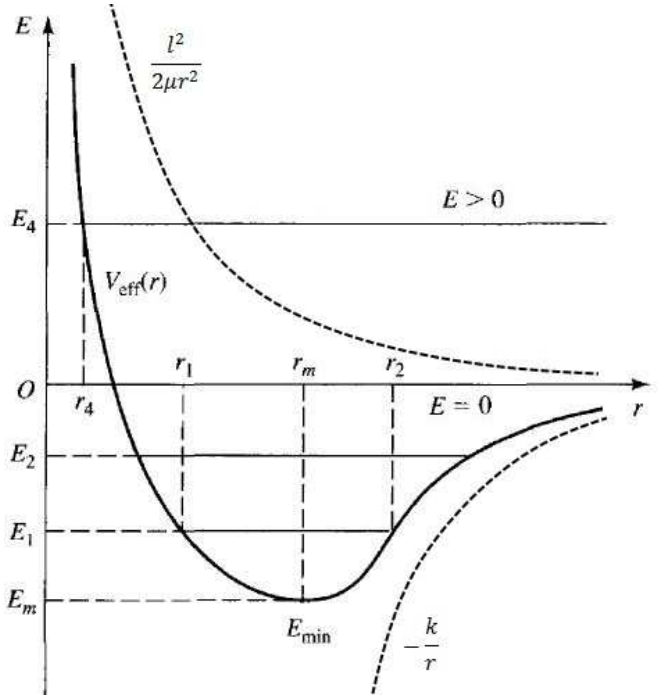


FIGURE 3.5 Equivalent one-dimensional potential for attractive inverse-square law of force.

Figure 3.5 shows the effective potential with the solid line and the two dashed lines showing the actual central potential term $-k/r$ and the centrifugal potential term $l^2/2\mu r^2$, which is a repulsive potential arising from angular momentum conservation. The relative sizes of the two terms determine if the effective potential is attractive or repulsive. The shape of the effective potential and the total energy of the system determine if the orbits are unbounded, bounded, or bounded and circular, and if the bounded orbits are periodic (closed)[1]. All unbounded orbits are open, but not all bounded orbits are closed, or in other words, periodic.

[1] In closed orbits, the particle eventually retraces its own footsteps. For example, motion of a particle in a circle about the center of force.

If the energy of the particle $E > 0$, like the case of $E = E_4$, the motion of the particle is unbounded with r_4 being the upper limit of r. A particle headed toward the center of force can come as far as r_4 and then must turn back, and may go back to infinity. In other words, there is a single turning point at $r = r_4$ (Figure 3.6a). The distance between E and V_{eff} is $\frac{1}{2}\mu\dot{r}^2$, i.e. proportional to the square of radial velocity, and becomes zero at the turning point $r - r_4$. Now, if the particle's energy $E - 0$, a roughly similar orbit behavior is obtained.

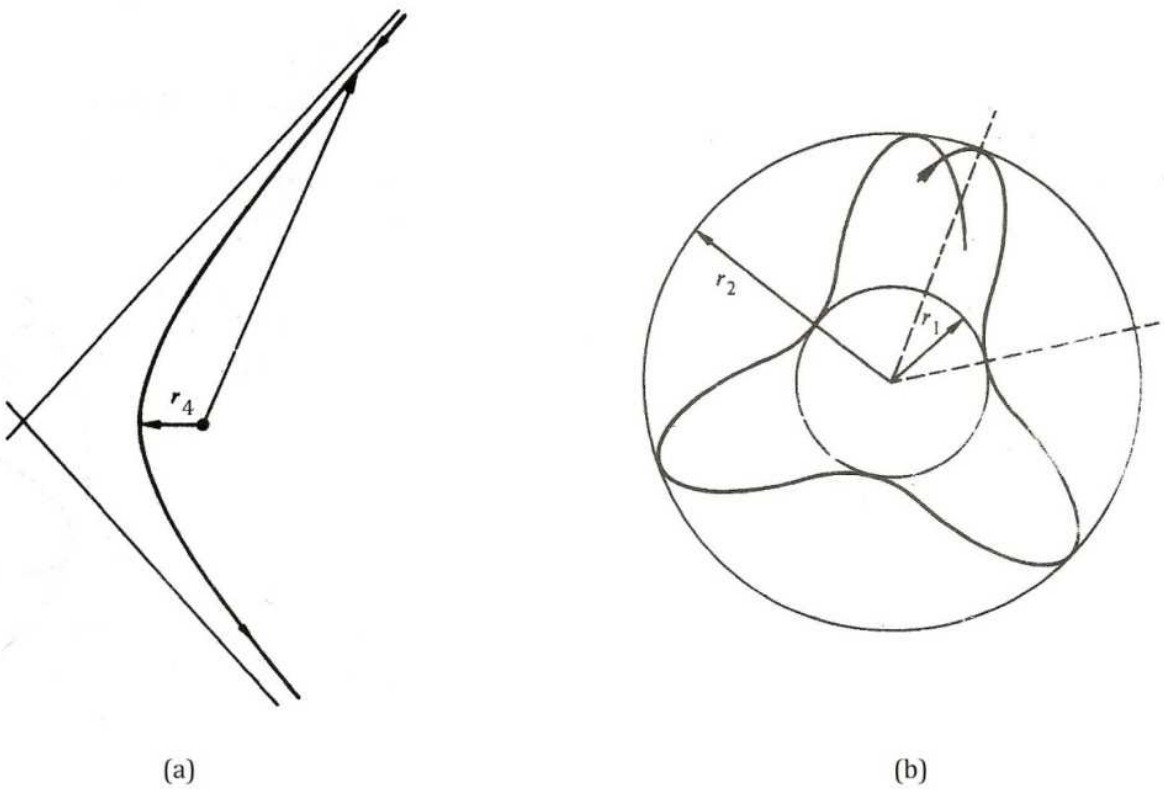


FIGURE 3.6 (a) The motion of particle when E > 0 corresponding to unbounded motion. (b) Motion of a particle with energy 0 > E > E_{min}, confined between two circles of radii r_1 and r_2.

But for any lower energy, i.e. $E < 0$, such as E_2 and E_1, the motion of the particle is bounded. Considering $E = E_1$ case, we can see that for the kinetic energy to be positive, in addition to a lower limit of $r = r_1$, there is also an upper limit of $r = r_2$. So, there are two turning points r_1 and r_2. Bounded system does not automatically imply that the orbits are closed, but that the orbits are contained between two circles of radii r_1 and r_2 with turning points always lying on the two circles (Figure 3.6b). If the energy $E = E_{min}$, is at the minimum of the effective potential, then the two bounds coincide, implying that there is no radial motion and the orbit is a circle of radius r_m. If the energy of the particle is less than the minimum energy E_{min}, no physically meaningful motion is possible, as in such a scenario the kinetic energy of the particle $\frac{1}{2}\mu\dot{r}^2$ becomes negative.

For the case of attractive inverse-square force law discussed above, we shall see in Section 3.7 that the shape of the orbit for particle energy $E > 0$ is a hyperbola, for $E = 0$ is a parabola, for $E < 0$ is a an ellipse and for $E = E_{min}$, where E_{min} is negative, the shape of the orbit is a circle.

Another commonly encountered force law is the isotropic harmonic law of force. The important features of the central force can also be understood by considering the case of a harmonic oscillator. For this case of linear restoring force,

$$F = -kr$$

For positive k, the minus sign ensures that the force is attractive and towards the center of the force. The potential energy for this force is given as

$$V = -\int F\,dr = \frac{1}{2}kr^2$$

Thus, the corresponding effective potential becomes

$$V_{eff} = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2} \qquad (3.21)$$

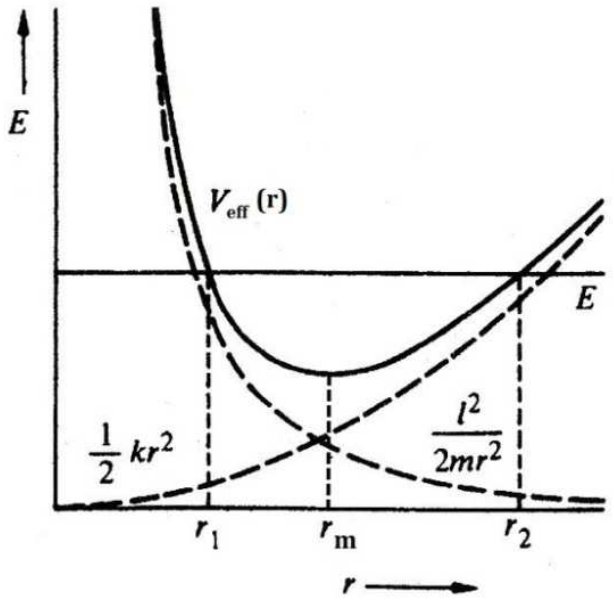


FIGURE 3.7 Equivalent one-dimensional potential for an isotropic harmonic law of force.

Figure 3.7 shows the effective potential with the solid line and the two dashed lines showing the actual central potential term $\frac{1}{2}kr^2$ and the centrifugal potential term $l^2/2\mu r^2$. As in the previous case, the relative sizes of the two terms determine if the effective potential is attractive or repulsive. For a given energy E, which is greater than the minimum energy $E_{min} = [V_{eff}]_{min}$, the particle oscillates between two extreme values, the lower limit of r_1

and the upper limit of r_2 of the radial distance r. These two points are the turning points of motion, where the kinetic energy of the particle $\frac{1}{2}\mu\dot{r}^2$ is zero. The motion is bounded for all physically possible energies and does not pass through the center of force. It can be seen that in this case, the shape of the orbit is an ellipse.

When the particle energy $E = E_{min} = \frac{1}{2}kr_m^2$, i.e. $V_{eff} = \frac{1}{2}kr^2$ and the angular momentum $l = 0$, the motion of the particle is bounded and describes circular motion, whose one-dimensional projection describes the simple harmonic motion. The radial distance r_m defines a circle of radius r_m.

Example 3: Apply the energy diagram method by making plots of V_{eff} vs. r to examine the feature of the orbit for the following attractive potential

$$V(r) = \frac{a}{r^3}$$

where a is a positive constant.

Solution: From the energy diagram shown in the following figure, we can see that for an energy E, there are two types of motions possible, depending on the initial value of r. If it is less than r_1, the motion will be bounded and r will always remain less than r_1, and the particle will pass through the center of force. On the other hand, if initial r is greater than r_2, then the motion will be unbounded and r_2, r_2 being the single turning point, will be the lower limit of r.

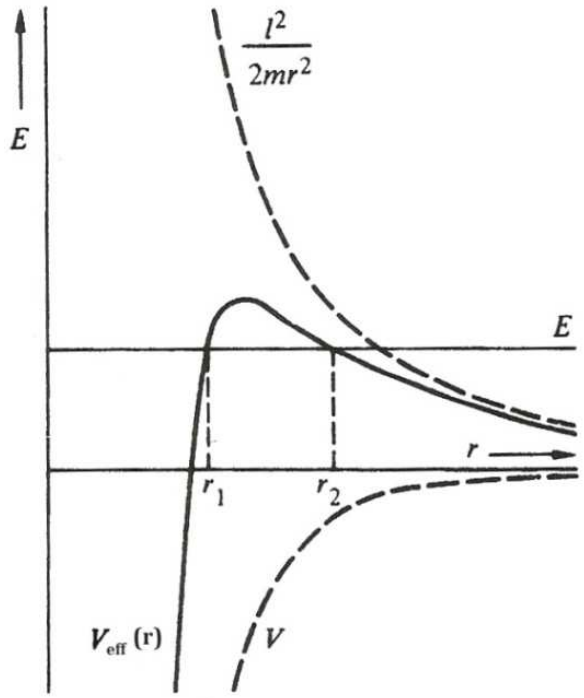


FIGURE 3.8 Equivalent one-dimensional potential for an attractive inverse-fourth law of force.

As for the initial value of r between r_1 and r_2, this is not physically possible because this would result in negative value of kinetic energy.

Self Assessment Question (SAQ) 1: Choose the correct option:

A particle moving under the influence of a central force $\boldsymbol{F(r) = kr}$ (where $\boldsymbol{r}$ is the position vector of the particle and k is a positive constant) has non-zero angular momentum. Which of the following curves is a possible orbit for this particle?

(a) A straight line segment passing through the origin.
(b) An ellipse with its center at the origin.
(c) An ellipse with one of the foci at the origin.
(d) A parabola with its vertex at the origin.

Self Assessment Question (SAQ) 2: Choose the correct option:

A planet of mass m moves in a circular orbit of radius r_0 in the gravitational potential $V(r) = -k/r$, where k is a positive constant. The orbital angular momentum of the planet is

(a) $2r_0km$ (b) $\sqrt{2r_0km}$ (c) r_0km (d) $\sqrt{r_0km}$

Paragraph for self assessment questions 3-4: Consider a comet of mass m moving in a parabolic orbit round the sun. The closest distance between the comet and the sun is b. Mass of the sun is denoted by M and the gravitational constant by G.

Self Assessment Question (SAQ) 3: Choose the correct option:

What is the angular momentum of the comet?

$(a)\ M\sqrt{Gmb}$ $(b)\ b\sqrt{GmM}$ $(c)\ G\sqrt{mMb}$ $(d)\ m\sqrt{2GMb}$

Self Assessment Question (SAQ) 4: Choose the correct option:

Which one of the following is TRUE for the above system?

(a) The acceleration of the comet is maximum when it is closest to the sun.
(b) The linear momentum of the comet is constant.
(c) The comet will return to the solar system after a specified period.
(d) The kinetic energy of the comet is a constant.

3.7 CONDITIONS FOR STABILITY AND CLOSURE OF ORBITS

If the potential function for the central force is of the form $V(r) = ar^{n+1}$, a being a positive constant, then for any given angular momentum l, the effective potential is given as

$$V_{eff} = ar^{n+1} + \frac{l^2}{2\mu r^2} \qquad (3.22)$$

The character of the circular orbit depends on whether the extreme value of V_{eff} is a minimum, or a minimum. If V_{eff} exhibits a minimum as shown in Figure 3.7, we can see that for a given value of l if the energy is a little higher than r_m that is required for a circular orbit, then the orbit will still be bounded, although not circular. On the other hand, if V_{eff} exhibits a maximum as shown in Figure 3.8, then the slightest raising of the energy above this value would result in the motion becoming unbounded. The circular orbit arising at a minimum is called *stable* and the circular orbit arising because of a maximum is termed as *unstable*. Thus, the stability of the circular orbit can be found from the sign of the second derivative of V_{eff}, the positive value signifying a minimum and a stable orbit while the negative second derivative indicates the presence of a maximum and therefore, an unstable circular orbit.

A stable orbit, therefore, occurs at $r - r_m$ when

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_m} = 0 \qquad and \qquad \left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_m} > 0$$

Thus,

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_m} = (n+1)ar_m^n - \frac{l^2}{\mu r_m^3} = 0$$

And

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_m} = (n+1)nar_m^{n-1} + 3\frac{l^2}{\mu r_m^4} = (n+3)\frac{l^2}{\mu r_m^4} > 0$$

Thus, a circular orbit under a central force is stable if $(n+3) > 0$ or

$$n > -3 \qquad (3.23)$$

If a circular orbit is stable, then a small raising of the energy above the value for a circular orbit results in slight variation of r about r_m and as a result, the particle executes a simple harmonic motion, an example of which is depicted in Figure 3.9.

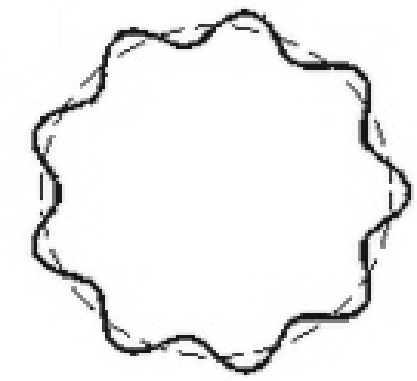


FIGURE 3.9 Orbit for motion in a central force deviating slightly from a circular orbit.

It has been found that the only values of n that result in closed circular orbits for all bound particles for more than first-order deviations from circularity, are $n = 1$ and $n = -2$ corresponding to the following familiar central forces:

— *Hooke's law* $(n = 1)$:

$$F(r) = -kr$$

— *Inverse-square law* $(n = 2)$:

$$F(r) = -\frac{k}{r^2}$$

The above result is enormously useful and has a very important consequence. Since, Hooke's law is not a realistic force law for all distances, because at large distances, $F \to -\infty$, the only possible alternative for closed orbits, therefore, is the inverse-square law of force. Thus, the

nature of the orbits in a gravitational field itself fixes the form of the force law and leads to the conclusion that *the gravitational force that exists in nature must vary as the inverse-square of the distance.*

3.8 KEPLER PROBLEM: INVERSE-SQUARE FORCE LAW

We have already seen that Kepler's second law follows directly from the conservation of angular momentum in conservative systems. Now, we shall see how Kepler's first and third law are consequences of the nature of gravitational potential, which follows from the inverse-square central force law. For this case, the force, which is attractive in nature, can be written as

$$F = -\frac{k}{r^2} = -ku^2$$

where k is known as the force constant.

$$\Rightarrow V = -\int F(r)\,dr$$

$$or \qquad V = \int \frac{k}{r^2}dr = -\frac{k}{r} = -ku$$

There are many ways to integrate the equation for the orbit. We directly use the expression (3.16b) and put $V = -ku$, to get

$$\varphi = \int \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu k u}{l^2} - u^2}} + \varphi' \qquad (3.24)$$

Here the integral is taken as indefinite and φ' is the constant of integration, whose value can be determined by the initial conditions.

The above indefinite integral is of the standard form,

$$\int \frac{dx}{\sqrt{\alpha - \beta x - \gamma x^2}} = \frac{1}{\sqrt{-\gamma}}\cos^{-1}\left(-\frac{\beta + 2\gamma x}{\sqrt{\beta^2 - 4\alpha\gamma}}\right)$$

To use this, we need to set,

$$\alpha = \frac{2\mu E}{l^2}, \qquad \beta = \frac{2\mu k}{l^2}, \qquad \gamma = -1$$

With these substitutes, equation (3.24) becomes

$$\varphi = -\cos^{-1}\left(\frac{\frac{l^2u}{\mu k} - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}}\right) + \varphi'$$

Replacing u by $1/r$ in the above equation, we get

$$\frac{1}{r} = \frac{\mu k}{l^2}\left[1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos(\varphi - \varphi')\right] \qquad (3.25a)$$

The constant of integration φ' can now be identified as one of the turning angles of the orbit. Comparing the above equation with the general equation of conics with one focus at the origin,

$$\frac{1}{r} = C[1 + \epsilon\cos(\varphi - \varphi')]$$

where ϵ is the eccentricity of the conic section, we find that the orbit is always a conic section with the eccentricity,

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \qquad (3.25b)$$

The value of ϵ determines the shape of the orbit, according to the following scheme:

- Ellipse: $0 < \epsilon < 1$ and $\mu k^2/2l^2 < E < 0$
- Circle: $\epsilon = 0$ and $E = -\mu k^2/2l^2$
- Parabola: $\epsilon = 1$ and $E = 0$
- Hyperbola: $\epsilon > 1$ and $E > 0$
- Not allowed: $\epsilon < 0$ and $E < -\mu k^2/2l^2$

The two particles in bound motion cannot leave each other, such as the earth and the sun, in which case the total energy $E < 0$. So, in the case of planetary motion, $\epsilon < 1$ and the orbit is elliptical. This justifies the *Kepler's first law*, which states that planets move in elliptical orbits with the sun at one focus.

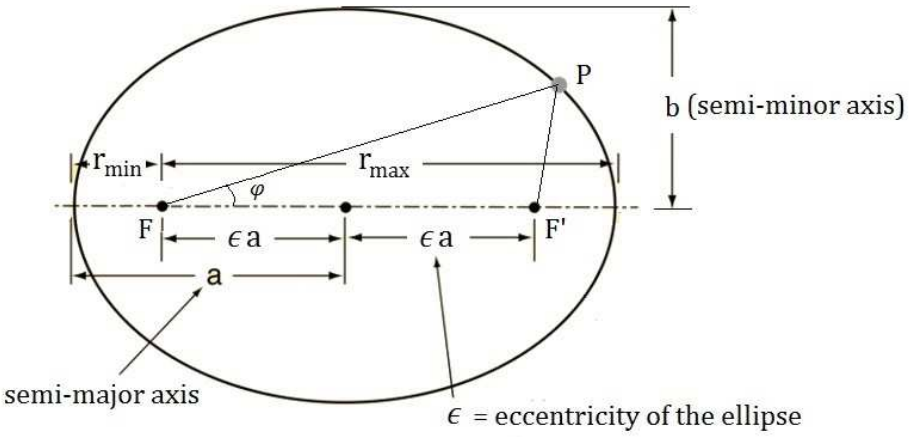


FIGURE 3.10 Definitions of different quantities in elliptical orbits.

From equation (3.25a),

$$r = \frac{l^2}{\mu k[1 + \epsilon \cos(\varphi - \varphi')]}$$

An ellipse is a curve traced by a point so that the sum of its distances from two foci F and F' is constant (Figure 3.10), i.e.

$$r_{max} + r_{min} = 2a \qquad (3.26)$$

where

$$r_{max} = \frac{l^2}{\mu k(1-\epsilon)}, \qquad r_{min} = \frac{l^2}{\mu k(1+\epsilon)}$$

Therefore, the length of semi-major axis in terms of the particle energy is equal to

$$a = \frac{1}{2}\frac{l^2}{\mu k}\left[\frac{1}{1-\epsilon} + \frac{1}{1+\epsilon}\right]$$

$$= \frac{l^2}{\mu k(1-\epsilon^2)} = \frac{l^2}{\mu k\left(1 - 1 - \frac{2El^2}{\mu k^2}\right)}$$

$$or \qquad a = -\frac{k}{2E} \qquad (3.27)$$

While the length of the semi-minor axis is given by

$$b = a\sqrt{1-\epsilon^2} \qquad (3.28)$$

Now, let us calculate the time period of planetary motion and see if it leads us to the Kepler's third law, which states that the square of the period of revolution is proportional to the cube of the semi-major axis. From the relation for areal velocity [equations (3.4b) and (3.3)], we have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\varphi} = \frac{l}{2\mu}$$

The total area covered in one period τ is then given by

$$A = \pi ab = \int_0^{\tau} \left(\frac{dA}{dt}\right) dt = \frac{l\tau}{2\mu}$$

$$or \quad \tau = \frac{2\pi\mu ab}{l} \tag{3.29}$$

Substituting the expressions for the semi-minor axis in the above equation, we get

$$\tau = \frac{2\pi\mu a^2\sqrt{1-\epsilon^2}}{l}$$

$$= \frac{2\pi\mu a^2\sqrt{1 \quad 1 \quad \frac{2El^2}{\mu k^2}}}{l}$$

From equation (3.27), substituting $E = -k/2a$ in the above equation, we get

$$\tau = \frac{2\pi\mu a^2}{\sqrt{a\mu k}} = 2\pi a^{3/2}\sqrt{\frac{\mu}{k}} \tag{3.30}$$

or $\tau^2 \propto a^3$, which is the *Kepler's third law*.

Since, the gravitational force between the sun and a planet, which is an example of the inverse-square law, is given as

$$F(r) = -\frac{GMm}{r^2}$$

where M is the mass of the sun, m is the mass of the planet and G is the gravitational constant, in astronomy, the constant k occurring in equation (3.30) is given by

$$k = GMm \tag{3.31}$$

3.9 SUMMARY

In this unit, we began with introducing the concept of reduced mass and how a two body central force problem can be reduced to a one body problem. Then we understood the different properties of the central force and we derived the Kepler's second law of planetary motion. This was followed by an overview of the concept of effective potential, and then we

derived the differential equation for an orbit. We also touched upon the conditions for stability and closure of orbits. And finally, we saw how the inverse-square force law relates to the Kepler's laws of planetary motion.

3.10 GLOSSARY

Acceleration - is the rate of change of velocity of an object with respect to time. An object's acceleration is the net result of any and all forces acting on the object, as described by Newton's Second Law.

Angular momentum – is the rotational equivalent of linear momentum. It is an important quantity in physics because it is a conserved quantity – the total angular momentum of a system remains constant unless acted on by an external torque. In three dimensions, the angular momentum for a point particle is a pseudovector $\mathbf{r}\times\mathbf{p}$, the cross product of the particle's position vector $\mathbf{r}$ (relative to some origin) and its momentum vector $\mathbf{p} = m\mathbf{v}$.

Central Force - is a force that points from the particle directly towards a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

Effective Potential – The effective potential (also known as effective potential energy) is a mathematical expression combining multiple (perhaps opposing) effects into a single potential. In classical mechanics it is defined as the sum of the 'opposing' centrifugal potential energy with the potential energy of a dynamical system.

Motion – is a change in position of an object over time.

Kinetic Energy – energy which a body possesses by virtue of being in motion.

Orbit – is the gravitationally curved trajectory of an object, such as the trajectory of a planet around a star or a natural satellite around a planet.

Planet – an astronomical body orbiting a star or stellar remnant that is massive enough to be rounded by its own gravity, and is not massive enough to cause thermonuclear fusion.

Potential Energy – the energy of a particle or system of particles derived from position, or condition, rather than motion.

Reduced Mass – the "effective" inertial mass appearing in the two-body problem of Newtonian mechanics. It is a quantity which allows the two-body problem to be solved as if it were a one-body problem.

Velocity – The velocity of an object is the rate of change of its position with respect to a frame of reference, and is a function of time. Velocity is equivalent to a specification of its speed and direction of motion.

3.11 TERMINAL QUESTIONS

1. Discuss the two-body central force problem. Show that the motion of two interacting particles is equivalent to the motion of a single particle in an external field.

2. What is meant by equation of motion and the first integrals?

3. State and prove the Kepler's laws of planetary motion.

4. Derive the differential equation for the orbit of a particle moving under the influence of a central force. Investigate the motion of the particle under the attractive inverse square law.

5. Choose the correct option:

A planet of mass m and an angular momentum L moves in a circular orbit in a potential $V(r) = -K/r$, where K is a constant. If it is slightly perturbed radially, the angular frequency of the radial oscillations is

$$(a)\ \frac{mK^3}{\sqrt{2}L^3} \qquad (b)\ \frac{mK^3}{L^3} \qquad (c)\ \frac{\sqrt{2}mK^3}{L^3} \qquad (d)\ \frac{\sqrt{3}mK^3}{L^3}$$

Paragraph for questions 6-7: Consider the motion of a particle in the potential field $V(x)$ as shown in the figure below.

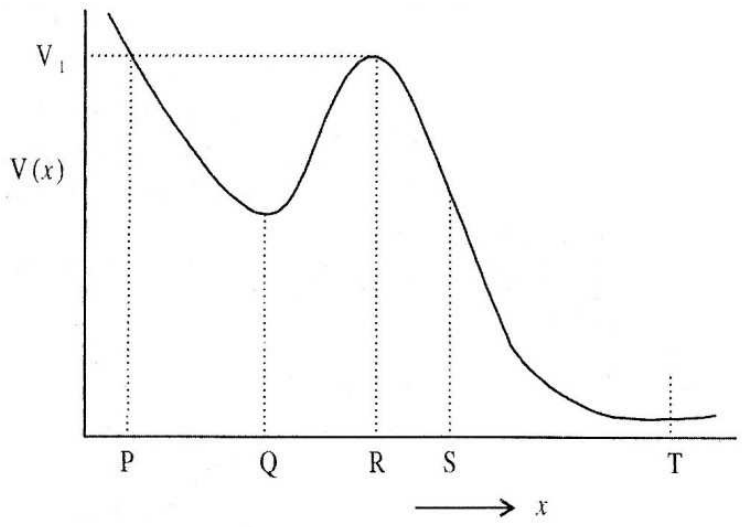


FIGURE 3.11

6. Choose the correct option:
Suppose the particle has a total energy $E = V_1$ as shown in the figure. Then the speed of the particle is zero when it is at point
(a) P (b) Q (c) S (d) T

7. Choose the correct option:
Which of the following statement is NOT correct about the particle?
(a) It experiences no force when its position corresponds to the point Q on the curve.

(b) It experiences no force when its position corresponds to the point R on the curve.
(c) Its speed is the largest when it is at S.
(d) It will be in a closed orbit between P and R, if $E < V_1$.

8. Choose the correct option:
A space station moving in a circular orbit around the earth goes into a new bound orbit by firing its engine radially outwards. The orbit is
(a) *a large circle* (b) *a smaller circle* (c) *an ellipse* (d) *a parabola*

3.12 ANSWERS

Selected Self Assessment Questions (SAQs):

1. (b)

2. (d)

3. (d)

4. (a)

Selected Terminal Questions:

5. (b)

6. (b)

7. (c)

8. (c)

3.13 REFERENCES

1. Classical Mechanics, Herbert Goldstein, Charles Poole, John Safko – Pearson Education, New Delhi
2. Classical Mechanics – System of Particles and Hamiltonian Dynamics, Walter Greiner – Springer-Verlag, New York
3. Classical Mechanics, John R. Taylor – University Science Books, New York

3.14 SUGGESTED READINGS

1. Classical Mechanics, H. M. Agrawal – New Age International, New Delhi
2. Classical Mechanics – System of Particles and Hamiltonian Dynamics, Walter Greiner – Springer-Verlag, New York
3. Classical Mechanics, John R. Taylor – University Science Books, New York

UNIT 4 CENTRAL FORCE MOTIONS - II

Structure

4.1 Introduction

4.2 Objectives

4.3 Escape Velocity

4.4 Geostationary Satellites

4.4.1 Angular Momentum

4.4.2 Law of Equal Areas

4.4.3 Launching Geostationary Satellites into Orbit

4.5 Orbital Transfers and Interplanetary Trajectories

4.6 Virial Theorem

4.7 Summary

4.8 Glossary

4.9 Terminal Questions

4.10 Answers

4.11 References

4.12 Suggested Readings

4.1 INTRODUCTION

In this unit, we will continue with the discussion from the last unit on Kepler laws of planetary motion and inverse square law. We will first understand the concept of escape velocity, and then move on to discuss the concepts of geosynchronous and geostationary orbits. Then we will see how a geostationary satellite is launched into its orbit. Then we will discuss about orbital transfers and interplanetary trajectories.

In this unit, we will also discuss the virial theorem.

4.2 OBJECTIVES

After studying this unit, you should be able to

- describe what is meant by escape velocity
- understand the concepts of geostationary and geosynchronous orbits
- describe how a geostationary satellite is launched into its orbit
- understand the concept of Hohmann transfer orbit
- discuss how interplanetary trajectories look like for both inner and outer planets
- discuss the virial theorem

4.3 ESCAPE VELOCITY

Consider a body of mass $m \ll M$ projected upward from the surface of the earth. Neglecting the air resistance, if the velocity of the projectile is greater than the *escape velocity*, the body will escape from the earth's sphere of influence into the outer space. If v is the velocity with which the body is projected upward, then the total energy of the body at the surface of the earth is given as

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

where R and M are the radius and the mass of the earth. Since, the gravitational force acting on the particle is given by $F = mg$, therefore, $mg = GMm/r^2$ or the acceleration due to gravity[2] $g = GM/r^2$. Using this relation, the energy of the body at the surface of the earth is given as

[2] Close to the surface of the earth, the acceleration due to gravity $g = GM/R^2 \approx 9.8\, m/s^2$.

$$E = \frac{1}{2}mv^2 - mgR$$

In order to escape, the trajectory has to be non-elliptical, which is possible only when $E \geq 0$. Therefore, if v_e is the escape velocity, then

$$\frac{1}{2}mv_e^2 \geq mgR$$

$$or \qquad v_e \geq \sqrt{2gR} \qquad (4.1)$$

It must be noted that the above condition for the escape velocity is independent of the angle of projection of the projectile. For $R = 6370\ km$, $g = 9.8\ m/s^2$, we get $v_e = 11.2\ km/s$.

One must understand that this is not a practical way of launching a satellite or a spacecraft and put them into orbits. In the subsequent sections, we will study briefly how this task is accomplished.

4.4 GEOSTATIONARY SATELLITES

A body that revolves constantly round a comparatively much larger body is called a satellite. Moon that revolves around the earth is an example of a natural satellite. Then there are man-made satellites or artificial satellites that are placed with the help of multistage rockets in stable orbits to revolve round the earth. The concepts that were discussed for the planetary orbits are also valid for satellites.

4.4.1 Geosynchronous Orbit

A geosynchronous orbit is an orbit around the earth with an orbital period of one sidereal[3] day, purposely matching the earth's sidereal rotation period (23 hours 56 minutes and 4.1 seconds), irrespective of its eccentricity and orientation with respect to the earth's equator. The synchronization of orbital and rotation period means that, an observer on earth will see an object in this orbit in the same position in the sky at the same time of the day, each day. In other words, for an observer on the surface of the earth, an object in the geosynchronous orbit returns to exactly the same place in the sky after a period of one sidereal day.

4.4.2 Geostationary Orbit

A special case of geosynchronous orbit is the geostationary orbit, which is a circular geosynchronous orbit at zero inclination (i.e., in the plane of the equator). With these additional constraints, an object in a geostationary orbit remains stationary as seen from all the points on the surface of the earth. This means that we can point a TV satellite antenna at a satellite in a geostationary orbit, it will continue to point at the geostationary satellite.

[3] Sidereal time refers to a time scale that is based on the earth's rate of rotation measured relative to the distant stars, i.e. the constellation of fixed stars, rather than the sun or the planets.

Let us now find the height h of the geostationary orbit above the earth's equator. Consider the case of a geostationary satellite of mass $m \ll M$, where M is the mass of the earth. Since, the satellite has the same period τ as the rotation of the earth, its orbiting speed v is given as

$$v = \frac{2\pi r}{\tau} \qquad (4.2)$$

where r is the radius of the orbit. The satellite in the orbit is only acted by gravitational force, which must provide the necessary centripetal force for it to move in a circular orbit. Therefore,

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \qquad (4.3)$$

Substituting v from (3.33) in equation (3.34), we get

$$r = R + \square = \sqrt[3]{\frac{GM\tau^2}{(4\pi)^2}} \qquad (4.4)$$

where the radius of earth $R = 6371\,km$. Substituting the values of $G = 6.674 \times 10^{-11}\ N.m^2/kg^2$, $M = 5.972 \times 10^{24}\ kg$, $\tau = 23 \times 3600 + 56 \times 60 + 4.1\ s$ in the above expression, we obtain the height of the geostationary orbit above the earth's equator as

$$h = 35786\ km$$

We could have also obtained equation (4.4) directly from Kepler's third law [equation (3.30)], where a becomes r in the case of circular orbit. Since, $M \gg m$, the reduced mass $\mu \cong m$.

4.4.3 Launching Geostationary Satellites into Orbit

There is a significant amount of knowledge and technology used to make sure that satellites enter their orbits in the most energy efficient ways possible. Certainly, it is possible to place a satellite directly into geostationary orbit, but this would take more energy and would not be practical.

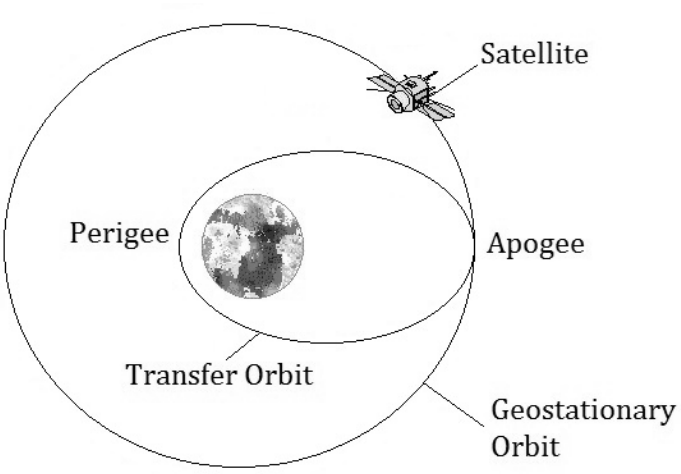


FIGURE 4.1 Transfer orbit for launching an artificial satellite into the geostationary orbit.

Figure 4.1 depicts a representative satellite launch method followed by space agencies such as ISRO and NASA. During the launch phase, the launch vehicle places the satellite into the transfer orbit, which is an elliptical orbit having its farthest point from earth known as apogee, at the geosynchronous elevation of 35,786 km and its nearest point from earth known as perigee, at an elevation of usually not less than 180 km. Alternatively, the satellite can first be placed into a low earth orbit and once in the correct position in this orbit, rockets are fired to put the satellite into the transfer orbit, where it simply coasts from perigee to apogee. Again, when the satellite reaches the required altitude (apogee), the rocket or booster is fired to retain it in the geostationary orbit with the correct velocity. This final step may not be a single step process but may require a few intermediate drift orbits before the orbit is fully circularized and its inclination is lowered to zero.

4.5 ORBITAL TRANSFERS AND INTERPLANATARY TRAJECTORIES

For simplicity, let us assume that the orbits of the planets around the sun to be circular and try to understand the different phases in the Mars Orbiter Mission, which can be envisaged as a rendezvous problem, i.e. a space probe on the earth has to go and meet the Martian orbit.

Let the radii of orbits of the earth and the Mars be r_E and r_M, respectively, with the sun as their common center as shown in Figure 4.2. In the *geocentric phase*, the spacecraft is injected into an elliptical transfer orbit by the launcher. One primary concern is to get the spacecraft to Mars, using the least amount of fuel, and hence a minimum energy transfer orbit or *Hohmann*[4] *transfer orbit* is used to send the spacecraft from the earth to the Mars. The elliptical transfer orbit is tangential to the earth's orbit at E and to the Mar's orbit at M.

[4] The fundamental assumption behind Hohmann transfer is that there is only one body which exerts a gravitational force on the body of interest, such as a spacecraft. This is also a good model for transferring an earth-based satellite from a low orbit to a geostationary orbit.

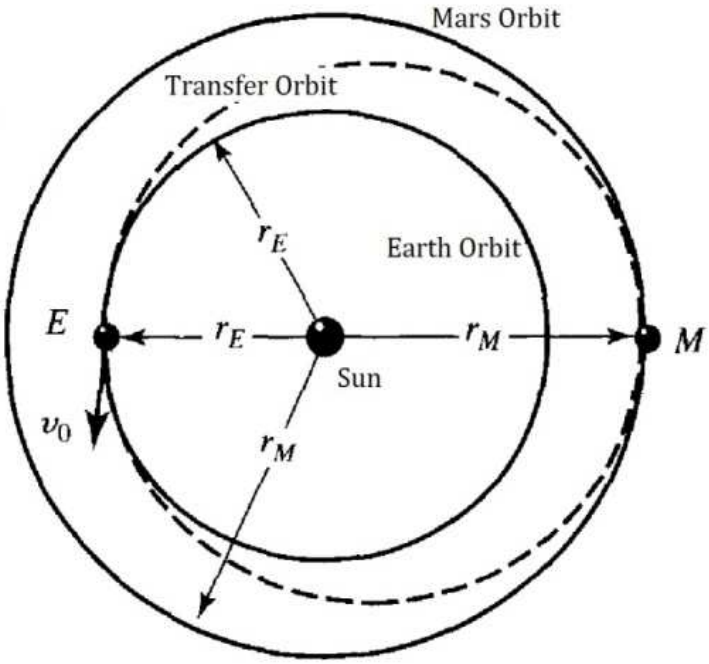


FIGURE 4.2 Transfer orbit from the earth to the Mars is tangential to the circular orbits of the earth and the Mars around the sun, at E and M, respectively.

As can be seen from Figure 4.2, the length of the major axis of the ellipse is given as

$$2a = r_E + r_M \tag{4.5}$$

The earth of mass m_E is going round the sun of mass M_S in a circular orbit with an orbital velocity of v_0. The gravitational force between them provides the necessary centripetal force for the earth to move in a circular orbit, therefore,

$$\frac{m_E v_0^2}{r_E} = \frac{GM_S m_E}{r_E^2}$$

$$\Rightarrow v_0^2 = \frac{GM_S}{r_E} \tag{4.6}$$

In the geocentric phase, the spacecraft is maneuvered into a departure hyperbolic trajectory with which it escapes from the earth's sphere of influence with earth's orbital velocity v_0 plus Δv boost. If v_1 is the velocity of the probe in the transfer orbit, then

$$v_1 = v_0 + \Delta v \tag{4.7}$$

At the point of departure E from the earth's sphere of influence, we can write the total energy E of the probe as the sum of the gravitational potential energy and the kinetic energy as

$$E = \frac{1}{2} m v_1^2 - \frac{GM_S m}{r_E} \tag{4.8}$$

For the transfer orbit, we can write

$$E = -\frac{k}{2a} = -\frac{GM_S m}{r_E \mid r_N} \tag{4.9}$$

Therefore,

$$v_1^2 = v_0^2 \frac{2r_M}{r_E + r_M} \tag{4.10}$$

And the gravitational boost needed for orbit transfer is given as

$$\Delta v = v_0 \left(\sqrt{\frac{2r_M}{r_E + r_M}} - 1 \right) \tag{4.11}$$

The sphere of influence of earth ends at about 918347 km from the surface of the earth, beyond which the perturbing force on the probe is due to the sun only and it enters the *heliocentric phase*. As the space probe leaves the earth's orbit and enters the transfer orbit, the probe's speed $v_1 > v_0$, the earth's orbital speed. Once, the probe is in the transfer orbit, it will simply coast from point E to point M, which is roughly one-half of an ellipse around the sun in about half the time period of the transfer orbit. Eventually it will intersect the orbit of Mars at the exact time that Mars is also there. According to Kepler's third law [equation (3.30)], $\tau^2 \propto a^3$, so we can write

$$\frac{\tau^2}{a^3} = \frac{\tau_E^2}{r_E^2}$$

$$or \quad \tau = \left(\frac{r_E + r_M}{2r_E} \right)^{3/2} \tau_E \tag{4.12}$$

where τ_E and τ are the time periods of the earth's orbit and the transfer orbit, respectively.

As the probe coasts to the other end of the ellipse, its speed continuously decreases as its distance from the sun increases based on the Kepler's second law. If v_2 is taken to be the speed of the probe when it reaches point M, then from energy conservation,

$$E = -\frac{GM_S m}{r_E + r_M} = \frac{1}{2} m v_2^2 - \frac{GM_S m}{r_M} \tag{4.13}$$

Therefore, from equations (3.44) and (3.37), we get

$$v_2^2 = v_0^2 \frac{2r_E^2}{r_M (r_E + r_M)} \tag{4.14}$$

The orbital speed of Mars round the sun is given as

$$v_M^2 = \frac{GM_S}{r_M} = v_0^2 \frac{r_E}{r_M} \qquad (4.15)$$

We can see that $v_2 < v_M$, i.e. the probe is moving slower than Mars when it arrives at Mars, hence, Mars approaching from behind will overtake the probe. To tackle this situation, the speed of the probe is to be increased from v_2 to v_M by giving it a Δv boost.

In the *Martian phase*, the probe finally arrives at the Mars sphere of influence, which is about 573473 km from the surface of the Mars. At the time, the space probe reaches the closest approach to the Mars (perigee), it is captured in the Martian orbit by imparting Δv retro[5], which is known as the Mars orbiter insertion maneuver.

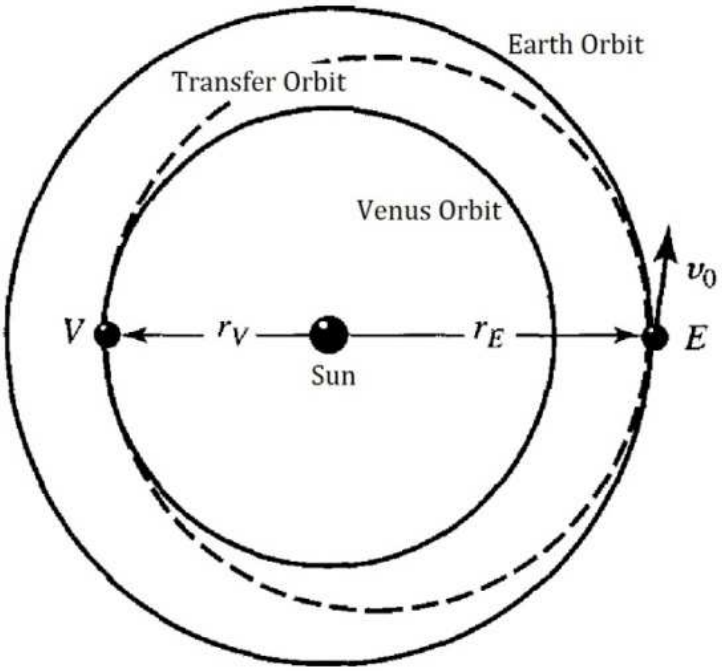


FIGURE 4.3 Transfer orbit from the earth to the Venus is tangential to the circular orbits of the earth and the Venus around the sun, at E and V, respectively.

The condition is a slightly different, if the space probe has to be transferred from the earth to one of the inner planets, like Venus as shown in Figure 4.3. When the probe is transferred from the earth's orbit round the sun to a smaller transfer orbit, it is slowed down from v_0 to v_1 through a retroburn. As the probe coasts from point E to point V of the elliptical transfer orbit to move closer to the sun, its speed continuously increases to v_2.

Example 1: A space vehicle A is moving round the earth in a circular orbit of radius r_1. Another space vehicle B is moving in a larger circular orbit of radius r_2. The vehicle A now

[5] Retro means backwards or behind. Retroburning means firing rockets in the opposite direction from the way they were fired to push the spacecraft forward. This thrust in the opposite direction slows the spacecraft down.

has to be docked with B. This is done by means of the Hohmann transfer ellipse between A and B (Figure 4.4). The transfer is achieved by means of a burst of speed Δv_A at A and a second burst of speed Δv_B at B to make sure that the satellite will remain in that orbit. Determine Δv_A and Δv_B.

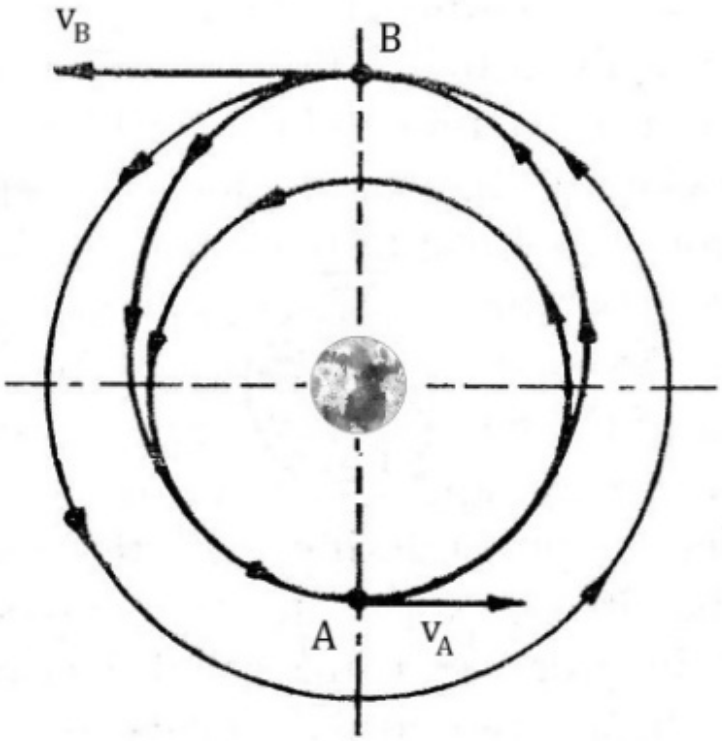


FIGURE 4.4 Hohmann transfer orbit and the docking of space vehicles.

Solution

For the two circular orbits, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow v = \sqrt{\frac{GM}{r}}$$

where M and m are the masses of the earth and the space vehicle. Therefore, the orbital speeds of space vehicles A and B are

$$v_{A0} = \sqrt{\frac{GM}{r_A}}, \qquad v_{B0} = \sqrt{\frac{GM}{r_B}} \tag{4.16}$$

where r_A and r_B are the radii of circular orbits of A and B, respectively.

From Kepler's law, we know that earth is at one of the foci of the transfer ellipse. Therefore, from Figure 4.4 we can see that the length of the semi-major axis of the transfer ellipse is given as

$$2a = r_A + r_B \tag{4.17}$$

At point A, while it is in the transfer orbit, the total energy of space vehicle A can be written as

$$E = \frac{1}{2}mv_A^2 - \frac{GMm}{r_A} \tag{4.18}$$

For the elliptical transfer orbit, we can also write the total energy as

$$E = -\frac{k}{2a} = -\frac{GMm}{r_A + r_B} \tag{4.19}$$

Therefore,

$$v_A^2 = v_{A0}^2 \frac{2r_B}{r_A + r_B} \tag{4.20}$$

Therefore, the speed burst required at point A is given as

$$\Delta v_A = v_A - v_{A0} = \sqrt{\frac{GM}{r_A}}\left(\sqrt{\frac{2r_B}{r_A + r_B}} - 1\right)$$

Similarly, the speed burst required at point B is given as $\Delta v_B = v_{B0} - v_B$, where the speed of the space vehicle at point B in the transfer orbit is v_B. As the space vehicle A coasts to the other end of the ellipse, its speed continuously decreases as its distance from the earth increases based on the Kepler's second law. From energy conservation,

$$E = -\frac{GMm}{r_A + r_B} = \frac{1}{2}mv_B^2 - \frac{GMm}{r_B}$$

Therefore,

$$v_B^2 = v_{B0}^2 \frac{2r_A}{r_A + r_B}$$

And

$$\Delta v_B = v_{B0} - v_B = \sqrt{\frac{GM}{r_B}}\left(\sqrt{\frac{2r_A}{r_A + r_B}} - 1\right)$$

Example 2: A planet has a circular orbit around a star of mass M. The star explodes, ejecting its outer envelope at a velocity much greater than the orbital motion of the planet, so that the mass loss may be considered instantaneous. The remnant of the star has a mass M' which is much greater than the mass of the planet m. What is the eccentricity of the planetary orbit after the explosion?

Solution:

Since, the mass of star >> mass of the planet, the reduced mass $\mu \approx m$. Before the explosion, since the orbit is circular, the eccentricity of the orbit is zero; i.e.

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(GMm)^2}} = 0$$

$$or \quad E = -\frac{m(GMm)^2}{2l^2} \tag{4.21}$$

where the angular momentum $l = mvR$, v being the orbiting speed of the planet and R being the radius of the orbit. For circular motion, the centripetal force is provided by the gravitational force between the star and the planet, i.e.

$$\frac{mv^2}{R} = \frac{GMm}{R^2}$$

$$\Rightarrow R = \frac{GM}{v^2}$$

$$or \quad R = \frac{l^2}{GMm^2} \tag{4.22}$$

The angular momentum is conserved after the explosion. However, the gravitational potential energy is reduced because of mass loss of the star from M to M'. Hence, the total energy of the system changes to E',

$$E' = E - \left[-\frac{G(M - M')m}{R}\right] \tag{4.23}$$

Therefore, the eccentricity of the orbit after the explosion is given by

$$\epsilon = \sqrt{1 + \frac{2E'l^2}{m(GM'm)^2}}$$

$$= \sqrt{1 + \frac{2\left[-\frac{m(GMm)^2}{2l^2} + \frac{G(M-M')m}{\left(\frac{l^2}{GMm^2}\right)}\right] l^2}{m(GM'm)^2}}$$

$$= \sqrt{1 + \left(\frac{M}{M'}\right)^2 \left(1 - \frac{2M'}{M}\right)}$$

4.6 VIRIAL THEOREM

Another property of central force motion can be derived from the virial theorem. Virial theorem was first given by Rudolf Clausius in 1870, and owes its name to the word "vires," the Latin word for force. Virial theorem is applicable to a general gravitationally bound system and relates the gravitational potential energy of a system to the kinetic energy, providing an insight into the stability of the system. Of course, in a more general system of this sort, for even a particle in an elliptical orbit the kinetic and potential energy change with time. This is why, the virial theorem refers to time averages of the kinetic and potential energy. The fact that the average value of the kinetic energy in a bound system gives a measure of the potential energy forms the basis for measurements of the missing mass, or dark matter, in galaxies and in clusters of galaxies.

Consider a general system of interacting particles with position vectors $\boldsymbol{r}_i$ and the applied forces $\boldsymbol{F}_i$ including forces of constraint, if any. The equations of motion are given as

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i \tag{4.24}$$

Clausius assigned the name virial to the quantity denoted $\boldsymbol{G}$ and defined it as the sum of the scalar product of each particle momentum $\boldsymbol{p}_i(t)$ by its position $\boldsymbol{r}_i(t)$,

$$G = \sum_i \boldsymbol{p}_i . \boldsymbol{r}_i \tag{4.25}$$

Therefore,

$$\frac{dG}{dt} = \sum_i \dot{\boldsymbol{p}}_i . \boldsymbol{r}_i + \sum_i \boldsymbol{p}_i . \dot{\boldsymbol{r}}_i \tag{4.26}$$

From equation (4.24), the first term on the RHS is

$$\sum_i \dot{\boldsymbol{p}}_i . \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i . \boldsymbol{r}_i$$

while the second term becomes

$$\sum_i m\dot{\boldsymbol{r}}_i.\dot{\boldsymbol{r}}_i = \sum_i mv_i^2 = 2T$$

Therefore, equation (4.26) reduces to

$$\frac{dG}{dt} - \sum_i \boldsymbol{F}_i.\boldsymbol{r}_i + 2T \tag{4.27}$$

Since, the virial theorem is a statement about the time-averaged motion of a mechanical system, we next take the time average of equation (4.27) over a finite time interval τ,

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt = \overline{\sum_i \boldsymbol{F}_i.\boldsymbol{r}_i} + \overline{2T}$$

$$or \quad \frac{1}{\tau}[G(\tau) - G(0)] = \overline{\sum_i \boldsymbol{F}_i.\boldsymbol{r}_i} + \overline{2T} \tag{4.28}$$

Now, if the system executes motion in a finite region of space, and the velocities of all the particles remain finite, then the virial G is bounded and the quantity on the LHS vanishes because if the motion is periodic, then τ can be chosen as the period and even if the motion is not periodic, by choosing τ large enough, LHS can be made as small as desired and consequently it may be reduced to zero. In both the cases,

$$\bar{T} = -\frac{1}{2}\overline{\sum_i \boldsymbol{F}_i.\boldsymbol{r}_i} \tag{4.29}$$

Equation (4.29) is called the *virial theorem* and the quantity $-\frac{1}{2}\overline{\sum_i \boldsymbol{F}_i.\boldsymbol{r}_i}$ is known as the *Virial of Clausius*. The theorem in this form is very useful in the kinetic theory of gases, in particular for obtaining the equation of state of imperfect gases.

In case, the forces are derivable from a potential

$$\boldsymbol{F}_i = -\frac{\partial V}{\partial \boldsymbol{r}_i}$$

the theorem becomes

$$\bar{T} = \frac{1}{2}\overline{\sum_i \frac{\partial V}{\partial \boldsymbol{r}_i}.\boldsymbol{r}_i} \tag{4.30}$$

Further, if V is a homogenous function of degrees $n+1$ of the coordinates r_i, i.e. $V = ar^{n+1}$, where the exponent is chosen so that the force law is of the form r^n, then from Euler's theorem on homogenous functions

$$\sum_i \frac{\partial V}{\partial r_i}.r_i = (n+1)V$$

so that equation (4.30) becomes

$$\bar{T} - \frac{n+1}{2}\bar{V} \tag{4.31}$$

Now, since for a conservative system, the total energy of the system is a constant of motion and $E = \bar{E} = \bar{T} + \bar{V}$, we get

$$\bar{T} = \frac{(n+1)E}{(n+3)}, \qquad \bar{V} = \frac{2E}{(n+3)} \tag{4.32}$$

Virial theorem, thus enables us to express both $\bar{T}$ and $\bar{V}$ in terms of the total energy of the system.

For the special case of any bound system of particles interacting by means of an *inverse-square force* ($n = -2$), equation (4.31) states that the average kinetic energy is half the average (negative) potential energy, i.e.

$$\bar{T} = -\frac{1}{2}\bar{V} \tag{4.33}$$

Example 3: For an elliptical central force motion, in an inverse-square law attractive force, show that

$$\oint p_r dr + \oint p_\varphi d\varphi = \pi k \sqrt{-\frac{2\mu}{E}}$$

Solution:

$$V = -\frac{k}{r}$$

Therefore,

$$L = T - V = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2$$

So,

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}, \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2\dot{\varphi}$$

Therefore,

$$\oint p_r dr = \int_0^\tau p_r \frac{dr}{dt} dt = \int_0^\tau \frac{1}{m} p_r^2 dt, \qquad \oint p_\varphi d\varphi = \int_0^\tau p_\varphi \frac{d\varphi}{dt} dt = \int_0^\tau \frac{1}{mr^2} p_\varphi^2 dt$$

Thus,

$$\oint p_r dr + \oint p_\varphi d\varphi = \int_0^\tau \left(\frac{p_r^2}{m} + \frac{p_\varphi^2}{mr^2} \right) dt = \int_0^\tau 2T dt$$

$$= \tau \frac{\int_0^\tau 2T dt}{\int_0^\tau dt} = \tau \overline{2T} \tag{4.34}$$

where $\bar{T}$ denotes the usual time average of T. From the virial theorem for force law of the form r^n [equation (4.32)], we have

$$\bar{T} = \frac{(n+1)E}{(n+3)}$$

Thus, for the inverse-square law, i.e. $n = -2$, the average kinetic energy $\bar{T} = -E$ and equation (4.34) takes the form,

$$\oint p_r dr + \oint p_\varphi d\varphi = -2\tau E \tag{4.35}$$

Further, for the inverse-square force law, we have

$$\tau = 2\pi a^{3/2} \sqrt{\frac{\mu}{k}}, \qquad where\ a = -\frac{k}{2E}$$

Hence,

$$-2\tau E = \pi k \sqrt{-\frac{2\mu}{E}}$$

Self Assessment Question (SAQ) 1: Choose the correct option:

An artificial satellite revolves about the earth at a height H above the surface. The orbital period so that a man in the satellite will be in a state of weightlessness is

(a) $2\pi\sqrt{g/R}$

(b) $2\pi\sqrt{R/g}$

(c) $\sqrt{g}/2\pi\sqrt{R}$

(d) None of the above

Self Assessment Question (SAQ) 2: Choose the correct option:

ISRO's probe *Mangalyaan* was sent recently to explore the planet Mars. The inter-planetary part of the trajectory is approximately a half-ellipse with the Earth (at the time of launch), Sun and Mars (at the time the probe reaches the destination) forming the major axis.

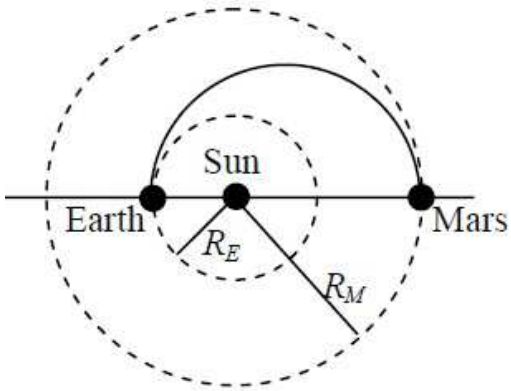


FIGURE 4.5

Assuming that the orbits of Earth and Mars are approximately circular with radii R_E and R_M, respectively (Figure 4.5), the velocity (with respect to the Sun) of the probe during its voyage when it is at a distance $r(R_E \ll r \ll R_M)$ from the Sun, neglecting the effect of Earth and Mars, is

(a) $\sqrt{2GM\dfrac{(R_E+R_M)}{r(R_E+R_M-r)}}$

(b) $\sqrt{2GM\dfrac{(R_E+R_M-r)}{r(R_E+R_M)}}$

(c) $\sqrt{2GM\dfrac{R_E}{rR_M}}$

(d) $\sqrt{\dfrac{2GM}{r}}$

Self Assessment Question (SAQ) 3: Choose the correct option:

A planet of mass m moves in the gravitational field of the sun (mass M). If the semi-major and the semi-minor axes of the orbit are a and b, respectively, the angular momentum of the planet is

(a) $\sqrt{2GMm^2(a+b)}$

(b) $\sqrt{2GMm^2(a-b)}$

(c) $\sqrt{2GMm^2ab/(a-b)}$

(d) $\sqrt{2GMm^2ab/(a+b)}$

Self Assessment Question (SAQ) 4: Choose the correct option:

A particle is moving in a spherically symmetric potential $V(r) = \alpha r^2$, where α is a positive constant. In a stationary state, the expectation value of the kinetic energy $\langle T \rangle$ of the particle is

(a) $\langle T \rangle = \langle V \rangle$ (b) $\langle T \rangle = 2\langle V \rangle$ (c) $\langle T \rangle = 3\langle V \rangle$ (d) $\langle T \rangle = 4\langle V \rangle$

4.7 SUMMARY

In this unit, we continued with the discussion of central force motions. We understood how the concepts that we learned in the previous unit can be applied to understand the idea of escpare velocity, geostationary and geosynchronous orbit and how a geostationary stationary satellite is launched into its orbit. We further learned about the Hohmann transfer orbit and interplanetary trajectories, for both the outer planets such as Mars, which is further away from the sun vis-avis the earth, and inner planets such as Venus, which are closer to the sun vis-à-vis the earth.

Finally, we ended our discussion with looking at the virial theorem, which is an important property of the central force motion and is applicable to a general gravitationally bound system and relates the gravitational potential energy of a system to the kinetic energy, providing an insight into the stability of the system.

4.8 GLOSSARY

Angular momentum – is the rotational equivalent of linear momentum. It is an important quantity in physics because it is a conserved quantity – the total angular momentum of a system remains constant unless acted on by an external torque. In three dimensions, the angular momentum for a point particle is a pseudovector $\mathbf{r}\times\mathbf{p}$, the cross product of the particle's position vector $\mathbf{r}$ (relative to some origin) and its momentum vector $\mathbf{p} = m\mathbf{v}$.

Angular velocity – the *angular velocity* of a particle is the time rate of change of its *angular* displacement relative to the origin. The SI unit of *angular velocity* is radians per second.

Central Force - is a force that points from the particle directly towards a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

Escape Velocity – is the minimum speed needed for an object to escape from the gravitational influence of a massive body. The escape velocity from Earth is about 11.186 km/s (6.951 mi/s; 40,270 km/h; 25,020 mph) at the surface.

G*eostationary* satellite – orbits the earth directly over the equator, approximately 22000 miles up. At this altitude, one complete trip around the earth (relative to the sun) takes 24 hours.

Geosynchronous satellite – A special case of geosynchronous satellite is the *geostationary* satellite, which has a *geostationary* orbit – a circular geosynchronous orbit directly above the Earth's equator.

G*eostationary* orbit – geostationary Earth orbit is a circular geosynchronous orbit 35,786 kilometres above the Earth's equator and following the direction of the Earth's rotation.

Geosynchronous orbit – is a high Earth orbit that allows satellites to match Earth's rotation. Located at 22,236 miles (35,786 kilometers) above Earth's equator, this position is a valuable spot for monitoring weather, communications and surveillance.

Motion – is a change in position of an object over time.

Kinetic Energy – energy which a body possesses by virtue of being in motion.

Orbit – is the gravitationally curved trajectory of an object, such as the trajectory of a planet around a star or a natural satellite around a planet.

Orbital Velocity – velocity sufficient to cause a natural or artificial satellite to remain in orbit. Inertia of the moving body tends to make it move on in a straight line, while gravitational force tends to pull it down. The orbital path, elliptical or circular, thus represents a balance between gravity and inertia. A cannon fired from a mountaintop will throw a projectile farther if its muzzle velocity is increased. If velocity is made high enough the projectile never falls to the ground. The surface of the Earth may be thought of as curving away from the projectile, or satellite, as fast as the latter falls toward it. The more massive the body at the centre of attraction, the higher is the orbital velocity for a particular altitude or distance. Near the surface of the Earth, if air resistance could be disregarded, orbital velocity would be about 8 km/s. The farther from the centre of attraction a satellite is, the weaker the gravitational force and the less velocity it needs to remain in orbit.

Planet – an astronomical body orbiting a star or stellar remnant that is massive enough to be rounded by its own gravity, and is not massive enough to cause thermonuclear fusion.

Potential Energy – the energy of a particle or system of particles derived from position, or condition, rather than motion.

Transfer Orbit – In *orbital* mechanics a *transfer orbit* is an intermediate elliptical *orbit* that is used to move a satellite or other object from one circular, or largely circular *orbit* to another. There are several types of *transfer orbits*, which vary in their energy efficiency and speed of *transfer*.

The Hohmann *transfer orbit* is an elliptical *orbit* used to *transfer* between two circular *orbits* of different radii in the same plane.

***Virial* Theorem** – The *virial* theorem states that, for a stable, self-gravitating, spherical distribution of equal mass objects (stars, galaxies, etc), the total kinetic energy of the objects is equal to minus 1/2 times the total gravitational potential energy.

4.9 TERMINAL QUESTIONS

1. State and prove the virial theorem.

2. Derive the equation for orbit of a particle moving under the influence of an inverse square central force field. Also calculate the time period of motion in elliptical orbit.

3. Fill in the blanks:

(i) The square of the period of revolution of the planet around the sun is proportional to the cube of the …….

(ii) If e is the eccentricity of the earth's orbit, the ratio of maximum and minimum speeds of the planet is ……

4. Show that earth's escape velocity is $11.2\ km/s$.

5. List down some of the uses of artificial satellites.

6. An artificial satellite is revolving around the earth at a distance of 620 km. Calculate the orbital velocity and the period of revolution. Radius of the earth is 6380 km and acceleration due to gravity at the surface of the earth is 9.8 m/s^2.

7. Choose the correct answer:
The radius of earth is approximately 6400 km. The height h at which the acceleration due to earth's gravity differs from g at the earth's surface by approximately 1% is

(a) $64\ km$ (b) $48\ km$ (c) $32\ km$ (d) $16\ km$

8. Calculate the height of an equatorial satellite which is always seen over the same point of earth's surface. (G = 6.66 x 10^{-11} SI units; Mass of earth M = 5.98 x 10^{24} kg)

9. Choose the correct answer:

A satellite is moving in a circular orbit around the Earth. If T, V and E are its average kinetic, average potential and total energies, respectively, then which one of the following options is correct?

(a) $V = -2T;\ E = -T$

(b) $V = -T, E = 0$

(c) $V = -E/2;\ E = T/2$

(d) $V = -3E/2;\ E = -T/2$

4.10 ANSWERS

Selected Self Assessment Questions (SAQs):

1. (b)

2. (b)

3. (d)

4. (a)

Selected Terminal Questions:

3. (i) semi-major axis of the ellipse

(ii) (1 + e)/(1 – e)

5. Artificial satellites are used in the following ways:

(i) Distant transmission of radio and TV signals.

(ii) To study upper regions of the atmosphere.

(iii) High altitude satellites for astronomical observations

(iv) Weather forecasting

(v) Earth measurements (gravitation and magnetic fields)

6. Radius of earth's satellite orbit r = R + h

= radius of earth + distance of satellite from the earth's surface

= 6380 + 620 = 7000 km

Therefore, the period of revolution is

$$T = \frac{2\pi r}{R}\sqrt{\frac{r}{g}}$$

$$= \frac{2\pi(7\times 10^6)}{6.38\times 10^6}\sqrt{\frac{(7\times 10^6)}{9.8}}$$

$$= 5775\ s$$

And the orbital velocity is

$$v = R\sqrt{\frac{g}{r}}$$

$$= 6.38\times 10^6\sqrt{\frac{9.8}{7\times 10^6}}$$

$$= 7550\ m/s$$

7. (c)

8. Let the height of the equatorial satellite by h. The equatorial satellite is seen over the same point of earth's surface, i.e. the angular velocity of satellite is the same as that of the earth itself.

Hence, the angular velocity of the satellite is

$$\omega = \frac{2\pi}{24\times 60\times 60} = 7.27\times 10^{-5}\ rad/s$$

Also,

$$\frac{GMm}{r^2} = m\omega^2 r$$

$$or \qquad r^3 = \frac{GM}{\omega^2} = 74.74\times 10^{21}$$

$$or \qquad r = 4.21\times 10^7\ m$$

Therefore, h = r – R = 3.57 x 10^4 km.

9. (a)

4.11 REFERENCES

4. Classical Mechanics, Herbert Goldstein, Charles Poole, John Safko – Pearson Education, New Delhi
5. Classical Mechanics – System of Particles and Hamiltonian Dynamics, Walter Greiner – Springer-Verlag, New York
6. Classical Mechanics, John R. Taylor – University Science Books, New York

4.12 SUGGESTED READINGS

4. Classical Mechanics, H. M. Agrawal – New Age International, New Delhi
5. Classical Mechanics – System of Particles and Hamiltonian Dynamics, Walter Greiner – Springer-Verlag, New York
6. Classical Mechanics, John R. Taylor – University Science Books, New York

UNIT 5 SCATTERING

Structure

5.1 Introduction

5.2 Objectives

5.3 Rutherford Alpha Scattering

5.4 Relationship between Scattering Angle and Impact Parameter

5.5 Scattering Cross-section

5.6 Two-body Elastic Collision

5.7 Summary

5.8 Glossary

5.9 Terminal Questions

5.10 Answers

5.11 References

5.12 Suggested Readings

5.1 INTRODUCTION

The *collision experiment,* or *scattering experiment,* is the single most powerful tool for investigating the structure of atomic and subatomic objects. In this type of experiment one fires a stream of projectiles, such as electrons or protons, at a target object—an atom or atomic nucleus, for example — and, by observing the distribution of "scattered" projectiles as they emerge from the collision, one can gain information about the target and its interactions with the projectile. Perhaps the most famous collision experiment was the discovery by Ernest Rutherford (1871-1937) of the structure of the atom: Rutherford and his assistants fired streams of *alpha* particles at a thin layer of gold atoms in a sheet of gold foil; by measuring the distribution of the scattered *alpha* particles, they were able to deduce that most of the mass of an atom is concentrated in a tiny, positively charged "nucleus" at the center of the atom. Since that time, most discoveries in atomic and subatomic physics (the discoveries of the neutron, of nuclear fission and fusion, of quarks, and many more) were made with the help of collision experiments, in which a stream of projectiles were directed at a suitable target and the outgoing particles carefully monitored.

You could imagine doing a scattering experiment with larger objects — scattering one billiard ball off another, or even a comet off the sun — but in these cases there are usually easier ways to find out about the target. Thus the main application of collision theory is at the atomic level and below. Since the correct mechanics for atomic and subatomic systems is quantum mechanics, this means that the most widely used form of collision theory is *quantum collision theory.* Nevertheless, many of the central ideas of quantum collision theory — total and differential scattering cross sections, lab and CM reference frames — already appear in the classical theory, which gives an excellent introduction to these ideas without the

complications of quantum theory. This, then, is the main purpose of this unit, to give an introduction to the main ideas of collision theory in the context of classical mechanics.

The scattering problem, in its one-body formulation, studies the scattering of particles by a center of force. Consider a uniform beam of particles all of the same mass and kinetic energy, incident upon a center of force. A particle in the beam will follow an *unbounded orbit*, since initially the particle is assumed to be at infinite distance from the center of force before it approaches it. Depending on the charge of the particle, it will either get attracted or repelled resulting in the deviation of its orbit from a straight-line trajectory. After passing the center of force, the force acting on the particle will in due course diminish and it will pick up the straight-line trajectory once again. In general, the final direction of the particle is not the same as its incident direction and the particle is said to be scattered.

We shall discuss the scattering problem using Rutherford's famous α-scattering experiment in which he bombarded very thin gold foil (~600 nm) with a stream of high energy α-particles from a radioactive source.

5.2 OBJECTIVES

After studying this unit, you should be able to

- explain what is meant by scattering angle and impact parameter and what is the relationship between the two
- describe the Rutherford alpha scattering experiment
- calculate the scattering cross-section and the differential scattering cross-section
- differentiate between the center of mass frame and the laboratory frame and describe scattering in both these frames

5.3 RUTHERFORD ALPHA SCATTERING

Rutherford's analysis of the observed α-scattering was based on the assumption that both the target nucleus contained in the thin gold foil and the incident α-particle, could be considered as point charges. The force was thus, taken to be Coulombic

$$F = \frac{1}{4\pi\epsilon_0}\frac{(ze)(Ze)}{r^2} \qquad (5.1)$$

where Ze is the charge on the nucleus and ze is the charge of the incident particle, which in our case is α-particle. Owing to the repulsive inverse-square type fundamental form of the interaction, the trajectory of the α-particle is a hyperbola with the nucleus at the outer focus (Figure 5.1). We assume that the nucleus is sufficiently massive to be not displaced by the encounter with the α-particle.

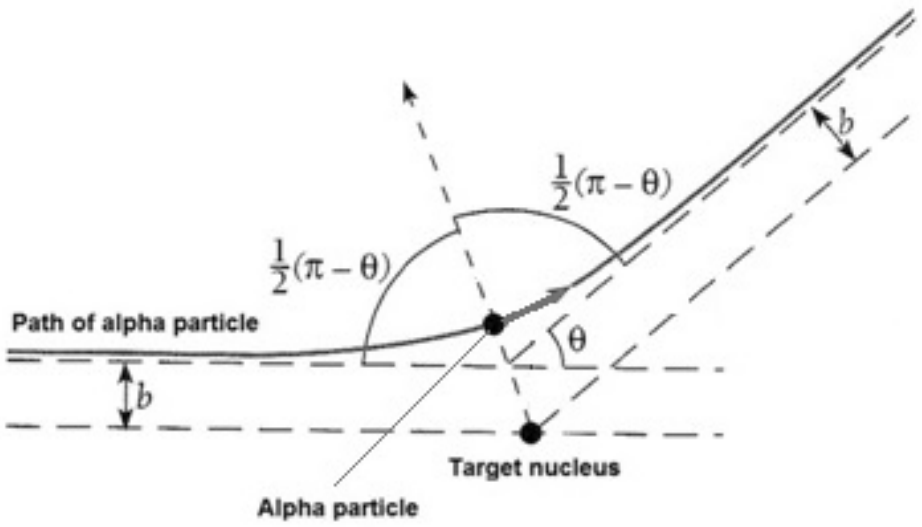


FIGURE 5.1 Relationship of orbit parameters and scattering angle in Rutherford alpha scattering.

The *impact parameter* b is the minimum distance at which the α-particle would approach the nucleus if there were no force between them. The Coulomb force exerted by the nucleus on the α-particle is along the radius vector r, basically a central force, therefore the angular momentum of the system must be conserved. So, if v is the incident particle speed, then

$$l = mvb = b\sqrt{2\mu E} \tag{5.2}$$

The scattering angle θ is the angle between the asymptotic incident direction of the α-particle and the asymptotic direction of the deflected α-particle. Since, the experiment is performed with an incident beam, instead of measuring θ for a single particle, we have to account for a number of particles scattered through various angles. Normally, detectors are placed at one or more scattering angles to count the particles scattered into small cones of solid angle subtended by the detectors. Thus, we consider the distribution of scattered particles and for that we define a quantity called the cross-section for scattering in a given direction as

$$\sigma(\Omega)d\Omega = \frac{number\ of\ particles\ scattered\ per\ unit\ time\ into\ solid\ angle\ d\Omega}{incident\ intensity} \tag{5.3}$$

where the incident intensity I gives the number of particle crossing unit area normal to the incident beam in unit time, $d\Omega$ is an element of solid angle in the direction Ω and $\sigma(\Omega)$ is called the differential scattering cross-section and has the dimensions of area.

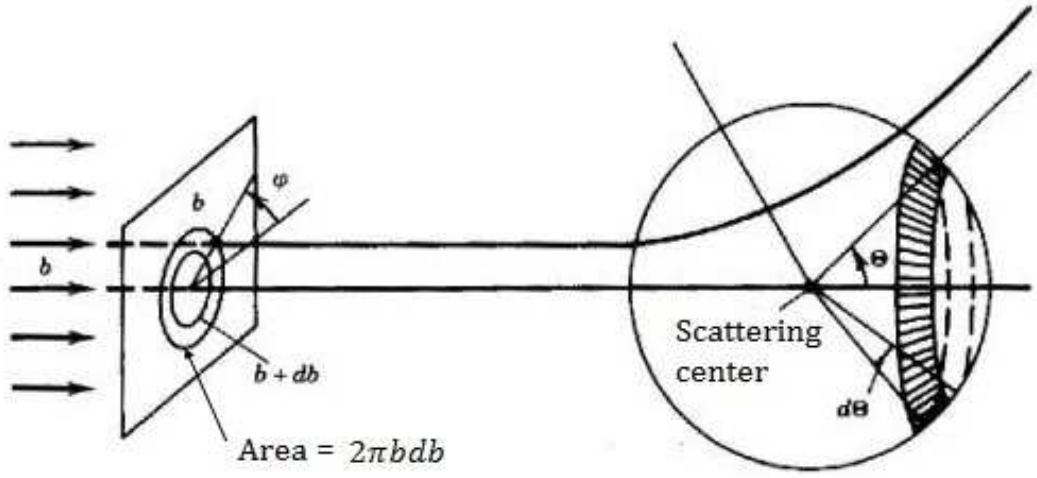


FIGURE 5.2 Detection of alpha particles scattered between θ and $\theta + d\theta$ in Rutherford experiment. A change in the impact factor has an associated change in the scattering angle.

For a central force, the distribution of particles does not depend on the azimuthal angle ψ because of the symmetry around the axis of the incident beam (Figure 5.2). Hence, the element of the solid angle can be written as

$$d\Omega = 2\pi \sin\theta \, d\theta \tag{5.4}$$

In an actual experiment, a detector is placed over a range of angles from θ to $\theta + \Delta\theta$ for different θ. The detector usually covers only a small angular range in φ because of symmetry about the beam axis. Thus, we need to find the number of particles scattered between θ and $\theta + d\theta$ that corresponds to the incident particles with impact parameters between b and $b + db$ as depicted in Figure 5.2. From equation (5.4), the number of particles scattered per unit time into solid angle $d\Omega$ is given as

$$N(\theta) = I\sigma(\theta)2\pi \sin\theta \, d\theta \tag{5.5}$$

The corresponding particles that pass through an area of $2\pi b db$ is given as

$$N(\theta) - I(2\pi b db) \tag{5.6}$$

From equations (5.5) and (5.6), the differential cross-section is given by

$$\sigma(\theta) - -\frac{b}{\sin\theta}\frac{db}{d\theta} \tag{5.7}$$

The minus sign in the equation indicates that as b increases θ decreases (Figure 5.1).

5.4 RELATIONSHIP BETWEEN SCATTERING ANGLE AND IMPACT PARAMETER

Comparing Coulomb force equation with the inverse-square force law

$$F = -\frac{k}{r^2}$$

we can write the value of the force constant as

$$k = -\frac{Zze^2}{4\pi\epsilon_0} \tag{5.8}$$

The eccentricity of the orbit is given by

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} = \sqrt{1 + \frac{2E\left(b\sqrt{2\mu E}\right)^2}{\mu\left(\frac{Zze^2}{4\pi\epsilon_0}\right)^2}}$$

$$or \qquad \epsilon = \sqrt{1 + \left(\frac{2Eb4\pi\epsilon_0}{Zze^2}\right)^2} \tag{5.9}$$

Since, $\left(\frac{2Eb4\pi\epsilon_0}{Zze^2}\right)^2 > 0$, the eccentricity $\epsilon > 0$ and hence, the orbit is a hyperbola (Figure 5.3).

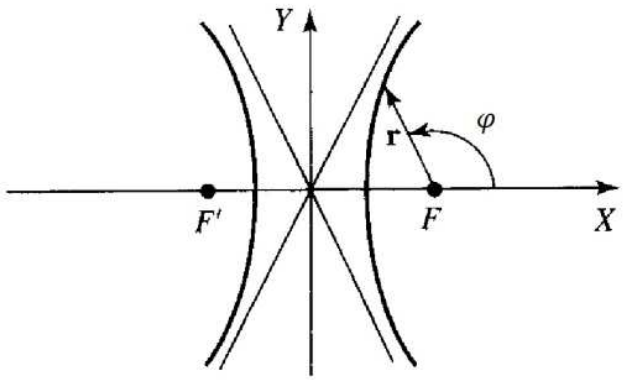


FIGURE 5.3 Shape of a hyperbolic orbit. F and F' are foci that represent the attractive and the repulsive force centers, respectively.

The shape of the orbit is given by the orbit equation,

$$\frac{1}{r} = \frac{\mu k}{l^2}\left[1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos(\varphi - \varphi')\right]$$

If φ' in the orbit equation is chosen to be π, then the periapis corresponds to $\varphi = 0$. And so the orbit equation for a hyperbola takes the form,

$$\frac{1}{r} = \frac{\mu k}{l^2}\left[1 - \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos\varphi\right] \qquad (5.10)$$

The hyperbolic path is always symmetric about the direction of the periapis (closest approach). As can be seen from Figure 5.1, the angle between the incoming asymptote and the direction of closest approach is given as

$$\varphi = \frac{1}{2}(\pi - \theta) \qquad (5.11)$$

The angle θ that the asymptotes make with the axes is obtained by substituting $r = \infty$ in the orbit equation and putting $\varphi - \theta$, since at infinite distance, $\varphi \to \theta$. That is, from equation (5.10),

$$1 - \epsilon\cos\varphi = 0$$

$$or \quad \cos\varphi = \frac{1}{\epsilon}$$

Therefore, using equation (5.11), we get

$$\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2} = \frac{1}{\epsilon}$$

$$or \quad \operatorname{cosec}^2\frac{\theta}{2} = 1 + \cot^2\frac{\theta}{2} = \epsilon^2$$

Using equation (5.9), therefore, we get the scattering angle and the impact factor as

$$\cot\frac{\theta}{2} = \frac{2Eb4\pi\epsilon_0}{Zze^2}$$

$$or \quad b = \frac{Zze^2}{8\pi\epsilon_0 E}\cot\frac{\theta}{2} \qquad (5.12)$$

This is the fundamental relation between the impact parameter b and the scattering angle θ, which shows that the scattering angle decreases as the impact factor increases, as depicted in Figure 5.4.

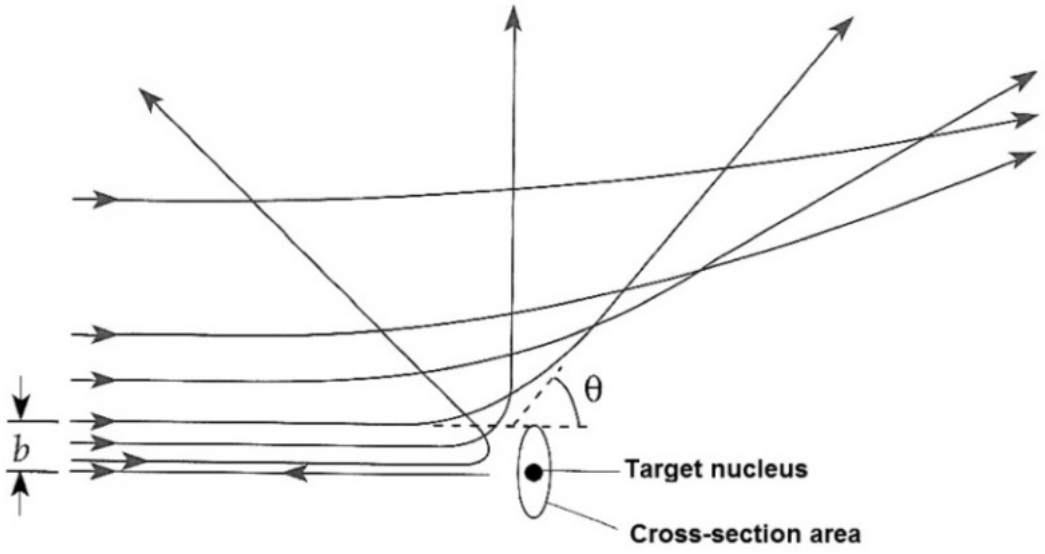


FIGURE 5.4 Scattering angle as a function of the impact parameter.

5.5 SCATTERING CROSS-SECTION

Differentiating equation (5.12) with respect to θ, we get

$$or \quad \frac{db}{d\vartheta} = \frac{Zze^2}{8\pi\epsilon_0 E}\text{cosec}^2\frac{\theta}{2} \tag{5.13}$$

Substituting b and $db/d\vartheta$ from equations (5.12) and (5.13) into equation (5.7), we get

$$\sigma(\theta) = \frac{1}{4}\left(\frac{Zze^2}{8\pi\epsilon_0 E}\right)^2 \text{cosec}^4\frac{\theta}{2} \tag{5.14}$$

This is the famous the Rutherford's scattering formula.

Quantum mechanics in the non-relativistic limit yields a result identical with the aforementioned classical prediction. Even when quantum-mechanical corrections are significant, it is often acceptable to use semi-classical approximation.

Example 1

Determine the differential scattering cross-section and the total scattering cross-section for the scattering of a particle by a rigid sphere of radius R.

Solution

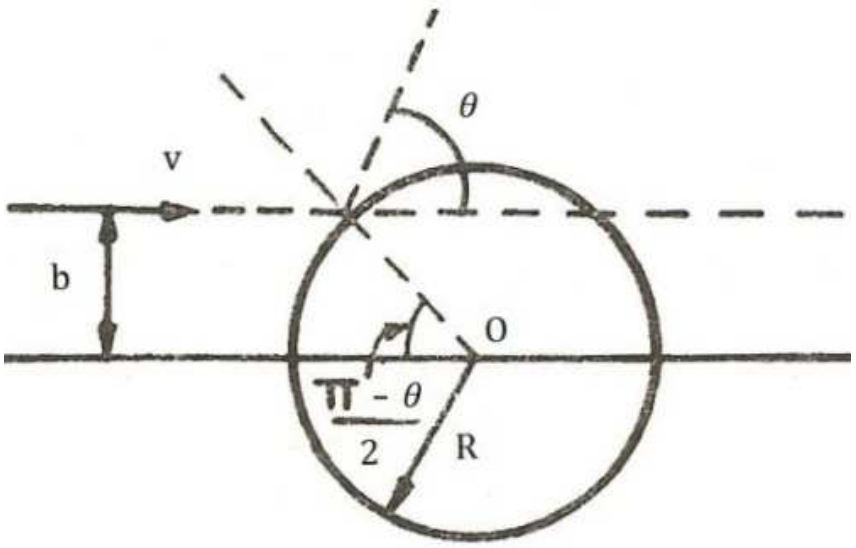


FIGURE 5.5 Scattering of a particle by a rigid sphere of radius R.

From Figure (5.5), we note that the impact parameter

$$b - R\sin\left[\frac{\pi-\theta}{2}\right] - R\cos\left(\frac{\vartheta}{2}\right)$$

Differentiating the above equation with respect to θ, we get

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$

Therefore, the differential cross-section is given as

$$\sigma(\theta) = \frac{b}{\sin\theta}\frac{db}{d\theta} = \frac{R\cos\left(\frac{\theta}{2}\right)}{\sin\theta}\left[\frac{R}{2}\sin\left(\frac{\theta}{2}\right)\right] = \frac{R^2}{4}$$

Hence, the total scattering cross-section is given as

$$\sigma_T = \int_0^{4\pi}\sigma(\Omega)d\Omega = \int_0^{2\pi}\int_0^{\pi}\sigma(\theta)\sin\vartheta\, d\theta d\psi$$

$$= 2\pi\int_0^{\pi}\sigma(\theta)\sin\theta\, d\vartheta = \pi R^2$$

Example 2

Show that for any repulsive central force, a formal solution for the angle of scattering can be expressed as

$$\varphi = \pi + \int_0^{u_0} \frac{p\,du}{\sqrt{1 - \frac{V}{E} - p^2u^2}}$$

where V is the potential energy, u = 1/r and u_0 corresponds to the turning point of the orbit. Write down the corresponding expression for φ for a force K/r^3.

Solution

For a central force,

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V = E$$

$$\Rightarrow \dot{r} = \left[\frac{2}{m}\left(E - V - \frac{1}{2}\frac{J^2}{mr^2}\right)\right]^{1/2}$$

But

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{J}{mr^2}$$

Therefore,

$$\frac{dr}{d\theta} - \frac{\dot{r}}{\frac{J}{mr^2}} - \frac{mr^2}{J}\left[\frac{2}{m}\left(E - V - \frac{1}{2}\frac{J^2}{mr^2}\right)\right]^{1/2}$$

$$or \quad d\theta = \frac{Jdr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{1}{2}\frac{J^2}{mr^2}\right)\right]^{1/2}}$$

$$or \quad \vartheta = \int \frac{Jdr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{1}{2}\frac{J^2}{mr^2}\right)\right]^{1/2}}$$

As $r = 1/u$, $dr = -du/u^2$. Integrating from 0 to u_0 (turning point), we obtain

$$\theta = -\int_0^{u_0} \frac{p\,du}{\left[1 - \frac{V}{E} - p^2u^2\right]^{1/2}}$$

where we have used $J = p(2mE)^{1/2}$.

Now,

$$\varphi = \pi \quad 2\theta = \pi \mid 2\int_0^{u_0} \frac{pdu}{\left[1 - \frac{V}{E} - p^2u^2\right]^{1/2}}$$

For $F = K/r^3$, $V = K/2r^2$ and then

$$\varphi = \pi + \int_0^{u_0} \frac{pdu}{\sqrt{1 - \frac{V}{E} - p^2u^2}}$$

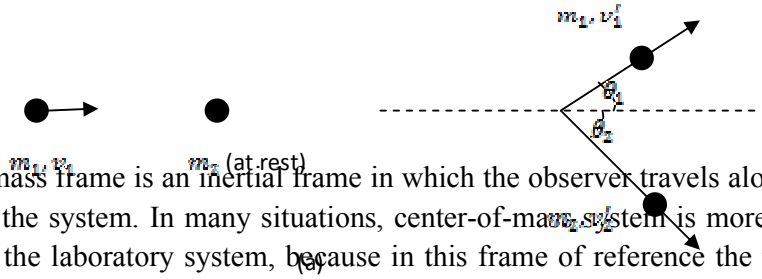
Self Assessment Question (SAQ) 1: Choose the correct option:

Rutherford's differential cross-section

(a) has the dimensions of area
(b) has the dimensions of solid angle
(c) is proportional to the square of the kinetic energy of the incident particle
(d) is inversely proportional to $\csc^4 \varphi/2$, where φ is the scattering angle

5.6 TWO-BODY ELASTIC COLLISION

In the previous section, we considered the one-body problem of the scattering of a particle by a fixed center of force. In practice, scattering always involves two bodies, where the second body is not fixed but recoils from its initial position as a result of the collision. Let us now look at the two-body elastic collision problem in both the center-of-mass and the laboratory frames of reference.



Center-of-mass frame is an inertial frame in which the observer travels along with the center-of-mass of the system. In many situations, center-of-mass system is more convenient to use rather than the laboratory system, because in this frame of reference the total momentum is zero and the collision of the reactants appear to undergo a head on collision and go off in opposite directions after collision.

The simplest problem that can be considered is a two-body collision. In Figure 5.6 a two-body collision problem is shown in *laboratory frame* and *center-of-mass frame*.

A particle of mass m_1 collides elastically with a particle of mass m_2 which initially is at rest in the laboratory frame of reference. The trajectory of m_1 is deflected through an angle θ_1 by

FIGURE 5.6 (a) Reaction dynamics of a binary nuclear reaction in laboratory frame, (b) in CM frame.

the collision. The scattering distribution is independent of the details of the interaction between the particles and is determined by the laws of conservation of energy and momentum.

We denote the initial velocities in the laboratory frame by v_1 and $v_2 = 0$, and the final velocities by v_1' and v_2'. The law of conservation of energy requires that in an elastic collision the total kinetic energy before and after the collision should be equal. Thus,

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

The law of conservation of momentum in the x-direction and the y-direction requires that

$$m_1v_1 = m_1v_1'\cos\theta_1 + m_2v_2'\cos\theta_2$$

$$0 = m_1v_1'\sin\theta_1 + m_2v_2'\sin\theta_2$$

Now, no doubt it's possible to solve the above three equations, nonetheless it is going to be tedious. It is far more convenient to view the collision in the center-of-mass frame. The position of the center-of-mass is defined by

$$R_{cm} = \frac{m_1r_1 + m_2r_2}{m_1 + m_2}$$

Differentiating with respect to time, we get

$$V_{cm} = \frac{m_1v_1}{m_1 + m_2}$$

We denote the initial velocities in the CM frame by u_1 and v_2, and the final velocities by u_1' and u_2'. We have the following relations between the velocities in the lab and CM frames:

$$v_1 = u_1 + V_{cm} \qquad v_2 = u_2 + V_{cm}$$

$$v_1' = u_1' + V_{cm} \qquad v_2' = u_2' + V_{cm}$$

Now, we know that the center-of-mass should remain at rest as no external force is acting and since the total momentum in this frame is zero, we don't need to write the momentum conservation equations. We can easily see that the energy is conserved when $u_1 = u_1'$ and $u_2 = u_2'$. The reaction dynamics in the CM frame is trivially simple and all scattering angles (θ) are allowed by the conservation laws. This is not true for $\theta_1$and $\theta_2$in the laboratory frame.

We can find the relation between θ and $\theta_1$by doing the following steps:

$$\tan\theta_1 = \frac{v_1'\sin\theta_1}{v_1'\cos\theta_1}$$

Since, the y-component of the final velocity of particle 1 is identical in the two frames. Also, $u_1 = u_1'$, therefore,

$$\tan\theta_1 = \frac{u_1' \sin\theta}{u_1' \cos\theta + V_{cm}} = \frac{\sin\theta}{\cos\theta + V_{cm}/u_1}$$

Also,

$$V_{cm} = \frac{m_1 v_1}{m_1 + m_2} = \frac{m_1(u_1 + V_{cm})}{m_1 + m_2}$$

Thus, we have

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

One can find that $0 < \theta < \pi$ when $m_2 > m_1$; $0 < \theta < \pi/2$ when $m_2 = m_1$; $0 < \theta < \theta_{max}$ for $m_2 < m_1$; where $\theta_{max} = \sin^{-1}(m_2/m_1)$.

Example 3

Assuming an α-particle scatters from an electron in the 600 nm gold foil, estimate the maximum scattering angle in a single encounter. [Given: Au mass number is 197 and its density is 19.3 g/cc]

Solution

Considering elastic collision between α-particle and e^- in the gold foil, the maximum momentum transfer occurs when the α-particle hits the e^- (at rest) head-on as shown in the figure below:

According to the conservation of linear momentum

$$M_\alpha \overrightarrow{v_\alpha} = M_\alpha \overrightarrow{v_\alpha'} + m_e \overrightarrow{v_e'}$$

Conservation of kinetic energy gives

$$\frac{1}{2} M_\alpha v_\alpha^2 = \frac{1}{2} M_\alpha (v_\alpha')^2 + \frac{1}{2} m_e (v_e')^2$$

An α-particle is an ionized helium atom (He^{++}), much more massive than the electron ($M_\alpha/m_e \approx 7000$). So, its velocity remains almost unchanged. On solving the above two equations, we get

$$v_\alpha' \cong v_\alpha \quad and \quad v_e' \cong 2v_\alpha$$

Thus, the maximum momentum change of the α-particle is

$$\Delta p_\alpha = M_\alpha v_\alpha - M_\alpha v_\alpha' = m_e v_e'$$

$$or \quad \Delta p_{max} = 2m_e v_\alpha$$

Based on the above vector diagram for α-particle momentum

$$\theta_{max} = \frac{\Delta p_\alpha}{p_\alpha} = \frac{length\ of\ arc}{radius}$$

$$= \frac{2m_e v_\alpha}{M_\alpha v_\alpha} = \frac{2m_e}{M_\alpha} = 2.7 * 10^{-4}\ rad = 0.016^0$$

Self Assessment Question (SAQ) 2: A particle of mass m_1 moving with velocity $\mathbf{u}_1$ is elastically scattered from another particle at rest. After the collision, the two particles move in the opposite direction with the same speed. Find the mass of the target.

Self Assessment Question (SAQ) 3: Choose the correct answer:

A bullet in motion hits and gets embedded in a solid block resting on a frictionless table. What is conserved?

(a) momentum and kinetic energy both
(b) momentum alone
(c) kinetic energy alone
(d) neither momentum nor kinetic energy.

5.7 SUMMARY

In this unit, we have studied the two-body collisions, first in the laboratory frame, which is a frame of reference centered on the laboratory in which the experiment (either real or thought experiment) is done, and then in the center-of-mass frame, which is a zero-momentum frame or CM frame of a system that is the unique inertial frame in which the total momentum of the system vanishes. We learned about the concepts of total cross-section and differential cross-section. We also studied the relationship between the impact parameter and the scattering angle.

5.8 GLOSSARY

Alpha Particle – a positively charged nuclear particle identical with the nucleus of a helium atom that consists of two protons and two neutrons.

Center of Mass – The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

Center-of-mass frame – the center-of-momentum frame (also zero-momentum frame or CM frame) of a system is the unique inertial frame in which the total momentum of the system vanishes. The *center of momentum* of a system is not a location (but a collection of relative momenta/velocities).

Central force –a force (possibly negative) that points from the particle directly towards a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

Coulomb's law or **Coulomb's inverse-square law** – is a law of physics that describes force interacting between static electrically charged particles. In its scalar form, the law is:

$$F - \frac{k_e q_1 q_2}{r^2}$$

where k_e is Coulomb's constant ($k_e = 8.99 \times 10^9$ N m^2 C^{-2}), q_1 and q_2 are the signed magnitudes of the charges, and the scalar r is the distance between the charges. The force of interaction between the charges is attractive if the charges have opposite signs (i.e., F is negative) and repulsive if like-signed (i.e., F is positive).

Cross-section – When two particles interact, their mutual cross section is the area transverse to their relative motion within which they must meet in order to scatter from each other. If the particles are hard inelastic spheres that interact only upon contact, their scattering cross section is related to their geometric size.

Elastic Scattering – a scattering of particles as the result of an elastic collision.

Impact Parameter – defined as the perpendicular distance between the path of a projectile and the center of a potential field created by an object that the projectile is approaching. It is often referred to in nuclear physics (see Rutherford scattering) and in classical mechanics.

Inelastic Scattering – a scattering of particles as the result of inelastic collision in which the total kinetic energy of the colliding particles changes.

Ion – atomic particle, atom, or a chemical radical bearing an electric charge, either positive or negative.

Kinematics – the branch of dynamics that studies the motion of a body or a system of bodies without consideration given to its mass or the forces acting on it.

Laboratory Frame – is a frame of reference centered on the laboratory in which the experiment (either real or thought experiment) is done. This is the reference frame in which the laboratory is at rest. Also, this is usually the frame of reference in which measurements are made, since they are presumed (unless stated otherwise) to be made by laboratory instruments. An example of instruments in a lab frame, would be the particle detectors at the detection facility of a particle accelerator.

Law of Conservation of Energy – Although the word energy does not possess an obvious definition, it is generally explained as the capacity to do work. Technically, energy is a scalar quantity associated with the state (or condition) of one or more objects. The principle that energy can be transformed from one type to another and transferred from one object to another, but the total amount is always the same, is called law of conservation of energy.

Law of Conservation of Linear Momentum – Conservation of linear momentum expresses the fact that a body or system of bodies in motion retains its total momentum, the product of mass and vector velocity, unless an external force is applied to it.

Linear Momentum – a property of a moving body that the body has by virtue of its mass and motion and that is equal to the product of the body's mass and velocity.

Scattering - a change in the direction of motion of a particle because of a collision with another particle.

5.9 TERMINAL QUESTIONS

1. What is differential scattering cross-section?

2. Discuss alpha scattering in Coulomb's field.

3. In Rutherford's scattering experiment 10^5 alpha particles are scattered at an angle of 2^0. Calculate the number of alpha particles scattered at an angle of 20^0.

4. What is meant by the laboratory system and the center-of-mass system in a two-body scattering problem?

5. Write short notes on:

   (i) Scattering Cross-section
   (ii) Impact Parameter
   (iii) Rutherford scattering

6. What is collision? What is the difference between elastic and inelastic collisions?
7. Choose the correct answer:

A body A strikes a body B at rest. If A and B have the same masses, after the collision:
(a) A and B move with the same velocities

(b) velocity of A is greater than that of B
(c) A is at rest while B moves
(d) none of the above

8. Choose the correct answer:

A completely inelastic collision is one in which the two colliding particles
(a) Are separated after collision
(b) Remain together after collision
(c) Split into small fragments flying in all directions
(d) None of the above

9. Which of the following statement(s) are true about collisions?
(a) Two colliding objects will exert equal forces upon each other even if their mass is significantly different.
(b) During a collision, an object always encounters an impulse and a change in momentum.
(c) During a collision, the impulse which an object experiences is equal to its velocity change.
(d) The velocity change of two respective objects involved in a collision will always be equal.
(e) While individual objects may change their velocity during a collision, the overall or total velocity of the colliding objects is conserved.
(f) In a collision, the two colliding objects could have different acceleration values.
(g) In a collision between two objects of identical mass, the acceleration values could be different.
(h) Total momentum is always conserved between any two objects involved in a collision.
(i) When a moving object collides with a stationary object of identical mass, the stationary object encounters the greater collision force.
(j) When a moving object collides with a stationary object of identical mass, the stationary object encounters the greater momentum change.
(k) A moving object collides with a stationary object; the stationary object has significantly less mass. The stationary object encounters the greater collision force.
(l) A moving object collides with a stationary object; the stationary object has significantly less mass. The stationary object encounters the greater momentum change.

5.10 ANSWERS

Selected Self Assessment Questions (SAQs):

1. (a)

2. Let the mass of the target be m_2. Let the particle and the target move with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, after the collision. Then,

$v_2 = -v_1$

As the collision is elastic, the laws of conservation of linear momentum and energy will be obeyed. The conservation of linear momentum gives

$m_1 u_1 = m_1 v_1 + m_2 v_2$

$= (m_1 - m_2) v_1$

$or \qquad v_1 - \frac{m_1}{m_1 - m_2} u_1$

For heavy target ($m_1 \ll m_2$), v_1 is opposite to u_1, i.e. the particle will be scattered backwards, while the target will move in the forward direction.

From the conservation of energy, we have

$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$or \qquad m_1 u_1^2 - (m_1 + m_2) v_1^2$

$= (m_1 + m_2) \left(\frac{m_1}{m_1 - m_2} \right)^2 u_1^2$

$or \qquad (m_1 - m_2)^2 = (m_1 + m_2) m_1$

$or \qquad m_1^2 + m_2^2 - 2 m_1 m_2 = m_1^2 + m_1 m_2$

Thus,

$m_2^2 = 3 m_1 m_2$

$or \qquad m_2 = 3 m_1$

3. (b)

Selected Terminal Questions:

3. The number of particles scattered per second at an angle φ is given by

$N_\varphi = C \csc^4 \varphi/2$

Therefore,

$$\frac{N_{20^0}}{N_{2^0}} = \frac{\csc^4 10^0}{\csc^4 1^0}$$

$$= \frac{\sin^4 1^0}{\sin^4 10^0}$$

For small angles, $\sin\theta \approx \theta$. Therefore,

$$\frac{N_{20^0}}{N_{2^0}} = \frac{1}{10^4}$$

$$or \quad N_{20^0} = \frac{10^5}{10^4}$$

$$or \quad N_{20^0} = 10$$

7. (c)

8. (b)

9. (a), (b), (f)

a. TRUE - In any collision between two objects, the colliding objects exert equal and opposite force upon each other. This is simply Newton's law of action-reaction.

b. TRUE - In a collision, there is a collision force which endures for some amount of time to cause an impulse. This impulse acts upon the object to change its momentum.

c. FALSE - The impulse encountered by an object is equal to mass multiplied by velocity change - that is, momentum change.

d. FALSE - Two colliding objects will only experience the same velocity change if they have the same mass and the collision occurs in an isolated system. However, their momentum changes will be equal if the system is isolated from external forces.

e. FALSE - This statement is mistaking the term velocity for momentum. It is momentum which is conserved by an isolated system of two or more objects.

f. TRUE - Two colliding objects will exert equal forces upon each other. If the objects have different masses, then these equal forces will produce different accelerations.

g. FALSE - It the colliding objects have different masses, the equal force which they exert upon each other will lead to different acceleration values for the two objects.

h. FALSE - Total momentum is conserved only if the collision can be considered isolated from the influence of net external forces.

i. FALSE - In any collision, the colliding objects exert equal and opposite forces upon each other as the result of the collision interaction. There are no exceptions to this rule.

j. FALSE - In any collision, the colliding objects will experience equal (and opposite) momentum changes, provided that the collision occurs in an isolated system.

k. FALSE - In any collision, the colliding objects exert equal and opposite forces upon each other as the result of the collision interaction. There are no exceptions to this rule.

l. FALSE - In any collision, the colliding objects will experience equal (and opposite) momentum changes, provided that the collision occurs in an isolated system.

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5.12 SUGGESTED READINGS

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UNIT 6 RIGID BODY DYNAMICS

Structure

6.1 Introduction

6.2 Objectives

6.3 Body and Space Reference Frames

6.3.1 Orthogonal Transformations

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6.4.1 Inertial Coefficients

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6.1 INTRODUCTION

In this chapter we develop the dynamics of a rigid body, one in which all inter-particle distances are fixed by internal forces of constraint. This is, of course, an idealization which ignores elastic and plastic deformations to which any real body is susceptible, but it is an excellent approximation for many situations, and vastly simplifies the dynamics of the very large number of constituent particles of which any macroscopic body is made. In fact, it reduces the problem to one with six degrees of freedom. While the ensuing motion can still be quite complex, it is tractable. In the process we will be dealing with a configuration space which is a group, and is not a Euclidean space.

6.2 OBJECTIVES

After studying this unit, you should be able to

- explain what is meant by degrees of freedom
- explain what are frames of reference, such as body and space frames of reference, or inertial and non-inertial frames of reference
- explain the orthogonality condition
- discuss Euler angles
- define moment of inertia and inertia tensor
- understand pseudo forces such as Coriolis force

6.3 BODY AND SPACE REFERENCE FRAMES

There are two coordinate systems that are generally used to describe the motion of a rigid body. A coordinate system, attached to the rigid body, is called a body-fixed frame and its axes are called the body set of axes. On the other hand, if the axes are fixed in the space

external to the body, the axes are called space set of axes, and such a coordinate system is called a space-fixed frame.

In Figure 6.1, *xyz* represents the space-fixed frame with origin O, while *x'y'z'* the body-fixed frame attached to the rigid body, with origin O' chosen to coincide with the center-of-mass of the rigid body. Let $\boldsymbol{R}$ be the radius vector from the origin of the frame *xyz* to the center-of-mass of the body. Obviously, 3 coordinates are needed to specify the origin O' of the body-fixed frame. Further, the orientation of the axes of this moving system, relative to the space-fixed system, is given by 3 angles. The 3 coordinates of O' and these 3 angles constitute 6 independent coordinates, which give complete configuration of the body at any given instant of time.

Next let us consider an arbitrary infinitesimal displacement of the rigid body, which is the sum of two parts:

I. An infinitesimal translation of the rigid body such that its center-of-mass moves from an initial position without any change in the orientation of the system *x'y'z'* with respect to the frame *xyz*.
II. An infinitesimal rotation about the center-of-mass O'.

If $\boldsymbol{R_p}$ is the position vector of an arbitrary point P in the rigid body with respect to the space-fixed frame *xyz* and $\boldsymbol{r}$ be its position vector with respect to the body-fixed frame *x'y'z',* then we can write the infinitesimal displacement of point P as

$$\boldsymbol{dR_p = dR + d\varphi \times r}$$

where $\boldsymbol{d\varphi}$ is the infinitesimal angle of rotation. Dividing both sides of the equation by time $\boldsymbol{dt}$ during which the infinitesimal displacement occurs, we get

$$\frac{\boldsymbol{dR_p}}{\boldsymbol{dt}} = \frac{\boldsymbol{dR}}{\boldsymbol{dt}} + \frac{\boldsymbol{d\varphi}}{\boldsymbol{dt}} \times \boldsymbol{r} \qquad (6.1)$$

FIGURE 6.1 Body-fixed frame x'y'z' and space-fixed reference frame xyz. P is an arbitrary point in the rigid body whose position vector with respect to the body-fixed frame is $\vec{r}$ and with respect to the space-fixed frame is $\overrightarrow{R_p}$. $\vec{R}$ is the position vector of the center-of-mass O' of the rigid body with respect to O, the origin of the space-fixed axes.

Here $\boldsymbol{dR/dt}$ is nothing but the instantaneous velocity of the center-of-mass with respect to the space-fixed frame, and $\boldsymbol{dR_p/dt}$ is the velocity of point P with respect to the space-fixed frame. $\boldsymbol{d\varphi/dt}$ is defined as the angular velocity $\boldsymbol{\omega}$ of the rotation of the body. Therefore, equation (6.1) can also be written as

$$\boldsymbol{V_p = V + \omega \times r} \qquad (6.2)$$

Instead of choosing the center-of-mass as the origin, even if some other point is chosen as the origin of the body-fixed frame, the translational velocity changes but the value of rotational velocity remains the same. So, we can conveniently choose O' (O' may be outside the body) such that it coincides with the origin of the space-fixed frame. In this case, the velocity $\boldsymbol{V} = 0$ so that the entire motion of the body at the given instant of time is a pure rotation about an axis passing through O'. It is obvious that when the rigid body rotates about a fixed point in space, it has only 3 DOFs.

There are many ways of specifying the orientation of the body-fixed axes *x'y'z'* with respect to the space-fixed axes *xyz* with common origin. A convenient way of doing this is to state the direction cosines. Direction cosines are the cosines of the angles between a line and the coordinate axes. Thus, the *x'* axis can be specified by its three direction cosines $\alpha_1, \alpha_2, \alpha_3$ with respect to the *x*, *y* and *z* axes, as follows:

$$\alpha_1 = \cos(\hat{i}', \hat{\imath}) = \hat{i}'.\hat{\imath} = \hat{\imath}.\hat{i}'$$

$$\alpha_2 = \cos(\hat{i}', \hat{\jmath}) = \hat{i}'.\hat{\jmath} = \hat{\jmath}.\hat{i}' \qquad (6.3)$$

$$\alpha_3 = \cos(\hat{i}', \hat{k}) = \hat{i}'.\hat{k} = \hat{k}.\hat{i}'$$

where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the three unit vectors along *x*, *y* and *z* respectively, while $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ are the three unit vectors along *x'*, *y'* and *z'* respectively. Thus, the vectors $\hat{i}'$ can be written as

$$\hat{i}' = \alpha_1\hat{\imath} + \alpha_2\hat{\jmath} + \alpha_3\hat{k} \qquad (6.4a)$$

Similarly, we can write $\hat{j}'$ and $\hat{k}'$ in terms of their respective direction cosines $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$,

$$\hat{j}' = \beta_1\hat{\imath} + \beta_2\hat{\jmath} + \beta_3\hat{k}$$

$$\hat{k}' = \gamma_1\hat{\imath} + \gamma_2\hat{\jmath} + \gamma_3\hat{k} \qquad (6.4b)$$

These set of 9 direction cosines then completely specify the orientation of the *x'*, *y'* and *z'* axes relative to the *x*, *y* and *z* axes. We may also invert the process and specify the *x*, *y* and *z* axes in terms of the *x'*, *y'* and *z'* axes. However, we must note that not all of the 9 direction cosines are independent, and relations between them arise. In fact only 3 are independent, because the unit vectors in the two Cartesian coordinate systems are orthogonal to each other and have a unit magnitude:

$$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$$

$$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1 \qquad (6.5)$$

with similar relations for $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$. Thus, we cannot take these 9 direction cosines as generalized coordinates to setup the Lagrangian equations of motion for the rigid body because they are not all independent. In order to have 3 generalized coordinates

corresponding to the 3 DOFs, we must use some set of 3 independent functions of the direction cosines. A number of such sets of 3 generalized coordinates are possible; the most commonly used being the Euler angles, introduced by Leonhard Euler to describe the orientation of a rigid body.

6.3.1 Orthogonal Transformations

Before, discussing the Euler angles, let us first derive the general orthogonality condition which should be satisfied for any orthogonal transformation, such as a transformation between 3-dimensional Cartesian coordinates as represented by equations (7.4). Equations (6.4) are a group of transformation equations that take us from a set of coordinates (*x*, *y*, *z)* to (*x'*, *y'*, *z')*, are an example of linear or vector transformation. In general, a vector transformation from a set of coordinates (x_1, x_2, x_3) to a new set (x_1', x_2', x_3') can be defined by the following equations:

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \qquad (6.6)$$

$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

where $a's$ are a set of coefficients. Equation (7.6) can be written, in short, as

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j \qquad i = 1,2,3. \qquad (6.7)$$

Relations between a_{ij}'s can be derived from very simple arguments. Both the primed and the unprimed coordinate systems are Cartesian, hence, the magnitude of a vector is given in terms of the sum of the squares of the components. Further, the magnitude of the vector is independent of the coordinate system used. This invariance of the magnitude can be written mathematically as

$$\sum_{i=1}^{3} x_i x_i = \sum_{i=1}^{3} x_i' x_i' \qquad (6.8)$$

Substituting x_i' from equation (6.7) into the above equation, we get

$$\sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3}\left[\sum_{j=1}^{3} a_{ij}x_j \sum_{k=1}^{3} a_{ik}x_k\right]$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\sum_{i=1}^{3} a_{ij}\,a_{ik}\right)x_j x_k$$

The above equality is satisfied, if and only if

$$\sum_{i=1}^{3} a_{ij}a_{ik} = 1 \quad when\ j = k$$
$$\sum_{i=1}^{3} a_{ij}a_{ik} = 0 \quad when\ j \neq k \qquad (6.9)$$

or in a more compact form,

$$\sum_{i-1}^{3} a_{ij}a_{ik} = \delta_{jk} \qquad j,k = 1,2,3 \qquad (6.10)$$

Equation (6.10) is known as the *orthogonality condition*, and any linear transformation that satisfies this condition is called an orthogonal transformation.

The array of transformation coefficients can be written as the elements of a matrix **A** called the matrix of transformation,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (6.11)$$

In order to see more clearly the usefulness of an orthogonal transformation, let us consider motion confined in a plane. Here we are restricted to 2-dimensional coordinate systems. Therefore, the matrix (7.11) simplifies to

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The 4 matrix elements are connected by 3 orthogonality conditions,

$$\sum_{i=1}^{2} a_{ij}a_{ik} = \delta_{jk} \qquad j,k = 1,2$$

It is therefore obvious, that only one independent rotation angle φ is needed to specify the transformation (Figure 7.2),

$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi$$

$$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$$

FIGURE 6.2 Rotation of the coordinate axes, as equivalent to two-dimensional orthogonal transformation.

So the matrix $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

The 3 orthogonality conditions can be expanded into equations, which then reduce to the mathematical identities, as follows:

$$a_{12}a_{11} + a_{21}a_{21} = 1 \rightarrow \cos^2\varphi + \sin^2\varphi = 1$$

$$a_{12}a_{12} + a_{22}a_{22} = 1 \rightarrow \sin^2\varphi + \cos^2\varphi = 1$$

$$a_{11}a_{12} + a_{21}a_{22} = 0 \rightarrow \cos\varphi \sin\varphi - \sin\varphi \cos\varphi = 0$$

The transformation matrix $\boldsymbol{A}$ can be thought of as an operator which acting on the unprimed system transforms it into the primed system. For a 3-dimensional transformation, we can write

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (6.12)$$

For any vector $\boldsymbol{r}$ and $\boldsymbol{r}'$, we may write

$$\boldsymbol{r}' = \boldsymbol{A}\boldsymbol{r} \qquad (6.13)$$

where both vectors are expressed in the same coordinate system.

Next, we consider two successive transformations transforming $\boldsymbol{r}$ to $\boldsymbol{r}'$ via transformation matrix $\boldsymbol{B}$

$$x_k' = \sum_j b_{kj} x_j \qquad (6.14a)$$

followed by a transformation from $\boldsymbol{r}'$ to $\boldsymbol{r}''$ via transformation matrix $\boldsymbol{A}$

$$x_i'' = \sum_k a_{ik} x_k' \qquad (6.14b)$$

Substituting x_k' from equation (6.14a) in equation (6.14b), we get

$$x_i'' = \sum_k a_{ik} \left(\sum_j b_{kj} x_j \right) = \sum_j \left(\sum_k a_{ik} b_{kj} \right) x_j = \sum_j c_{ij}\, x_j$$

where $c_{ij} = \sum_k a_{ik} b_{kj}$. Thus, we can see that the successive application of two orthogonal linear transformations is equivalent to a third linear transformation. The transformation matrix $\boldsymbol{C}$ of the latter is the product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} \qquad (6.15)$$

It can be shown that the transformation generated by the matrix $\boldsymbol{C}$ is also orthogonal if those due to $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal. It should be noted here that this matrix operation is not commutative,

$$\boldsymbol{BA \neq AB}$$

However, matrix multiplication is associative,

$$\boldsymbol{(AB)C = A(BC)}$$

Another useful concept is the inverse transformation, which transforms the primed coordinates back to the unprimed coordinates. Let the corresponding matrix be denoted by $\boldsymbol{A^{-1}}$ with matrix elements designated by a'_{ij}. Then the transformation equation is

$$x_i = \sum_j a'_{ij} x'_j \qquad (6.16)$$

Equation (6.16) should be consistent with equation (6.7). Therefore, we have

$$x_i = \sum_j a'_{ij} \left(\sum_k a_{jk} x_k \right)$$

$$= \sum_k \left(\sum_j a'_{ij} a_{jk} \right) x_k$$

This is true, if and only if

$$\sum_j a'_{ij} a_{jk} = 1 \quad when \ i = k$$

$$\sum_j a'_{ij} a_{jk} = 0 \quad when \ i \neq k \qquad (6.17)$$

or in a more compact form,

$$\sum_j a'_{ij} a_{jk} = \delta_{ik} \qquad (6.18)$$

In matrix notation, equation (6.18) can be written as

$$\boldsymbol{A^{-1}A = 1} \qquad (6.19)$$

where $\mathbf{1}$ is known as the identity matrix:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $\boldsymbol{A}^{-1}$ is the inverse matrix of $\boldsymbol{A}$. It is also easy to check that $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ commute, i.e.

$$\boldsymbol{A^{-1}A = AA^{-1} = 1}$$

The matrix obtained by interchanging the rows and the columns is known as the *transposed matrix* and is represented by $\tilde{\boldsymbol{A}}$. For orthogonal matrix,

$$\tilde{\boldsymbol{A}}\boldsymbol{A} = \boldsymbol{1} \qquad (6.20)$$

which is identical with the set of orthogonality conditions, equation (6.10), and can be verified by direct expansion. Another important property of orthogonal matrix is that its determinant can be described by the following condition

$$|\boldsymbol{A}|^2 = \boldsymbol{1} \qquad (6.21)$$

which implies that the determinant of an orthogonal matrix can only have the value $+1$ or -1. Orthogonal transformations with determinant $+1$ are called *proper*, while those with the determinant -1 are called *improper*. The transformations representing rigid body motion is restricted to matrices having the determinant $+1$.

6.3.2 Euler Angles

We can carry out the transformation from a given Cartesian coordinate system to another (in our case from space set of axes to body set of axes) by means of 3 successive rotations performed in a specific sequence. The most commonly used set of 3 angles are the Euler angles, typically denoted by φ, θ and ψ. First, the space set of axes is rotated counterclockwise through an angle φ, known as the *precession angle*, about the z-axis to produce (x_1, y_1, z_1) as shown in Figure 6.3a. Next, the system is rotated counterclockwise through an angle θ, known as the *nutation angle*, about x_1 axis to produce (x_2, y_2, z_2) as shown in Figure 6.3b. The x_2-axis is at the intersection of the *y*-*z* and y_2-z_2 planes and is known as the *line of nodes*. Finally, the system is rotated counterclockwise by an angle ψ, called the body angle, about z_2 axis to produce the desired system of (x', y', z') as shown in Figure 6.3c.

FIGURE 6.3 Rotations defining the Euler angles.

The elements of the complete transformation matrix $\boldsymbol{A}$ for transforming (x, y, z) to (x', y', z') can be obtained by writing $\boldsymbol{A}$ as the triple product of successive rotations.

$$\boldsymbol{A = BCD} \qquad (6.22)$$

The first rotation about the z-axis can be described by a matrix $\boldsymbol{D}$,

$$\boldsymbol{D} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.23a}$$

and the transformation equations can be written as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{6.23b}$$

Similarly, the $\boldsymbol{C}$ transformation corresponds to a rotation about y_1-axis, with the matrix given by

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{6.24a}$$

and the transformation equations can be written as

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{6.24b}$$

For the final rotation, the $\boldsymbol{B}$ matrix is given by

$$\boldsymbol{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.25a}$$

and the transformation equations can be written as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \tag{6.25b}$$

Therefore, the product matrix $\boldsymbol{A} = \boldsymbol{BCD}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \tag{6.26a}$$

and the transformation equations can be written as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (6.26b)$$

The inverse transformation from the body set of axes to the space axes is given by the $\boldsymbol{A^{-1}}$ matrix, which from equation (6.20) is equal to the transposed matrix $\tilde{\boldsymbol{A}}$, i.e.

$$\boldsymbol{A^{-1}} = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & \sin\theta\sin\varphi \\ \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & -\sin\theta\cos\varphi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix} \qquad (6.27)$$

6.4 ANGULAR MOMENTUM OF A RIGID BODY AND INERTIA TENSOR

Since, a rigid body can be approximated by a system of particles, if the origin of the body-fixed frame is chosen to be the center-of-mass, then the angular momentum of the body consists of two terms, one arising from the translation of the center-of-mass and the other from the rotation about the center-of-mass.

The total angular momentum about a reference point is given by

$$\boldsymbol{L} = \sum_i r_i \times (m_i v_i)$$

From equation (6.2), the velocity $\boldsymbol{v_i}$ *of the* i^{th} *particle can be written as*

$$\boldsymbol{v_i} = \boldsymbol{V} + \boldsymbol{\omega} \times r_i$$

For a rotating body, with its center-of-mass at rest, $\boldsymbol{V} = 0$, *therefore, the total angular momentum can be written as*

$$\boldsymbol{L} = \sum_i m_i r_i \times (\boldsymbol{\omega} \times r_i) \qquad (6.28)$$

By the vector identity, the vector triple product $\boldsymbol{A \times (B \times C) = (A.C)B - (A.B)C}$, *the above equation becomes*

$$\boldsymbol{L} = \sum_i m_i [\boldsymbol{\omega} r_i^2 - r_i (r_i . \boldsymbol{\omega})] \qquad (6.29)$$

All particles in the rigid body have the same angular velocity $\boldsymbol{\omega}$, so the result (6.29) can be written in terms of the 3 components of $\boldsymbol{\omega}$. If $\boldsymbol{L_x}$, $\boldsymbol{L_y}$ *and* $\boldsymbol{L_z}$ *are the components of angular momentum along the x, y and z axes, respectively, then*

$$L_x = \sum_i m_i \left[r_i^2 \omega_x - x_i (x_i \omega_x + y_i \omega_y + z_i \omega_z) \right]$$

$$or \qquad L_x = \omega_x \sum_i m_i (r_i^2 - x_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i \qquad (6.30a)$$

Similarly,

$$or \qquad L_y = -\omega_x \sum_i m_i y_i x_i + \omega_y \sum_i m_i (r_i^2 - y_i^2) - \omega_z \sum_i m_i y_i z_i \qquad (6.30b)$$

$$or \qquad L_z = -\omega_x \sum_i m_i z_i x_i - \omega_y \sum_i m_i z_i y_i + \omega_z \sum_i m_i (r_i^2 - z_i^2) \qquad (6.30c)$$

7.4.1 Inertial Coefficients

We see in the expression (6.30a) for L_x *involves 3 quantities:*

$$\sum_i m_i (r_i^2 - x_i^2), \qquad -\sum_i m_i x_i y_i, \qquad -\sum_i m_i x_i z_i$$

These quantities depend on the distribution of the mass in the body and on the instantaneous orientation of the body relative to the x, y, z coordinate axes. Thus, they will depend on time. We call these quantities inertial coefficients or moments of inertia:

$$I_{xx} = \sum_i m_i (r_i^2 - x_i^2)$$

$$I_{xy} = -\sum_i m_i x_i y_i \qquad (6.31)$$

$$I_{xz} = -\sum_i m_i x_i z_i$$

Similarly, for the inertial coefficients which enter into L_y *and* L_z*. We could then write the components of equation (6.30) in the following form:*

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \qquad (6.32)$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

We can see that for a rigid body of arbitrary shape and mass distribution, the angular momentum $\boldsymbol{L}$ *is not simply a scalar multiple of the angular velocity* $\boldsymbol{\omega}$. In other words, $\boldsymbol{L}$ *is not in general in the direction of* $\boldsymbol{\omega}$.

In matrix notation, equation (6.32) could be written as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad (6.33a)$$

$$or \qquad L = I\omega \qquad (6.33b)$$

The 9 inertial coefficients of the 3×3 matrix may be regarded as components of a single entity I, called the *inertia tensor*. Since, $I_{xy} = I_{yx}$ *etc.,* I *is a symmetric tensor.*

In Cartesian system, a tensor T *of* N^{th} *rank consists of* 3^N *components. A tensor of zero rank has one component, which is invariant under orthogonal transformation. Hence, a scalar is a tensor of zero rank. A tensor of first rank has 3 components and is completely equivalent to a vector. The inertia tensor is of second rank having* $3^2 = 9$ *components. The 9 components of the tensor of second rank transform under an orthogonal transformation of coordinates,*

$$T' = ATA \qquad (6.34)$$

where $\tilde{A}$ *is the transpose of matrix* A.

Until now, we took a rigid body to be composed of discrete particles. In case of a continuous body, the summation sign in the inertial coefficients is replaced by volume integration. If $\rho(r)$ *is the density at* r*, we can rewrite the inertial coefficients in the integral form as*

$$I_{xx} = \int \rho(r)(r^2 - x^2)dV, \qquad I_{xy} = -\int \rho(r)xydV, \qquad etc. \qquad (6.35)$$

where dV *is the volume element.*

6.5 ROTATIONAL KINETIC ENERGY

The kinetic energy of a rigid body referred to the center-of-mass at rest is called the rotational kinetic energy of the body. Let a rigid body be rotating about a fixed point in the body with an angular velocity ω. Then the rotational kinetic energy of the rigid body made of discrete particles is given by

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

$$= \frac{1}{2}\sum_i v_i.(m_i v_i)$$

Since $v_i = \boldsymbol{\omega} \times r_i$, *we get*

$$T = \frac{1}{2}\sum_i (\boldsymbol{\omega} \times r_i).(m_i v_i)$$

Using the vector identity, $(\boldsymbol{A} \times \boldsymbol{B}).\boldsymbol{C} = \boldsymbol{A}.(\boldsymbol{B} \times \boldsymbol{C})$, the above equation can be written as

$$T = \frac{1}{2}\sum_i \boldsymbol{\omega}.(r_i \times m_i v_i)$$

Since, the angular velocity $\boldsymbol{\omega}$ is the same for all the particles of the rigid body, we can take it out of the summation, i.e.

$$T = \frac{1}{2}\boldsymbol{\omega}.\sum_i r_i \times (m_i v_i)$$

Since, $L = \sum_i r_i \times (m_i v_i)$ *about a fixed point, the above equation can be written as*

$$T = \frac{1}{2}\boldsymbol{\omega}.\boldsymbol{L} \qquad (6.36)$$

The kinetic energy is a scalar quantity and can be written in the form

$$T = \frac{1}{2}\boldsymbol{\omega}.\boldsymbol{L} = \frac{1}{2}\omega_x L_x + \frac{1}{2}\omega_y L_y + \frac{1}{2}\omega_z L_z$$

$$= \frac{1}{2}\omega_x(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) + \frac{1}{2}\omega_y(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) + \frac{1}{2}\omega_z(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)$$

$$or \quad T = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2 + I_{xy}\omega_x\omega_y + I_{yz}\omega_y\omega_z + I_{xz}\omega_x\omega_z \qquad (6.37a)$$

In a more compact form, the rotational kinetic energy may be written as

$$T = \frac{1}{2}\sum_{j,k} I_{jk}\omega_i\omega_j \qquad (6.37b)$$

6.6 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA

If we conveniently choose the body-fixed axes in such a way that the off-diagonal coefficients vanish and only the diagonal coefficients remain in the inertia tensor, then such preferred set of axes are termed as the *principal axes*, or *eigenvectors of the inertia tensor*

and the corresponding moments of inertia as the *principal moments of inertia,* or *eigenvalues of the inertia tensor*. Thus, the diagonal form of the inertia tensor is written as

$$I_{diagonal} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{6.38}$$

where I_1, I_2 *and* I_3 *are the principal moments of inertia. If* ω_1, ω_2, ω_3 *are the components of the angular velocity and* L_1, L_2, L_3 *those of angular momentum about the principal axis, then equation (6.33) connecting* $\mathbf{L}$ *and* $\boldsymbol{\omega}$ *reduces to*

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{6.39}$$

or $L_1 = I_1\omega_1$, $L_2 = I_2\omega_2$ *and* $L_3 = I_3\omega_3$.

And so the rotational kinetic energy (6.37) may be written as

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 \tag{6.40}$$

In general, if the rotation is about one of the principal axes, say $\boldsymbol{\omega} = \omega_1\hat{\omega_1}$ *with* $\omega_2 = \omega_3 = 0$, *then* $L_1 = I_1\omega_1$, $L_2 = L_3 = 0$ *and* $L = I_1\boldsymbol{\omega}$. *Thus, we can say that for rotations about a principal axis, the angular momentum is co-linear with the angular velocity.*

The angular momentum $\mathbf{L}$ *and angular velocity* $\boldsymbol{\omega}$ *each will have 3 components along the axes of any arbitrary body-fixed coordinate system. We know that if the body is rotating about it principal axis, we have* $\mathbf{L} = I\boldsymbol{\omega}$, *i.e.*

$$L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} = I(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k})$$

But the general relation (6.33) also holds. Therefore, we have

$$L_x = I\omega_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I\omega_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I\omega_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

This is a system of equations that can be rewritten as

$$\begin{pmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \tag{6.41}$$

A trivial, but not useful solution exists when $\omega_x = \omega_y = \omega_z = 0$. *To get more useful and nontrivial solutions, the determinant of the system should be zero.*

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-I \end{vmatrix} = 0 \qquad (6.42)$$

This determinant leads to a polynomial of third order in I. *Equation (6.42) is a secular or characteristic equation, whose 3 roots give the desired principal moments. A body whose three principal moments of inertia are all different, i.e.* $I_1 \neq I_2 \neq I_3$, *is called an asymmetrical top. If any two are equal, then the body is called a symmetrical top, and if all three are equal, i.e.* $I_1 = I_2 = I_3$, *is called a spherical top.*

Now, the question arises – how are the principal axes found? There is no simple way of determining the principal axes for an asymmetrical top. For bodies that posses some symmetry, such as a symmetric top we can determine the principal axes by inspection. The principal axes are then the symmetry axis corresponding to the single root and any two perpendicular axes in the plane normal to the symmetry axis indicated by a double root to the secular equation, since all directions perpendicular to the axis of symmetry are alike.

If the principal axes can't be found by inspection, it is possible to find them by substituting a corresponding value of I, say $I = I_1$ *back into the equations (6.41). Three equations result for the components of* $\hat{a}$, *whose ratios give the direction of* $\hat{a}$ *or the direction cosines of the corresponding principal axis in whatever body-fixed coordinate system we are using , i.e.*

$$\hat{a} = \frac{\left(\hat{\imath} + \frac{\omega_y}{\omega_x}\hat{\jmath} + \frac{\omega_z}{\omega_x}\hat{k}\right)}{\sqrt{1 + \left(\frac{\omega_y}{\omega_x}\right)^2 + \left(\frac{\omega_z}{\omega_x}\right)^2}} \qquad (6.43)$$

Hence, we can determine the direction of *the principal axis corresponding to* I_1. *Similarly, if we substitute* I_2 *or* I_3, *we may find the direction of the corresponding principal axis by using equations (6.41). The magnitude of the angular velocity is arbitrary and we are free to take any value for it.*

6.7 EULER'S EQUATIONS OF MOTION FOR A RIGID BODY

The equation of motion of a rotating body is given by

$$\left(\frac{dL}{dt}\right)_{inertial} = \tau \qquad (6.50)$$

The above equation holds in an inertial reference frame.

The body-coordinate system rotates with an angular velocity ω. The time derivative of angular momentum in the body-fixed rotating reference frame and the space-fixed inertial frame are related as

$$\left(\frac{d\boldsymbol{L}}{dt}\right)_{inertial} = \frac{d\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L} \qquad (6.51)$$

Therefore,

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L} \qquad (6.52)$$

where $\boldsymbol{L}$ is observed in the rotating frame.

We suppose that the Cartesian axes in the rotating frame lie along the principal axes 1, 2, 3. *From equations (6.39), we have* $L_1 = I_1\omega_1$, $L_2 = I_2\omega_2$ *and* $L_3 = I_3\omega_3$. Therefore, the component of (6.52) along axis 1 is given by

$$\left(\frac{d\boldsymbol{L}}{dt}\right)_1 + (\boldsymbol{\omega} \times \boldsymbol{L})_1 = \frac{dL_1}{dt} + \omega_2 L_3 - \omega_3 L_2$$

$$= I_1\frac{d\omega_1}{dt} + \omega_2 I_3\omega_3 - \omega_3 I_2\omega_2 = \tau_1$$

On rearranging the above equation and writing out the equations for the components of (6.52) along axes 2 and 3, we get

$$I_1\frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 = \tau_1$$

$$I_2\frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 = \tau_2 \qquad (6.53)$$

$$I_3\frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2 = \tau_3$$

These equations are known as the Euler equations for the motion of a rigid body with one point fixed. They are a good starting point for rotating-body problems.

6.8 ROTATING FRAMES

An inertial frame of reference also known as Galilean reference frame is a reference frame that describes time and space homogeneously, isotropically and in a time-independent manner. All inertial frames have zero acceleration and are in a state of constant, rectilinear motion with respect to one another.

The accelerated frames are called as non-inertial frames because in such a frame, a free particle will seem to have an acceleration. A coordinate system attached to the earth is an example of a non-inertial reference frame because the earth rotates and is accelerated with respect to the sun.

FIGURE 6.4 Cartesian axes of an inertial frame (S) and a uniformly rotating frame (S') about their common z-axis and origin with an angular speed ω with respect to S.

Consider a frame S'(x', y', z') rotating with an angular velocity $\boldsymbol{\omega} = (0,0,\omega)$ about the z-axis relative to an inertial frame S(x, y, z). For simplicity, we assume that both the coordinate systems have a common origin O and a common z-axis (Figure 8.1).

The relationship between the coordinates of the two frames, $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{r'} = (x', y', z')$ is as follows:

$$x' = x\cos\omega t + y\sin\omega t \tag{6.54}$$

$$y' = y\cos\omega t - x\sin\omega t \tag{6.55}$$

$$z' - z \tag{6.56}$$

Now, for a free particle of mass *m* in frame S, the Lagrangian is given by

$$L = \frac{1}{2}m\dot{r}^2 \tag{6.57}$$

We substitute the expressions (6.54) – (6.56) into the Lagrangian (6.57) to find L in terms of the rotating coordinates,

$$L = \frac{1}{2}m[(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}^2] = \frac{1}{2}m(\dot{r}' + \boldsymbol{\omega}\times\boldsymbol{r}')^2 \tag{6.58}$$

To construct the Lagrangian equations of motion in the rotating frame, taking the derivatives of equation (6.58), we get

$$\frac{\partial L}{\partial \boldsymbol{r}'} = m(\boldsymbol{r}'\times\boldsymbol{\omega}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}'}\right) = m(\ddot{\boldsymbol{r}}' + \boldsymbol{\omega}\times\dot{\boldsymbol{r}}')$$

Therefore, the Lagrange's equation in the rotating frame becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}'}\right) - \frac{\partial L}{\partial \boldsymbol{r}'} = m[\ddot{\boldsymbol{r}}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') + 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}'] = 0$$

$$or \qquad m\ddot{\boldsymbol{r}}' = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') - 2m\boldsymbol{\omega}\times\dot{\boldsymbol{r}}' \tag{6.59}$$

The term on the LHS corresponds to the product of mass and the inertial acceleration, that is, the actual external forces applied to the system. The first and the second terms on the RHS are recognized as the *centrifugal* and the *Coriolis* forces, respectively. Both these forces are examples of *pseudo* or *fictitious forces* that we have seen earlier and are a consequence of the

reference frame, rather than any actual interaction. Thus, the particle moves in the rotating frame in accordance with the Newton's laws provided we add the two fictitious forces.

6.9 SUMMARY

In this unit, we studied the rigid body dynamics. We began with the discussion of body and space reference frames. This was followed by talking about orthogonal transformations and Euler Angles. Then we derived the angular momentum of a rigid body and the inertia tensor, and looked at the inertial coefficients.

We derived the equation for the rotational kinetic energy of a rigid body and learned about the principal axes and the principal moments of inertia. We then went on to study the Euler's equation of motion for a rigid body before ending our discussion with understanding the difference between an inertial reference frame and a non-inertial frame of reference. Rotating frame is an example of non-inertial frame. Therefore, a particle moves in the rotating frame in accordance with the Newton's laws provided we add the two fictitious forces – Coriolis force and the centrifugal force.

6.10 GLOSSARY

Angular Momentum – The angular momentum of a rigid object is defined as the product of the moment of inertia and the angular velocity. It is analogous to linear momentum and is subject to the fundamental constraints of the conservation of angular momentum principle if there is no external torque on the object.

Coriolis effect – an effect whereby a mass moving in a rotating system experiences a force (the *Coriolis force*) acting perpendicular to the direction of motion and to the axis of rotation. On the earth, the effect tends to deflect moving objects to the right in the northern hemisphere and to the left in the southern and is important in the formation of cyclonic weather systems.

Euclidean Space - Euclidean space encompasses the two-dimensional Euclidean plane, the three-dimensional space of Euclidean geometry, and certain other spaces. It is named after the Ancient Greek mathematician Euclid of Alexandria.

Euler Angles – The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system. They can also represent the orientation of a mobile frame of reference in physics or the orientation of a general basis in 3-dimensional linear algebra.

Kinetic energy – energy which a body possesses by virtue of being in motion.

Rotational energy or angular kinetic energy is kinetic energy due to the rotation of an object and is part of its total kinetic energy.

Moment of Inertia – The moment of inertia, otherwise known as the angular mass or rotational inertia, of a rigid body is a tensor that determines the torque needed for a desired angular acceleration about a rotational axis.

Pseudo Force – A Pseudo force (also called as fictitious force) is an apparent force that acts on all masses whose motion is described using a non-inertial frame of reference frame, such as rotating reference frame.

Reference Frame – A frame of reference (or reference frame) consists of an abstract coordinate system and the set of physical reference points that uniquely fix (locate and orient) the coordinate system and standardize measurements.

Rigid Body – a rigid body is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it.

Tensor – Tensors, defined mathematically, are simply arrays of numbers, or functions, that transform according to certain rules under a change of coordinates. In physics, tensors characterize the properties of a physical system.

A tensor may be defined at a single point or collection of isolated points of space (or space-time), or it may be defined over a continuum of points. In the latter case, the elements of the tensor are functions of position and the tensor forms what is called a tensor field. This just means that the tensor is defined at every point within a region of space (or space-time), rather than just at a point, or collection of isolated points.

A tensor may consist of a single number, in which case it is referred to as a tensor of order zero, or simply a scalar.

6.11 TERMINAL QUESTIONS

1. Define Euler's angles and obtain an expression for the complete transformation matrix.

2. What do you mean by inertia tensor? Give its physical significance.

3. Explain what do you understand by principal axes and the principal moments of inertia.

4. Derive an expression for the rotational kinetic energy of a rigid body.

5. What are body and space coordinate systems in relation to the motion of a rigid body?

6. How many generalized coordinates are needed to specify the motion of a rigid body?

7. Choose the correct answer:

For a rigid body, angular momentum vector and the angular velocity vector are

(a) Always in the same direction
(b) Always in different directions
(c) Not always in the same direction
(d) None of the above

8. What do you mean by non-inertial frame of reference?

9. What are fictitious forces? Define centrifugal force.

10. Differentiate between real and fictitious forces. What is Coriolis force?

11. Choose the correct option:

Non-inertial frames are

(a) accelerated frames
(b) unaccelerated frames
(c) are those frames in which a force-free particle moves with constant velocity
(d) cannot be rotating frames

12. Choose the correct option:

A particle is at rest in a rotating frame. The pseudo force acting on the particle in the rotating frame is

(a) Zero
(b) Only the centrifugal force
(c) Only the Coriolis force
(d) Both centrifugal and the Coriolis forces

13. Choose the correct option:

In the most general case, which one of the following quantities is NOT a second order tensor?

(a) Stress (b) Strain (c) Moment of inertia (d) Pressure

14. Choose the correct option:

An annulus of mass *M* made of a material of uniform density has inner and outer radii *a* and *b*, respectively. Its principle moment of inertia along the axis of symmetry perpendicular to the plane of the annulus is:

(a) $\frac{1}{2}M\frac{(b^4+a^4)}{(b^2-a^2)}$

(b) $\frac{1}{2}M\pi(b^2 - a^2)$

(c) $\frac{1}{2}M(b^2 - a^2)$

(d) $\frac{1}{2}M(b^2 + a^2)$

6.12 ANSWERS

Selected Terminal Questions:

6. Six

7. (c)

11. (a)

12. (b)

13. (b)

14. (d)

6.13 REFERENCES

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6.14 SUGGESTED READINGS

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UNIT 7: Hamilton's Equations of Motion

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7.1. Objectives

After finishing this chapter students will be able to know

- The Hamilton's equation of motion,
- Concept of the cyclic coordinates and conservation theorems,
- Routh's procedure
- cyclic coordinates and association of these coordinates with conserved quantities.
- To solve the numerical problems based on the Hamiltonian formulation

7.2. Introduction

There are three major formulations of classical mechanics; Newtonian formulation, Lagrangian formulation and the Hamiltonian formulation. The Hamiltonian formulation is the further development of Lagrangian formulation. In which n 2^{nd} order Lagrange Equations of motion are replaced by the 2n 1^{st} order time dependent equations of motion.

In further section you will learn to derive these Hamilton's equations As you have understood the Newtonian and Lagrangian formulations. In further discussion you will be familier with the Hamiltonian formulations and the Hamilton's equations of motion which is the heart of whole formulation.

Newtonian and Lagrangian formulations are naturally associated with configuration space, while the Hamiltonian formulation is the natural description for working in phase space. Hamilton's approach arose in 1835 in his unification of the language of optics and mechanics.

7.3.Hamilton's Equations of Motion:-Hamilton's equations of motion can be derived in the following two different ways:

(1) From the Hamiltonian of the system
(2) From the variational principle

7.3.1. From the Hamiltonian of the system let us first derive the Hamilton's equations from the Hamiltonian of the system. Generally Hamiltonian is the function of generalized coordinates, generalized momenta and time, i.e., H=H(q,p,t). And relation between the Lagrangian, L and the Hamiltonian, H of a system is defined as

$$H=\sum_k p_k \dot{q}_k - L(q,\dot{q},t) \quad (7.1)$$

Where q stands for $q_1, q_2, \ldots, q_k, .., q_n$

Differentiating equation(7.1), we have

$$dH=\sum_k [p_k d\dot{q}_k + \dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k] - \frac{\partial L}{\partial t} dt \quad (7.2)$$

now as you know that $p_k = \frac{\partial L}{\partial \dot{q}_k}$ when we replace the value of p_k in the first term of above equation then the first and fourth terms on the right side within the square bracket will together vanishes. Hence we a new expression as;

$$dH=\sum_k [\dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k] - \frac{\partial L}{\partial t} dt \quad (7.3)$$

Again, H=H(q,p,t)

Taking the differential of H

$$dH=\sum_k (\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k) + \frac{\partial H}{\partial t} dt \quad (7.4)$$

now comparing equations, (7.3) and (7.4), we get

$$\dot{q}_k - \frac{\partial H}{\partial p_k}, \quad k=1,2,\ldots., n \quad (7.5)$$

$$\dot{p}_k = \frac{\partial H}{\partial q_k}, \quad k = 1,2,3, \ldots, n \quad (7.6)$$

And $$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (7.7)$$

Equations (7.5) and (7.6) are called Hamilton's equations of motion. They are also called the canonical equations of motion. They constitute a set of 2n first order differential equations replacing the n second order Lagrange differential equations.

Hamilton's equations are applicable to holonomic conservative systems.

7.3.2.From the variational principle

There are so many variational principles exist in theoretical physics. Hamilton's variational principle can be treated as special case of general formulation of variational principles and can be stated as ,

$$\delta I = \delta \int_{t_1}^{t_2} L(q,\dot{q},t) dt = 0 \quad (7.8)$$

Here $q_k(t)$ and hence $\dot{q}_k(t)$ is to be varied such that

$$\delta q_k(t_1) = \delta q_k(t_2)=0 \quad (7.9)$$

Which refers to paths in configuration space. In Hamilton's formalism, the integral I has to be evaluated over the trajectory of the system point in phase space, and the varied paths must be in the neighbourhood of this phase space trajectories, we have to express the integrand of the integral I as a function of the independent coordinates p an q and their time derivatives. This can be achieved only by replacing L in equation (7.9) by the following expression

$$L(q,\dot{q},t)=\sum_k p_k \dot{q}_k - H(q,p,t) \quad (7.10)$$

So from equation(7.8), (7.9) and (7.10) we have,

$$\delta S = \delta \int_{t_1}^{t_2} [\sum_k p_k \dot{q}_k - H(q, p, t)]\, dt = 0 \qquad (7.11)$$

Where q(t) is varied subject to $\delta q_k(t_1) = \delta q_k(t_2) = 0$. And $p_k(t)$ is varied without any end-point reaction . Since the original variational principle is modified to suit phase space, it is known as **modified Hamoilton's principle**.

Carrying out the variations in equation (11) we have

$$\int_{t_1}^{t_2} [\sum_k p_k \delta\dot{q}_k + \dot{q}_k \delta p_k - \frac{\partial H}{\partial q_k}\delta q_k - \frac{\partial H}{\partial p_k}\delta p_k]\, dt = 0 \qquad (7.12)$$

Now manipulating and then integrating the first term in above equation we have

$$\int_{t_1}^{t_2} p_k \delta\dot{q}_k dt = \int_{t_1}^{t_2} p_k \delta \frac{d}{dt} q_k dt = \int_{t_1}^{t_2} p_k \frac{d}{dt}\delta q_k dt$$

$$= [p_k \delta q_k]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k \delta q_k dt$$

Now the term $[p_k \delta q_k]_{t_1}^{t_2}$ vanishes at the end-points t_1 and t_2 therefore,

$$\int_{t_1}^{t_2} p_k \delta\dot{q}_k dt = -\int_{t_1}^{t_2} \dot{p}_k \delta q_k dt \qquad (7.13)$$

Substituting equation (7.13) in equation (7.12), we get the following expression;

$$\int_{t_1}^{t_2} [\sum_k \{-\left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right)\delta q_k\} + (\dot{q}_k - \frac{\partial H}{\partial p_k})\delta p_k]\, dt = 0 \qquad (7.14)$$

Since the modified Hamilton's principle is a variational principle in phase space, δq's and δp's are the arbitrary and therefore the coefficients of δq_k and δp_k in equation (7.14) must vanish separately. Hence we get the following two equations;

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad (7.15)$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad (7.16)$$

(where k=1,2,...., n)

Thus, Hamilton's principle gives an independent method for obtaining Hamilton's equations of motion without a prior Lagrangian formulation.

7.4.Hamilton's equations in different coordinate systems

7.4.1. **In Cartesian coordinates**-

The kinetic energy of the particle $T=\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

Potential energy of the particle, V=V(x,y,z)

Lagrangian, $L=T-V=\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$- V(x,y,z)

Generalized momentum, $p_k = \frac{\partial L}{\partial \dot{q}_k}$

Hence, $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

Or $\dot{x} = \frac{p_x}{m}$, similarly, $\dot{y} = \frac{p_y}{m}$, $\dot{z} = \frac{p_z}{m}$

Hamiltonian, H=$\sum_k p_k \dot{q}_k - L$

For,k=x,y,z

H=$p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$+ V(x,y,z)

H= $\frac{p_x^2}{m} + \frac{p_x^2}{m} + \frac{p_x^2}{m} - \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x,y,z)$

H=$\frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x,y,z)$ (7.17)

Hamilton's equations are $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

The Hamilton's equations in Cartesian are (using equations (7.17),

$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$, and $\dot{p}_x = -\frac{\partial H}{\partial x}$ => $\dot{p}_x = -\frac{\partial V}{\partial x}$

$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$, and $\dot{p}_y = -\frac{\partial H}{\partial y}$ => $\dot{p}_y = -\frac{\partial V}{\partial y}$

$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$, and $\dot{p}_z = -\frac{\partial H}{\partial z}$ => $\dot{p}_z = -\frac{\partial V}{\partial z}$

Now since p=m$\ddot{x}$, Hence the equations of motion, will be

$m\ddot{x} = -\frac{\partial V}{\partial x}$ (7.18)

$m\ddot{y} = -\frac{\partial V}{\partial y}$ (7.19)

$m\ddot{z} = = -\frac{\partial V}{\partial z}$ (7.20)

7.4.2. In polar coordinates: If r and θ are polar coordinates of a particle of mass m, then

x=rcos θ, and y=rsin θ, therefore, $\dot{x}$=$\dot{r}$cos θ-r$\dot{\theta}$sin θ and $\dot{y}$=$\dot{r}$sin θ+r$\dot{\theta}$cos θ

Now kinetic energy, T=$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$= $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, and L=T-V therefore

L=$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$-V(r,$\theta$)

Generalized momenta, $p_r = \frac{\partial L}{\partial \dot{r}}$ =m$\dot{r}$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ =m$r^2\dot{\theta}$

Hence $\dot{r} = \frac{p_r}{m}$ and, $\dot{\theta} = \frac{p_\theta}{mr^2}$

Thus, H=$\sum_k p_k \dot{q}_k - L$

Or, H=$p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ + V(r,θ)

By manipulating this expression we get,

H=$\frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right)$ + V(r,θ) (7.21)

Now by using the Hamilton's equations, we get (using equation (7.16),

Using, $\dot{r} = \frac{\partial H}{\partial p_r}$, we have $\dot{r} = \frac{p_r}{m}$ and by using, $\dot{p}_r = -\frac{\partial H}{\partial r}$, we have, $\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$

Similarly

Using, $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$, we have, $\dot{\theta} = \frac{p_\theta}{mr^2}$ and by using, $\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$, we have, $\dot{p}_\theta = -\frac{\partial V}{\partial \theta}$

7.4.3. In Cylindrical Coordinates: in cylindrical coordinates, x=rcos θ, y=rsin θ, z=z

Hence, L=T-V = $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ - V(x,y,z)

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$ - V(r,θ,z)

Generalized momentum, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$, $p_z = \frac{\partial L}{\partial \dot{z}}$

Hence $\dot{r} = \frac{p_r}{m}$ and, $\dot{\theta} = \frac{p_\theta}{mr^2}$, $\dot{z} = \frac{p_z}{m}$

H=$\sum_k p_k \dot{q}_k - L$, where, k = r,θ,z.

Thus

H= $(p_r\dot{r} + p_\theta\dot{\theta} + p_z\dot{z}) - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$ + V(r,θ, z)

$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right)$ + V(r,θ, z)

Hamilton's equations are $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

Therefore, Using, $\dot{r} = \frac{\partial H}{\partial p_r}$, we have $\dot{r} = \frac{p_r}{m}$ and by using, $\dot{p}_r = -\frac{\partial H}{\partial r}$, we have, $\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$

Similarly

Using, $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$, we have, $\dot{\theta} = \frac{p_\theta}{mr^2}$ and by using, $\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$, we have, $\dot{p}_\theta = -\frac{\partial V}{\partial \theta}$

Using, $\dot{z} = \frac{\partial H}{\partial p_z}$, we have, $\dot{z} = \frac{p_z}{m}$, and by using, $\dot{p}_z = -\frac{\partial H}{\partial z}$, we have, $\dot{p}_z = -\frac{\partial V}{\partial z}$

7.4.4. Spherical Coordinates:

In spherical polar coordinate,

x=rcosθsinφ, y= rsinθsinφ, and z = rcosφ

L=T-V = $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ - V(x,y,z)

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\sin^2\theta\dot{\varphi}^2)$ - V(r,θ,z)

Now proceeding as previous cases, we can write,

H=$\frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right)$ + V(r,θ,z)

Now applying the Hamilton's equations($\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$), we get the following expressions, $\dot{r} = \frac{p_r}{m}$, $\dot{p}_r = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3\sin^2\theta} - \frac{\partial V}{\partial r}$

$$\dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{p}_\theta = \frac{p_\varphi^2 \cos\theta}{mr^2 \sin^3\theta} - \frac{\partial V}{\partial \theta}$$

$$\dot{\varphi} = \frac{p_\varphi}{mr^2 \sin^2\theta}, \qquad \dot{p}_\varphi = -\frac{\partial V}{\partial \varphi}$$

7.5. Cyclic Coordinates as we know that a cyclic or ignorable co-ordinate as one that does not appear explicitly in the Lagrangian of a system. If the coordinate q_k is not appearing in the Lagrangian,$(\partial L/\partial q_k)=0$ and since $H=\sum_k p_k \dot{q}_k - L$, then

$$\frac{\partial H}{\partial q_k} = \frac{\partial}{\partial q_k}(\sum_k p_k \dot{q}_k - L)$$

Or $$\frac{\partial H}{\partial q_k} = \frac{\partial}{\partial q_k}(\sum_k p_k \dot{q}_k) - \frac{\partial}{\partial q_k}(L)$$

First term on r.h.s. of above equation will become zero since there is differentiation of $(p_k \dot{q}_k)$ w.r.t. q_k and the second term will become zero due to properties of cyclic coordinates,i.e., from $(\partial L/\partial q_k)=0$

Therefore above expression will provide us, $\frac{\partial H}{\partial q_k} = 0$ (7.22)

Hence, q_k will not appear in the Hamiltonian also. It means we can write

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = 0$$

And this expression directly tell us that

p_k=constant

That is , the momentum conjugate to a generalised co-ordinate which is cyclic is conserved. Or in other way, the generalized momentum conjugate to a cyclic coordinate is conserved during the motion.

7.5.1. Conservation theorems The theorems of conservation of linear and angular momentum are the special cases of the general principle for cyclic coordinates in the Lagrangian formulation.

7.5.2. Conservation of Linear momentum-The Lagrange's equation of motion for a generalized coordinate q_k is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = 0 \qquad (7.23)$$

where $L = T - V$.

Suppose q_k represents a translation of the entire system along a given direction. We consider a conservative system so that V is not a function of velocities and T is not a function of position. Therefore, $\frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial q_k} = 0$

Now we can write equation (6.23) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \dot{p}_k$$

$$-\left(\frac{\partial V}{\partial q_k}\right) = G_k \qquad (7.24)$$

Or, $G_k = \dot{p}_k$ (7.25)

These G_k and p_k are the components of the total force F and total linear momentum p of the system along the direction of the translation dq_k. For example, if the system is given a translation along X-axis, then $dq_k = dx$ and $G_k = F_x$ and $p_x = P_x$. This can be shown as follows;

$dr_i = dq_k\hat{\mathbf{x}} = dx\hat{\mathbf{x}}$

Where $\hat{x}$ is a unit vector along X-axis.

This gives $\frac{\partial r_i}{\partial q_k} = \frac{\partial r_i}{\partial x} = \hat{\mathbf{x}}$

As we have learnt that, the component of generalized force is given by;

$G_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_i \mathbf{F}_i \cdot \hat{\mathbf{x}} =$

$\hat{\mathbf{x}} \cdot \sum_i \mathbf{F}_i = \hat{\mathbf{x}} \cdot \mathbf{F} = F_x$.

Now since $T = \sum_i \frac{1}{2} m_i \dot{r}_i^2$

Therefore, $p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_k}$

Or $p_k = \sum_i m_i \dot{r}_i \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}} \cdot \sum_i m_i \dot{r}_i = \hat{\mathbf{x}} \cdot \mathbf{P} = P_x$

Hence equation (7.23) represents the equation of motion for the direction of translation. Now suppose that the translation coordinate q_k is cyclic. This means that q_k is not appearing in L= T-V.

Then $\frac{\partial L}{\partial q_k} = -\left(\frac{\partial V}{\partial q_k}\right) = 0$. therefore from equations (7.24) and (7.25), we get $G_k = \dot{p}_k = 0$

Or p_k = constant

For X- direction, $F_x = \dot{P}_x = 0$ or P_x = constant.

This is the well known conservation theorem for linear momentum. Thus in absence of a given component of applied force, the corresponding component of linear momentum is conserved.

7.5.3. Conservation of Angular momentum- Let us consider that the generalized coordinate q_k represents rotation

θ and dq_k represents a rotation $d\theta$, then the Lagrange equation can be written as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = -\left(\frac{\partial V}{\partial q_k}\right)$$

Because V is independent of $\dot{q}_k$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \dot{p}_k$$

And $-\left(\frac{\partial V}{\partial q_k}\right)$=$G_k$

hence $\dot{p}_k = G_k$ (7.25a)

Here we want to show that for a rotational coordinate q_k, the generalized force G_k is the component of the total applied torque τ about the axis of rotation and the generalized momentum p_k is the component of the total angular momentum J about the same axis.

Now, $G_k=\sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_k}$ (7.25b)

Here, $dq_k=d\theta$ is an infinitesimal rotation of the position vector r_i of the particle of the system about Z-axis such that the magnitude of r_i remains constant.

From figure 7.2, the infinitesimal small distance $|dr_i|$ is

$| dr_i| = r_i \sin\varphi dq_k = r_i \sin\varphi d\theta$

Or $d\mathbf{r_i} = dq_k(\hat{\mathbf{z}} \times \mathbf{r_i})$

Where $\hat{\mathbf{z}}$ is a unit vector along the axis of rotation.

Therefore, $\frac{\partial r_i}{\partial q_k} = \hat{\mathbf{z}} \times \mathbf{r_i}$

Thus $G_k = \sum_i F_i \cdot \hat{\mathbf{z}} \times \mathbf{r_i} = \sum_i \hat{\mathbf{z}} \cdot \mathbf{r_i} \times F_i = \hat{\mathbf{z}} \cdot \sum_i \tau_i = \hat{\mathbf{z}} \cdot \boldsymbol{\tau} = \tau_z$

Where $\sum_i \tau_i = \boldsymbol{\tau}$ is the total applied torque and $\tau_i = \hat{\mathbf{z}} \cdot \boldsymbol{\tau}$ is the component of the total torque $\boldsymbol{\tau}$ along Z-axis.

Similarly,

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \mathbf{v_i} \cdot \hat{\mathbf{z}} \times \mathbf{r_i}$$

$$= \sum_i \hat{\mathbf{z}} \cdot \mathbf{r_i} \times m_i \mathbf{v_i} = \hat{\mathbf{z}} \cdot \sum_i \mathbf{r_i} \times \mathbf{P_i} = \hat{\mathbf{z}} \cdot \sum_i \mathbf{J_i} = \hat{\mathbf{z}} \cdot \mathbf{J} = J_z \qquad (7.26)$$

Where $\sum_i \mathbf{J_i} = \mathbf{J}$ is the total angular momentum and $J_z = \hat{\mathbf{z}} \cdot \mathbf{J}$ is the component of the total angular momentum J along Z-axis.

Thus equation (7.25a) represents the equation of motion about the axis of rotation ($\dot{j}_z = \tau_z$).

Now, if the rotation coordinate q_k is cyclic, it will not appear in the Lagrangian L or V and hence

$$G_k = -\frac{\partial V}{\partial q_k} = 0$$

Therefore, from equation (7.25a), we have $G_k = \dot{p}_k = 0$ or $\tau_z = \dot{j}_z = 0$

Or J_z =constant (7.27)

This is the theorem of conservation of angular momentum which states that in absence of a given component of applied torque along an axis, the corresponding component of angular momentum along the same axis is conserved.

7.5.4. conservation of energy: In the Lagrangian formulation one may expect the deduction of the theorem of conservation of the total energy for a system where the potential energy is a function of position only. in fact we shall see the theorem of conservation of total energy is a special case of a more general conservation theorem. Consider a general Lagrangian L of a system is given by

$$L = L(q_1, q_2, \ldots, q_k, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k, \ldots, \dot{q}_n, t)$$

We denote it for our convenience by

$$L = L(q_k, \dot{q}_k, t)$$

The total time derivative of L is

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \qquad (7.28)$$

From Lagrange equations, we have $\left(\frac{\partial L}{\partial q_k}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$ (7.29)

From equations (7.28) and (7.29) we have,

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t}$$

or

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial t}$$

or, $$\frac{d}{dt}(\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L) = -\frac{\partial L}{\partial t} \quad (7.30)$$

hence equation (7.30) can be written as

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \quad (7.31)$$

If the Lagrangian L does not depend on time t explicitly, then $\frac{\partial L}{\partial t}$ =0.

So that $\frac{dh}{dt} = 0$ i.e.,

h= constant. (7.32)

thus when the Lagrangian is not explicit function of time, the energy function is the constant of motion. It is one of the first integrals of the motion and is called jacobi's integral. But as we know $p_k = \frac{\partial L}{\partial \dot{q}_k}$. Hence equation (7.30) can be written as

$$\frac{d}{dt}(\sum_k \dot{q}_k p_k - L) = -\frac{\partial L}{\partial t} \quad (7.33)$$

Or $$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad (7.33)$$

Where H is called the Hamiltonian function. Hence

$$H = (\sum_k \dot{q}_k p_k - L) \quad (7.34)$$

In general, the Hamiltonian function H is the function of Generalized coordinates,q_k generalized momenta, p_k

And time t. i.e.,

$$H=H(q_1,q_2,....,q_k,....,q_n, p_1, p_2,..., p_k,..., p_n, t) \quad (7.35)$$

Or $$H=H(q_k, p_k, t) \quad (7.36)$$

It is to be seen that the energy function h is identical in value with the Hamiltonian H. It is given a different name and symbol h because h is a function of q_k, $\dot{q}_k$ and t, while H that of q_k, p_k, and t.

If t does not appear in the Lagrangian L explicitly, then $\frac{\partial L}{\partial t} = 0$ hence from equation (7.33);

$\frac{dH}{dt} = 0$ or H= constant, where the value of H is $\sum_k \dot{q}_k p_k - L$.

Thus, if the time t does not appear in the Lagrangian L explicitly, we see that the Hamiltonian H is constant in time i.e., conserved. This is the conservation theorem for the Hamiltonian of

the system. Under special circumstances, the Hamiltonian H is equal to the total energy E of the system. In-fact, this is the case in most of the physical problems.

Physical Significance: the Hamiltonian takes a special form, if the system is conservative i.e., the potential energy V is independent of velocity coordinates $\dot{q}_k$ and the transformation equations for coordinates do not contain time explicitly, i.e.,

$r_i = r_i(q_1,q_2,....,q_k,....,q_n)$

for conservative system $\partial V/\partial \dot{q}_k = 0$

hence $p_k = \partial L/\partial \dot{q}_k = \partial(T-V)/\partial \dot{q}_k = \partial T/\partial \dot{q}_k$

so $H= \sum_k \dot{q}_k p_k - L = [\sum_k \left(\frac{\partial T}{\partial \dot{q}_k}\right) \dot{q}_k - L]$ (7.37)

if r_i does not depend on time t explicitly, then the kinetic energy T is a homogeneous quadratic function. It is easy to show that $\sum_k \frac{\partial \dot{T}}{\partial \dot{q}_k} q_k = 2T$ (7.38)

in fact, for a natural conservative system neither T nor v contains any explicit time dependence and hence the Lagrangian. And T is a homogeneous quadratic function of the time derivatives $\dot{q}_k$. Hence from equations (7.37) and (7.38), we have

$H= 2T-L$

$=2T-(T-V)$

$= T+V= E$, constant (7.39)

Thus the Hamiltonian H represents the total energy E of the system and is conserved, provided the system is conservative and T is a homogeneous quadratic function.

7.6.Routh's Procedure: It has been remarked that the Hamiltonian formulation is not particularly helpful in the direct solution of mechanical problems. Often we can solve the 2n first-order equations only by eliminating some of the variables, for example, the p variables, which speedily leads back to the second-order Lagrangian equations of motion. But an important exception should be noted. The Hamiltonian procedure is especially adapted to the treatment of problems involving cyclic coordinates. **Routhian mechanics is a sterile hybrid, combining some of the features of both the Lagrangian and the Hamiltonian formulations developed by Edward John Routh.** Correspondingly, the Routhian is the function which replaces both the Lagrangian and Hamiltonian functions. It is a function of mixed variables q_k, $\dot{q}_k$, and p_k, where the number of q_k coordinates is n, the number of degrees of freedom and the rest n velocity like independent($\dot{q}_k$'s and p_k's) variables are shared in between $\dot{q}_k$'s and p_k's. When there are some cyclic coordinates in the Lagrangian,

the construction of Routhian is useful. If the first s out of n coordinates are cyclic in L then the Routhian for the system under consideration is defined as

$R=R(q_1,q_2,\ldots,q_n;p_1,p_2\ldots,p_s;\dot{q}_{s+1},\ldots,\dot{q}_n;t)$

$= (\sum_{k=1}^{s} p_k\dot{q}_k - L(q_1,q_2,\ldots,q_n, \dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n,t)$ (7.40)

Where p_k(k=1,2,.....,s) are constants of motion

Now $dR=\sum_{k=1}^{n}\frac{\partial R}{\partial q_k}dq_k + \sum_{k=1}^{s}\frac{\partial R}{\partial p_k}dp_k + \sum_{k=s+1}^{n}\frac{\partial R}{\partial \dot{q}_k}d\dot{q}_k + \frac{\partial R}{\partial t}dt$ (7.41)

$$dR=\sum_{k=1}^{n}(p_k d\dot{q}_k + \dot{q}_k dp_k) - \sum_{k=1}^{n}\left(\frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k\right) - \frac{\partial L}{\partial t}dt$$

or, $dR=-\sum_{k=s+1}^{n} p_k d\dot{q}_k + \sum_{k=1}^{s}\dot{q}_k dp_k - \sum_{k=1}^{n}\dot{p}_k dq_k - \frac{\partial L}{\partial t}dt$ (7.42)

comparing equations (7.31) and (7.32) for dR, we have for the first s coordinates

$$\dot{q}_k = \frac{\partial R}{\partial p_k}, \quad \text{and} \quad \dot{p}_k = -\frac{\partial R}{\partial q_k}$$ (7.43)

For the rest n-s coordinates, i.e, from k=s+1 to k=n, we get

$$\dot{p}_k = -\frac{\partial R}{\partial q_k}$$

$$p_k = -\frac{\partial R}{\partial \dot{q}_k}$$

Substituting for p_k in first of above, we get,

$\frac{d}{dt}(\frac{\partial R}{\partial \dot{q}_k}) - \frac{\partial R}{\partial q_k}=0$, for k=s+1,.....,n (7.44)

For t variable, $\frac{\partial R}{\partial t} = -\frac{\partial L}{\partial t}$ (7.45)

Thus for the first s coordinates, which are supposed to be cyclic, equations (7.43) are similar to the Hamilton's equations of motion, where one has to replace H by R. They would conserve momenta p_k(k=1,2.....,s) since R is cyclic in the corresponding $q_k(k = 1,2,\ldots,s)$. Therefore one may express R as

$R=R(q_{s+1},\ldots,q_n;\alpha_1,\alpha_1,\ldots\ldots,\alpha_s;\dot{q}_{s+1},\ldots\ldots,\dot{q}_n;t)$ (7.46)

Where the momenta $p_1,p_2,\ldots,p_s$ are constants $\alpha_1,\alpha_2,\ldots,\alpha_s$(which are to be determined from the initial conditions).

Rest of the n-s conditions (k=s+1,....., n) $q_{s+1},\ldots,q_n$ are not cyclic and only the variables in R are the n-s conditions and the corresponding n-s generalised velocities. these coordinates

satisfy in equation (7.44) Lagrange type equations of motion in R instead of L. Thus the Routhian function R effectively behaves like a lagrangian of a system, having the number of degrees of freedom n-s. In conclusion, in Routh's procedure, a problem with some cyclic and remaining non-cyclic coordinates may be solved in two steps:

(1) Solve Hamilton's equations for cyclic coordinates with the Routhian R as the Hamiltonian of the system, and
(2) Solve Lagrange's equations for non-cyclic coordinates with the Routhian as the Lagrangian of the system.

7.7. Advantages over the Hamiltoian formulation over Lagrangian Formulations:

Hamilton's equations consist of 2n first-order differential equations, while Lagrange's equations consist of n second-order equations. However, Hamilton's equations usually don't reduce the difficulty of finding explicit solutions; They still offer some advantages, since important theoretical results can be derived because coordinates and momenta are independent variables with nearly symmetric roles. Hamilton's equations have another advantage over Lagrange's equations that if a system has a symmetry, such that a coordinate does not occur in the Hamiltonian, the corresponding momentum is conserved, and that coordinate can be ignored in the other equations of the set. Effectively, this reduces the problem from n coordinates to (n-1) coordinates. In the Lagrangian framework, of course the result that the corresponding momentum is conserved still follows immediately, but all the generalized velocities still occur in the Lagrangian - we still have to solve a system of equations in n coordinates.but there are some areas too of physics where Lagrangian formulation is preferred than the Hamiltonian one, for example the quantum field theory(QFT) has been described in terms of Lagrangian formulation.

7.8. Worked Examples

Ex.1. Obtain the Hamilton's equations for a simple pendulum. Solution: We use the generalised co-ordinate. For evaluating potential energy, the energy corresponding to the mean position is taken as zero.

The velocity of the bob, $v=l\dot{\theta}$

Kinetic energy, $T= \frac{1}{2}ml^2\dot{\theta}^2$

Potential energy, $V= mgl(1-\cos\theta)$

Lagrangian, $L=T-V= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

Now by applying $p_k = \frac{\partial L}{\partial \dot{q}_k}$, we may get,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \rightarrow \quad \dot{\theta} = \frac{p_\theta}{ml^2}$$

Hamiltonian $H(\theta, p_\theta) = \dot{\theta}p_\theta - L$

$$= \frac{p_\theta^2}{2ml^2} - \{\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\}$$

$$= \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$$

Now applying Hamilton's equations, $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

We have $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$, $\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$

$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = -\frac{g\sin\theta}{l}$

since θ is small, $\sin\theta \simeq \theta$ and the above equation reduces to

This is standard differential equation; the solution of this equation is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Ex.2. Charged particle moving in an electromagnetic field. obtain the equations of motion from Hamilton's equations.

The Lagrangian L for a charged particle in an electromagnetic field is given by

L=T-qϕ+q(**v.A**)

$$= \frac{1}{2}\sum_k mv_k^2 + q\sum_k v_k A_k - q\phi$$

The canonical momenta, $p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial v_k} = mv_k + qA_k$

Now, $H = \sum_k p_k\dot{q}_k - L = \sum_k(mv_k^2 + qv_kA_k)$ - L

$$H = mv^2 + q(\mathbf{v}.\mathbf{A}) - \frac{1}{2}mv^2 + q\phi - q(\mathbf{v}.\mathbf{A})$$

$$= \frac{1}{2}mv^2 + q\phi$$

From equation expression of p_k, we have, $v_k = \frac{p_k}{m} - \frac{q}{m}A_k$, then H can be expressed as

$$H = \sum_k \frac{1}{2m}(p_k - qA_k)^2 + q\phi$$

Or, $H = \frac{1}{2m}(p - qA)^2 + q\phi$

Hence Hamilton's equations are

$v_k = \frac{\partial H}{\partial p_k} = \frac{1}{m}(p_k - qA_k)$or

And by using $\dot{p}_k = -\frac{\partial H}{\partial q_k}$, we have

Ex.3. Apply the Hamilton's equations in the study of the Compound Pendulum.
Solution: Let us consider compound pendulum be suspended from a pivot point such that its centre of mass is oscillating in the vertical plane which is the plane of the paper.

Now the Lagrangian,

L= T- V= $\frac{1}{2}I\dot{\theta}^2$+ mglcos θ

Generalized momenta $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$

so

H=$\sum_k p_k \dot{q}_k - L = p_\theta\dot{\theta}$- $\frac{1}{2}I\dot{\theta}^2$- mglcos θ

Or H= $(p_\theta^2/2I) - Mglcos\theta$

Now using Hamilton's equations $\dot{q}_k = \frac{\partial H}{\partial p_k}$,

and $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ we get

$\dot{\theta} = \frac{\partial H}{\partial p_\theta}$ or $\dot{\theta} = \frac{p_\theta}{I}$ whence $\dot{p}_\theta = I\ddot{\theta}$ and , $\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$ or , $\dot{p}_\theta = -mgl\ sin\theta$

Equating these two values of $\dot{p}_\theta$ we have,

$I\ddot{\theta}$=- Mgl sinθ

When θ is small then sin$\theta \cong \theta$ and the equation of motion is

$\ddot{\theta}+\frac{Mgl}{I}\theta = 0$

This equation represents simple harmonic motion, whose periodic time is given by

T= $\sqrt{\frac{I}{Mgl}}$

Ex. 4. Describe the Hamiltonian and Hamilton's equations for an ideal spring-mass arrangement.
Solution: L= $\frac{1}{2}m\dot{x}^2$- $\frac{1}{2}kx^2$, where k is the force constant of the spring.

$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ or $\dot{x} = \frac{p_x}{m}$

H=$\sum_k p_k \dot{q}_k - L$

= $p_x\dot{x}$-[$\frac{1}{2}m\dot{x}^2$- $\frac{1}{2}kx^2$]

$= \frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$

Or $H= \frac{p_x^2}{2m}+\frac{1}{2}kx^2$

Hamilton's equations are $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$ and $\dot{p}_x = -\frac{\partial H}{\partial x} = -kx$

Thus the equation of motion of spring-mass system is

$m\ddot{x} = -kx$

or $\ddot{x} + \frac{k}{m}x = 0$

Ex. 5. Using Hamilton's equations of motion, Show that the Hamiltonian.

$H=\frac{p^2}{2m}e^{-rt} + \frac{1}{2} m \omega^2x^2e^{rt}$, leads to the equation of motion of a damped harmonic oscillator, $\ddot{x} + r\dot{x} + \omega^2x = 0$ (where r is a constant).

Solution:

Equations of motion are

$\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$

Here q = x, therefore

$\dot{x} = \frac{\partial H}{\partial x} = \frac{p}{m}e^{-rt}$ and $\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2xe^{rt}$

Now $\dot{x} = \frac{p}{m}e^{-rt}$ from here we will have, $p = m\dot{x}e^{rt}$ and $\dot{p} = m\ddot{x}e^{rt} + mr\dot{x}e^{rt}$

Now equating the above two values of $\dot{p}$ we will have $m\ddot{x}e^{rt} + mr\dot{x}e^{rt} = -m\omega^2xe^{rt}$

or $\ddot{x} + r\dot{x} + \omega^2x = 0$

which is the desired equation of damped harmonic oscillator.

Ex.6. Determine the Hamiltonian corresponding to the Lagrangian $L=a\dot{x}^2 + b\dot{y}^2-kxy$.

Solution: $H=\sum_k p_k\dot{q}_k - L$, or

$H = p_x\dot{x} + p_y\dot{y} - a\dot{x}^2 - b\dot{y}^2 + kxy$ (since, $L=a\dot{x}^2 + b\dot{y}^2-kxy$)

Now since $p_x = \frac{\partial L}{\partial \dot{x}}$ hence, $p_x = 2a\dot{x}$ and $p_y = 2b\dot{y}$, substituting these values of $\dot{x}$ and $\dot{y}$ in above expression of Hamiltonian, we get

$$H= p_x \left(\frac{p_x}{2a}\right) + p_y \left(\frac{p_y}{2b}\right) - a\left(\frac{p_x}{2a}\right)^2 - b\left(\frac{p_y}{2b}\right)^2 + kxy$$

$$= \frac{p_x^2}{2a} + \frac{p_y^2}{2b} - \frac{p_x^2}{4a} - \frac{p_y^2}{4b} + kxy$$

Or $H = \frac{p_x^2}{4a} + \frac{p_y^2}{4b} + kxy$

Ex.7. Find the Routhian for the Lagrangian L, given by
$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$, where $\mu = \frac{mM}{m+M}$.

Solution:

$R = \sum_{k=1}^{s} p_k \dot{q}_k - L$

Since θ is the cyclic coordinate,

$R = p_\theta \dot{\theta} - \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$

But $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$ =constant,

Hence, $\dot{\theta} = \frac{p_\theta}{\mu r^2}$

Hence, $R = -\frac{1}{2}\mu\dot{r}^2 + \frac{p_\theta^2}{2\mu r^2} - \frac{GMm}{r}$

The last two terms in R give the effective potential for r-motion.

7.9. Summary: In this chapter we have discussed the Hamiltonian formulation of classical mechanics thoroughly. We also discussed a hybrid formulation of Lagrangian formulation and Hamiltonian mechanics which was called the Routh's procedure. We illustrated various worked examples to get enhance the understanding of each topic thoroughly. We studied the concept of cyclic coordinates and their consequences in the classical mechanics. In going through the cyclic coordinates we took three subtopics the conservation of linear momentum, conservation of angular momentum and the conservation of energy respectively. Hence we learned that how the knowledge of cyclic coordinates makes easy of the mechanical problems. At last of the unit we have compared the two formulations; Lagrangian and Hamiltonian and found the Hamiltonian formulation best but still we cannot replace the Lagrangian formulation in some special topics of physics like the modern field theories.

7.10. Model questions

A. Multiple Choice Questions

(1) If the Lagrangian of a particle moving in one dimensions is given by $L = \frac{\dot{x}^2}{2x} - V(x)$. the Hamiltoian is

(a) $\frac{1}{2}xp^2 + V(x)$ (b) $\frac{\dot{x}^2}{2x} + V(x)$

(b) $\frac{1}{2}\dot{x}^2 + V(x)$ (d) $\frac{p^2}{2x} + V(x)$

(3) The Lagrangian for a simple pendulum is given by, $L=\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$,

Hamilton's equations are given by

(a) $\dot{p}_\theta$= -mgl sinθ, $\dot{\theta} = \frac{p_\theta}{ml^2}$ (b) $\dot{p}_\epsilon$= mgl sinϵ, $\dot{\theta} = \frac{p_\vartheta}{ml^2}$

(c) $\dot{p}_\vartheta$= -m$\ddot{\theta}$, $\dot{\theta} = \frac{p_\theta}{m}$ (d) $\dot{p}_\theta$=-$(\frac{g}{l})\theta$, $\dot{\theta} = \frac{p_\theta}{m}$

(4) Hamilton's equations of motion are

(a) $\dot{q}_k = \frac{\partial H}{\partial p_k}$; $p_k = -\frac{\partial H}{\partial q_k}$ (b) $\dot{q}_k = -\frac{\partial H}{\partial p_k}$; $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

(c) $q_k = \frac{\partial H}{\partial p_k}$; $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ (d) $\dot{c}_k = \frac{\partial H}{\partial p_k}$; $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

(5) The Lagrangian for a simple pendulum is given by, $L=\frac{1}{2}m(l^2\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mgl\cos\theta$. which of the following is conserved

(a) $\dot{\varphi}\sin^2\theta$ (b) $\dot{\varphi}\sin\theta$

(c) $\dot{\varphi}/\sin\theta$ (d) $\dot{\varphi}/\sin^2\theta$

(6) Hamilton's equations are

(a) 2n 1st order time dependent equations (b) n 1st order time dependent equations

(c) n 2nd order time dependent equations (d) 2n 2nd order time dependent equations

(7) Which of the following set of phase-space trajectories is not possible for a particle obeying Hamilton's equations of motion?

(8) The Lagrangian of a particle moving in a plane under the influence of a central potential is given by $L = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$ - V(r). The generalized momenta corresponding to r and θ are given by

(a) m$\dot{r}$ and mr$^2\dot{\theta}$ (b) m$\dot{r}$ and mr $\dot{\theta}$

(c) m$\dot{r}^2$ and mr$^2\dot{\theta}$ (d) m$\dot{r}^2$ and mr$^2\dot{\theta}^2$

(9) The Hamiltonian corresponding to the Lagrangian L=a$\dot{x}^2$ + b $\dot{y}^2$-kxy is

(a) $\frac{p_x^2}{2a}+\frac{p_y^2}{2b}$ + kxy (b) $\frac{p_x^2}{4a}+\frac{p_y^2}{4b}$ - kxy

(c) $\frac{p_x^2}{4a}+\frac{p_y^2}{4b}$ + kxy (d) $\frac{p_x^2+p_y^2}{4ab}$ + kxy

(10). The Hamiltonian of relativistic particle of rest mass m and momentum p is given by H=$\sqrt{p^2+m^2}+V(x)$, in units in which the speed of light c = 1. The corresponding Lagrangian is

(a) L = $m\sqrt{1+x^2}$- V(x) (b) L= $-m\sqrt{1-x^2}$ - V(x)

(c) L = $\sqrt{1-mx^2}$- V(x) (d) L= $\frac{1}{2}mx^2$ – V(x)

(11). The Hamiltonian of a classical particle moving in one dimension is H= $\frac{p^2}{2m}$ +αq^4. Where α is a positive constant and p and q are its momentum and position respectively. Given that its total energy E≦E_0. the available volume of phase space which depends on E_0 as

(a) $E_0^{3/4}$ (b) E_0

(c) $\sqrt{E_0}$ (d) is independent of E_0

Answers: (1)-a, (2)-b, (3)-d, (4)-a, (5)-a, 6-b, (7)-a, (8)-c, (9)-c,(10)-b, (11)-a

Answers of self-check Questions: (1).it is the function of q, p and t., **(2).**see any graduate level maths book / Wikipedia.(**3**). Linear momentum, (**4**). Angular momentum, (**5**). No,it is only when the system is conservative (**6**). All are best since use of any one of them depends on the situation of the system and on convenience.

B. Short Answer type Questions

1. What are the advantages of Hamiltonian formulation over the Lagrangian formulation?
2. Define the phase –space and conjugate variables
3. Define the cyclic coordinates
4. Write the Hamiltonian function and Hamilton's equations
5. What are the advantages of Hamiltonian formulations over the Lagrangian formulations
6. What is the first integral in the Hamiltonian formulation

C.Long Answer type questions

(1) Derive the Hamilton's equations of motion by the Hamiltonian of the system and the variational principle

(2) Define the cyclic coordinates and apply this concept in the principle of conservation of linear momentum, angular momentum and energy.

(3) Apply the Hamiltonian formulation to illustrate the equations of motion to for a charged particle moving in an electromagnetic field.

(4) Apply the Hamilton's equations to describe the motion of a simple harmonic oscillator and a compound pendulum.

(5) A Hamiltonian of one degree of freedom has the form

$H= \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{ba}{2}q^2e^{-\alpha t}(\alpha + b\, e^{-\alpha t}) + \frac{k}{2}q^2$.

Where α,a,b, and k are constants.

(a) Find a Lagrangian corresponding to this Hamiltonian.

(b) Find an equivalent Lagrangian that is not explicitly dependent on time.

(c) What is the Hamiltonian corresponding to this second Lagrangian, and what is the relationship between the two Hamiltonian?

(6) The Lagrangian for a system can be written as

$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2 + y^2}$.

Where a, b, c, f, g, and k are constants. What is the Hamiltonian? What quantities are conserved?

(7) A dynamical system has the Lagrangian

$L = \dot{q}_1^2 + \frac{q_2^2}{\alpha + bq_1^2} + k_1q_1^2 + k_2\dot{q}_1\dot{q}_2$

Where a, b, k_1, k_2 are constants. Find the equations of motion in the Hamiltonian formulation.

7.11.References and books suggested

- J.C Upadhyaya,Classical Mechanics, Himalaya Publishing House
- G.Aruldhas, Classical Mechanics, PHI Learning private Limited.
- H.Goldstein, Classical Mechanics, Addison-Wesley
- Wikipedia

Unit 8 : Canonical Transformations

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8.1 Aims and Objectives

In this unit you will go through a new type of variation •-variation, rather than δ-variation. And we use this variation in deriving the principle of least action. You will be familiar with a transformation called the canonical transformations and their need in classical mechanics. You will be able to know a new concept that is the Poisson Bracket and it's applications in writing the equations of motion in terms of Poisson brackets. Then you will learn how we can form the various forms of Poisson bracket with the help of canonical transformations. You will learn how to apply these transformations in actual mechanical problems like harmonic oscillator.

8.2 Introduction In Hamiltonian mechanics, a canonical transformation is a change of canonical coordinates (q, p, *t*) → (Q, P, *t*) that preserves the form of Hamilton's equations. This is sometimes known as form invariance. It need not preserve the form of the Hamiltonian itself. Canonical transformations are useful in their own right, and also form firstly,the basis for the Hamilton–Jacobi equations which is a useful method for calculating conserved quantities and secondly, Liouville's theorem which itself the basis for 'classical statistical mechanics'.

8.3 Δ-Variation- so far you have been learned about in δ –variation, firstly, the variation of the path allows for variations in the coordinate q_k at constant t and secondly the varied path and the correct path have the same end points i.e., $\delta q_k(t_1) = \delta q_k(t_2)=0$

While in Δ- variation time as well as position coordinates are allowed to vary. At the end points of the path, the position coordinates are kept fixed, while changes in the time are allowed. The Δ- variation of a coordinate q_k is shown in figure 8.1 while figures 8.2 and 8.3 represent the d-variation and δ-variation respectively. The end points of the path A and B after time Δt take the position A' and B' so that there is no change in the position coordinates, that is, $\bullet q_k(1) = \bullet q_k(1) = 0$. A point P on the actual path now goes over to the point P', on the varied path with the correspondence, given by;

$q_k \rightarrow q_k' = q_k + \bullet q_k = q_k + \delta q_k + \dot{q}_k \Delta t$.

Where δ-variation is that where change in position coordinates and time both are kept fixed at the end points.

The Δ- variation of any function $f = f(q_k, \dot{q}_k, t)$ is given by

$$\Delta f(q_k, \dot{q}_k, t) = \sum_k \left[\frac{\partial f}{\partial q_k}\Delta q_k + \frac{\partial f}{\partial \dot{q}_k}\Delta \dot{q}_k\right] + \frac{\partial f}{\partial t}\bullet t$$

$$= \sum_k \frac{\partial f}{\partial q_k}[\delta q_k + \dot{q}_k \Delta t] + \sum_k \frac{\partial f}{\partial \dot{q}_k}[\delta \dot{q}_k + \ddot{q}_k \Delta t] + \frac{\partial f}{\partial t}\bullet t$$

$$= \sum_k \left[\frac{\partial f}{\partial q_k}\delta q_k + \frac{\partial f}{\partial \dot{q}_k}\delta \dot{q}_k\right] + \left[\sum_k \left(\frac{\partial f}{\partial q_k}\dot{q}_k + \ddot{q}_k \frac{\partial f}{\partial \dot{q}_k}\right) + \frac{\partial f}{\partial t}\right] \bullet t$$

$$= \delta f + \bullet t \frac{df}{dt}$$

Relation between these two types of variations can be written as-

$$\Delta = \delta + \Delta t \frac{d}{dt} \qquad (8.1)$$

8.4. Principle of least Action- According to the principle of least action for a conservative system

$$\Delta \int_{t_1}^{t_2} \sum_k p_k q_k dt = 0 \qquad (8.2)$$

Where Δ represents variation of path which allows time as well as position coordinates to vary. As we know that Hamilton's principle function S, is given by

$S = \int_{t_1}^{t_2} L dt$, the Δ-variation of S is

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \left[\delta + \Delta t \frac{d}{dt}\right] \int_{t_1}^{t_2} L dt$$

$$= \delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \Delta t d(L)$$

$$=\delta \int_{t_1}^{t_2} L dt + [L\Delta t]_{t_1}^{t_2}$$

$$= \delta \int_{t_1}^{t_2} L dt + [L\Delta t]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right] dt + [L\Delta t]_{t_1}^{t_2} \quad (8.3)$$

In the present case $\delta q_k \neq 0$ at the end points, hence $\delta \int_{t_1}^{t_2} L dt$ is not equal to zero.Now, according to Lagrange's equations, we have $\left(\frac{\partial L}{\partial q_k}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$ (8.4)

Also, $\delta \dot{q}_k = \frac{d}{dt}[\delta q_k]$ (8.5)

Hence, $\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k = \frac{d}{dt}(p_k \delta q_k)$ (8.6)

Again we know that, $\Delta = \delta + \Delta t \frac{d}{dt}$, hence

$\Delta q_k = \delta q_k + \Delta t \frac{dq_k}{dt}$, or $\delta q_k = \Delta q_k - \Delta t \dot{q}_k$ or $p_k \delta q_k = p_k \Delta q_k - \Delta t \dot{q}_k p_k$

Hence, $\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k = \frac{d}{dt}(p_k \delta q_k) = \frac{d}{dt}(p_k \Delta q_k - \dot{q}_k p_k \Delta t)$ (8.7)

hence $\Delta S = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right] dt + [L\Delta t]_{t_1}^{t_2}$ (8.8)

using equations (7.3) and (7.8), we have

$$\Delta S = \int_{t_1}^{t_2} \sum_k [d(p_k \Delta q_k) - d(p_k \dot{q}_k \Delta t)] + [L\Delta t]_{t_1}^{t_2}$$

or $$\Delta S = \sum_k [p_k \Delta q_k]_{t_1}^{t_2} - \sum_k [p_k \dot{q}_k \Delta t]_{t_1}^{t_2} + [L\Delta t]_{t_1}^{t_2} \quad (8.9)$$

First term on r.h.s will become zero as $\Delta q_k = 0$ at the end points.

Therefore equation (8.9) will become

$$\Delta \int_{t_1}^{t_2} L dt = [(L - \sum_k p_k \dot{q}_k)\Delta t]_{t_1}^{t_2}$$

$$= -[H\Delta t]_{t_1}^{t_2} \qquad \text{(since } H = \sum_k p_k \dot{q}_k - L\text{)}$$

$$= -\Delta \int_{t_1}^{t_2} H dt \quad \text{(for conservative system), then}$$

$$\Delta \int_{t_1}^{t_2} (H + L) dt = 0$$

Or, $\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k \, dt = 0$, which is known as principle of least action.

There are some other forms of principle of least action which are described as following-

8.4.1.Fermat's Principle-for a conservative system, the Hamiltonian is constant and the potential energy is independent of time. So that

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial (T-V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k$$

Therefore, $\sum_k p_k \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$

By substituting this value of $\sum_k p_k \dot{q}_k$ in expression of principle of least action, we get

$$\Delta \int_{t_1}^{t_2} 2T dt = 0, \qquad (8.10)$$

This is the another form of principle of least action. If there is no external force acting on the system, then total kinetic energy T as well as total energy H will be conserved. In this case the equation (8.10) will become $\Delta \int_{t_1}^{t_2} dt = 0$, or $\Delta(t_2 - t_1) = 0$

or, $(t_2 - t_1)$= an extremum (8.11)

here we see that the word least is misnomer since it is not always true rather sometime it take longer time than the normal. This form of the principle of least action is the same as Fermat's principle in geometrical optics, which states that a ray of light travels between two points along such a path that the time taken is the extremum.

8.4.2. Jacobi's form of the principle of least action: when transformation equations do not involve time, the kinetic energy of a system can be expressed as a homogeneous quadratic function of the velocities,i.e,

$$T = \frac{1}{2} \sum_{kl} M_{kl} \dot{q}_k \dot{q}_l \qquad (8.12)$$

we construct a configuration space for which M_{kl} coefficients form the metric tensor and the element of path length $d\rho$ in this space is defined as

$$d\rho^2 = \sum_{kl} M_{kl} \dot{q}_k \dot{q}_l \qquad (8.13)$$

so that, $[d\rho/dt]^2 = \sum_{kl} M_{kl} \dot{q}_k \dot{q}_l \qquad (8.14)$

from equations (8.12) and (8.13), we get

$$T = \frac{1}{2} \left(\frac{d\rho}{dt}\right)^2 \qquad (8.15)$$

Hence $dt=d\rho/\sqrt{2T}$ (8.16)

Hence the principle of least Action, $\Delta\int_{t_1}^{t_2} 2Tdt = 0 = \Delta\int_{t_1}^{t_2}\sqrt{2T}d\rho$ (8.17)

But H=T+V(q), total energy is constant for conservative system. Thus, the principle of least Action takes the form

$$\Delta\int_{t_1}^{t_2}\sqrt{2[H-V(q)]}d\rho = 0 \quad (8.18)$$

This is known as Jacobi's form of the least action principle

8.4.3. Principle of least Action in terms of arc length of the particle trajectory- if the system contains only particle of mass m, its kinetic energy is given by

$$T=\frac{1}{2}mv^2=\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \quad (8.19)$$

Where ds is the element of arc traversed by the particle in time dt. From this expression, we have

$$dt = \sqrt{m/2T}ds$$

so that the principle of least Action , $\Delta\int_{t_1}^{t_2} 2Tdt = 0$ can be written as

$$\Delta\int_{t_1}^{t_2} 2T\sqrt{m/2T}ds = 0$$

Or, $$\Delta\int_{t_1}^{t_2}\sqrt{2mT}ds = 0$$

Or, $$\Delta\int_{t_1}^{t_2}\sqrt{2m(H-V)}ds = 0 \quad (8.20)$$

This equation represents the principle of least Action in terms of arc length of the particle trajectory. This equation is similar to the Jacobi's form of the principle of least action.

8.5 Canonical Transformations- In several problems, we may need to change one set of position and momentum coordinates into another set of position and momentum coordinates. Suppose that q_k and p_k are the old position and momentum coordinates and Q_k and P_k are the new ones. Let these coordinates be related by the following transformations:

$P_k=P_k(q_1, q_2, \ldots q_n, p_1, p_2 \ldots p_n, t)$ and $Q_k = Q_k(q_1, q_2, \ldots q_n, p_1, p_2 \ldots p_n, t)$ (8.21)

Now if there exist a Hamiltonian H' in the new coordinates such that

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \quad (8.22a)$$

$$\dot{Q}_k = \frac{\partial H'}{\partial p_k} \quad (8.22b)$$

where $$H'=\sum_{k=1}^{n} P_k\dot{Q}_k - L' \quad (8.23)$$

and L' substituted in the Hamilton's principle, $\delta \int_{t_1}^{t_2} L'dt = 0$ (8.24)

gives the correct equations of motion in terms of the new coordinates P_k and Q_k, then the transformations (8.21) are known as canonical or contact transformations

8.5.1. Legendre Transformations This is a mathematical technique used to change the basis from one set of coordinates to another. If f(x,y) is function of two variables x and y, then the differential of this function can be written as

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \quad \text{or, } df=udx+vdy \tag{8.25}$$

where $u=\frac{\partial f}{\partial x}, v=\frac{\partial f}{\partial y}$ (8.26)

Now, we want to change the basis from (x,y) to (u,v) so that u is now an independent variable and x is a dependent one. Let f' be a function of u and y such that

$$f'=f-ux \tag{8.27}$$

Then, $$df'=df-udx-xdu \tag{8.28}$$

Substituting for df, we get,

$$df'=udx+vdy-udx-xdu$$

Or $$df'=vdy-xdu \tag{8.29}$$

But f' is a function of u and y, therefore

$$df'=\frac{\partial f'}{\partial u}du+\frac{\partial f'}{\partial y}dy \tag{8.30}$$

Comparing equations (8.29) and (8.30), we get

$$x=-\frac{\partial f'}{\partial u} \quad \text{and} \quad v=\frac{\partial f'}{\partial y} \tag{8.31}$$

these are the necessary relations for Legendre transformations.

8.5.2.Generating Functions For canonical transformations, the Lagrangian L in p_k, q_k coordinates and L' in P_k, Q_k coordinates must satisfy the Hamilton's principle, i.e.,

$$\delta \int_{t_1}^{t_2} Ldt = 0 \text{ and } \delta \int_{t_1}^{t_2} L'dt = 0 \tag{8.32}$$

But $L = \sum_{k=1}^{n} p_k \dot{q}_k - H$ and $L' = \sum_{k=1}^{n} P_k \dot{Q}_k - H'$ (8.33)

$$\delta \int_{t_1}^{t_2} [\Sigma_k \; p_k \dot{q}_k - H \;]dt = 0 \tag{8.34}$$

And $\delta \int_{t_1}^{t_2} \sum_k [P_k \dot{Q}_k - H'] dt = 0$ (8.35)

Subtracting equation (7.35) from equation (7.34), we get

$$\delta \int_{t_1}^{t_2} [(\sum_k p_k \dot{q}_k - H) - (\sum_k P_k \dot{Q}_k - H')] dt = 0 \quad (8.36)$$

In δ-variation process, the condition $\delta \int f\, dt = 0$ is to be satisfied, in general, by $f = \frac{dF}{dt}$, where F is an arbitrary function. Therefore,

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = 0 \quad (8.37)$$

Where, $dF/dt = L - L'$ (8.38)

Or, $\frac{dF}{dt} = (\sum_k p_k \dot{q}_k - H) - (\sum_k P_k \dot{Q}_k - H')$ (8.39)

The function F is known as the generating function. The first function in equation (8.39) is a function of p_k, q_k and t and the second as a function of P_k, Q_k and t. F is therefore , in general, subjected to the transformation equations (8.21) and therefore F may be regarded as the function of (2n+1) variables, comprising t and any 2n of the p_k,q_k, P_k, Q_k. Thus we see that F can be written as a function of (2n+1) independent variables in the following four forms:

(i) $F_1(q_k, Q_k, t)$, (ii) $F_2(q_k, P_k, t)$
(ii) $F_3(p_k, Q_k, t)$, (iv) $F_4(p_k, P_k, t)$

First form :$F_1(q_k, Q_k, t)$- if we choose this form, i.e.,

$F_1 = F_1\ (q_1, q_2, ..., q_k, ..., q_n, Q_1, Q_2, ..., Q_k, ..., Q_n, t)$ (8.40)

Then $\frac{dF_1}{dt} = \sum_k \frac{\partial F_1}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k} \dot{Q}_k + \frac{\partial F_1}{\partial t}$ (8.41)

subtracting equation (8.41) from equation (8.39), we write,

$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right) \dot{q}_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right) \dot{Q}_k + H' - H - \frac{\partial F_1}{\partial t} = 0$$

Or,

$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right) dq_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right) dQ_k + \left[H' - H - \frac{\partial F_1}{\partial t}\right] dt = 0 \quad (8.42)$$

As q_k, Q_k and t may be regarded as independent variables,

$$p_k = \frac{\partial F_1(q_k, Q_k, t)}{\partial q_k}$$

$$P_k = -\frac{\partial F_1(q_k, Q_k, t)}{\partial Q_k}, \text{ and } H' - H = \frac{\partial F_1(q_k, Q_k, t)}{\partial t} \quad (8.43)$$

In principle, first equation of equation (8.43) may be solved to give

$Q_k=Q_k(q_k, p_k, t)$ (8.44)

Substituting this in the second equation of equation (8.43), we get,

$P_k=P_k(q_k, p_k, t)$ (8.45)

Equations (8.44) and (8.45) are the transformation equations (8.21). Thus we find that transformation equations can be derived from a knowledge of the function F. This is why F is known as the generating function of the transformation.

Second Form: $F_2(q_k,P_k,t)$- If the generating function is of the type $F_2(q_k,P_k,t)$, then it can be dealt with by affecting a Legendre transformation of $F_1(q_k, Q_k,t)$.

The Legendre transformation: f'=f-ux , where u= $\frac{\partial f}{\partial x}$

hence, since

P_k=- $\frac{\partial F_2}{\partial Q_k}$, we have u= - P_k, x=Q_k, f'=F_2 and f = F_1.

Therefore, $F_2(q_k,P_k,t) = F_1(q_k,Q_k,t) + \sum_k P_k Q_k$ (8.46)

Substituting the value of F_1 from above equation in equation (8.39), we have

$$(\sum_k p_k\dot{q}_k - H) - (\sum_k P_k\dot{Q}_k - H') = \frac{dF_2}{dt} = \frac{d}{dt}[F_2 - \sum_k P_k Q_k]$$

Or $\frac{dF_2}{dt} = \sum_k p_k\dot{q}_k + \sum_k P_k\dot{Q}_k + H' - H$ (8.47)

Total time derivative of $F_2(q_k,P_k,t)$ is

$$\frac{dF_2}{dt} = \sum_k \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k}\dot{P}_k + \frac{\partial F_2}{\partial t}$$ (8.48)

From equations (8.47) and (8.48) $p_k=\frac{\partial F_2}{\partial q_k}$, $Q_k = \frac{\partial F_2}{\partial P_k}$ and $H' - H= \frac{\partial F_2}{\partial t}$ (8.49)

Third form:$F_3(p_k,Q_k,t)$- now we can relate this type of generating function to F_1 by a Legendre transformation in view of the first relation in form third, $p_k= \frac{\partial F_2}{\partial q_k}$. Here u = p_k, x=q_k, f'=F_3 and f=F_1. therefore,

$F_3(p_k,Q_k,t) = F_1(q_k,Q_k,t) - \sum_k p_k q_k$

Or, $F_1(q_k,Q_k,t) = F_3(p_k,Q_k,t) + \sum_k p_k q_k$

Hence from equation (8.21),

$(\sum_k p_k\dot{q}_k \quad H) \quad (\sum_k P_k\dot{Q}_k \quad H') = \frac{dF_3}{dt} = \frac{d}{dt}[F3(pk,Qk,t) + \sum_k p_k q_k]$

$$\frac{dF_3}{dt} = -\sum_k \dot{p}_k q_k - \sum_k P_k \dot{Q}_k + H' - H \qquad (8.50)$$

Also, $$\frac{dF_3}{dt} = \sum_k \frac{\partial F_3}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k}\dot{Q}_k + \frac{\partial F_3}{\partial t} \qquad (8.51)$$

Therefore the new transformation equations are

$$q_k = -\frac{\partial F_3}{\partial p_k}, \quad P_k = -\frac{\partial F_3}{\partial Q_k} \text{ and } \quad H' - H = \frac{\partial F_3}{\partial t} \qquad (8.52)$$

Fourth form:$F_4(p_k,P_k,t)$- Using Legendre transformations, the generating function $F_4(p_k,P_k,t)$ can be connected to $F_1(q_k,Q_k,t)$ as

$F_4(p_k,P_k,t) = F_1((q_k,Q_k,t) + \sum_k P_k Q_k - \sum_k p_k q_k$

Using equation (8.21)

$$(\sum_k p_k\dot{q}_k - H) - (\sum_k P_k\dot{Q}_k - H') = \frac{d}{dt}[(F_4 - \sum_k P_k Q_k + \sum_k p_k q_k)]$$

Or, $$\frac{dF_4}{dt} = -\sum_k \dot{p}_k q_k - \sum_k \dot{P}_k Q_k + H' - H \qquad (8.53)$$

But $$\frac{dF_4}{dt} = \sum_k \frac{\partial F_4}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_4}{\partial p_k}\dot{P}_k + \frac{\partial F_4}{\partial t} \qquad (8.54)$$

A comparison of the above two equations gives,

$$q_k = -\frac{\partial F_4}{\partial p_k}, \quad Q_k = \frac{\partial F_4}{\partial P_k}, \quad H' - H = \frac{\partial F_4}{\partial t} \qquad (8.55)$$

8.5.3.Table.Summary of Generating functions-

Generating Function	Transformation Equations	
$F_1(q,Q,t)$	$p_k = \frac{\partial F_1(q_k,Q_k,t)}{\partial q_k}$	$P_k = -\frac{\partial F_1(q_k,Q_k,t)}{\partial Q_k}$
$F_2(q,P,t)$	$p_k = \frac{\partial F_2(q_k,P_k,t)}{dq_k}$	$Q_k = \frac{\partial F_2(q_k,P_k,t}{\partial P_k}$
$F_3(p,Q,t)$	$q_k = -\frac{\partial F_3(p_k,Q_k,t)}{\partial p_k}$	$P_k = -\frac{\partial F_3(p_k,Q_k,t)}{\partial Q_k}$
$F_4(p,P,t)$	$q_k = -\frac{\partial F_4(p_k,P_k,t)}{\partial p_k}$	$Q_k = \frac{\partial F_4(p_k,P_k,t)}{\partial P_k}$

8.5.4.Procedure for application of canonical transformations:- we note that the relation between H and $\mathbf{H'}$ in all the cases has the same form i.e., $\mathbf{H' - H} = \frac{\partial F}{\partial t}$. If F has no explicit time dependence, then $\frac{\partial F}{\partial t}$=0 and hence

$$\mathbf{H' = H} \tag{8.56}$$

Thus, when the generating function has no explicit time dependence, the new Hamiltonian $\mathbf{H'}$ is obtained from the old Hamiltonian H by substituting for p_k, q_k in terms of the new variables P_k, Q_k. Further we note that the time t has been treated as an invariant parameter of the motion and we have not made any provision for a transformation of the time coordinate along with the other coordinates.

If in the new set of coordinates (P_k, Q_k, t) all coordinates Q_k are cyclic, then $\dot{P}_k = -\frac{\partial H'}{\partial Q_k}$=0 or, P_k=constant, say α_k (8.57)

If the generating function F does not depend on time t explicitly and H is a constant of motion then $\mathbf{H'}$ will also a constant of motion.

Hence, $\mathbf{H}(q_k, p_k) = \mathbf{H'}(Q_k, P_k) = \mathbf{H'}(P_k) = \mathbf{H'}(\alpha_1, \alpha_2, \ldots, \alpha_n)$

Hamilton's equations for Q_k are

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k} = \frac{\partial H'}{\partial \alpha_k} = \omega_k \tag{8.58}$$

Where ω_k's are functions of the α_k's only and are constant in time.

Equation (8.58) has the solution

$$Q_k = \omega_k t + \beta_k \tag{8.59}$$

8.5.5. Condition for canonical transformation

Suppose=F$(q_k, Q_k,)$ then obviously $\frac{\partial F}{\partial t}$=0, and H= $\mathbf{H'}$

$p_k = \frac{\partial F_1(q_k, Q_k, t)}{\partial q_k}$	$P_k = -\frac{\partial F_1(q_k, Q_k, t)}{\partial Q_k}$

Also dF = $\sum_k \frac{\partial F_2}{\partial q_k} dq_k + \sum_k \frac{\partial F_2}{\partial Q_k} dQ_k$

Or, dF = $\sum_k p_k dq_k - \sum_k P_k dQ_k$ (8.60)

The L.H.S. of above equation is an exact differential, hence for a given transformation to be canonical, the R.H.S must be an exact differential also.

8.5.6. Harmonic Oscillator In case of a Harmonic Oscillator, the Hamiltonian in terms of q and p coordinates can be expressed as

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m}$$

$$\text{Or, } H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \qquad (8.61)$$

Let us consider the generating function, given by

$$F_1(q,Q,t) = \frac{1}{2}\, m\omega q^2 \cot Q$$

$p_k = \frac{\partial F_1(q_k, Q_k, t)}{\partial q_k}$	$P_k = -\frac{\partial F_1(q_k, Q_k, t)}{\partial Q_k}$

$$\text{Hence, } p = m\omega q \cot Q, \qquad P = \frac{m\omega q^2}{2\sin^2 Q} \quad H' = H \qquad (8.62)$$

$$\text{Hence, } q = \sqrt{\frac{2P}{m\omega}}\sin Q, \quad p = \sqrt{2m\omega P}\cos Q \qquad (8.63)$$

Now, the transformation, H' is obtained by using equation (8.62) and (8.63) i.e.,

$$H' = H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) = \frac{1}{2m}(2m\omega P\cos^2 Q + 2m\omega P\sin^2 Q)$$

$$H' = H = \omega P \qquad (8.64)$$

Since the Hamiltonianis cylic in Q, the conjugate momentum P is constant. In fact $H = H' = E$ is the constant energy E so that

$$P = E/\omega \qquad (8.65)$$

Then the equation of motion for Q reduces to the simple form

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$

With the solution

$$Q = \omega t + \varphi \qquad (8.66)$$

Where φ is a constant of integration

Equations (8.63), (8.65) and (8.66) we have

$$q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \varphi) \qquad (8.67)$$

which is the solution of a harmonic oscillator.

8.5.7. Identity transformation

Let us consider the generating function of the form, $F=\sum_i q_i P_i$

This generating function looks like the second form, hence when we determine the p_i and Q_i, we get

$p_i = \partial F_2 / \partial q_i = P_i$ and $Q_i = \partial F_2 / \partial P_i = q_i$ therefore

again since F_2 is independent of time therefore $H' = H$

Thus the new and old variables are separately equal and hence F generates the identity transformation.

8.5.8. Bilinear invariant condition according to this condition, if a transformation (q_k,p_k) coordinates to (Q_k,P_k) coordinates is canonical, then bilinear form $\sum_k(\delta p_k\, dq_k - \delta q_k\, dp_k)$ remains invariant. This means that

$$\sum_k(\delta p_k\, dq_k - \delta q_k\, dp_k) = \sum_k(\delta P_k\, dQ_k - \delta Q_k\, dP_k) \qquad (8.68)$$

Proof- from Hamilton's canonical equations, we have

$\dot{q}_k = \frac{\partial H}{\partial p_k}$ or $dq_k = \frac{\partial H}{\partial p_k} dt$

and $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ or $dp_k = -\frac{\partial H}{\partial q_k} dt$, similarly, $dQ_k = \frac{\partial H}{\partial p_k} dt$ and $dP_k = -\frac{\partial H}{\partial Q_k} dt$

since δp_k and δq_k are arbitrary,

$$\sum_k \delta p_k\left(dq_k - \frac{\partial H}{\partial p_k} dt\right) - \sum_k \delta q_k\left(dp_k + \frac{\partial H}{\partial q_k} dt\right) = 0 \qquad (8.69)$$

Obviously in order to satisfy this equation, the coefficients of δp_k and δq_k must be zero and we may get Hamilton's equations again. Hence equation (8.68) is correct.

Equation (7.68) can be written as

$$\sum_k(\delta p_k\, dq_k - \delta q_k\, dp_k) - \sum_k\left(\frac{\partial H}{\partial p_k}\delta p_k + \frac{\partial H}{\partial q_k}\delta q_k\right) dt = 0$$

Or, $$\sum_k(\delta p_k\, dq_k - \delta q_k\, dp_k) - \delta H dt = 0 \qquad (8.70)$$

Similarly, for $H' = H$, when F does not depend on time,

$$\sum_k(\delta P_k\, dQ_k - \delta Q_k\, dP_k) - \delta H dt = 0 \qquad (8.71)$$

Eliminating $\delta H dt$ from equations (8.70) and (8.71), we obtain

$\sum_k(\delta p_k\, dq_k - \delta q_k\, dp_k) = \sum_k(\delta P_k\, dQ_k - \delta Q_k\, dP_k)$, which poves the statement.

8.5.9.Infinitesimal contact transformations Those transformations in which the new set of coordinates (Q_k,P_k) differ from the old set (q_k,p_k) by infinitesimals i.e., $Q_k = q_k+\delta q_k$ and $P_k = p_k+ \delta p_k$, are called infinitesimal contact transformations. As we know that the generating function $F_2=\sum_k q_k P_k$ generates the identity transformation i.e, $Q_k = q_k$ and $P_k = p_k$. The generating function, giving an infinitesimal change in the variables, can be readily written as

$$F_2=\sum_k q_k P_k + \epsilon G(q_k, P_k) \qquad (8.76)$$

Where ϵ is an infinitesimal parameter of the transformation and $G(q_k, P_k)$ is arbitrary. Now putting the value of F_2 in the following equations

$$p_k=\frac{\partial F_2}{\partial q_k},\ Q_k = \frac{\partial F_2}{\partial P_k} \text{ and } H' - H= \frac{\partial F_2}{\partial t}$$

We get $p_k=P_k+\epsilon \frac{\partial G}{\partial q_k},\ Q_k = q_k+\epsilon \frac{\partial G}{\partial P_k}$, and $H' = H$ (8.77)

Therefore, $P_k-p_k = \delta p_k = -\epsilon \frac{\partial G}{\partial q_k}$, and $Q_k-q_k = \delta q_k = \epsilon \frac{\delta G}{\partial p_k}$ (8.78)

Since the difference, (P_k-p_k) is infinitesimal. We can replace P_k by $p_k$in the derivative and also $G(q_k,P_k)$ by $G(q_k,p_k)$. So that the equations (8.78) are;

$$\delta p_k = -\epsilon \frac{\partial G}{\partial q_k}, \text{ and } \delta q_k = \epsilon \frac{\partial G}{\partial p_k}. \qquad (8.79)$$

In case of infinitesimal contact transformations, the description is transferred to the G instead of the original generating function F. Thus G is the new generating function which generates the infinitesimal contact transformation.

Let us consider a special case in which $\epsilon = dt$, and G=H. Equations (8.79) can be written by using Hamilton's equations of motion as

$$\delta p_k = -dt \frac{\partial H}{\partial q_k} = dt\dot{p}_k = dp_k$$

$$\text{and } \delta q_k = dt \frac{\partial H}{\partial p_k} = dt\dot{q}_k = dq_k \qquad (8.80)$$

these changes in the conjugate variables represent an infinitesimal change in coordinate in time dt. Equations (8.80) give thus a transformation from the variables q_k, p_k at time t to q_k+dq_k, p_k+dp_k at time t+dt. Hence the motion of the system in a small time dt can be described by an infinitesimal canonical transformation generated by the Hamiltonian H of the system. And the motion of the system in a finite interval of time is described by a succession of infinitesimal canonical transformations generated by the same Hamiltonian. In other words, the motion of a system corresponds to the continuous evolution of canonical

transformation. Thus we can say that the Hamiltonian of the system is the generator of the motion of the system in phase space with time.

8.6. Poisson Bracket: Hamilton's equations of motion for $\dot{q}$ and $\dot{p}$ give the time evolution of the coordinates and momenta of a system in phase space. Using these equations, we can find the equation of motion for any function F(q,p) in terms of what is known as Poisson Brackets. They are similar to commutator brackets in quantum mechanics and provide a bridge between classical mechanics and quantum mechanics.

The Poisson Bracket of any two functions F(q,p,t) and G(q,p,t) with respect to the canonical variables(q,p), written as $[F,G]_{q,p}$, is defined by

$$[F, G]_{q,p} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right) \qquad (8.81)$$

8.6.1. Fundamental Poisson brackets: The following three brackets are called the fundamental Poisson brackets.

First. $[q_j, q_k]_{q,p} = 0$

Proof: $[q_j, q_k]_{q,p} = \sum_i \left(\frac{\partial q_j}{\partial q_i}\frac{\partial q_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial q_k}{\partial q_i}\right) = 0$

(Since $\left(\frac{\partial q_k}{\partial p_i}\right) = \left(\frac{\partial q_j}{\partial p_i}\right) = 0$)

Second. $[p_j,p_k]_{q,p}=0$

Proof: $[p_j,p_k]_{q,p}=\sum_i \left(\frac{\partial p_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial p_k}{\partial q_i}\right) = 0$

Third. $[q_j,p_k]_{q,p}=\delta_{jk}$

Proof: $[q_j,p_k]_{q,p}=\sum_i \left(\frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i}\right)$ the second term inside the bracket is zero and $\left(\frac{\partial q_j}{\partial q_i}\right)=\delta_{ij}$, $\frac{\partial p_k}{\partial p_i} =\delta_{ik}$, hence, $[q_j,p_k]_{q,p}= \delta_{jk}$

8.6.2. Fundamental Properties of poisson brackets Let F,G and S be functions of canonical variables(q,p) and time. The following fundamental identities can be obtained from the Poisson bracket defined as $[F,G]_{q,p} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$;

(1). [F, F]=0

Proof:

$[F,F]= \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial F}{\partial q_i}\right)=0$

(2). [F,C]=0, where C is constant

(3). [F,G]=-[G,F]

Proof:

$$[F,G]=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)=-\sum_i\left(\frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}-\frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}\right)=-[G,F]$$

(4). [F,G+S]=[F,G]+[F,S]

Proof:

$$[F,G+S]=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial (G+S)}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial (G+S)}{\partial q_i}\right)$$

$$=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)+\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial S}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial S}{\partial q_i}\right)$$

=[F,G]+[F,S]

(5). [F,GS]=[F,G]S+G[F,S]

Proof:

$$[F,GS]=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial (GS)}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial (GS)}{\partial q_i}\right)$$

$$=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}S+G\frac{\partial F}{\partial q_i}\frac{\partial S}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}S-G\frac{\partial F}{\partial p_i}\frac{\partial S}{\partial q_i}\right)$$

$$=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)S+\sum_i G\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

=[F,G]S+G[F,S]

(6). One of the important properties of Poisson bracket is the **Jacobi identity** for any three functions: Which states that ‘‘**the sum of the cyclic permutations of the double Poisson Bracket of three functions is zero’’. i.e.,**

[F,[G,S]]+[G,[S,F]]+[S,[F,G]]=0. Which we will study in further section of this unit.

(7). $\frac{d}{dt}[F,G]=[\frac{dF}{dt},G]+[F,\frac{dG}{dt}]$

Proof: $\frac{d}{dt}\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\quad\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$

$$=\sum_i\left[\frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial p_i}+\frac{\partial F}{\partial q_i}\frac{d}{dt}\left(\frac{\partial G}{\partial p_i}\right)-\frac{d}{dt}\left(\frac{\partial F}{\partial p_i}\right)\frac{\partial G}{\partial q_i}-\frac{\partial F}{\partial p_i}\frac{d}{dt}\left(\frac{\partial G}{\partial q_i}\right)\right]$$

$$=\sum_i\left[\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial p_i}-\frac{\partial}{\partial p_i}\left(\frac{dF}{dt}\right)\left(\frac{\partial G}{\partial q_i}\right)+\left(\frac{\partial F}{\partial q_i}\right)\frac{\partial}{\partial p_i}\left(\frac{dG}{dt}\right)-\frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{dG}{dt}\right)\right]$$

$=[\frac{dF}{dt},G]+[F,\frac{dG}{dt}]$

A pair of functions for which the Poisson Bracket [F,G]=0 are said to be commute with each other.

8.6.3.Equations of motion in Poisson Bracket form-

Let us consider a function F which is a function of q's, and p's and time t:

F=F(q,p,t) from here we can write

$$\frac{dF}{dt}=\sum_i\left(\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial p_i}\dot{p}_i\right)+\frac{\partial F}{\partial t}$$

Replacing $\dot{q}_i$ and $\dot{p}_i$ using Hamilton's equations

$$\frac{dF}{dt}=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right)+\frac{\partial F}{\partial t}$$

$$\frac{dF}{dt}=[F,H]+\frac{\partial F}{\partial t} \qquad (8.82)$$

Which is the equation of motion of F in terms of Poisson bracket. In equation (8.82), H is the Hamiltonian of the system. If F is replaced by q_j and p_j, then equation (8.82) gives,

$\dot{q}_j=[q_j,H]$ and $\dot{p}_j=[p_j,H]$ (when q_j and p_j do not depend explicitly on t) (8.83)

These two equations constitute the canonical equations of motion in Poisson bracket form.

8.6.4.Poisson Bracket and Constants of motion-

One of the important uses of Poisson Brackets is finding the integrals of motion. Let us consider the equation, $\frac{dF}{dt}=[F,H]+\frac{\partial F}{\partial t}$. For F to be constant of motion, dF/dt must be equal to zero. Hence, $[F,H]+\frac{\partial F}{\partial t}=0$ (8.84)

If the constant of motion F does not contain t explicitly, above equation reduces to

$$[F,H]=0 \qquad (8.85)$$

i.e., when the constant of motion does not contain t explicitly, its Poisson bracket with the Hamiltonian of the system vanishes. Conversely, the Poisson brackets of constants of motion with the Hamiltonian H must be zero.

8.7.Poisson Theorem- One of the important property of Poisson bracket is Poisson theorem **according to this theorem, the Poisson bracket of any two constants of the motion is also a constant of motion.** Or mathematicall,which states that if F(q,p,t) and G(q,p,t) are two constants of motion, then [F,G] is also a constant of motion. i.e.,

[F,G]=constant (8.86)

Since F and G are constants of motion

$\frac{dF}{dt}$ =[F,H]+ $\frac{\partial F}{\partial t}$=0 or, $\frac{\partial F}{\partial t}$= -[F,H] (8.87)

$\frac{dG}{dt}$ =[G,H]+ $\frac{\partial G}{\partial t}$=0 or, $\frac{\partial G}{\partial t}$ = -[G,H] (8.88)

Now, let us consider the time derivative of the Poisson bracket [F,G].(With the help of equation, $\frac{dF}{dt}$ = [F,H]+ $\frac{\partial F}{\partial t}$)

$\frac{d}{dt}[F,G]$ =[[F,G],H]+ $\frac{\partial}{\partial t}[F,G]$

Now using expression, $\frac{d}{dt}[F,G]$=[$\frac{dF}{dt}$,G]+[F,$\frac{dG}{dt}$],

$\frac{d}{dt}[F,G]$=[[F,G],H]+[$\frac{\partial F}{\partial t}$, G] +[F, $\frac{\partial G}{\partial t}$]

Now replacing $\frac{\partial F}{\partial t}$ and $\frac{\partial G}{\partial t}$ from equations (7.87) and (7.88) in above expression, we get

$\frac{d}{dt}[F,G]$=[[F,G],H]+[[H,F],G]+[[G,H],F]

Now right side of above expression will become zero by applying the Jacobi's identity.

Hence, $\frac{d}{dt}[F,G]$=0

It means that, $[F,G]$=constant.

8.8.Jacobi's identity One of the important properties of Poisson bracket is the Jacobi identity for any three functions as we stated earlier that the sum of the cyclic permutations of the double Poisson Bracket of three functions is zero, i.e.,

[F,[G,S]]+[G,[S,F]]+[S,[F,G]]=0

Proof: The proof of this theorem has a lengthy algebraic process, there are some short cut methods too to prove this identity but those methods comprise of some other identities. It might possible that the reader might not familiar with those identities so we will go through this identity via same lengthy procedure this time.

Let us start from an expression in Poisson bracket form [F,[G,K]]-[G,[F,K]]

$=[F, \sum_i \left(\frac{\partial G}{\partial q_i}\frac{\partial K}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial K}{\partial q_i}\right)] - [G, \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial K}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial K}{\partial q_i}\right)]$

$=[F, \sum_i (\frac{\partial G}{\partial q_i}\frac{\partial K}{\partial p_i})]-[F, \sum_i (\frac{\partial G}{\partial p_i}\frac{\partial K}{\partial q_i})] - [G, \sum_i (\frac{\partial F}{\partial q_i}\frac{\partial K}{\partial p_i})] + [G, \sum_i (\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial q_i})]$

Now, using the property[F,GK]=[F,G]K+[F,K]G, we have

$[F,[G,K]]-[G,[F,K]]=[F, \sum_i \frac{\partial G}{\partial q_i}] \sum_i \frac{\partial K}{\partial p_i} + [F, \sum_i \frac{\partial K}{\partial p_i}] \sum_i \frac{\partial G}{\partial q_i}$-[F,$\sum_i \frac{\partial G}{\partial p_i}$]$\sum_i \frac{\partial K}{\partial q_i}$-[F, $\sum_i \frac{\partial K}{\partial q_i}] \sum_i \frac{\partial G}{\partial p_i}$ –
[G, $\sum_i \frac{\partial F}{\partial q_i}] \sum_i \frac{\partial K}{\partial p_i}$-[G, $\sum_i \frac{\partial K}{\partial p_i}] \sum_i \frac{\partial F}{\partial q_i}$+[G, $\sum_i \frac{\partial F}{\partial p_i}] \sum_i \frac{\partial K}{\partial q_i}$+[G, $\sum_i \frac{\partial K}{\partial q_i}] \sum_i \frac{\partial F}{\partial p_i}$

$=\sum_i \{-\frac{\partial K}{\partial q_i}([\frac{\partial F}{\partial p_i}, G] + [F, \frac{\partial G}{\partial p_i}]) + \frac{\partial K}{\partial p_i}([\frac{\partial F}{\partial q_i}, G] + [F, \frac{\partial G}{\partial q_i}])\} + \sum_i \{\frac{\partial G}{\partial q_i}[F, \frac{\partial K}{\partial p_i}] - \frac{\partial G}{\partial p_i}[F, \frac{\partial K}{\partial q_i}])$-
$\frac{\partial F}{\partial q_i}([G, \frac{\partial K}{\partial p_i}] + \frac{\partial F}{\partial p_i}[G, \frac{\partial K}{\partial q_i}]\}$

$=\sum_i [-\frac{\partial K}{\partial q_i}\frac{\partial}{\partial p_i}[F,G] + \frac{\partial K}{\partial p_i}\frac{\partial}{\partial q_i}[F,G]] + 0$ (since Using the identity $\frac{\partial}{\partial x}$[F,G]=[$\frac{\partial F}{\partial x}$, G]+[F,$\frac{\partial G}{\partial x}$])

= - [K,[F,G]]

Thus, [F,[G,K]]+[G,[K,F]]+[K,[F,G]]=0

Which is called the Jacobi's identity.

8.8.1.The Canonical invariance of Poisson bracket- Probably the most important property of poisson bracket is that it is invariant under canonical transformation. This means that if (q,p) and (Q,P) are two canonically conjugate sets, then

$[F,G]_{q,p} = [F,G]_{Q,P}$ (8.89)

Where F and G are any pair of functions of (q,p) or (Q,P). The (q,p) and (Q,P) sets are related by a canonical transformation of the type given in equation as

$Q_i = Q_i(q,p,t)$ And $P_i = P_i(q,p,t)$

The Poisson bracket of the functions F and G with respect to the (q,p) set is given by

$[F,G]_{q,p} = \sum_k (\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k})$ (8.90)

Let us consider the function, G=G(Q,P)

$\frac{\partial G}{\partial p_k} = \sum_i (\frac{\partial G}{\partial Q_i}\frac{\partial Q_i}{\partial p_k} + \frac{\partial G}{\partial P_i}\frac{\partial P_i}{\partial p_k})$ (8.91)

$$\frac{\partial G}{\partial q_k} = \sum_i \left(\frac{\partial G}{\partial Q_i}\frac{\partial Q_i}{\partial q_k} + \frac{\partial G}{\partial P_i}\frac{\partial P_i}{\partial q_k}\right) \quad (8.92)$$

Substituting equations (8.91) and (8.92) in equation (8.90), we have,

$$[F,G]_{q,p} = \sum_k \left[\frac{\partial F}{\partial q_k}\sum_i\left(\frac{\partial G}{\partial Q_i}\frac{\partial Q_i}{\partial p_k} + \frac{\partial G}{\partial P_i}\frac{\partial P_i}{\partial p_k}\right) - \frac{\partial F}{\partial p_k}\sum_i\left(\frac{\partial G}{\partial Q_i}\frac{\partial Q_i}{\partial q_k} + \frac{\partial G}{\partial P_i}\frac{\partial P_i}{\partial q_k}\right)\right] \quad (8.93)$$

$$= \sum_k \sum_i \left(\frac{\partial G}{\partial Q_i}\left(\frac{\partial F}{\partial q_k}\frac{\partial Q_i}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial Q_i}{\partial q_k}\right) + \frac{\partial G}{\partial P_i}\left(\frac{\partial F}{\partial q_k}\frac{\partial P_i}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial P_i}{\partial q_k}\right)\right)$$

$$= \sum_i \left\{\frac{\partial G}{\partial Q_i}[F,Q_i]_{q,p} + \frac{\partial G}{\partial P_i}[F,P_i]_{q,p}\right\} \quad (8.94)$$

Replacing F by Q_k and G by F

$$[Q_k,F]_{q,p} = \sum_i \left\{\frac{\partial G}{\partial Q_i}[Q_k,Q_i]_{q,p} + \frac{\partial F}{\partial P_i}[Q_k,P_i]_{q,p}\right\} \quad (8.95)$$

Since $[Q_k, Q_i]=0$ and $[Q_k, P_i]=\delta_{ik}$

$$[Q_k,F]_{q,p} = -\frac{\partial F}{\partial P_k} \quad \text{or} \quad [F,Q_i]_{q,p} = -\frac{\partial F}{\partial P_i} \quad (8.96)$$

In the same way replacing F by P_k and G by F in the same expression, we have

$$[P_k,F]_{q,p} = -\frac{\partial F}{\partial Q_k} \quad \text{or} \quad [F,P_i]_{q,p} = \frac{\partial F}{\partial Q_i} \quad (8.97)$$

Substituting equations (8.96) and (8.97) in equation (8.95), we have

$$[F,G]_{q,p} = \sum_i \left(\frac{\partial F}{\partial Q_i}\frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial Q_i}\right)$$

$$= [F,G]_{Q,P}$$

Thus, Poisson bracket description of mechanics is invariant under a canonical transformation. Therefore, a canonical transformation can be defined as one that preserves the Poisson bracket description of mechanics.

8.9.Lagrange Brackets- In addition to Poisson bracket, other canonical invariants exist. One such invariant is the Lagrange bracket.

The Lagrange bracket of any two functions F(q,p) and G(q,p) with respect to (q,p) variables, written as $\{F,G\}_{q,p}$, is defined as

$$\{F,G\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial F}\frac{\partial p_i}{\partial G} - \frac{\partial p_i}{\partial F}\frac{\partial q_i}{\partial G}\right) \quad (8.98)$$

The lagrange brackets are invariant under canonical transformations. That is,

$$\{F,G\}_{q,p} = \{F,G\}_{Q,P} \quad (8.99)$$

Hence, the subscripts (q,p) or (Q,P) may be dropped.

From equation (8.98), $\{G,F\}_{q,p} = -\sum_i \left(\frac{\partial q_i}{\partial G}\frac{\partial p_i}{\partial F} - \frac{\partial p_i}{\partial G}\frac{\partial q_i}{\partial F}\right)$

$$\{F,G\}_{q,p} = -\{G,F\}_{q,p} \quad (8.100)$$

If we take $F=q_k$ and $G = q_l$, from equation (7.98)

$$\{q_k , q_l\} = \sum_i \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_i}{\partial q_l} - \frac{\partial p_i}{\partial q_k}\frac{\partial q_i}{\partial q_l}\right)$$

$$= 0 \quad (8.101)$$

In the same way, $\{p_k , p_l\} = 0$ (8.102)

Taking $F=q_k$ and $G= p_l$ in equation (7.98),

$$\{q_k , p_l\} = \sum_i \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_i}{\partial p_l} - \frac{\partial p_i}{\partial q_k}\frac{\partial q_i}{\partial p_l}\right) = \sum_i \frac{\partial q_i}{\partial q_k}\frac{\partial p_i}{\partial p_l}$$

$$= \delta_{ik}\delta_{il} = \delta_{kl} \quad (8.103)$$

Equations (8.98) to (8.103) are called the fundamental lagrange brackets.

The definitions of Poisson and Lagrange brackets clearly indicate some kind of inverse relationship between the two. The relation between the two is given by

$$\sum_{l=}^{2n} [r_l, r_i][r_l, r_j] = \delta_{ij} \quad (8.104)$$

Lagrange brackets do not obey Jacobi's identity.

8.10 worked example

Ex.1 show that the transformation

$P=\frac{1}{2}(p^2 + q^2), Q = \tan^{-1}\frac{q}{p}$ is canonical.

The transformation will be canonical, if pdq-PdQ is an exact differential. Here

$dQ=\frac{(pdq-qdp)}{p^2+q^2}$

therefore, pdq- PdQ $= pdq- \frac{1}{2}(p^2 + q^2)\frac{(pdq-qdp)}{(p^2+q^2)}$

$= \frac{1}{2}(pdq+qdp)$

$=d(\frac{1}{2}pq)$

= an exact differential

This means that the given transformation is canonical.

Ex.(2): Show that the transformation $Q = \frac{1}{p}$, and $P = qp^2$ is canonical.

Solution: $Q=\frac{1}{p}$, therefore, $dQ=\frac{\partial Q}{\partial p}dp+\frac{\partial Q}{\partial q}dq$

Or, $dQ=\frac{\partial}{\partial p}(1/p)dp+\frac{\partial}{\partial q}(1/p)dq = -\frac{1}{p^2}dp$ (8.72)

$\delta Q=\frac{\partial Q}{\partial p}\delta p+\frac{\partial Q}{\partial q}\delta q =-\frac{1}{p^2}\delta p$ (8.73)

Similarly, $dP=p^2dq +2qpdp$ (8.74)

And $\delta P = p^2\delta q +2qp\ \delta p$ (8.75)

Hence, $\delta P dQ - \delta Q dP= (p^2\delta q +2qp\ \delta p)\ (-\frac{1}{p^2}dp) -(-\frac{1}{p^2}\delta p)\ (p^2dq +2qpdp)$

$= -\delta q dp-\frac{2q}{p}\delta p\ dp+ \delta p dq+ \frac{2q}{p}\delta p\ dp$

$= \delta p dq- \delta q dp.$

Therefore, the bilinear form is invariant and hence the transformation is canonical.

Ex.3 Using the Poisson Bracket, show that the transformation

$Q=\sqrt{2P}\sin Q$ and $p= \sqrt{2P}\cos Q$ is canonical.

Solution: from the definition of Poisson bracket, it is obvious that [Q,Q]=0 and [P,P]=0.

From the given data we have

$\tan Q= \frac{q}{p}$ and $2P = q^2 +p^2$

$\sec^2 Q\frac{\partial Q}{\partial q}= \frac{1}{p}$ or $\frac{\partial Q}{\partial q} = \frac{\cos^2 Q}{p}$

$\sec^2 Q\frac{\partial Q}{\partial p}= -\frac{q}{p^2}$ or $\frac{\partial Q}{\partial p} = -\frac{q}{p^2}\cos^2 Q$

$\frac{\partial p}{\partial q} = q$ and $\frac{\partial p}{\partial p} = p$

$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial p}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial p}{\partial q}$

$= (\frac{\cos^2 Q}{p})p + \frac{q^2}{p^2}\cos^2 Q$

$= \cos^2 Q(1 + \frac{q^2}{p^2})$

$= \cos^2 Q(1 + \tan^2 Q)$

$=1$

Hence, the transformation is canonical.

Ex.4 For what values of α and β

$Q = q^{\alpha}\cos\beta p$ and $P= q^{\alpha}\sin\beta p$ represent a canonical transformation. Also find the generator of the transformation.

Soution: $\frac{\partial Q}{\partial q} = \alpha q^{\alpha-1}\cos\beta p$ $\frac{\partial P}{\partial q} = \alpha q^{\alpha-1}\cos\beta p$

$\frac{\partial Q}{\partial p} = -\beta q^{\alpha}\sin\beta p$ $\frac{\partial p}{\partial p} = \beta q^{\alpha}\cos\beta p$

$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial p}{\partial p}$

$= \alpha\beta q^{2\alpha-1}\cos^2\beta p + \alpha\beta q^{2\alpha-1}\sin^2\beta p$

$= \alpha\beta q^{2\alpha-1}$

For the transformation to be canonical, this must be equal to 1. Hence,

$\alpha\beta q^{2\alpha-1}=1$

R.H.S of this equation is dimensionaless. Hence, $q^{2\alpha-1}$ should be. Therefore,

$2\alpha-1=0$

Or $\alpha = ½$ and

from $\alpha\beta =1$ gives $\beta = 2$.

Hence $Q = q^{½}\cos\beta p$ $P= q^{½}\sin\beta p$

$Q = q^{½}\cos\beta p$ gives the value of q as; $q = \frac{Q^2}{\cos^2 2p}$. we see that q is a function of p and Q so the generating function must be of the type

$F= F_3(p,Q)$. Hence $q = -\frac{\partial F_3}{\partial p}$ so $F_3 = -\int q\,dp$

$F_3 = -\int \frac{Q^2}{\cos^2 2p}dp = -\frac{1}{2}Q^2\tan 2p$

8.13 Summary- you have learned in this unit a different type of variation from δ-variation which is called the Δ-variation and you learned the principle of least action. In this unit you also learned the canonical transformation and need of such type of transformations in classical mechanical problems thoroughly. Along with an infinitesimal contact transformation also has been discussed. you learned the generating functions and the reason why they are called so. You also learned an important concept the Poisson bracket which has a close association with quantum mechanics and how these commutators has been revolutionised the modern physics. You also learned how one can write the equations of motion in terms of the Poisson Brackets. A bracket which is different from Poisson bracket called the Lagrange brecket and its properties are also discussed.

8.14. Model Questions

Multiple Choice Questions

(1) Phase-space refers to

(a) Position coordinates (b) momentum coordinates

(b) Both position and momentum coordinates (c) None of these

(2) The Poisson bracket of [$|\vec{r}|,|\vec{p}|$] has the value

(a) $|\vec{r}||\vec{p}|$ (b) $\hat{r}.\hat{p}$ (c) 3 (d) 1

(3) The Lagrangian for a simple pendulum is given by, L= $\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$. The Poisson Bracket between θ and $\dot{\theta}$ is given by

(a) $[\theta,\dot{\theta}] = 1$ (b) $[\theta,\dot{\theta}] = \frac{1}{ml^2}$

(c) $[\theta,\dot{\theta}] = \frac{1}{m}$ (d) $[\theta,\dot{\theta}] = \frac{g}{l}$

(4) Let (p,q) and (P,Q) be two pairs of canonical variables. The transformation

Q = $q^{\alpha}\cos(\beta p)$, P = $q^{\alpha}\sin(\beta p)$, is canonical for

(a) $\alpha = 2, \beta = 1/2$ (b) $\alpha = 2, \beta = 2$ (c) $\alpha = 1, \beta = 2$ (d) $\alpha = 1/2, \beta = 2$

(5)The Hamilton's canonical equation of motion in terms of Poisson Brackets are

(a) $\dot{q}$ =[q,H]; $\dot{p}$ =[p,H] (b) $\dot{q}$ =[H, q]]; $\dot{p}$ =[H, p]

(b) $\dot{q}$ =[H, p]; $\dot{p}$ =[H, p] (d) $\dot{q}$ =[p, H]; $\dot{p}$ =[q, H]

(6) which of the following is the correct representation of the Poisson bracket and commutator relationship

(a) $[F, H] = -i/\hbar[\hat{F}, \hat{H}]$ (b) $[F,H] = [\hat{F}, \hat{H}]$

(c) $[F, H] = -[\hat{F}, \hat{H}]$ (d) $[F,H] = i/\hbar[\hat{F}, \hat{H}]$

Answers

(1)-b, (2)-b, (3)-b, (4)-d, (5)-a, (6)-a

Answers for self-check:, (1). See the article 7.3,(2) search in google, (3) search in google, (4) read the article7.5.2, (5). pairs of variables, related by the Heisenberg uncertainty principle, (6). Commutator, (7). Are associated with the conserve quantities,(8). yes, can be derived by the use of some higher mathematical identities, like a research article a concise proof by R.P.malik, (9) search in google.

Short answer type Questions

1. Define the canonical transformations
2. Define the Poisson bracket. What are the use of Poisson brackets.
3. What is Jacobi's identity. Where this identity is used.

Long answer type questions

1. Define canonical transformations and obtain the transformation equations corresponding to all possible generating functions.
2. What is generating function?. Obtain canonical transformation equations corresponding to first two types of generating functions
3. Derive equations of motion in terms of Poisson's bracket. Prove Jacobi identity

8.15.References

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UNIT 9 HAMILTON-JACOBI THEORY

Structure REPLACE 8 BY 9

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9.1 INTRODUCTION

A branch of classical variational calculus and analytical mechanics in which the task of finding extremals (or the task of integrating a Hamiltonian system of equations) is reduced to the integration of a first-order partial differential equation — the so-called Hamilton–Jacobi equation. The fundamentals of the Hamilton–Jacobi theory were developed by W. Hamilton in the 1820s for problems in wave optics and geometrical optics. In 1834 Hamilton extended his ideas to problems in dynamics, and C.G.J. Jacobi (1837) applied the method to the general problems of classical variational calculus.

Hamilton-Jacobi equation is an alternative formulation of classical mechanics, equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics and Hamiltonian mechanics. Hamilton–Jacobi equation is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely.

Hamilton–Jacobi equation is also the only formulation of mechanics in which the motion of a particle can be represented as a wave. One can say that in this manner, Hamilton–Jacobi equation fulfilled a long-held aim of theoretical physics dating back to Johann Bernoulli in the 18th century, of finding an analogy between the propagation of light and the motion of a particle. The wave equation followed by mechanical systems is similar to, but not identical with, Schrödinger's equation; for this reason, Hamilton–Jacobi equation is considered the "closest approach" of classical mechanics to quantum mechanics.

9.2 OBJECTIVES

After studying this unit, you should be able to

- understand the Hamilton-Jacobi equation
- explain how the Hamilton-Jacobi equation compares with other formulations of classical mechanics
- explain how the Hamilton–Jacobi equation is useful in identifying conserved quantities for mechanical systems
- discuss action-angle variables
- discuss harmonic oscillator in the Hamilton–Jacobi theory
- describe motion of a particle in a central force field using the Hamilton–Jacobi theory

9.3 HAMILTON-JACOBI THEORY

We have already discussed the time-dependent canonical transformation, which maps the coordinates of a system at a given fixed time to their values at a later time t. Now, we can consider the reverse transformation, mapping

$$\{q(t), p(t)\} \rightarrow \{Q = q_0, P = p_0\}$$

But then $\dot{Q} = 0$, $\dot{P} = 0$ and the Hamiltonian, which generates these trivial equations of motion is $K = 0$.

9.3.1 Hamilton-Jacobi Equation

Consider a canonical transformation from *q-p* basis to *Q-P* basis, changing the Hamiltonian $H(q,p,t)$ to $K(Q,P,t)$; then the equation of motion in terms of the new Hamiltonian are given by

$$\dot{Q}_j = \frac{\partial K}{\partial P_j},$$

$$\dot{P}_j = -\frac{\partial K}{\partial Q_j}, \tag{9.1}$$

$$K = H + \frac{\partial F_j}{\partial t}$$

where F_j is the generating function of the transformation.

If the new variables Q_j and P_j are cyclic variables, then $\dot{Q}_j = 0$ and $\dot{P}_j = 0$, i.e.

$$Q_j = constant,$$
$$P_j = constant \tag{9.2}$$

Since, K contains $Q's$ and $P's$ as constants but t is not constant, therefore, let $K = 0$. Then the generating function F satisfies the following equation

$$H(q,p,t) + \frac{\partial F_j}{\partial t} = 0 \tag{9.3}$$

It is convenient to take F_j as $F_2(q,p,t)$ type. Then

$$p_j = \frac{\partial F_2}{\partial q_j}$$

and equation (9.3) assumes the form

$$H\left(q_1, q_2 \ldots, q_n, \frac{\partial F_2}{\partial q_1}, \frac{\partial F_2}{\partial q_2} \ldots, \frac{\partial F_2}{\partial q_n}, t\right) + \frac{\partial F_2}{\partial t} = 0 \tag{9.4}$$

Equation (9.4) is known as the *Hamilton-Jacobi equation*, which is usually written in terms of the Hamilton's principle function S (or action) integral.

$$H\left(q_1, q_2 \ldots, q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2} \ldots, \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{9.5}$$

This is a first order partial differential equation in S involving $(n+1)$ independent variables $q_1, q_2 \ldots, q_n$ and t. Thus

$$S = S(q_1, q_2 \ldots, q_n, t)$$

In order to understand the physical significance of S, we write the total time derivative of S as

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j}\dot{q}_j + \frac{\partial S}{\partial t}$$

$$= \sum_j p_j \dot{q}_j - H = L$$

where

$$p_j = \frac{\partial S}{\partial q_j}$$

and

$$\frac{\partial S}{\partial t} = -H$$

Thus,

$$S = \int L dt + constant \qquad (9.6)$$

9.3.2 Complete Integral of the Hamilton-Jacobi Equation

The Hamilton-Jacobi equation is a first order partial differential equation of *n* independent coordinates $q_1, q_2 \ldots, q_n$ and t. Therefore, for a system with *n* degrees of freedom, a complete integral of this equation contains $(n+1)$ constants as $\alpha_1, \alpha_2 \ldots, \alpha_n, \alpha_{n+1}$.

In Hamilton-Jacobi equation, only $\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2} \ldots, \frac{\partial S}{\partial q_n}$ and $\frac{\partial S}{\partial t}$ type derivatives are involved. One of these constants, say $\alpha_{n+1} = \alpha_0$ is taken as additive constant. The complete integral of the Hamilton-Jacobi equation is

$$S = S(q_1, q_2 \ldots, q_n, \alpha_1, \alpha_2 \ldots, \alpha_n, t) + \alpha_0$$

The additive constant α_0 is neglected as it plays no physically significant role.

Consider a canonical transformation from old variables (q, p) to new variables (Q, P), taking $S(q, \alpha, t)$ as the generating function. The constants $\alpha_1, \alpha_2 \ldots, \alpha_n$ are treated as the new momenta. Then

$$p_j = \frac{\partial S}{\partial q_j} \tag{9.7}$$

$$Q_j = \beta_j = \frac{\partial S}{\partial \alpha_j} \qquad (i = 1,2,3 \dots) \tag{9.8}$$

$$K = H + \frac{\partial S}{\partial t} \tag{9.9}$$

where $\beta_1, \beta_2 \dots, \beta_n$ are the new coordinates.

Since, S satisfies the Hamilton-Jacobi equation, so $K = 0$. Hence, the Hamilton's canonical equations in the new variables are

$$\dot{\alpha}_n = -\frac{\partial K}{\partial \beta_j} = 0, \qquad \beta_j = \frac{\partial H}{\partial \alpha_j} = 0 \tag{9.10}$$

which gives

$$\alpha_j = constant, \qquad \beta_j = constant \tag{9.11}$$

Thus, the new coordinates β_j are constants and could be initial values of $q_1, q_2 \dots, q_n$.

Equation (9.8) can also be written as

$$q_j = q_j(\alpha, \beta, t) \tag{9.12}$$

$$and \qquad p_j = p_j(\alpha, \beta, t) \tag{9.13}$$

Thus, we get the solutions of the problem in equations (9.12) and (9.13). The Hamilton-Jacobi equation is valid when the constraints of the problem are holonomic.

9.3.3 Hamilton's Characteristic Function

When the system is conservative, i.e. the Hamiltonian H does not involve time explicitly, then $H = E$ and the Hamilton-Jacobi equation can be written as

$$H\left(q_1, q_2 \dots, q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2} \dots, \frac{\partial S}{\partial q_n}\right) + \frac{\partial S}{\partial t} = 0 \tag{9.14}$$

We can solve the above equation by separating the t from the q dependence and we may write

$$S(q,\alpha,t) = W(q,\alpha) - \alpha_1 t \tag{9.15}$$

where α_1 is the separation constant independent of q and t, but not necessarily of p (which may be constant).

As the momenta

$$p_j = \frac{\partial S}{\partial q_j}$$

we can write

$$p_j = \frac{\partial S}{\partial q_j} = \frac{\partial W}{\partial q_j} \tag{9.16}$$

and the Hamilton-Jacobi equation becomes

$$H\left(q_1, q_2 \ldots, q_n, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2} \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1 = constant \tag{9.17}$$

where $\alpha_1 = E$ $(constant)$.

The physical significance of W may be understood by estimating dW/dt. Thus,

$$\frac{dW}{dt} = \sum_j \frac{\partial W}{\partial q_j}\dot{q}_j = \sum_j p_j \dot{q}_j \tag{9.18}$$

On integration, we get

$$W = \int \sum_j p_j \dot{q}_j \, dt$$

$$or \quad W = \int \sum_j p_j \, dq_j = S_0 \tag{9.19}$$

where S_0 is the action.

We now consider a canonical transformation in which the new momenta are the constants α_j and $\alpha_1 = H$. Let $W(q,p)$ be a part of the generating function for this transformation. Then

$$p_j = \frac{\partial S}{\partial q_j} = \frac{\partial W}{\partial q_j}$$

$$Q_j = \frac{\partial S}{\partial P_j} = \frac{\partial W}{\partial \alpha_j} \tag{9.20}$$

with the condition that $H(q_j, p_j) = \alpha_1$.

$$or \qquad H\left(q, \frac{\partial W}{\partial q_j}\right) = \alpha_1 \tag{9.21}$$

But the function W is time independent. Therefore,

$$K = H = \alpha_1 \tag{9.22}$$

Then the canonical equations for Q_j and P_j are

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} = 1 \qquad for\ i = 1$$

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} = 0 \qquad for\ i \neq 1 \tag{9.23}$$

and

$$\dot{P}_j = \frac{\partial K}{\partial Q_j} = 1$$

$$or \qquad P_j = \alpha_j \tag{9.24}$$

Integrating equation (9.23) using equations (9.20), we get

$$Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \frac{\partial W}{\partial E} \tag{9.25a}$$

$$Q_j = \beta_j = \frac{\partial W}{\partial \alpha_j} \qquad (j \neq 1) \tag{9.25b}$$

Thus, we see that the function $W(q_j, \alpha_j)$ generates a canonical transformation in which all the new coordinates except one are cyclic. Q_1 is not a constant of motion. This shows that t and E are canonical conjugate variables.

9.3.4 Comparison with other Formulations of Mechanics

Hamilton-Jacobi equation is a single, first-order partial differential equation for the function S of the N generalized coordinates $q_1 ... q_N$ and the time t. The generalized momenta do not appear, except as derivatives of S. Remarkably, the function S is equal to the classical action.

For comparison, in the equivalent Lagrange equations of motion of Lagrangian mechanics, the conjugate momenta also do not appear; however, those equations are a system of N, generally second-order equations for the time evolution of the generalized coordinates. Similarly, Hamilton's equations of motion are another system of 2N first-order equations for the time evolution of the generalized coordinates and their conjugate momenta $p_1 ... p_N$.

Since, Hamilton-Jacobi equation is an equivalent expression of an integral minimization problem such as Hamilton's principle, Hamilton-Jacobi equation can be useful in other

problems of the calculus of variations and, more generally, in other branches of mathematics and physics, such as dynamical systems, symplectic geometry and quantum chaos.

9.4 APPLICATIONS OF HAMILTONIAN-JACOBI METHOD

9.4.1 Harmonic Oscillator Problem

Consider a one-dimensional harmonic oscillator. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{9.26}$$

where k is the force constant.

The Hamilton-Jacobi equation is given as

$$H\left(q_j, \frac{\partial S}{\partial q_j}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{9.27}$$

By setting

$$p = \frac{\partial S}{\partial q}$$

and taking H from equation (9.26), we get

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2} + \frac{\partial S}{\partial t} = 0 \tag{9.28}$$

Since, the explicit dependence of S on t is involved only in the last term, a solution of the above equation may be found in the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \tag{9.29}$$

where α is a constant of integration. With this choice of the solution, t can be eliminated from equation (9.28) and it reduces to

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{kq^2}{2} = \alpha \tag{9.30}$$

$$or \quad \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 = \left(\alpha - \frac{kq^2}{2}\right)$$

$$or \quad \frac{\partial W}{\partial q} = \sqrt{2m\left(\alpha - \frac{kq^2}{2}\right)}$$

$$or \qquad W = \sqrt{mk} \int \sqrt{\frac{2\alpha}{k} - q^2}\, dq$$

From equation (9.29),

$$S - \sqrt{mk} \int \sqrt{\frac{2\alpha}{k} - q^2}\, dq - \alpha t \tag{9.31}$$

We know

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}$$

Therefore,

$$p = \sqrt{mk} \sqrt{\frac{2\alpha}{k} - q^2} \tag{9.32}$$

Now,

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t$$

Therefore,

$$t + \beta = \frac{\partial W}{\partial \alpha} = \sqrt{\frac{m}{k}} \int \frac{dq}{\sqrt{\frac{2\alpha}{k} - q^2}}$$

Putting

$$\frac{2\alpha}{k} = a^2$$

and

$$q = a \sin\theta$$

in the above equation, we get

$$\iota + \beta - \frac{\partial W}{\partial \alpha} - \sqrt{\frac{m}{k}} \int \frac{a \cos\theta}{\sqrt{a^2 - a^2 \sin^2\theta}} d\theta$$

$$or \qquad t + \beta = \sqrt{\frac{m}{k}}\, \theta$$

$$or \quad t+\beta = \frac{\theta}{\omega}$$

$$where\ \omega = \sqrt{\frac{k}{m}}$$

$$or \quad t+\beta = \frac{1}{\omega}\sin^{-1}\frac{q}{a}$$

$$or \quad \frac{q}{a} = \sin\omega(t+\beta)$$

$$or \quad q - \sqrt{\frac{2\alpha}{k}}\sin\omega(t+\beta) \tag{9.33}$$

The above equation is the familiar solution for a harmonic oscillator. The constants α and β must be connected with the initial conditions q_0 and p_0 at time $t = 0$. The dimension of β is time.

Let the particle be at $q = 0, p = p_0$ at $t = 0$. From equation (9.32)

$$p_0 = \sqrt{mk}\sqrt{\frac{2\alpha}{k} - 0} = \sqrt{2mE}$$

$$or \quad \alpha = E \tag{9.34}$$

The constant α is therefore, the initial total energy of the system, while β is time. The conjugate momenta of the generalized coordinate in time is the energy.

From equation (9.33)

$$0 = \sqrt{\frac{2\alpha}{k}}\sin\omega\beta$$

$\beta = 0$ under the given initial conditions. So now the solution becomes

$$q = \sqrt{\frac{2E}{k}}\sin\omega t \tag{9.35}$$

One can show that $S = \int L dt$ on the basis of the above equation in agreement with the general relation (9.32).

9.4.2 Motion of a Particle in a Central Force Field

Consider a particle of mass *m* moving in a central force field. The Hamiltonian of the particle is given by

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\varphi^2}{r^2}\right] + V(r) \tag{9.36}$$

Since, H does not involve time, the Hamiltonian is equal to the total energy, $H = E$. Then, the Hamilton-Jacobi equation is written as

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] + V(r) = 0 \tag{9.37}$$

The solution of equation (9.37) is

$$S - W(r, \varphi, \alpha_r, \alpha_\varphi) - Et \tag{9.38}$$

where W is the Hamilton's characteristic function, which satisfies the Hamilton-Jacobi equation. Therefore, we get

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2\right] + V(r) = E \tag{9.39}$$

The function W can be written as $W = W_r(r) + W_\varphi(\varphi)$. Substituting it in equation (9.39), we get

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W_\varphi}{\partial \varphi}\right)^2 = 2m(E - V)$$

$$or \qquad \left(\frac{\partial W_\varphi}{\partial \varphi}\right)^2 = r^2\left[2m(E - V) - \left(\frac{\partial W_r}{\partial r}\right)^2\right] \tag{9.40}$$

Now, let

$$\frac{\partial W_\varphi}{\partial \varphi} = \alpha_\varphi^2$$

Integrating it, we get

$$W_\varphi = \alpha_\varphi \varphi \tag{9.41}$$

Also, from (9.40), we get

$$\alpha_\varphi^2 = r^2\left[2m(E - V) - \left(\frac{\partial W_r}{\partial r}\right)^2\right]$$

$$or \quad \frac{\partial W_r}{\partial r} = \sqrt{2m(E-V) - \frac{\alpha_\varphi^2}{r^2}}$$

$$or \quad W_r = \int \sqrt{2m(E-V) - \frac{\alpha_\varphi^2}{r^2}}\, dr \tag{9.42}$$

Therefore, the Hamiltonian characteristic function is given as

$$W = W_r + W_\varphi = \int \sqrt{2m(E-V) - \frac{\alpha_\varphi^2}{r^2}}\, dr + \alpha_\varphi \varphi \tag{9.43}$$

Using equation (9.25), the transformation equations become

$$t + \beta_1 = \frac{\partial W}{\partial E} = \frac{\partial W_r}{\partial E} = \int \frac{m dr}{\sqrt{2m(E-V) - \frac{\alpha_\varphi^2}{r^2}}} + \alpha_\varphi \varphi \tag{9.44}$$

and

$$\beta_\varphi = \frac{\partial W}{\partial \alpha_\varphi} = \varphi - \int \frac{\alpha_\varphi dr}{r^2 \sqrt{2m(E-V) - \frac{\alpha_\varphi^2}{r^2}}} \tag{9.45}$$

Using equation (9.45), φ can be expressed as

$$\varphi = \beta_\varphi + \int \frac{\alpha_\varphi dr}{r^2 \sqrt{2m(E-V) - \frac{\alpha_\varphi^2}{r^2}}} \tag{9.46}$$

Let $\beta_\varphi = \varphi_0$ and changing to a new variable $u = 1/r$, equation (9.46) reduces to the following form, which gives the equation of the orbit.

$$\varphi = \varphi_0 - \int \frac{du}{\sqrt{\frac{2m}{\alpha_\varphi^2}(E-V) - u^2}} \tag{9.47}$$

9.5 ACTION-ANGLE VARIABLES

Consider a conservative system whose motion is periodic and the Hamilton-Jacobi equation is separable. The periodic motion is determined by the characteristics of the phase space trajectory. For simplicity, let us consider a system with one degree of freedom, in which case the phase space is a two-dimensional plane. Two cases arise, each representing periodic motion:

1. In the first type, the system retraces its path periodically and the phase trajectory is closed as shown in Figure 9.1a. Both q and p are then periodic functions of time having the same frequency. This type of motion is often referred by the astronomical name *libration*. An example of such a periodic motion is that of a one-dimensional harmonic oscillator.

FIGURE 9.1 Phase space trajectory for periodic motion of one-dimensional systems; (a) libration, (b) rotation.

2. In the second type of periodic motion, the phase space orbit is such that the variable p is periodic in q, with a period of q_0, as shown in Figure 9.1b. This kind of motion implies that the system configuration basically remains unchanged, when q is increased by q_0, and the system is said to undergo a *rotation*, as opposed to libration. Now, in the phase space, the system does not travel in a closed orbit but only p is a periodic function of q. An example of this kind of periodic motion is that of a rigid body such as a heavy symmetrical top constrained to rotate about a given axis. Here the coordinate q is the angle of rotation φ, and $q_0 = 2\pi$. Increasing q by 2π then produces no real change in the state of the system, and the values of q are unbounded and increase indefinitely.

   Another familiar example of periodic motion is that of a simple pendulum in a gravitational field, through which we can see both kinds of periodic motion – libration and rotation. When the total energy is less than mgl, where m and l are the mass and the length of the pendulum respectively, the phase trajectories are closed elliptical curves, and the pendulum oscillates back and forth, retracing its path. However, when the total energy is greater than mgl, even at the highest point the pendulum's velocity is non-zero and it continues its circular motion without reversal of direction.

Many interesting physical problems such as in astronomy require the frequencies of the periodic motion rather than the detailed dynamical behavior of the system. An elegant and powerful technique of handing such system is provided by a variation of the Hamilton-Jacobi procedure.

We introduce a new variable known as action variable J_i for each degree of freedom. The action variable J_i corresponding to the pair of separation variables (q_j, p_j) is defined as

$$J_j = \oint p_j dq_j \tag{9.48}$$

where the integration is to be carried over a complete period of libration or of rotation, as the case may be.

We consider the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(q, \alpha, t) = 0$$

whose solution $S(q, \alpha, t)$ is given by

$$S(q, \alpha, t) = W(q_1, q_2, \ldots, q_n, \alpha_1, \alpha_2, \ldots \alpha_n) - \alpha_1 t \tag{9.49}$$

where $\alpha_1 = E = H$.

Using separation of variable method, we may write the above equation as

$$S(q, \alpha, t) = \sum_j W_j(q_j, \alpha_1, \alpha_2, \ldots \alpha_n) - \alpha_1 t \tag{9.50}$$

Now, from the property of S, we have

$$p_j = \frac{\partial S}{\partial q_j} = \frac{\partial W_j}{\partial q_j} \tag{9.51}$$

Substituting this value of p_j in equation (9.48), we get

$$J_j = \oint \frac{\partial W_j}{\partial q_j} dq_j \tag{9.52}$$

It is clear that J_j is a function of constants $\alpha_1, \alpha_2, \ldots \alpha_n$. Therefore, for $j = 1, 2, \ldots, n$ we have $J_j = J_j(\alpha_1, \alpha_2, \ldots \alpha_n)$. We may invert these n equations to express each α as a function of $J_1, J_2, \ldots J_n$. Thus, the characteristic function W can be written as

$$\alpha_1 = H = H(J_1, J_2, \ldots J_n) \tag{9.53}$$

The variable J_i has the dimensions of angular momentum. J_j is action variable and its conjugate variable is angle ω_j. We call (J, ω) as action-angle variables. The function W is used as a generating function for a canonical transformation from (q, p) to (ω, J).Then

$$\omega_j = \frac{\partial W(q, J)}{\partial J_j}, \qquad J_j = \frac{\partial W(q, J)}{\partial q_j} \tag{9.54}$$

When action-angle variables are used, the Hamilton's canonical equations are given by

$$\frac{\partial H}{\partial \omega_j} = -\dot{J}_j = 0,$$
$$\frac{\partial H}{\partial J_j} = -\omega_j = \nu_j(J_j) \tag{9.55}$$

where ν_j are constant functions of J_j, J_j's are constants and ω_j can be given by

$$\omega_j = \nu_j t + \beta_j \tag{9.56}$$

Here β_j are the constants of integration. Thus, ω_j is a linear function of time t.

Let us determine the change in a given variable, say ω_1, when the coordinate q_1 goes through a complete cycle:

$$\Delta\omega_1 = \oint d\,\omega_1$$

$$= \oint \frac{\partial \omega_1}{\partial q_j}\, dq_j$$

$$= \oint \frac{\partial^2 W}{\partial q_j \partial J_1}\, dq_j$$

$$= \frac{\partial}{\partial J_1} \oint \frac{\partial W}{\partial q_j}\, dq_j$$

$$or \qquad \Delta\omega_1 = \frac{\partial J_j}{\partial J_1} = \begin{cases} 1 & if\ j = 1 \\ 0 & if\ j = 2,3,\ldots,n \end{cases} \tag{9.57}$$

Thus, ω_1 increases by unity when q_1 goes through a complete cycle, but remains unaffected when other variables change. Again from equation (9.56), we have

$$\Delta\omega_1 = 1 = \nu_1 \Delta t$$

$$or \qquad \nu_1 = \frac{1}{\Delta t} = \frac{1}{\tau} \tag{9.58}$$

where $\tau = \Delta t$ is the time period.

It is evident from the above equation that ν_1 is the frequency of the periodic motion of q_1. Similarly, ν_j is the frequency of the periodic motion of q_j. Thus, using action-angle

variables, we can determine the frequencies of a periodic motion without solving for a complete solution.

9.5.1 Linear Harmonic Oscillator using Action-Angle Variables

We apply the action-angle variable method to the linear harmonic oscillator problem, whose Hamilton is given by

$$H - \frac{p^2}{2m} + \frac{1}{2}kq^2 - E$$

Writing

$$p = \frac{\partial S}{\partial q}$$

the Hamilton-Jacobi equation is written as

$$\left\{\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}kq^2\right\} + \frac{\partial S}{\partial t} = 0 \qquad (9.58)$$

with

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t \qquad (9.59)$$

where $\alpha = E$.

Therefore, the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}kq^2 = E$$

$$or \quad \left(\frac{\partial W}{\partial q}\right)^2 = 2mE - mkq^2$$

$$or \quad W = \sqrt{mk}\int\sqrt{\frac{2E}{k} - q^2}\,dq \qquad (9.60)$$

Using equation (9.59), the integration constant β is obtained as

$$\beta = \frac{\partial S}{\partial E} = \frac{\partial W}{\partial E} - t$$

$$= \sqrt{mk}\left(\frac{1}{k}\right)\int\frac{dq}{\sqrt{\frac{2E}{k} - q^2}} - t$$

$$or \qquad \beta + t = \sqrt{\frac{m}{2E}} \int \frac{dq}{\sqrt{1 - \frac{kq^2}{2E}}} \tag{9.61}$$

On integration, we get

$$\beta + t = \sqrt{\frac{m}{E}} \sin^{-1}\left(q\sqrt{\frac{k}{2E}}\right)$$

$$= \frac{1}{\omega} \sin^{-1}\left(q\sqrt{\frac{m\omega^2}{2E}}\right)$$

$$or \qquad q = \sqrt{\frac{2E}{m\omega^2}} \sin \omega(t + \beta) \tag{9.62}$$

where $\omega = \sqrt{k/m}$.

Then

$$p = \frac{\partial W}{\partial q} = \sqrt{mk}\sqrt{\frac{2E}{k} - q^2} \tag{9.63}$$

Substituting equation (9.62) into (9.63), the momentum p is given by

$$p = \sqrt{mk}\sqrt{\frac{2E}{k} - \frac{2E}{k}\sin^2 \omega(t + \beta)}$$

$$or \qquad p = \sqrt{2mE} \cos \omega(t + \beta) \tag{9.64}$$

Now, the action variable J is expressed as

$$J - \oint p dq - \sqrt{mk} \oint \sqrt{\frac{2E}{k} - q^2} dq$$

$$or \qquad J = \sqrt{2mE} \oint \sqrt{1 - \frac{kq^2}{2E}} dq \tag{9.65}$$

Let

$$q = \sqrt{\frac{2E}{k}} \sin\varphi$$

then

$$dq = \sqrt{\frac{2E}{k}} \cos\varphi \, d\varphi$$

Substituting them in equation (9.65), the equation becomes

$$J = 2E\sqrt{\frac{m}{k}} \int_0^{2\pi} \oint \cos^2\varphi \, d\varphi$$

$$or \qquad J = 2\pi E\sqrt{\frac{m}{k}} = \frac{2\pi E}{\omega} \tag{9.66}$$

Thus, we get

$$H = E = \frac{J}{2\pi}\sqrt{\frac{k}{m}} = \frac{J\omega}{2\pi} \tag{9.67}$$

Therefore, the frequency is

$$\dot{\omega} = \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega}{2\pi} \tag{9.68}$$

and the time period is

$$\tau = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega} \tag{9.69}$$

We find that $\omega(t + \beta) = 2\pi\omega$.

Then equations (9.62) and (9.64) take the form

$$q = \sqrt{\frac{J}{\pi m\omega}} \sin 2\pi\omega \tag{9.70}$$

$$or \qquad p = \sqrt{\frac{mJ\omega}{\pi}} \cos 2\pi\omega \tag{9.71}$$

Equations (9.70) and (9.71) give the transformation equations from the canonical variables (ω, J) to the canonical variables (q, p).

9.6 SUMMARY

In this unit, we studied the Hamilton-Jacobi equation, which is an alternative formulation of classical mechanics, equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics and Hamiltonian mechanics. We studied how the Hamilton-Jacobi equation compares with other formulations of classical mechanics and is useful in identifying conserved quantities for mechanical systems. We then discussed the linear harmonic oscillator and the motion of a particle in a central force field in the Hamilton–Jacobi theory.

In the end, we studied the action-angle variables and .discussed its application in understanding the linear harmonic oscillator.

9.7 GLOSSARY

Action-angle Coordinates – In classical mechanics, action-angle coordinates are a set of canonical coordinates useful in solving many integrable systems. The method of action-angles is useful for obtaining the frequencies of oscillatory or rotational motion without solving the equations of motion.

Angular Frequency – is a scalar measure of rotation rate. It refers to the angular displacement per unit time or the rate of change of the phase of a sinusoidal waveform, or as the rate of change of the argument of the sine function.

Angular Momentum – The angular momentum of a rigid object is defined as the product of the moment of inertia and the angular velocity. It is analogous to linear momentum and is subject to the fundamental constraints of the conservation of angular momentum principle if there is no external torque on the object.

Degrees of Freedom – In physics, the degree of freedom (DOF) of a mechanical system is the number of independent parameters that define its configuration. It is the number of parameters that determine the state of a physical system and is important to the analysis of systems of bodies in mechanical engineering, aeronautical engineering, robotics, and structural engineering.

Differential Equation - an equation involving derivatives of a function or functions.

An ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives. The term ordinary is used in

contrast with the term partial differential equation which may be with respect to more than one independent variable.

Frequency – the number of occurrences of a repeating event per unit of time.

Generalized Coordinates – refers to the parameters that describe the configuration of the system relative to some reference configuration. These parameters must uniquely define the configuration of the system relative to the reference configuration.

Generating Function – Generating functions which arise in Hamiltonian mechanics are quite different from generating functions in mathematics. In physics, a generating function is, loosely, a function whose partial derivatives generate the differential equations that determine a system's dynamics.

Harmonic Oscillator – In classical mechanics, a harmonic oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force, F, proportional to the displacement, x.

Holonomic System – a system may be defined as holonomic if all constraints of the system are holonomic. For a constraint to be holonomic it must be expressible as a function:

i.e. a holonomic constraint depends only on the coordinates x_j and time t. It does not depend on the velocities or any higher order derivative with respect to t.

A constraint that cannot be expressed in the form shown above is a non-holonomic constraint.

Libration – a type of reciprocating motion in which an object with a nearly fixed orientation repeatedly rotates slightly back and forth.

Periodic Motion – motion repeated in equal intervals of time.

Simple harmonic motion is a special type of periodic motion or oscillation motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

Theory – A theory is a contemplative and rational type of abstract or generalizing thinking, or the results of such thinking. The word has its roots in ancient Greek, but in modern use it has taken on several related meanings.

In modern science, the term "theory" refers to scientific theories, a well-confirmed type of explanation of nature, made in a way consistent with scientific method, and fulfilling the criteria required by modern science.

9.8 TERMINAL QUESTIONS

1. Briefly discuss the Hamilton-Jacobi theory.
2. Apply Hamilton-Jacobi theory to solve the one-dimensional harmonic oscillator problem.
3. What are action-angle variables?
4. Explain how action-angle variables can be used to obtain the frequencies of periodic motion and hence, determine the frequency of one-dimensional harmonic oscillator.

9.9 REFERENCES

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9.10 SUGGESTED READINGS

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2. Classical Mechanics – System of Particles and Hamiltonian Dynamics, Walter Greiner – Springer-Verlag, New York
3. Classical Mechanics, John R. Taylor – University Science Books, New York

UNIT 10 SMALL OSCILLATIONS

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10.1 INTRODUCTION

Every oscillatory motion is simple harmonic motion in character. When a system is displaced from its equilibrium position, oscillation occurs. It means that oscillation refers to any periodic motion moving at a distance about the equilibrium position and repeats itself over and over for a period of time.Oscillation in a system is caused due to restoring forces that come into play, during displacement of an object from its equilibrium position. Restoring forces can do both positive and negative work. Such forces acting on the system are conservative in which the potential depends only on the spatial coordinates; it is independent of the time and velocity

In this chapter, we will study the concept of small oscillations about a stable equilibrium point. The description of motion about a stable equilibrium is one of the most important problems in physics. We will use the Lagrange's method to solve the equations of motion that emphasizes related to small oscillations. While dealing with the formulation of small oscillations, we will discuss the concepts of normal frequency, normal modes and normal coordinates of a system.In this unit, we will discuss the behavior of a system under forced vibrations.

10.2 OBJECTIVES

After studying this unit, you should be able to

- understand concepts of potential energy
- define equilibrium systems
- formulate and solve small oscillation problems
- understand and calculate normal coordinates
- understand and calculate normal modes of vibrations
- understand driven oscillations
- apply Lagrange method to solve the problems related to small oscillations

10.3 POTENTIAL ENERGY

Potential energy is energy which results from position or configuration. It can also be defined as the capacity of doing work as a result of its position in a field. The field can be a gravitational field,an electric fieldor a magnetic field. An example of potential energy can be explained by considering a mechanical system say spring.

Figure1(a) shows the mass attached to an unstretched spring, which means the potential energy is zero. When the spring is displaced slightly from its "stable" position by a distance x, it undergoes oscillation, as can be seen in Figure 1(b). The cause of oscillation is the restoring force that comes into play. For most mechanical systems, when the system is not too far from the equilibrium, the restoring force is proportional to the displacement ($F = -kx$). Here k is a spring constant.F in the definition of potential energy is the force exerted by the force field. The potential energy V is equal to the work one must do against that force to move an object from the V=0 reference point to the position x. The force you must exert to move it must be equal but oppositely directed, and that is the source of the negative sign.

The negative sign on the derivative shows that if the potential V increases with increasing x, $V = -\int_{x=0}^{x} \vec{F}.\overrightarrow{dx}$, the force will tend to move it toward smaller x to decrease the potential energy.

$$F(x) = -\frac{dV}{dx}$$

10.4 EQUILIBRIUMS

As we are studying the small oscillations experienced by various systems, it is important to have an idea of conditions of equilibrium. Therefore, we will study the following equilibrium systems:

10.4.1 Stable Equilibrium

If a slight displacement of the system from its position of equilibrium results only in small bounded motion about the point of equilibrium, then system is said to be in a stable equilibrium.
Examples: a book lying on a horizontal surface, bar Pendulum at rest, a suspension galvanometer at its zero position. The minimum potential energy is also an important requisite for stable equilibrium. Therefore, the bounded motion is possible only when the potential at equilibrium is minimum.

10.4.2Unstable Equilibrium

If a slight displacement of the system from its equilibrium results in unbounded motion, then the system is said to be in an unstable equilibrium.Examples: a pencil standing on its point, stick in vertically standing position. The unstable equilibrium is not characterized by minimum potential energy.

Let us discuss the concepts of stable and unstable equilibrium using potential energy curve. Consider a particle executing bound and unbound motion. The potential energy of the particle as a function of displacement is shown in figures (2) and (3), respectively.

Figures 2 and 3 show the variation of potential energy with distance. From Figure2, it can be explained that in the case of a stable equilibrium point, the potential energy is at a local minimum, which means point x_0 is the point of minimum potential energy (P.E.).Any external force will insert a small amount of energy into the body, which will move the body until it stops at a point where all that extra energy has been converted into potential energy. After this, the body will once again move back towards the stable equilibrium point, and that small amount of extra energy will oscillate between kinetic and potential energy, causing the object to oscillate about the stable equilibrium point until the extra energy has dissipated.

In the case of an unstable equilibrium, the potential energy is at the maximum value at the point of equilibrium, and does not know which direction to fall in (Figure3). An external force that pushes it off the equilibrium point will give the body a direction to fall in, and the potential energy will decrease while the kinetic energy will increase, until the body either reaches a new equilibrium point or else crashes into another object.

10.4.3 Static Equilibrium

The state of zero kinetic energy continues for an indefinite period, and the immediate surroundings of the system are not changing with time. Example: a stone at the bottom of a valley.

10.4.4 Dynamic Equilibrium

The net force on the system is zero and the system continues with zero kinetic energy, but the immediate surroundings of the system change with time in such a way that it exerts the balancing force on the system.Example: charge neutrality of atoms.Molecuels and solids make them exert zero electrical force on one another, but each of them is in dynamic equilibrium.

10.5 FORMULATION OF SMALL OSCILLATIONS

Before the discussion and formulation of small oscillations, we understood the concept of P.E. and how it behaves in stable and unstable equilibriums. For discussion of small oscillations, we consider a conservative system in which the potential energy is a function of position only. Also the constraints that depend on time will be excluded from this formulation.

As we studied in previous section, a system of particles is said to be in *stable* equilibrium if all the particles are and remain at rest. The generalized forces acting on each particle must vanish:

$$Q_k = -\left(\frac{\partial V}{\partial q_k}\right)_0 = 0 \qquad \text{.....(1)}$$

This equation yields the values q_{0j} of the generalized co-ordinates that the particles have in equilibrium configuration. If we assume the equilibrium to be *stable*, the potential energy must be a minimum when evaluated at these q_{0j}. If displacements of the generalized coo-ordinates from their equilibrium value be denoted by τ_j then

$$q_k = q_{0k} + \tau_k \qquad \text{.....(2)}$$

In equation (2), q_{0j} are fixed and$\tau_k$areconsideredas generalized co-ordinates of the system. Expanding P.E. aboutq_{0j},

$$V(q_1, \dots q_n) = V(q_{01}, q_{02}, \cdots q_{0n}) + \sum_k \left(\frac{\partial V}{\partial q_j}\right)_0 \tau_k + \frac{1}{2}\sum_{k,l}\left(\frac{\partial^2 V}{\partial q_k}\partial q_l\right)_0 \tau_k \tau_l + \cdots \text{.....(3)}$$

The terms linear in $\tau_k$vanishes as a result of equation (1),because of stability requirement.The first term in the series is the P.E. of the equilibrium position, which may be arbitrarily, be chosen as reference level wih the P.E. zero there. We are therefore left with the quadratic term as the first approximation to V:

$$V = \frac{1}{2}\sum_{k,l}\left(\frac{\partial^2 V}{\partial q_k \partial q_l}\right)_0 \tau_k \tau_l = \frac{1}{2}\sum_{k,l} V_{kl}\tau_k\tau_l \text{.....(4)}$$

The coefficients

$V_{kl} = \left(\frac{\partial^2 V}{\partial q_k \partial q_l}\right)_0 = \left(\frac{\partial^2 V}{\partial q_l \partial q_k}\right)_o = V_{lk}$are constants and are characteristic of equilibrium configuration. We calculate the generalized force on the k^{th}particle

$$Q_j = \frac{-\partial V}{\partial q_j} = -\frac{\partial V}{\partial u_j} = -\sum_k V_{jk}u_k$$

The force changes any generalized co-ordinate depends on the displacement of all the others coordinates and we have the correct form of the potential energy that is given by (4).

A similar series expansion can be obtained for the kinetic energy (K.E.).The expression in generalized co-ordinates is a homogeneous quadratic function of generalized velocities:

$$T = \frac{1}{2}\sum_{k,l} m_{kl}\dot{q}_k\dot{q}_l = \frac{1}{2}\sum_{k,l} m_{kl}\dot{\tau}_k\dot{\tau}_l \quad \text{.....(5)}$$

Where, the coefficients $m_{kl}$are given by

$$m_{kl} = m_{lk} = \sum_i m_i\left(\frac{\partial x_i}{\partial q_k}\right)\left(\frac{\partial x_i}{\partial q_l}\right) = m_{il}(q_1, q_2, \cdots q_n) \quad \text{.....(6)}$$

Equation 5 is a quadratic in the small quantities$\dot{\iota}_k$, we cannot use the higher order terms and therefore choose the lowest order approximation to T by taking

Denoting the constant value of m_{kl}functions at equilibrium by T_{kl}, wecan therefore write kinetic energy as

$$T = \frac{1}{2}\sum_{k,l} T_{kl}\dot{\tau}_k\dot{\tau}_l \quad \text{.....(7)}$$

Near the equilibrium configuration of the system, Lagrangian is given as:

$L = \frac{1}{2}\sum_{k,l}(T_{kl}\dot{\tau}_k\dot{\tau}_l - V_{kl}\tau_k\tau_l)$ (as L=T-V).....(8)

Lagrange's equations of motion in our notation appear as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tau}_k}\right) - \frac{\partial L}{\partial \dot{\iota}_k} = 0$$

Taking the $\tau's$ as the general coordinates, the Lagrangian of equation (8) leads to following equation of motion

$\sum_l (T_{kl}\ddot{\tau}_l + V_{ki}\tau_l) = 0$(k=1,...., n);.....(9)

The equation (9) will involve, in general all coordinatesτ_k, and it is this set of simultaneous differential equations that must be solved to obtain the motion near equilibrium.

The co-ordinates τ_{lk}may be used to define a column matrix $\boldsymbol{\tau}$ and in matrix notation equations (9) appear as

$$\begin{bmatrix} T_{11}T_{12}\cdots T_{1n} \\ T_{21}\cdots\cdots\cdots \\ \cdots \\ T_{n1}\cdots\cdots T_{nn} \end{bmatrix}\begin{bmatrix} \ddot{\tau}_1 \\ \ddot{\tau}_2 \\ \cdots \\ \ddot{\tau}_n \end{bmatrix}+\begin{bmatrix} V_{11}V_{12}\cdots V_{1n} \\ V_{21}\cdots\cdots\cdots \\ \cdots \\ V_{n1}\cdots\cdots V_{nn} \end{bmatrix}\begin{bmatrix} \tau_1 \\ \tau_2 \\ \cdots \\ \tau_n \end{bmatrix}=0$$

or$\mathbf{T\ddot{\tau}+V\tau=0}$ ….. (10)

An equation (10) is the equation of motion for a system displaced from a stable equilibrium.

10.6DETERMINATION OF NORMAL FREQUENCIES AND EIGEN VALUE PROBLEM

The equations (9) and (10) in the previous topic represents the equations of motion near equilibrium or when a system is displaced slightly from the equilibrium and is linear differential equations with constant coefficients. These equations are solved by trying an oscillatory solution of the form

$$\tau_l = Ca_l e^{-i\omega t} \qquad \text{…..(11)}$$

This is a harmonic oscillator solution which equations of type (9) and (10) must have. Here Ca_l provides the complex amplitude of the oscillation for each coordinate. It is a scalar factor. On substituting equation (11) in to equation (9), it leads to the following equation for the unknown a_l :

$$\sum_l (V_{kl}-\omega^2 T_{kl})a_l = 0,\ \ (j=1,2,\ldots n) \qquad \text{…..(12)}$$

Equations (12) constitute n linear homogenous functions for the $a_k's$. The equations can be written by expanding as follows:

$$(k=1),\quad (V_{11}-\omega^2T_{11})a_1+(V_{12}-\omega^2T_{12})a_2+\cdots(V_{1n}-\omega^2T_{1n})a_n=0$$

$$(k=2),\quad (V_{21}-\omega^2T_{21})a_1+(V_{12}-\omega^2T_{22})a_2+\cdots(V_{2n}-\omega^2T_{2n})a_n=0$$

… … … … …

$$(k=n),\quad (V_{n1}-\omega^2T_{n1})a_1+(V_{n2}-\omega^2T_{n2})a_2+\cdots(V_{nn}-\omega^2T_{nn})a_n=0$$

The above written equations can have a nontrivial solution only if the determinant of the coefficient vanishes:

$$\begin{bmatrix} (V_{11}-\omega^2T_{11}) & (V_{12}-\omega^2T_{12}) & \cdots \\ (V_{21}-\omega^2T_{21}) & (V_{22}-\omega^2T_{22}) & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}=0 \qquad \text{…..(13)}$$

By solving the determinantal condition in equation (13), an algebraic equation of the n^{th} degree is obtained for the frequency ω^2. The roots $(\omega_1^2\cdots\omega_n^2)$ of the determinant provide the frequency for which equation (11) represents the correct solution for equations of motion

(9)and (10). These values of ω are the **normal frequencies**or the **eigenfrequencies**of the system. For each of these values of ω^2, equations (12) may be solved for the amplitudes of $\mathbf{a}_l$, precisely, for (n-1) of the amplitudes in terms of remaining $\mathbf{a}_l$.Equation (12) may also be written as

$$(\mathbf{Va} - \omega^2\mathbf{Ta}) = \mathbf{0} \qquad \text{…..(14)}$$

Equations (14) can also be expressed as

$\mathbf{Va} = \omega^2\mathbf{Ta}$and is alsoknown as secular equation.

where, **a**is now a column matrix of n components:

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \dots \\ \mathbf{a}_n \end{bmatrix}$$

The above equation is not in the form of an eigenvalue equation as **Va**is not equal toa constant times **a** but a constant times **Ta**.If we make**T** invertible, one can achieve an eigenvalueequation

$$\mathbf{T^{-1}Va} = \omega^2\mathbf{Ia} \text{…..(15)}$$

Since we have n homogeneous equations, and haven solutions for ω^2. Letus denote the m^{th} mode frequency by $\omega^2 = \lambda_m$. Thus the vector **a** corresponding to thismode be written as

$$\mathbf{a}_m = \begin{bmatrix} \mathbf{a}_{m1} \\ \mathbf{a}_{m2} \\ \dots \\ \mathbf{a}_{mn} \end{bmatrix}$$

Thus we have

$$\mathbf{Va}_m = \lambda_m\mathbf{Ta}_m \qquad \text{…..(16)}$$

Taking conjugate of this equation and changing the index m to o, we get

$\mathbf{V\tilde{a}}_o = \lambda_m\mathbf{T\tilde{a}}_o$…..(17)

On multiplying equation (16) by $\tilde{\mathbf{a}}_k$,we get

$$\lambda_m = \frac{\tilde{a}_m \mathbf{V} a_m}{\tilde{a}_m \mathbf{T} a_m} \qquad \text{…..(18)}$$

Here, we have used $\tilde{\mathbf{a}}$ is used to denote the transpose of the matrix**a**.
From equations (16) and (17), we get,

$$\tilde{\mathbf{a}}_o\mathbf{Va}_m = \lambda_m\tilde{\mathbf{a}}_o\mathbf{Ta}_m$$
$$\tilde{\mathbf{a}}_o\mathbf{Va}_m = \lambda_o\tilde{\mathbf{a}}_o\mathbf{Ta}_m$$

Thus we get

$$(\lambda_m - \lambda_o)\tilde{\mathbf{a}}_o \mathbf{T} \mathbf{a}_m = 0$$

Therefore, if the eigenvalues are non-degenerate, i.e. if $\lambda_o \neq \lambda_m$, we get the orthogonality condition

$$\tilde{\mathbf{a}}_o \mathbf{T} \mathbf{a}_m = 0 \quad \text{.....(19)}$$

As equations (15) doesn't uniquely determine $\mathbf{a}$, we define normalization condition as

$$\tilde{\mathbf{a}}_o \mathbf{T} \mathbf{a}_o - 1 \quad \text{.....(20)}$$

If we form all the eigenvectors $\mathbf{a}_o$ into a square matrix $\mathbf{A}$, then equations (19) and (20) can be combined to form one matrix equation:

$$\tilde{\mathbf{A}}\mathbf{T}\mathbf{A} = \mathbf{I} \quad \text{.....(21)}$$

Example 1: Consider two masses m_1 and m_2 connected by three springs of spring constant k in a two coupled oscillator. The masses of the system are equal, such that $m_1 = m_2 = m$. Calculate the frequency of vibration of the system. (*Problem from Classical Mechanics,Gupta, Kumar, Sharma*)

Solution: The system can be visualized by two equal masses joined by identical springs (same stiffness constant k) to each other and to fixed walls as shown in Figure (5). We assume the system to be ideal i.e., free from friction. The motion is thus undamped and takes place along the straight line of their configuration. The generalized co-ordinates $x_1$and $x_2$of each mass are measured from their equilibrium positions. The masses are equal such that, $m_1 = m_2 = m$.

The kinetic energy of the system is given by

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \quad \text{.....(1)}$$

Lagrangian for the system, L=T-V, where the potential energy V of the system is the sum of potential energies of the two masses plus the potential energy resulting from interaction between them which is a function of relative displacement(x_2-x_1) of the particles. To sum up:

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_2 - x_1)^2 \text{.....(2)}$$

and

$$L = \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2\right) - \left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_2 - x_1)^2\right) \quad \text{.....(3)}$$

Lagrange's equations of motion then follow immediately:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

or

$$m\ddot{x}_1 + m\omega_0^2 x_1 - m\omega_5^2(x_2 - x_1) = 0 \qquad \text{.....(4)}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

or

$$m\ddot{x}_2 + m\omega_0^2 x_2 - m\omega_0^2(x_2 - x_1) = 0 \qquad \text{.....(5)}$$

since $\left(\frac{dV}{dx}\right)_{x_0} = 0$ at a point x_0 of stable equilibrium.

Since zero level of potential energy is arbitrary, we can choose it at the point x_0, making $(V)_{x0} = 0$. Then, to a first approximation, we can write

$$V(x) - \frac{1}{2}\left(\frac{\partial^2 V}{\partial x^2}\right)_{x_0}(x - x_0)^2 - \frac{1}{2}k(x - x_0)^2$$

Where,

$$k - \left(\frac{\partial^2 V}{\partial x^2}\right)_{x_0}$$

where, we have set $k = m\omega_0^2$and $m_1=m_2=m$. Equations (4) and (5) arethe set of simultaneous linear differential equations with constant coefficients and therefore, we try solutions characteristic of such equation, viz,

$$x_1 = Ae^{\alpha t}, x_2 = Be^{\alpha t} \qquad \text{.....(6)}$$

where A and B are constants. If we substitute these into equations (4) and (5), we obtain two homogeneous simultaneous equations for the determination of constants A and B;

$$(\alpha^2 + 2\omega_0^2)A - \omega_0^2 B = 0,$$

$$-\omega_0^2 A + (\alpha^2 + 2\omega_0^2)B = 0 \qquad \text{.....(7)}$$

Such a set will have a solution if and only if the determinant of coefficients of A and B vanishes that is

$$\begin{vmatrix} \alpha^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \alpha^2 + 2\omega_0^2 \end{vmatrix} = 0 \qquad \text{.....(8)}$$

Expanding and solving forα, we get the pairs of roots of α, viz.

$$\alpha = \pm i\omega_0, \pm i\sqrt{3}\omega_0 \qquad \text{.....(9)}$$

Since the equations (4) and (5) are linear, the sum of four solutions is, obtained upon putting the four values of α, will also be a solution and will contain four arbitrary constants. It will be, in effect, the general solution, by the superposition principle. With a slight rearrangement and keeping in view that x_1 and x_2 must correspond to real physical situation (in other words x_1 and x_2 are real), we obtain*,

The general solutions for x_1 and x_2, after having substituted the four values of α given by equation (9), become of the forms

$$x_1 = a_1 e^{i\omega_0 t} + a_2 e^{-i\omega_0 t} + a_3 e^{i\sqrt{3}\omega_0 t} + a_4 e^{-i\sqrt{3}\omega_0 t}$$

$$x_2 = h_1 e^{i\omega_0 t} + h_2 e^{-i\omega_0 t} + h_3 e^{i\sqrt{3}\omega_0 t} + h_4 e^{-i\sqrt{3}\omega_0 t}$$

where a's and b's are each four arbitrary constants mentioned in this paragraph. Now take up the part with a single frequency, say ω_0 in x_1 and let

$$a_1 = \frac{A_1}{2} e^{i\phi_1}, a_2 = \frac{A_1}{2} e^{-i\phi_1}$$

Here two new constants A_1 and $\varphi_1$are introduced and each is real whereas a_1 and a_2 could be complex numbers also. We thus have for this part:

$$f_1(\omega_0 t) = \frac{A_1}{2} e^{i\omega_0 t} e^{i\phi_1} + \frac{A_1}{2} e^{-i\omega_0 t} e^{-i\phi_1} = \frac{A_1}{2} e^{i(\omega_0 t+\phi_1)} + \frac{A_1}{2} e^{-i(\omega_0 t+\phi_1)} =$$

$$= A_1 \cos(\omega_0 t + \phi_1)$$

$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_2 = B_1 \cos(\omega_0 t + \phi_1) + B_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

The constants A and B are, however, connected by the conditions equation (7):

$$B_1 = \frac{2\omega_0^2 + \alpha_1^2}{\omega_0^2} A_1, B_2 = \frac{2\omega_0^2 + \alpha_2^2}{\omega_0^2} A_2$$

Hence the general solution assumes the form

$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{3}\omega_0 t + \phi_2) \text{and}$$

$$x_2 = A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{3}\omega_0 t + \phi_2) \ldots..(10)$$

It is noted from eqations (10) that the motion of each co-ordinate is a superposition of two harmonic vibrations of frequencies ω_0 and $\sqrt{3}\omega_0$. The frequencies of oscillation are the same for both x_1 and x_2 but their relative amplitudes are different.

10.7FREQUENCIES OF FREE VIBRATIONS, THEORY OF SMALL OSCILLATIONS, NORMAL MODES

10.7.1Free Vibrations When a system is slightly displaced form equilibrium; the equations of motion for such a system is given by equation (9) or (10).We have already studied the system under the topic of formulation of small oscillations.

$\sum_l (T_{kl}\ddot{\tau}_l + V_{kl}\tau_l) = 0$ (k=1,…., n)

or $\mathbf{T\ddot{\tau} + V\tau = 0}$

These equations of motion are satisfied by oscillatory solutions given in equation (11).

$\tau_l = Ca_l e^{-i\omega t}$

The solution provided by equation (11) is not for one frequency only, but in general for a set of n frequencies. Thus, a complete solution of equations of motion involves a superposition of oscillations with all the allowed frequencies $\omega_1, \omega_2, \ldots \ldots \omega_n$. A system displaced slightly from equilibrium position performs small oscillations about equilibrium with these frequencies. The solution of these equations are therefore, referred as frequencies of **free vibration** or the **resonant frequencies** of the system. The general solution of the equations of motion may be now written as:

$\tau_k = C_m a_{km} e^{-i\omega_m t}$…..(22)

Here, ω_m represent the set of n frequencies, C_m is the complex scale factor for each resonant frequency, $a_{lm}$is the eigenvector element (make up columns & rows of "eigenvector matrix" **A**). For each eigenvalue $\lambda_m = \omega_m{}^2$. It may be found that for each solution of λ_m there aretwo resonant frequencies$\pm\omega_m$.

10.7.2Normal coordinates

Certain solutions of equations of motion are such that there is one single frequency involved; the coordinates used in such solutions are called normal coordinates. Thus, generalized coordinates, each coordinate corresponds to a vibration of the entire system with only one frequency is referred to as normal coordinates.

We define the new set of coordinates

$\tau_k = \sum_l a_{kl}\gamma_l$ …..(23)

The equations (23) can be written in single column matrix form as

$\boldsymbol{\tau = A\gamma}$…..(24)

The P.E. in matrix notation can be written from equation (4):

$V = \frac{1}{2}\tilde{\tau}\mathbf{V}\tau$…..(25)

Now, the transpose of τgives

$\tilde{\tau} = \widetilde{A\gamma} = \tilde{\gamma}\tilde{A}$…..(26)

From equations, (24), (25) and (26), the P.E.is given as

$\mathbf{V = \frac{1}{2}\tilde{\gamma}\tilde{A}VA\gamma}$….. (27)

If a diagonal matrix σ is introduced, then

$\lambda_{om} = \lambda_m \sigma_{om}$

Thus, we get

$$\sum_{l=1}^{n=1} V_{kl}\, a_{lm} = \sum_{l} T_{kl}\, a_{lm}\, \lambda_{om}$$

In terms of matrix form, $\mathbf{Va = Ta\sigma}$

The above equation after multiplying by $\tilde{\mathbf{a}}$ **becomes**

$\mathbf{\tilde{a}Va = \tilde{a}Ta\sigma}$

or

$\mathbf{\tilde{a}Va = \sigma}$

that can also be written as

$\mathbf{\tilde{A}VA = \sigma}$

As $\mathbf{A}$ diagonalizes $\mathbf{V}$ by equation $\mathbf{\tilde{A}VA = \sigma}$, P.E. can be written as:

$\mathbf{V = \frac{1}{2}\tilde{\gamma}\sigma\gamma = \frac{1}{2}\lambda_m \gamma_m^{\,2} = \frac{1}{2}\omega_m^{\,2}\gamma_m^{\,2}}$…..(28)

Similarly, the K.E. is given by using equation (7),

$\mathbf{T = \frac{1}{2}\tilde{\dot{\gamma}}\tilde{A}TA\dot{\gamma}}$…..(29)

Using equation (21), equation (29) reduces to

$\mathbf{T = \frac{1}{2}\tilde{\dot{\gamma}}\dot{\gamma} = \frac{1}{2}\dot{\gamma}_m\dot{\gamma}_m = \frac{1}{2}\dot{\gamma}_m^{\,2}}$…..(30)

Equations (28) and (30) are the P.E. and K.E. in new coordinates.

Now the Lagrangian using new set of coordinates will be,

$\mathbf{L - T - V - \frac{i}{2}\dot{\gamma}_m^{\,2} - \frac{1}{2}\omega_m^{\,2}\gamma_m^{\,2}}$ …..(30)

Using equation (30) and substituting in the equation below will result,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}_m}\right) - \frac{\partial L}{\partial \gamma_m} = 0$$

$\ddot{\gamma}_m + \omega_m^{\,2}\gamma_m = 0$ …..(31)

Equation (31) is thesolution of the equation of motion with the new coordinates. The solution of the equation is given by:

$$\gamma_m = C_m e^{-i\omega_m t} \quad \text{.....(32)}$$

From equation (32), each of new coordinate is simply a periodic function involving only one of the resonant frequencies. Thus γ_m are the normal coordinates of the system.

The periodic function can also be expressed as :

$$\gamma_m = A_m \cos\omega_m t + \sin\omega_m t \quad \text{if } \omega_m{}^2 > 0 \quad \text{.....(33)}$$

10.7.3Normal Modes of Vibration

Each normal coordinate corresponds to a vibration of the system with only one frequency, and these components of the oscillation are known as **normal modes of vibration**. System A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation. The free motion described by the normal modes takes place at the fixed frequencies. These fixed frequencies of the normal modes of a system are known as its natural frequencies. When relating to music, normal modes of vibrating instruments (strings, airpipes, drums etc.) are called harmonics or overtones.

From equation (33), the periodic oscillation is provided by

$\gamma_m = A_m \cos\omega_m t + \sin\omega_m t$, such that $m = (1, 2, \ldots, n)$; ω_m is the normal frequency

The above equation can be written as:

$$\gamma_1 = A_1 \cos\omega_1 t + \sin\omega_1 t$$

$$\gamma_2 = A_2 \cos\omega_2 t + \sin\omega_2 t$$

$$\ldots\ldots\ldots\ldots\ldots \quad \text{.....(34)}$$

A and B are the arbitrary constants. Now, if we choose all theconstants are zero, except A_1 and B_1. Then only γ_1 will vary periodically will time. Such a system is said to be vibrating in the normal mode.

Equation (34) can be re-written as:

$$\gamma_m = A_m(\cos\omega_m t + \varphi_m) \quad \text{.....(35)}$$

Here, φ_m is the arbitrary phase.

Example 2:A system consists of two identical simple pendula, each of mass m,length l and coupled by a massless spring of force constant k, as shown in Figure 5. They move in a vertical plane and, in equilibrium position.The two pendula are identical. Calculate the normal frequencies of vibration and equations of motion of the system.(*Problemfrom Classical Mechanics, Gupta, Kuma, Sharma*)

Solution:Let the anglesθ_1 and θ_2 as the generalized co-ordinates, the kinetic and potential energy can be expressed as

$$^{*}T = \frac{1}{2} ml^2 \left(\dot{\theta_1}^2 + \dot{\theta_2}^2 \right)$$

or

$$2T = \begin{pmatrix} \dot{\theta_1} & \dot{\theta_2} \end{pmatrix} \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta_1} \\ \dot{\theta_2} \end{pmatrix}$$

giving,

$$T = \left(T_{jk} \right) = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \text{.....(1)}$$

Further,*

$$V = \frac{1}{2} \left[(mgl \mid kl^2)(\theta_1^2 \mid \theta_2^2) \quad 2kl\theta_1\theta_2 \right]$$

that can also be expressed as:

$$2V = (\theta_1 \theta_2) \begin{pmatrix} mgl + kl^2 & -kl^2 \\ kl^2 & mgl \mid kl^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

giving,

$$V = \left(V_{jk} \right) = \begin{pmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 \end{pmatrix} \qquad \text{.....(2)}$$

From equations (1) and (2),

$$|V - \omega^2 T| = \begin{pmatrix} mgl + kl^2 - \omega^2 ml^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - \omega^2 ml^2 \end{pmatrix} = 0 \qquad \text{.....(3)}$$

On solving, we get,

$$(mgl + kl^2 - \omega^2 ml^2)^2 = k^2 l^4$$

or

$$(mgl + kl^2 - \omega^2 ml^2) = \pm kl^2$$

or

$$\omega^2 = \frac{1}{ml^2}[mgl + kl^2 \pm kl^2]$$

giving,

$$\omega_1 = \sqrt{\frac{g}{l}} \text{ and } \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}} \quad \text{.....(4)}$$

Here ω_1 and ω_2 are the normal frequencies of vibration.

*Expressions of T and V are derived in the following way:

Let x_1 and x_2 be the horizontal displacements and y_1 and y_2 be the vertical displacements of each mass point from the position of equilibrium. Then

$$T = \frac{1}{2}m(\dot{x_1}^2 + \dot{x_2}^2) + \frac{1}{2}m(\dot{y_1}^2 + \dot{y_2}^2)$$

$$V - mg(y_1 + y_2) + \frac{1}{2}k(x_2 - x_1)^2$$

From Figure (5)

$$x_1 = l\sin\theta_1 \text{ and } x_2 = l\sin\theta_2$$

$$y_1 = l(1 - \cos\theta_1) \text{ and } y_2 = l(1 - \cos\theta_2)$$

Putting values of $\dot{x}_1$, $\dot{x}_2$, $\dot{y}_1$ and $\dot{y}_2$ etc. in the expressions for T and V, we get

$$T = \frac{1}{2}ml^2\left(\dot{\theta_1}^2 + \dot{\theta_2}^2\right)$$

On expanding cosine terms and retaining only upto squared terms, as $\theta_1$and θ_2 are small, then

$$V = mgl[(1 - \cos\theta_1) + (1 - \cos\theta_2)] + \frac{1}{2}kl^2(\sin\theta_2 - \sin\theta_1)^2$$

$$V = mgl\left[\left(1 - 1 + \frac{\theta_1^2}{2}\right) + \left(1 - 1 + \frac{\theta_2^2}{2}\right)\right] + \frac{1}{2}kl^2(\theta_2 - \theta_1)^2$$

$$V = \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) + \frac{1}{2}kl^2(\theta_1^2 + \theta_2^2) - kl^2\theta_1\theta_2$$

$$V = \frac{1}{2}[(mgl + kl^2)(\theta_1^2 + \theta_2^2) - 2kl^2\theta_1\theta_2]$$

Suppose normal co-ordinate $\gamma_1$corresponds to ω_1, and γ_2 corresponds to ω_2. Then from equations(32), we c equations of motion in terms of normal co-ordinates as

$$\ddot{\gamma}_1 + \omega_1^2 \gamma_1 = 0$$

$$\ddot{\gamma}_2 + \omega_2^2 \gamma_2 = 0 \qquad \text{.....(5)}$$

Now, we will write γ_1 and γ_1 in terms of generalised co-ordinates θ_1 and θ_2. From equation (23) of normal coordinates, the generalized coordinate θ can be written as:

$$\theta_k = \sum_l{}' a_{kl}\, \gamma_l$$

thus, we get

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \text{.....(6)}$$

We need to calculate the components $a_{11}$and $a_{21}$of eigenvector a_1; and $a_{12}$and a_{22} of eigenvector a_2, respectively,

For a_1 we put $\omega=\omega_1$ in eq. (3) which becomes,

$$\begin{pmatrix} mgl + kl^2 - \omega_1^2 m l^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - \omega_1^2 m l^2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

Putting $\omega_1^2 = \frac{g}{l}$ from equation (4), we get

$$\begin{pmatrix} kl^2 & -kl^2 \\ -kl^2 & kl^2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

or

$$kl^2 a_{11} - kl^2 a_{21} = 0$$

$$-kl^2 a_{11} + kl^2 a_{21} = 0$$

Giving $a_{11} = a_{21} = \alpha$ (say)

For eigenvector a_2, we put $\omega^2=\omega_2^2= \frac{g}{l} + \frac{2k}{m}$ in equation (3) to obtain

$$\begin{pmatrix} mgl + kl^2 - \left(\frac{g}{l} + \frac{2k}{m}\right) m l^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - \left(\frac{g}{l} + \frac{2k}{m}\right) m l^2 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0 \qquad \text{.....(7)}$$

or

$$\begin{pmatrix} -kl^2 & -kl^2 \\ -kl^2 & -kl^2 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$$

or

$$-kl^2 a_{12} - kl^2 a_{22} = 0$$

$$kl^2 a_{12} \quad kl^2 a_{22} = 0 \qquad \text{.....(8)}$$

or

$a_{12} = -a_{22} = \beta$ (say)

Therefore,

$$\mathbf{a} = \begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix}$$

Its transpose will be

$$\tilde{\mathbf{A}} = \begin{pmatrix} \alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

So that from equation (21) ,we can write

$$\tilde{\mathbf{A}}\mathbf{T}\mathbf{A} = \mathbf{I}$$

$$\begin{pmatrix} \alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} ml^2\alpha & ml^2\beta \\ ml^2\alpha & -ml^2\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} ml^2\alpha^2 + ml^2\alpha^2 & ml^2\alpha\beta - ml^2\alpha\beta \\ ml^2\alpha\beta - ml^2\alpha\beta & ml^2\beta^2 + ml^2\beta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 2ml^2\alpha^2 & 0 \\ 0 & 2ml^2\beta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

giving

$$2ml^2\alpha^2 = 1$$

$$2ml^2\beta^2 = 1$$

or

$$\alpha = \sqrt{\frac{1}{2ml^2}} = \beta$$

so that

$$a_{11} = a_{21} = \frac{1}{\sqrt{2ml^2}}$$

and

$$a_{12} = -a_{22} = \frac{1}{\sqrt{2ml^2}}$$

Therefore from equation (6),

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2ml^2}} & \frac{1}{\sqrt{2ml^2}}\\ \frac{1}{\sqrt{2ml^2}} & -\frac{1}{\sqrt{2ml^2}}\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$$

$$\theta_1 = \frac{\gamma_1}{\sqrt{2ml^2}} + \frac{\gamma_2}{\sqrt{2ml^2}}$$

$$\theta_2 = \frac{\gamma_1}{\sqrt{2ml^2}} - \frac{\gamma_2}{\sqrt{2ml^2}}$$

giving

$$\gamma_1 = \sqrt{2ml^2}(\theta_1 + \theta_2)$$

$$\gamma_2 = \sqrt{2ml^2}(\theta_1 - \theta_2) \qquad \text{.....(9)}$$

Thus equations (4), (5) and (9) are related to the parallel pendula.

In order to interpret modes of vibration,put γ_2 equal to zero and discuss the mode associated with $\gamma_1$and hence with ω_1. Similarly, we put γ_1 equal to zero and discuss mode with ω_2. Let us first put

$$\gamma_2 = 0$$

Then from equation (9),

$\theta_1=\theta_2$

Which means the two pendula oscillate in phase and frequency of the motion is

$$\omega_1 = \sqrt{\frac{g}{l}}$$

Obviously k, the spring constant is not occurring in ω_1, the two pendula can vibrate as if they are independent i.e., there is no stretching or compressing of the spring during the motion justifying the inphase condition.

Now we put,

$$\gamma_1 = 0$$

Then from equation (9),

$\theta_1= - \theta_2$

i.e. they are out of phase which is obvious because the frequency associated with this mode involves spring constant.

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}},$$

10.8FORCED VIBRATIONS, DISSIPATION

10.8.1Forced Vibrations

As we studied in the earlier sections that free vibration occur when the system is displaced from its equilibrium position such that the system oscillates by itself. Very often, however the system is set into oscillations by an external driving force that continues to act on the system after$t = 0$. The frequency of such forced oscillation is then determined by the frequency of the driving force.

In this case if F_l is the generalized force corresponding to the coordinates τ_i, then the generalized force Q_k for the normal coordinateγ_k is given by

$$Q_k = a_{lk}\Gamma_l \qquad \text{.....(35)}$$

Using Lagrange equation for non-potential generalized forces,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \qquad \text{.....(36)}$$

Here $Q_k = \sum_l F_l \frac{\partial r_l}{\partial q_k}$ is the non-potential force. An Example of a non-potential force is frictional force, which is proportional to the velocity of the particle, and, is given by $F = -kv$.The frictional forces are generally given as:

$$\overrightarrow{F} = -k_x v_x i - k_y v_y j - k_z v_z k,$$

where k_x, k_y, k_z are positive quantities. Such non-potential forcesin the Lagrangian formalism can be included by defining Rayleigh Dissipation Function **R**and is given as:

$$R = \frac{1}{2}\sum_k (k_x v_x^{\,2} + k_y v_y^{\,2} + k_z v_z^{\,2}) \qquad \text{.....(37)}$$

The force is given as:

$F_k = \frac{\partial R}{\partial v_{kx}}$ i.e. $F_k = -\nabla_{vk} R$

Where the gradient is taken with respect to velocity field and i is the particle index. Work done by the system in overcoming the resistive force is given as:

$$dW = -\vec{F}.\overrightarrow{dr} = -\vec{F}.v dt = -2R dt$$

Thus,

$\frac{dW}{dt} = -2R$.....(38)

Now, the generalized force Q_k can be written as:

$$Q_k = \sum_l F_l \frac{\partial r_l}{\partial q_k} = \sum_l F_l \frac{\partial \dot{r}_l}{\partial \dot{q}_k} = -\sum_l \nabla_{vl} R \frac{\partial \dot{r}_l}{\partial \dot{q}_k} = -\frac{\partial R}{\partial \dot{q}_k}$$.....(39)

The Lagrange equation now becomes by incorporating an additional term

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = Q_k$$.....(40)

Here, $Q_k$are non-potential forces which are not derivable either from a potential or from a dissipation function.

10.8.2Driven Oscillations

If $V(q,t)$ is the time dependent potential along with the harmonic potential, then

$$V(q,t) = V(0,t) + q\frac{\partial V(q,t)}{\partial q} = -qF(t)$$

Here, $V(0,t) = 0$ is considered as the reference potential. The Lagrangian in this case is given as:

$$L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 + qF(t)$$

The Euler Lagrange equation gives

$$m\ddot{q} + k\dot{q} = F(t)$$

The driving force is sinusoidal, $F = F_0 \cos\omega t$, with frequency ω .On introducing the velocity dependent damping, the equation of motion becomes,

$$\ddot{q} + 2\epsilon\dot{q} + {\omega_0}^2 q = \frac{F_0}{m}e^{i\omega t}$$.....(41)

Here,${\omega_0}^2 = \frac{k}{m}$

The solution of equation (41) has two parts,a solution of the homogeneous equation$\ddot{q} + 2c\dot{q} + {\omega_0}^2 = 0$and a particular solution. The equation is solved by a particular solution by putting $q = Ae^{-i\omega t}$. Thus we get,

$\dot{q} = -i\omega q$ and$\ddot{q} = -\omega^2 q$. .....(42)

Substituting equations (42) in equation (41), we get

$$(-\omega^2 + 2i\omega c + {\omega_0}^2)Ae^{-i\omega t} = \frac{F_0}{m}e^{i\omega t}$$

Thus, $A = \frac{F_0}{m}\frac{1}{{\omega_0}^2 - \omega^2 - 2i\omega\epsilon}$

The motion is oscillatory with the frequency same as that of the driving frequency.

10.9 SUMMARY

In this unit, you have studied about the concepts of small oscillations. In order to understand about oscillations, concept of equilibrium was stated. The basic definition of potential energy with respect to equilibrium was also studied. You have also studied the formulation of small oscillations for a system using $\mathbf{T\ddot{\tau} + V\tau = 0}$. If you are provided with a system with small oscillations, you can form equations of motion using Lagrangian method , L=T-V and hence $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} - 0$. You also learnt to solve the equation of motion using oscillatory solution. Furthermore, you studied calculating eigenfrequencies using equation $\mathbf{(Va - \omega^2 Ta) = 0}$and hence studied tocalculatenormal frequency, normal modes and normal coordinates of a system. The different examples in the form of problems are also discussed such as, coupled oscillator and parallel pendula. You have also studied about the free and driven oscillations. Few solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are also provided.

10.10 GLOSSARY

Applied -	Forces imposed to the system by coupling elements (force actuators and springs) and forces which can be described by physical laws (gravity force, etc.)
Constraint forces -	The constraint forces are imposed to the system by kinematic constraint elements (joints, bearings, actuators that prescribe motion of the system)
Constrained particle -	A particle whose motion is limited and constrained
Constraint -	A restriction on the motion of a system
Cycle -	It is the sequence of a periodic quantity during one period
Damping -	The process related to dissipation of energy
Degrees of freedom -	The number of independent generalized coordinates required to uniquely define the configuration of a system
Equilibrium -	A state of a body which is at rest or undergoes a uniform motion
Forced vibration -	Vibration of a system caused by a sustained excitation
Free vibration -	Vibration over an interval of time during which the system is free from excitation
Lagrange's equations -	A tool for deriving equation of motion of a mechanical system based on K.E, P.E. and generalized coordinates
Period -	The time taken to complete one cycle of the periodic motion
Periodic motion -	An oscillatory motion having the same pattern after a definite time interval
Resonance -	It is a state of vibration occurring if the external frequency coincides with the natural frequencies

10.11 REFERENCES

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2. Classical Mechanics, Gupta, Kumar, Sharma, A Pragati Edition

3. Classical Mechanics, S.N. Biswas, Books and Allied Ltd.

4. Classical Mechanics, Small Oscillations, Dipan Kumar Ghosh, UM-DAE Centre for Excellence in Basic Sciences

5.L.D. Landau and E.M. Lifshitz, Mechanics, Pergamon Press

10.12 SUGGESTED READINGS

1.Classical Dynamics: A Contemporary Approach J.V. Jose and E.J. Saletan, Cambridge University Press,
2. Classical Mechanics, N.C. Rana and P.S. Joag, Tata McGraw-Hill Education Pvt. Ltd.

10.13 TERMINAL QUESTIONS

10.13.1 Short Answer type

1. What do you understand by stable and unstable equilibria?
2. What is the condition of stable equilibrium in terms of potential energy?
3. Explain the following:

(a) Normal coordinates (b) Normal modes (c) Free vibration

10.13.2Long Answer type

1. Consider a system with N generalized coordinates $q_k$describedby a mass $m_k$and a potential V. The kinetic energy of the system is given by T. Obtain the formulation of small oscillations for the defined system. How would you calculate the normal frequency for such system?
2. Consider a triatomic molecule (eg. CO_2). Such molecule have linear structures with two masses “m”at both the ends and a mass “M”.Assume that the masses are connected by springs (as shown in the Figure 6). Calculate the normalfrequencies of such molecule.

3. Two masses m_1 and m_2 resting on a smooth surface are joined together by a spring of negligible mass and spring constant k. Length of the spring at rest is l_0. Find the normal

frequency. Aaume that the motion remains one dimensional. (*Classical Mechanics, Gupta, Kumar, Sharma*).

4. A particle of mass m is moving along the positive x direction under a potential,

$$V = \frac{1}{2}kr^2 + \frac{\beta}{2r^2}$$

If the particle is slightly displaced from its equilibrium position, calculate the potential at stable equilibrium.

10.13.3 Numerical type question

1. A spring of force constant 1 Nm^{-1} is connected between two identical simple pendulums, each of length 5 m. Calculate the period of other pendulum if one is damped. The mass of each bob is 0.3 kg (g= 10m/s^2). (*Problem from Classical Mechanics. H. Goldstein, C. Poole and J.Safko*).
2. A mass of 0.300 kg is placed on a vertical spring and the spring stretches by 1m. It is then pulled down an additional 0.5 m and then released. Find (i) the force constant, (ii) the angular frequency,and (iii) the time period, (g= 10m/s^2).
3. Calculate the force constantof harmonic oscillator if its resting position potential energy is 6 joule and total energy is 10 joule. The amplitude given is 2m. (*Problem from Classical Mechanics. H. Goldstein, C. Poole and J.Safko*).

10.14 ANSWERS

10.14.2Long Answer type

1. **(Hint):** Topics 9.5 and 9.6 of this unit
2. **(Hint):**

$$L = T - V$$

The Lagrangian can be written as:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}M\dot{x}_2^2 \text{ and } V = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_1)^2$$

$$\text{So, } L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}M\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_1)^2$$

Form the matrices of T and V and write the secular equation

$$(V - \omega^2 T) = 0$$

This will provide three values ofω, that are

$\omega_1^2 = 0$,

$\omega_2^2 = \frac{k}{m}$ and $\omega_3^2 = \frac{k}{m} + \frac{2k}{M}$

3. $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$

$V = \frac{1}{2}k(x_1 - x_2)^2$

These can be written as,

$2T = (\dot{x}_1 \dot{x}_2)\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$

$2V = (x_1 x_2)\begin{pmatrix} c & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

The matrices for K.E. and P.E. can be written as:

$[T] = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ and

$[V] = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$

The secular equation is given as $(V - \omega^2 T) = 0$. Thus putting T and V in this equation, we get

$$\begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_1 \end{bmatrix} = 0$$

$\omega_1^2 = 0$ and $\omega_2^2 = k\left(\frac{m_1 + m_2}{m_1 m_2}\right)$

4. $V = \frac{1}{2}kr^2 + \frac{\beta}{2r^2}$

For particle at stable equilibrium,

$$\left|\frac{\partial V}{\partial r}\right|_{r=r_0} = 0$$

Thus

$$\left|\frac{\partial V}{\partial r}\right|_{r=r_0} = kr - \frac{\beta}{r^3} = 0$$

Thus,

$$r_0 = \left(\frac{\beta}{r}\right)^{\frac{1}{4}}$$

Putting in the value of potential equation $[V]_{r=r_0} = \frac{1}{2}kr^2 + \frac{\beta}{2r^2}$

$$[V]_{r=r_0} = \frac{1}{2}k\left[\left(\frac{\beta}{r}\right)^{\frac{1}{4}}\right]^2 + \frac{\beta}{2\left[\left(\frac{\beta}{r}\right)^{\frac{1}{4}}\right]^2}$$

or

$$[V]_{r=r_0} = \frac{1}{2}k\left(\frac{\beta}{r}\right)^{\frac{1}{2}} + \frac{\beta}{2\left(\frac{\beta}{r}\right)^{\frac{1}{2}}}$$

or

$$[V]_{r=r_0} = \frac{1}{2}\frac{1}{\left(\frac{\beta}{r}\right)^{\frac{1}{2}}}\left[k\frac{\beta}{r} + \beta\right]$$

10.14.3 Numerical type question

1. If one of the pendulums is damped, the equation of motion of the other pendulum is given as:

$$m\frac{d^2x}{dt^2} = -kx - \frac{mgx}{l}$$

which can be written as,

$$m\frac{d^2x}{dt^2} + kx + \frac{mgx}{l} = 0$$

or

$\frac{d^2x}{dt^2} + \left[\frac{k}{m} + \frac{g}{l}\right]x = 0$, it represents the simple harmonic motion equation of motion.

and the time period is given as:

$$T = \frac{2\pi}{\left[\frac{k}{m} + \frac{g}{l}\right]} = \frac{2\pi}{\left[\frac{1}{0.3} + \frac{10}{3}\right]}$$

On solving,

$$T = 1.17 \text{ sec}$$

2. The spring constant can be given as:

(i) $k = \frac{F}{x} = \frac{mg}{x} = \frac{0.3 \times 10}{1}$

thus,

$k - 3$ N/m

(ii) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3}{0.3}}$

or $\omega = 3.16 \text{ rad/sec}$

(iii) $T = \frac{2\pi}{\omega} = \frac{2 \times 3.14}{3.16} = 1.98$

thus,

$T - 1.98 \text{ sec}$

3. Potential energy (P.E.) at rest = 6 J

Potential energy at maximum displacement = 10 J

Gain in P.E. = 10 J - 6 J = 4J

thus,

$$P.E. = \frac{1}{2}kx^2 = 4J$$

$\frac{1}{2}k(2)^2 = 4J$, amplitude $x = 2 \text{ m}$

thus, $k = 2 \text{ N/m}$